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Robust Optimization of Mixed CVaR STARR Ratio using Copulas

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Abstract

We introduce the robust optimization models for two variants of stable tail-adjusted return ratio (STARR), one with mixed conditional value-at-risk (MCVaR) and the other with deviation MCVaR (DMCVaR), under joint ambiguity in the distribution modeled using copulas. The two models are shown to be computationally tractable linear programs. We apply a two-step procedure to capture the joint dependence structure among the assets. We first extract the filtered residuals from the return series of each asset using AutoRegressive Moving Average Glosten Jagannathan Runkle Generalized Autoregressive Conditional Heteroscedastic (ARMA-GJR-GARCH) model. Subsequently, we exploit the regular vine copulas to model the joint dependence among the transformed residuals. The tree structure in the regular vines is accomplished using Kendall's tau. We compare the performance of the proposed two robust models with their conventional counterparts when the joint distribution in the latter is described using Gaussian copula only. We also examine the performance of the obtained portfolios against those from the Markowitz model and multivariate GARCH models using the rolling window analysis. We illustrate the superior performance of the proposed robust models than their conventional counterpart models on excess mean returns, Sortino ratio, Rachev ratio, VaR ratio, and Treynor ratio, on three data sets comprising of indices across the globe.

Keywords: Portfolio optimization, robust portfolio optimization, STARR ratio, mixed conditional value-at-risk, ARMA-GJR-GARCH model, regular vine copula

1. Introduction

Since the beginning of the era of modern portfolio theory with the pioneering work of Markowitz [53], where the measure of risk in investment is taken to be the standard

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deviation of return distribution, the literature has seen a proliferation of several reward-risk optimization models. These models have a common aim to find an optimal portfolio of maximum return and minimum risk. The bi-objective models generally are transformed into single objective problems of maximizing return given an upper bound on risk or minimizing risk given a lower bound on the return or maximizing the combined return-risk utility function for a given risk-aversion parameter. The optimal solutions of the resulting problems are sensitive to the choice of the upper bound, lower bound, and risk-averse parameter, respectively.

Another extensively studied approach in the very same context is the reward-risk ratio optimization. Most of the ratio optimization models can be transformed to linear or convex optimization problems under some mild conditions and hence possess a computational advantage. The first such ratio is Sharpe ratio [69] defined as the ratio of the mean excess return from the risk-free rate of return to the standard deviation of the portfolio's returns. However, in the financial markets, the downside risk causes more pain than the upside gain. The big meltdowns in the equity markets are known to leave a significant imprint on wealth. The key to winning in equities investment is knowing how to avoid substantial losses and provide the downside risk protection to the portfolio. With this view, the mean excess return to the downside deviation of the portfolio's return is defined and called Sortino ratio ([71], [72]). The stable tail-adjusted return ratio (STARR ratio) [54] which measures the ex-ante risk-adjusted return by including the downside risk in the form of expected tail loss more popularly known by conditional value-at-risk (CVaR), can be an excellent proxy to limit the downside risk. The Rachev ratio, the generalized Rachev ratio [14], and Omega ratio [48] are some other reward-risk ratios which use downside risk. Among these, the first two use CVaR.

Unlike the Rachev and its generalized case where their optimization models result in the mixed-integer linear programs, the optimization problem of maximizing STARR ratio is a linear program. This computational advantage constitutes one of the reasons for us to focus on the STARR ratio in the present research. Another of our recent work [36] defines two variants of the STARR ratio and study their application to the problem of enhanced indexing.

Portfolio returns typically follow a multivariate distribution whose validity not only depends on the correct estimation of the marginal distribution of each of its constitutes but also on how correctly one captures the joint dependence structure among them. Now it is well recognized that the dependence structure among the financial assets is asymmetric, heavy-tailed, and non-linear. Copula functions are efficient tools to model these stylized facts in the joint dependence structure.

