Modeling the Dynamics of Oculomotor System in Three Dimensions

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***Abstract*— Gaze control in three dimensions is achieved by the geometrical arrangement of the extra-ocular muscles. For head-fixed saccadic eye movements, the ocular rotation axis is constrained according to Listing's law. In this paper, an ocular model is developed to control gaze. Quaternions associated with eye positions and rotations are used to precisely describe the axis of rotation.**

Keywords: Eye movements, Listing's law, Rotations, Quaternions

I. Introduction

The optimal visual resolution in retina is achieved for targets projected on the fovea (central part of the retina). Since the fovea has a small diameter (less than a degree), whenever the visual system starts exploring the visual target, the gaze line must be precisely aligned with the object. Such a movement in gaze is controlled by the saccadic system. Although in general, goal-directed gaze shifts may be achieved by many different combinations of eye-, head-, and body rotations, the study here is focused on the head fixed saccades.

If the eye is moved from one fixation to another, in theory, there are unlimited ways to orient the axis about which the eye rotates in 3-D space. But in reality, eye is constrained in its torsional freedom. This was first stated by Donders (1847), i.e., for steady fixation with the head upright, the actual positions of the eye are restricted in such a way that there is only one eye position for every gaze direction. This restricts the three-dimensional space of all possible orientations to a two-dimensional subspace. Listing and Helmholtz further investigated and determined to which two-dimensional subspace the eye is restricted. Listing's law, a specific case of more general Donders' law, states that any physiologic eye orientation can be reached from a particular eye position known as the primary position, by rotation around a single axis, and that all such possible axes lie in a single plane known as Listing's plane.

Unless the trajectory follows a radial line passing through the primary position, the rotation axis used to move the eye from one position to another, obeying Listing's law, tilts out of Listing's plane. Experiments done on normal human subjects and rhesus monkeys (see [11] and [3]) confirm this notion. Listing's and Donder's laws and the rotation axis corresponding to an eye movement can be precisely described using quaternions and rotation vectors.

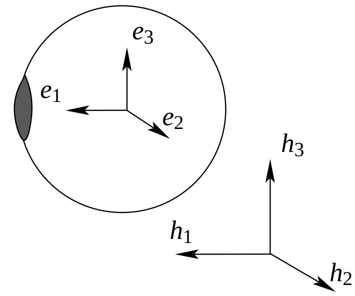

Fig. 1. Head-fixed $\{h_1,h_2,h_3\}$ and eye-fixed $\{e_1,e_2,e_3\}$ right-handed coordinate systems.

II. Rotation Matrices, Quaternions and Rotation Vectors

In order to represent eye positions, a reference position (see [2]) is defined first. This reference position is usually defined as the position the eye assumes when the subject is looking straight ahead keeping the head upright. The current eye position is defined as a 3D rotation from the reference position to the current eye position. According to Euler's theorem, each eye position can be achieved by a single rotation from the reference position (Euler, 1775).

A. Rotation matrices

In order to define 3-D movements, we first establish a head-fixed $\{h_1,h_2,h_3\}$ and an eye-fixed $\{e_1,e_2,e_3\}$ right-handed coordinate systems (see Figure 1). When the eye is in the reference position, h_1 coincides with the line of sight, h_2 with the interaural axis and h_3 with the vertical axis. The eye-fixed coordinate system coincides with the head-fixed coordinate system when the eye is in the reference position. A 3D rotation of the eye-fixed coordinate system from the reference position to any new position can be described by

$$e_i = \mathbf{R}h_i \tag{1}$$

where $\mathbf{R}$ describes the rotation about a space-fixed axis and belongs to the *special orthogonal* group $SO(3)$. To illustrate, the matrix

$$\mathbf{R} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

generates a rotation around the h_1 axis through an angle of θ.

B. Quaternions and Rotation Vectors

An efficient way of characterizing a rotation of the eye is to use a vector, with the direction of the vector given by the axis of the rotation, its length proportional to the angular measure of the rotation, and the orientation is given by the *right-hand rule*. Two such descriptions are used in the oculomotor literature: quaternions and rotation vectors.

The set of quaternions, along with addition and multiplication operations, form a *ring* or rather a *non-commutative division ring* emphasizing the fact that the quaternion product, in general, is non commutative, and also that the multiplicative inverse exists for every non-zero element from the set (see [6]). Let the space of quaternions be $\mathbb{Q}$. Each $a \in \mathbb{Q}$ can be written as

$$a_0\vec{\mathbf{1}} + a_1\vec{\mathbf{i}} + a_2\vec{\mathbf{j}} + a_3\vec{\mathbf{k}}$$

. $\mathbf{i} = (0,1,0,0), \mathbf{j} = (0,0,1,0), \mathbf{k} = (0,0,0,1)$ satisfy $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1 = \mathbf{ijk}$. $a_1\vec{\mathbf{i}} + a_2\vec{\mathbf{j}} + a_3\vec{\mathbf{k}}$ is called the vector part and $a_0\vec{\mathbf{1}}$ its scalar part. The vector $a_1\vec{\mathbf{i}} + a_2\vec{\mathbf{j}} + a_3\vec{\mathbf{k}}$ will be identified with $(a_1, a_2, a_3) \in \mathbf{R}^3$ and the $\vec{\mathbf{1}}$ can be dropped from the scalar part, i.e. simply write it as a_0. Thus we have maps,

$$\mathbf{vec} : \mathbb{Q} \longrightarrow \mathbb{R}^3, \;\; a \longmapsto (a_1, a_2, a_3),$$

and

$$\mathbf{scal} : \mathbb{Q} \longrightarrow \mathbb{R}, \;\; a \longmapsto a_0.$$

Space $\mathcal{Q}$ of unit quaternions will be identified with the unit sphere in $\mathbb{R}^4$, i.e., S^3. Each $q = (q_0, q_1, q_2, q_3) \in S^3$ can be associated with a rotation by an angle $\theta \in [0, \pi]$ about an axis $\mathbf{u}$ in the following form (see [7]),

$$\begin{aligned} q_0 &= \cos(\theta/2) \\ |\mathbf{q}| &= \sqrt{q_1^2 + q_2^2 + q_3^2} = \sin(\theta/2) \\ \mathbf{u} &= \frac{\mathbf{q}}{|\mathbf{q}|}. \end{aligned}$$

Then,

$$q = \cos(\theta/2) + \sin(\theta/2)\mathbf{u}.$$

Quaternion rotation operator $L_q(\mathbf{v})$ corresponding to a rotation of a vector $\mathbf{v}$ about an axis $\mathbf{u}$ by an angle θ to obtain a vector $\mathbf{w}$ can be described as

$$w = L_q(\mathbf{v}) = qvq^{-1}$$

where $w = 0 + \mathbf{w}$ and $v = 0 + \mathbf{v}$ are pure quaternions and the product is defined as the quaternion product, i.e., for two quaternions $p = p_0 + \mathbf{p}$ and $q = q_0 + \mathbf{q}$

$$pq = p_0q_0 - \mathbf{p}\cdot\mathbf{q} + p_0\mathbf{q} + q_0\mathbf{p} + \mathbf{p}\times\mathbf{q}.$$

The order of the multiplication is important and the product qp would lead to a different rotation.

The rotation vector $\mathbf{r}$ which corresponds to the quaternion q can be described as follows.
There exists a map

$$\begin{gathered} \chi : \mathcal{Q} \longrightarrow \mathbb{R}^3 \\ (a,b,c,d) \longmapsto \tan(\theta/2)\mathbf{n} \end{gathered}$$

or

$$(a,b,c,d) \longmapsto (b/a, c/a, d/a); a \neq 0.$$

The inverse map also exists

$$\begin{gathered} \chi^{-1} : \mathbb{R}^3 \longrightarrow \mathcal{Q} \\ (r_1, r_2, r_3) \longmapsto \frac{(1, r_1, r_2, r_3)}{\sqrt{1 + r_1^2 + r_2^2 + r_3^2}}. \end{gathered}$$

The rotation vector corresponding to the rotation matrix $\mathbf{R}$ can be obtained from the elements of the rotation matrix as

$$\mathbf{r} = \frac{1}{1 + (r_{11} + r_{22} + r_{33})}\begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix}.$$

III. Donder's law and Listing's law

Eye positions can be described by rotation vectors (or quaternions), the coordinates of which are in the head-fixed reference frame. This means that any 3D eye position is obtained by the rotation vector for the rotation from the reference position to the current position (reference position is chosen as the straight ahead gaze direction when head is fixed).

Donder's law states that the amount of torsion of the eye when looking at a target keeping head stationary, is uniquely determined by the gaze direction (Donders, 1848), i.e.for $\mathbf{r} = (r_1, r_2, r_3)$,

$$r_1 = \text{function}(r_2, r_3).$$

Listing's law further states the amount of this torsion. This simply means that all rotation vectors corresponding to 3D eye position lie on a plane (see [11]), i.e.,

$$r_1 = 0.$$

This amounts to a statement that all eye positions have quaternion representations $q \in S^3$ with $q_1 = 0$. Thus the Listing's plane coincides with $h_2 - h_3$ plane of the head-fixed coordinate system.