Copula models work sequentially in two steps: marginal distribution modeling, and dependence modeling. The marginal distribution modeling specifies the functional form of the marginal distribution of returns of each financial asset accounting for the detailed stylized features and the dependence modeling determines the best copula function to capture the actual dependence structure among them. However, a single copula function could not encapsulate the dynamical changes in the trends of markets. In this paper, taking motivation from the studies on robust optimization under data ambiguity[1], we propose to extend our recent work [36] on portfolio selection by introducing robustness concerning the change in the copula functions in the two variants of the STARR ratio optimization models.

In [36], we present two variants of STARR ratio with mixed CVaR (MCVaR) and deviation MCVaR (DMCVaR). The MCVaR is the weighted sum of multiple CVaRs at different confidence levels while DMCVaR is the deviation version of MCVaR obtained by replacing the random variable in MCVaR by its dispersion from its expected value. The MCVaR and DMCVaR inherit more information than a simple CVaR while enjoying all the theoretical properties of the latter. We can approximate the optimization of the STARR ratio with MCVaR and DMCVaR by linear programs using finite realizations from the return distribution of portfolio. We propose to study the worst case analysis for both the variants of the STARR ratio optimization using the robust optimization framework concerning copulas. The models involve maximization of the minimum (or worst case) value of the STARR ratios computed over the multiple prior, and the copula functions enable to capture the dependence structure among returns.

We follow a two-step procedure to model the dependence structure among the returns of assets. We first fit the marginal distribution in the returns of each asset by ARMA-GJR-GARCH [2] model. We then use the regular vines on the transformed residuals of the marginals and accomplish the tree structure on them using Kendall's tau.

To carry out the performance analysis of the proposed models, we consider the conventional STARR models in which only the multivariate Gaussian copula is used to model the residuals in the returns. We also compare the performance of the portfolios from the proposed models with that of the Markowitz model and multivariate GARCH models, specifically symmetric Dynamic Conditional Correlation (DCC) GARCH and asymmetric DCC (aDCC) GARCH models using the rolling window scheme. The empirical study is carried out on three data sets comprising of prices of various global indices from different periods.

Contributions

The contribution of the paper lies in studying the worst case analysis for the reward-risk ratio in the framework of robust portfolio optimization concerning the change in copula. We present the robust versions of two STARR ratios models with MCVaR and DMCVaR measures. The empirical analysis on three different datasets shows the superior performance of the proposed worst-case models over their conventional counterparts (when only Gaussian copula is applied), Markowitz model and multivariate GARCH models, on several performance indicators including, excess mean return from the $1/m$ naive portfolio, Sortino ratio, Rachev ratio, VaR ratio, and Treynor ratio. To the best of our knowledge,

[1]The probability model is unknown under data ambiguity [51]

[2]AutoRegressive Moving Average Glosten Jagannathan Runkle Generalized AutoRegressive Conditional Heteroscedasticity

this research is the first attempt to apply the robust optimization concerning copula to the STARR ratio optimization problems.

The remaining organization of the paper is as follows. Section 2 presents the literature on reward-risk ratios, copulas, and robust mean-risk formulations. Section 3 narrates the copula theory along with the regular vines. Section 4 introduces the robust optimization models for two variants of STARR ratio and their equivalent linear programming problems. Section 5 explains the methodology and data used in the empirical analysis. Section 6 presents the performance analysis of the proposed models, followed by the conclusion in Section 7.

2. Literature Review

In this section, we first survey the literature on the reward-risk ratio portfolio optimization models. We then go on to provide the related research on copulas and vine copulas. Finally, we briefly touched upon the existing works on robust optimization with data ambiguity. The documentation on these topics is enormous and beyond the scope of the paper to cite all. We shall be referring to only those which are strictly relevant to our present study.