IV. 3D model of the eye

The block diagram of the model proposed here is shown in Figure 2. Inputs to the quaternion calculator $\mathbf{Q}$ are the initial and final gaze directions and the axis of rotation is determined according to the Listing's law as discussed earlier. This information is used in generating the activations $\mathbf{a}$ by the motor-neurons. The proposed approach here is to use a lookup table or an artificial neural network based training algorithm.

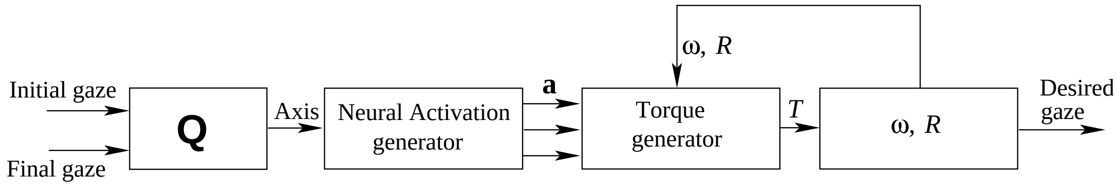

Fig. 2. Block diagram of the ocular model

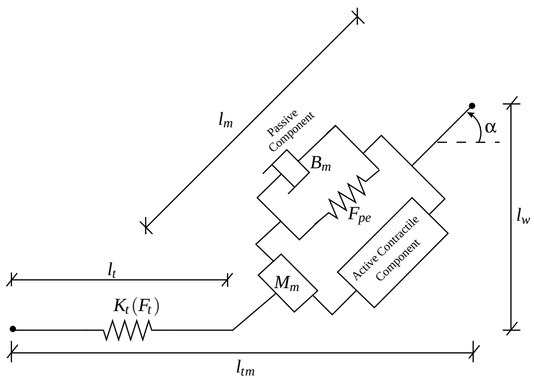

Fig. 3. Hill-type model of the musculotendon complex.

The torque $\mathbf{T}$ is generated from the active forces in musculotendons. Musculotendons are modeled according to Hill's approach (see [4],[12] and Figure 3). The muscle of length l_m is in series and off-axis by a pennation angle α (we assume $\alpha = 0$ here) with the tendon of length l_t. The total length of the musculotendon complex is l_{tm}. The muscle has two main components: an active force generator and a parallel passive component. The passive component consists of a parallel elastic element (F_{pe}) which describes the passive muscle elasticity and a damping component which corresponds to the passive muscle viscosity (B_m). The active component generates the active force for the muscle, which is the product of length-tension relation $f_l(l_m)$, velocity-tension relation $f_v(\dot{l}_m)$, and the activation level $a(t)$ [12].

Once the mass of the muscle is ignored, the force balance for the musculotendon complex becomes

$$F_t = F_{act} + F_{pe} + B_m \dot{l}_m \tag{2}$$

where F_t, F_{act}, F_{pe} are the tendon force, the active, and the passive forces in the muscle whereas l_m is the length of the muscle.

Assuming the musculotendon leaves the eye globe tangentially through the extraocular pulleys (see [1] and [9]) and attaches to the annulus of zin at the back of the eye ball, the torque from each muscle can be written as

$$\rho \times F_t.$$

where ρ is the radial vector to the point on the eye globe where the musculotendon leaves the eye tangentially.

In order to obtain the model of the eye, let $\mathbf{R}$ be the rotation matrix as given in equation (1). Let x_e and x_h be the coordinates of an arbitrary vector with respect to eye-fixed and head-fixed coordinate systems. These two sets of coordinates are related by the linear transformation

$$x_h = \mathbf{R} x_e. \tag{3}$$

Moreover, denoting $\omega_h(t) = \mathrm{col}(\omega_{h1}, \omega_{h2}, \omega_{h3})$ (respectively $\omega_e(t) = \mathrm{col}(\omega_{e1}, \omega_{e2}, \omega_{e3})$) be the angular velocity in the head-fixed frame (respectively in the eye-fixed frame) and $\mathbf{R}(t)$ be the value at time t of the matrix $\mathbf{R}$, one can write the *kinematic equation*(see [5])

$$\dot{\mathbf{R}}(t) = \Omega_e \mathbf{R}(t) \tag{4}$$

where

$$\Omega_e = \begin{bmatrix} 0 & \omega_{e3} & -\omega_{e2} \\ -\omega_{e3} & 0 & \omega_{e1} \\ \omega_{e2} & -\omega_{e1} & 0 \end{bmatrix}$$

The eye is subjected to external torques due to the action of extraocular muscles. The momentum balance equation yields

$$J\dot{\omega}_e + B\omega_e + K\int_0^t \omega_e dt = \mathbf{T}_e \tag{5}$$

where J, B, and K, are the moment of inertia of the eye ball, viscous and elastic coefficients corresponding to the resistive forces from surrounding tissues respectively and Γ_e is the resultant of the external torques in the eye-fixed frame. This can be written in the head-fixed coordinate frame, using the equation (3), or more precisely, using

$$\omega_e = \mathbf{R}^T \omega_h.$$

Then equation (5) becomes

$$J\dot{\omega}_h = J\Omega_e\omega_h + \mathbf{T}_h - B\omega_h - \mathbf{R}K\int_0^t \omega_e dt \tag{6}$$

which is commonly known as the *dynamic equation*.