2.1. Reward-Risk Ratio

Unlike mean-risk models, the maximization of reward-risk ratio does not require prior knowledge of risk aptitude of an investor to produce an optimal ending portfolio. With the progress of risk management tools, several researchers ([30], [31], [73]) develop different ratio optimization models. The famous ratios include Treynor ratio [74], Sharpe ratio [69], Sortino ratio [72], Omega ratio [48], VaR ratio, STARR ratio ([64] and references therein), Rachev ratio [14] and generalized Rachev ratio. Sortino ratio improves the Sharpe ratio by incorporating square root of semi-variance to measure volatility and thus penalizing only the under-achievement of a portfolio from the mean return. On the other hands, the VaR, STARR, Rachev and generalized Rachev ratios focus on the risk of extreme losses. Among these, the STARR ratio is studied widely due to the computational advantage it offers on account of being a quasi-concave function.

Stoyanov et al. [73] discuss a variety of reward-risk ratios in detail. They categorized ratios in the groups of coherent and aggressive coherent ratios according to the coherent conditions fulfilled by the reward and risk functions separately. Further, they discuss the computational complexity of each of the ratio depending upon the concavity and convexity properties satisfy by the reward and risk functions.

Rachev et al. [63] compare momentum strategies based on various reward-risk ratios with cumulative return approach. They report that although the cumulative return based strategy produces a maximum return, it does not pay attention to risk and thus fails to provide better risk-adjusted returns compare to the reward-risk based strategy. They also specify that ratios using alternative risk measures perform better than the Sharpe ratio due to the non-normality in the assets return (see, [30] also).

Cvitanić et al. [22] extend the ratio optimization model from a single period to a multi-period investment horizon. They consider a continuous-time complete-market settings and employ martingale methods to solve variance-minimizing policy subject to a constraint that the expected terminal wealth equals some given level, sitting on the initial date.

Among the recent applications of ratio optimization, include Omega ratio optimization by Sharma et al. [68] and Guastaroba et al. [38]. Sharma et al. [68] replace the fixed threshold with a distribution based limit in Omega ratio and study its worst-case analysis under mixed, box, and ellipsoidal uncertainty sets. Guastaroba et al. [38] apply the Omega ratio to the problem of enhanced indexing.

The rationality of portfolio return largely depends on how one captures the correct dependence structure among its constituent assets. The multivariate Gaussian distribution is commonly used to accomplish this task. The two limitations in using it are its symmetric nature and utilizing linear correlation to measure linear dependence. In fact, returns from assets have historically shown a propensity not to follow the Gaussian distribution. Studies ([6], [21], [40]) indicate that asset returns exhibit stronger co-movements in the down markets than the up markets and hence the dependence among them is asymmetric, nonlinear,[3] negatively skewed and fatter tails. The multivariate Gaussian distribution fails to capture these stylized features. The Gaussian distribution also fails to describe the marginal distributions because of asymmetric and leptokurtic returns, and time-varying volatility in them.

Copulas are multivariate distribution functions having one-dimensional marginal distributions uniformly distributed on the interval $[0, 1]$. Copulas capture the dependence structure in the marginal distributions of the random variables instead of working directly with the random variables themselves. The flexibility of copulas and their easy calibration make them widely applicable to problems of finance. The Basel II accord and Solvency II directives, the two respective regulatory frameworks for banks and insurance, emphasize utilizing copulas for aggregating various forms of risks. The Basel II accord encourages banks to maintain a minimum capital level calculated from their market risk, credit risk, and operational risk, taken together ([1]). Banks and financial institutions usually establish a high-dimensional joint distribution to model their risks computed by percentile of loss distributions at some confidence level. Since copulas can work directly on percentile measures of the loss distributions, they are more suitable for aggregating the financial risks.

The mathematical and statistical perspectives of copulas can be traced in the studies of Nelsen [57] and Joe [42]. Embrechts et al. [26] apply copulas to capture the real dependence instead of merely using correlation which is applicable only for elliptical distributions. Cherubini et al. [20] apply copulas to study credit risk analysis and pricing of exotic derivatives.

Jondeau and Rockinger [44] present copula-GARCH models[4] to investigate dependence

[3]Linear correlation cannot explain concepts like co-monotonicity or rank correlation.

[4]GARCH models are used to address the excess kurtosis and conditional heteroscedasticity ([27], [15]) in marginal distribution modeling.

among global financial markets. Hu [40] considers a mixed copula model to obtain various possible patterns of dependence structure across the financial markets. Munnix and Schafer [56] use the copula approach to figure out the tail-dependency among the constituents of S&P 500.

Patton [61] analyzes the time-varying copulas where the dependence parameter is modeled by an AutoRegressive Moving Average (ARMA) process to include the dynamic dependence structure among the financial assets. With a similar motive of analyzing the time-varying dependent structure, several authors ([21], [62], [66]) propose regime switching models.

2.2. Vine Copulas

Albeit a rich collection of copula families available for bivariate copulas, the choice of an appropriate copula is rather limited in high dimension. The Student's t and Gaussian are two widely used multivariate copulas which fail to capture asymmetry and fat tail dependency on the financial data.

Joe [41] proposes a flexible graphical model (tree-like structures) to obtain multivariate copula using a cascade of bivariate copulas. The so obtained copulas are called vine copulas, and the process of getting them is called a pair copula construction (PCC). Bedford and Cooke ([10], [11]) and Kurowicka and Cooke [50] further develop the vine copulas.

Czado [23] and Kurowicka and Joe [43] provide good reviews on vine copulas, including empirical applications. The vine copulas are known to perform better than the alternative traditional multivariate models to capture the dependence structure between the random variables specially in the financial market (see, [2], [3], [33]). Fink et. al.[32] use R-vine Markov-switching model with different, pre-defined R-vine structures, to find periods of "normal" and "abnormal" regimes within a data set consisting of North-American, European and Asian markets indices.

Arreola et al. [7] apply the pair vine copula models to examine the dependence risk characteristics of three portfolios, each having 20-stocks, from the retail, manufacturing and gold-mining equity sectors of the Australian market. The R-vine model is found best to capture the multivariate dependence structure of stocks in the retail and gold-mining portfolios. De Backer et al. [24] and Kraus and Czado [49] use vine copula models for quantile regression.

Furthermore, vine copulas have been successfully used to study the contagion effect in the financial market. BenSaïda [13] develops a tractable Markov regime-switching C-vine and D-vine models to investigate the contagion effect between the Eurozone and the U.S. sovereign debt markets. Recently, Goel and Mehra [35] apply the time-varying vine copula model to study the contagion in financial deep stress periods.

However, no single copula can encapsulate the dynamical trends and changes across different market scenarios. The work of Kakouris and Rustem [45] motivates us to study the worst case analysis using copulas in the framework of robust optimization.

2.3. Robust Optimization

Robust optimization is one of the expedient methods to handle data uncertainty (caused by estimation errors) and data ambiguity (cause due to unknown underlying probability model) in the model. Here, we do not review the literature on data uncertainty but instead focus on the research writings related to data ambiguity as the present study falls in this domain.

Ben-Tal and Nemirovski [12] develop a robust convex optimization problem when the data ambiguity is described by an ellipsoid and show that such problems admit a computational tractable robust counterpart.

Goldfarb and Iyengar [37], Halldorsson and Tutuncu [39], and Lu [52] propose the robust counterpart to mean-variance portfolio optimization. Goldfarb and Iyengar [37] study three types of uncertainty in mean and the covariance matrix of asset returns. The resulting problems are semi-definite programs or second-order cone programs.

Ghaoui et al. [34] analyze the worst case of value-at-risk (VaR) when only the partial information of the underlying distribution is available. They consider two instances of uncertainty sets namely polytopic uncertainty when the moment pair (mean-variance) is only known to belong to a given polytope and component-wise bounds on the moment pairs.

Zhu and Fukushima [76] study the worst-case CVaR under mixture, box, and ellipsoidal uncertainty sets. The associated problems are linear programs for mixture and box uncertainty sets, and second-order cone programs for the ellipsoidal case. Fabozzi et al. ([29]) review robust portfolio optimization using mean, VaR, and CVaR risk measures. Kapos et al. [46] and Sharma et al. [68] study the worst case of Omega ratio under mixture, box, and ellipsoidal uncertainty sets.