The equations (4) and (6) describe the control of the attitude of the eye. Though the state space for the system, counting the number of equations, seems in $\mathbb{R}^{12}$, $\mathbf{R}$ belongs to the 3-dimensional embedded submanifold $SO(3)$ of $\mathbb{R}^{3\times 3}$. Thus the natural state space of the system given by equations (4) and (6) is the 6-dimensional smooth manifold (see [5])

$$N = SO(3) \times \mathbb{R}^3.$$

V. RESULTS

Simulation results given here are for the final block of the proposed model shown in Figure 2. The input torque $T = (T_1, T_2, T_3)$ is chosen to be a pulse of duration 20*ms* and an amplitude of 50 units(gramf cm) in T_2 and T_3 components and 0 for T_1. The sudden onset of the saccade was visible with angular velocity components shown in Figure 4. The position vector is shown in Figure 5 with sampled intermediate positions.

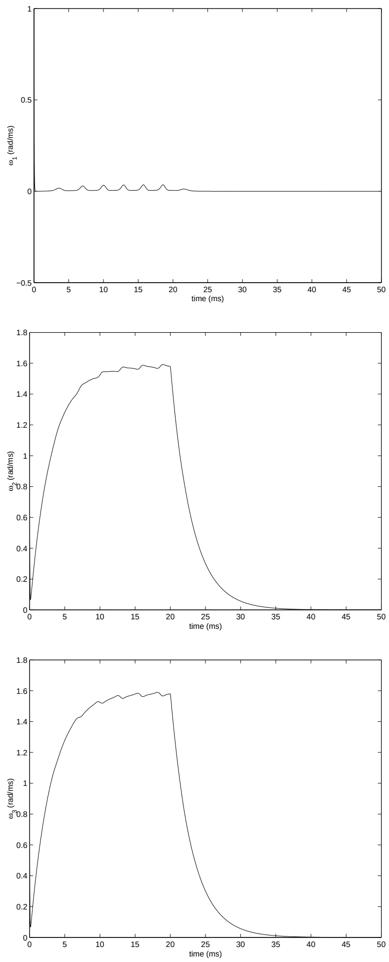

Fig. 4. Angular velocity components : $\omega = (\omega_1, \omega_2, \omega_3)$.

VI. CONCLUSION

Three-dimensional analysis of eye movements needs a whole new approach compared to planer analysis (for planer models, see [10] and [8]). This is due to the fact that 3D rotations follow a different set of kinematic properties which could elegantly be analyzed using quaternions and rotation vectors. Quaternions and rotations vectors are merely a computational tool and no assumptions needed regarding whether the oculomotor system uses them to represent eye

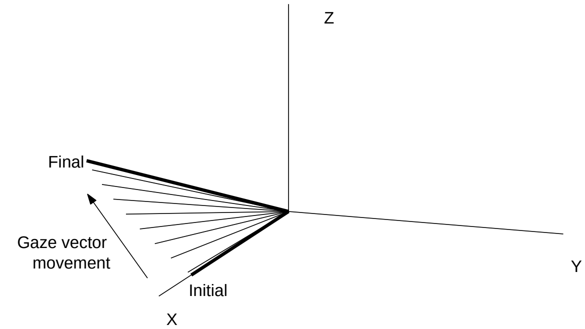

Fig. 5. Movement of the gaze vector.

positions and rotations. Also the Donder's and Listing's laws are better explained using them.

Results show that the dynamic model of the eye with torques as the input, depicts the 3D eye movements. More work is being carried out to explore the generation of the torque. As shown in Figure 2, the ω and $\mathbf{R}$ values need to be fed back to the torque generating module in order to calculate $\dot{l}_m$ in equation 2. Motorneurons which generate the activations to drive the musculotendons and in turn the eye, are fed with the rotation axis information computed by the quaternion block $\mathbf{Q}$ according to the Listing's law discussed earlier. As a summary the quaternion based approach reveals the kinematic structure imposed on the oculomotor system by its computational tasks and gives a clear mathematical basis to model the 3D eye movements.

VII. REFERENCES

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