Kakouris and Rustem [45] perform worst-case analysis of CVaR under mixture copula distribution. The resulting problem is a linear program. They use Gaussian copula along with three Archimedean copulas in the mixture setting to cover a broad spectrum of possible dependencies.

Kara et al.[47] analyze return-risk trade-off by introducing robust CVaR under parallelepiped uncertainty. Their methodology provides a more stable portfolio allocation and reduces the portfolio risk. Ozmen et al.[60] develop theory and methods for a robust multivariate adaptive regression spline for modeling uncertain data regarding polyhedral uncertainty.

Øksendal and Sulem [58] study a robust optimal portfolio selection problem in a non-Markovian setting governed by a backward stochastic differential equation with jumps. In another study, Øksendal and Sulem [59] use stochastic control theory and obtain new results connecting the primal and its convex dual, both in non-robust and robust setting.

Baltas et al. [9] focus on robust control problems of parabolic stochastic partial differential equations under model uncertainty.

3. Multivariate Modeling

Understanding and quantifying dependence among assets are at a core of all modeling efforts in financial econometrics; the root task always is to model the joint distribution of returns of assets. Copulas are potent tools to model dependence among several random variables. Modeling dependence via copulas can be viewed as a two-step procedure. The first step is to identify the marginal model of the assets returns and second step is to find the joint distribution via copulas. First, we present the time series model for marginal distribution and then discuss the vine copula model in brief.

3.1. ARMA-GJR-GARCH model for marginal distribution

Mohammadi and Su [55] observe heavy tails and autocorrelation in asset returns. The volatility also tends to increase more following a substantial price drop than following a price rise of the same magnitude, known as the leverage effect. Advanced GARCH models such as exponential-GARCH (EGARCH), GJR-GARCH, TGARCH, FIGARCH, IGARCH [75], to name a few, are proposed over the years to capture the stylized features in asset returns.

In this paper, we use the ARMA(p, q)-GJR-GARCH(1,1)-skewed-t-model to estimate the marginal distributions of return series for all assets. In words of Brooks [16] "a GARCH (1,1) model is sufficient to capture the volatility clustering in data, and rarely any higher order model is estimated or even entertained in the academic finance literature"; we stick to take order 1 in the GARCH term in our proposed strategy. The following describes the ARMA(p, q)-GJR-GARCH(1,1) model:

$$r_t = \mu + \sum_{i=1}^{p} \phi_i r_{t-i} + \sum_{j=1}^{q} \theta_j \epsilon_{t-j} + \epsilon_t \,, \tag{1}$$

$$\epsilon_t = \sigma_t z_t \,, \tag{2}$$

$$\sigma_t^2 = \omega + a\sigma_{t-1}^2 + b\epsilon_{t-1}^2 + dI_{t-1}\epsilon_{t-1}^2 \,, \tag{3}$$

where $\omega > 0,\ a,\ b \geq 0,\ a + d \geq 0$, and the indicator function I_{t-1} is given by:

$$I_{t-1} = \begin{cases} 1, & \text{if} \quad \epsilon_{t-1} < 0 \\ 0, & \text{if} \quad \epsilon_{t-1} \geq 0. \end{cases}$$

The ARMA(p, q) in equation (1) indicates that the current movement of return r_t can be explained by a constant term μ, p lags of its own past movement $r_{t-1}, \ldots, r_{t-p}$, and q lags of the residual terms $\epsilon_{t-1}, \ldots, \epsilon_{t-q}$. In equation (2), the residual is defined as the product of the conditional volatility σ_t and the standardized innovation z_t. We assume that z_t follows a skewed-t distribution to capture both skewness and high kurtosis. Equation (3) defines the evolution of the conditional volatility where I_{t-1} captures the leverage effect. The parameters a and b represent magnitude (or symmetric) and conditional variance, respectively. A higher value of b indicates that the conditional variance will take a long time to die out under a market stress and vice versa ([5]). The parameter d reflects the leverage effect.

3.2. Copulas

An n dimensional copula function C is given by:

$$C(u_1, \dots, u_n) = P(U_1 \leq u_1, \dots, U_n \leq u_n), \tag{4}$$

where $U_1, \dots, U_n$ are $U[0,1]$ distributed random variables and $u_i \in I = [0,1],\ i = 1, \dots, n$.

We recall the famous Sklar's theorem which provides a relationship between distribution functions and copulas.

Theorem 3.1. Sklar's Thorem*[70]*

Let $F_1, \dots, F_n$ be the marginal distributions of an n-dimensional distribution function F. Then there exists an n-dimensional copula C such that

$$F(x_1, \dots, x_n) = C(F_1(x_1), \dots, F_n(x_n)), \quad \forall\, x = (x_1, \dots, x_n) \in \mathbb{R}^n. \tag{5}$$

If $F_1, \dots, F_n$ are all continuous then C is unique; otherwise, C is uniquely determined on $RangeF_1 \times \dots \times RangeF_n$. Conversely, if C is an n-dimensional copula and $F_1, \dots, F_n$ are 1-dimensional distribution functions, then the function F defined in (5) is an n-dimensional distribution function with marginal distributions $F_1, \dots, F_n$. More precisely, we have,

$$C(u_1, \dots, u_n) = F(F_1^{-1}(u_1), \dots, F_n^{-1}(u_n)), \tag{6}$$

where $F_i^{-1}(\cdot)$ denotes the quasi inverse of F_i, $i = 1, \dots, n$.

3.3. Vine Copulas

Let f be the multivariate probability density function (pdf) with distribution function F. Also, consider $f_1, \dots, f_n$ be the marginal pdfs of f with their respective distribution functions $F_1, \dots, F_n$. The copula density $c(u_1, \dots, u_n)$, for $u_i = F_i(x_i),\ i = 1, \dots, n$, corresponding to the copula $C(u_1, \dots, u_n)$, is given by

$$c(u_1, \dots, u_n) = \frac{\partial^n C(u_1, \dots, u_n)}{\partial u_1 \dots \partial u_n} = \frac{f(x_1, \dots, x_n)}{\prod_{i=1}^{n} f_i(x_i)}. \tag{7}$$

Therefore, we have,

$$f(x_1, \dots, x_n) = c(F_1(x_1), \dots, F_n(x_n)) \prod_{i=1}^{n} f_i(x_i). \tag{8}$$

Also, we know that the joint density function in terms of the conditional probability density function $f(\cdot \mid \cdot)$ is given by

$$f(x_1, \dots, x_n) = \prod_{i=2}^{n} f(x_i \mid x_1, \dots, x_{i-1}) \times f_1(x_1). \tag{9}$$

Using (8) and (9), for $n = 2$, we can represent $f(x_i \mid x_j)$ as follows:

$$f(x_i \mid x_j) = c_{ij}(F_i(x_i), F_j(x_j)) \times f_i(x_i), \tag{10}$$

where $c_{ij}(\cdot, \cdot)$ denotes a bivariate copula density.

We use the notation $c_{i,j|i_1,i_2,\ldots,i_m}$ to denote $c_{i,j}(F(x_i \mid x_{i_1}, \ldots, x_{i_m}), F(x_j \mid x_{i_1}, \ldots, x_{i_m}))$ for distinct indices i, j, i_1, $i_2, \ldots, i_m$, satisfying $i < j$ and $i_1 < i_2 < \ldots < i_m$, where $F(\cdot \mid \cdot)$ is the conditional distribution function of $f(\cdot \mid \cdot)$.

Applying (10) recursively, we obtain the following expression for $f(x_i \mid x_1, x_2, \ldots, x_{i-1})$:

$$\begin{aligned} f(x_i \mid x_1, x_2, \ldots, x_{i-1}) &= c_{1,i|2,3,\ldots,i-1} \times f(x_i \mid x_2, \ldots, x_{i-1}) \\ &= \prod_{j=1}^{i-2} c_{j,i|j+1,\ldots,i-1} \times c_{i-1,i} \times f_i(x_i). \end{aligned} \tag{11}$$

Using (11) in (9), we finally get the following PCC:

$$f(x_1, \ldots, x_n) = \prod_{j=1}^{n-1} \prod_{i=1}^{n-j} c_{i,i+j|i+1,\ldots,i+j-1} \times \prod_{s=1}^{n} f_s(x_s). \tag{12}$$

Since the decomposition of joint density in (9) is not unique, there exist many such iterative PCCs in (12).

Bedford and Cooke ([10], [11]) introduce Regular vines (R-vines) to organize the possible decomposition of the joint density function f. The definitions and results stated in the remaining section are taken from [10], unless otherwise cited.

R-vine: An n-dimensional R-vine $V = \{T_1, \ldots, T_{n-1}\}$ is a sequence of $n-1$ linked trees with nodes N_i and edges E_i, $i = 1, \ldots, n-1$, such that

1. tree T_1 has nodes $N_1 = \{1, \ldots, n\}$ and set of edges E_1;
2. for $i \geq 2$, tree T_i has nodes $N_i = E_{i-1}$;
3. for $i = 2, \ldots, n-1$ and edge $e = (a, b) \in E_i$, it must hold that $|a \cap b| = 1$ (proximity condition).

The proximity condition ensures that the two nodes are connected in tree T_i if they have been edges incident to a common node in tree T_{i-1} (that is, nodes in the succeeding tree are edges in the preceding tree).

R-vine copula: (F, V, B) is an R-vine copula specification if $F = (F_1, \ldots, F_n)$ is a vector of continuous invertible marginal distributions, V is an n-dimensional R-vine, and $B = \{B_e : e \in E_i, i = 1, \ldots, n-1\}$ is the set of bivariate copulas on edges of trees $T_1, \ldots, T_{n-1}$.

The unique density function for an R-vine copula specification is given by:

$$f(x_1,\ldots,x_n)=\prod_{i=1}^{n-1}\prod_{e\in E_i} c_{C_{e,a},C_{e,b}|D_e}(F_{C_{e,a}|D_e}(x_{C_{e,a}}\mid x_{D_e}),F_{C_{e,b}|D_e}(x_{C_{e,b}}\mid x_{D_e}))\times\prod_{s=1}^{n} f_s(x_s), \quad (13)$$

where D_e is the conditioning set of edge $e=(a,b)$ and $C_{e,a},C_{e,b}$ are the conditioned sets of e.

4. Models Formulation

In this section, we present our linear programming formulations for the robust case concerning mixture copulas for the two STARR ratio models. We first present our notations.

4.1. Notations

Consider a portfolio P of n-assets $P=(w_1,\ldots,w_n)'$, where w_i is the decision variable denoting the proportion of total budget to be allocated to the i-th asset, $i=1,\ldots,n$, and $'$ denotes the transpose. The set of admissible portfolios be denoted by

$$W=\{w: w^{'}=(w_1,\ldots w_n),\ w_i\geq 0,\ i=1,\ldots,n\}.$$

Let the investment horizon Γ be partitioned into an equal number of time points T to observe the jth outcome (or a particular realization), $j=1,\ldots,T$, of each stock. Let x_{ij} be the j-th outcome of i-th stock with probability p_j and let $x=[x_{ij}]_{n\times T}$. Let x_j denotes the j-th column of x, then $x=(x_1,\ldots,x_T)$. The mean return of i-th stock is $\mu_i=\sum_{j=1}^{T}p_jx_{ij}$, $i=1,\ldots,n$. The j-th realization of return for the portfolio P can be obtained as $X_j=\sum_{i=1}^{n}w_ix_{ij}=w'x_j$, with probability p_j, $j=1,\ldots,T$.

Throughout the paper, the benchmark index is a random variable denoted by I, and its realizations are I_j, $j=1,\ldots,T$. We shall also be using the notation $\zeta^+=\max\{\zeta,0\}$.

4.2. STARR Ratio Model with Mixed CVaR and Mixed Deviation CVaR

Motivated by the twin facts that MCVaR and DMCVaR enjoy several desirable features of an ideal risk measure ([36]), and the appropriateness of capturing the downside risks, we propose to study the robust optimization models of STARR ratio with MCVaR and DMCvaR.

For a positive $CVaR_\delta$ measure[5], the STARR ratio at δ-confidence level, $\delta\in(0,1)$, is defined as follows:

$$STARR_\delta(X)=\begin{cases}\dfrac{E(X)-E(I)}{CVaR_\delta(I-X)}, & \text{if} \quad E(X)>E(I)\\ 0, & \text{if} \quad E(X)\leq E(I),\end{cases}$$

[5]Theoretically, CVaR of a loss variable -Y can assume any sign, however, in practice, it is a positive unless the portfolio never yields a negative return.

where,

$$CVaR_\delta(-Y) = (1/(1-\delta)) \int_{-Y \geq VaR_\delta} -Y dF_{-Y}(r),$$

$$VaR_\delta(-Y) = \min\{r : F_{-Y}(r) = Pr(-Y \leq r) \geq \delta,\ r \in \mathbb{R}\},$$

and $F_{-Y}(r)$ is distribution of the random loss variable $-Y$.

For a positive coherent risk measure $CVaR_\delta(-Y)$, the STARR ratio is a coherent ratio in the sense that it has a positive coherent reward measure $E(X) - E(I)$ ([64]).

In [36], we propose STARR ratio with MCVaR and STARR ratio with DMCVaR (in notation, MCVaR$^\triangle$). The MCVaR is the weighted sum of CVaR values at different confidence levels while MCVaR$^\triangle$, the deviation version of MCVaR, is obtained by replacing the random variable in MCVaR by its dispersion from the mean value.

Let α be the grid of m distinct confidence levels $\alpha_k,\ k = 1, \ldots, m$, with $0 \leq \alpha_m < \ldots < \alpha_1 < 1$. The MCVaR$_\alpha$ of the random variable $-Y$ is defined as follows:

$$MCVaR_\alpha(-Y) = \lambda_1 CVaR_{\alpha_1}(-Y) + \ldots + \lambda_m CVaR_{\alpha_m}(-Y),$$

where $\lambda_k > 0,\ k = 1, \ldots, m$, and $\sum_{k=1}^{m} \lambda_k = 1$.

The CVaR is the most studied coherent risk measure from the class of distortion risk measures [8]. Distortion risk measure is associated with a non-decreasing function $g : [0, 1] \to [0, 1]$, called the distortion function, such that $g(0) = 0$ and $g(1) = 1$. The distortion function for the CVaR measure is continuous and concave (not strictly concave), but not differentiable. Since the distortion function is concave, CVaR is also a spectral risk measure [18]. Spectral risk measure [4] is described by a risk spectrum which assigns higher weights to smaller quantiles. Since any convex combination of CVaRs generates a spectral risk measure [18], MCVaR and MCVaR$^\triangle$ are also spectral risk measures.

For $E(X) > E(I)$, the STARR ratios with MCVaR$_\alpha$ and MCVaR$_\alpha^\triangle$ are defined as follows:

$$MSTARR_\alpha(X) = \frac{E(X) - E(I)}{MCVaR_\alpha(I - X)}\,, \tag{14}$$

$$MSTARR_\alpha^\triangle(X) = \frac{E(X) - E(I)}{MCVaR_\alpha^\triangle(I - X)} = \frac{E(X) - E(I)}{MCVaR_\alpha(I - X - E(I - X))}\,. \tag{15}$$

Note that unlike $MCVaR_\alpha(I-X)$ in the denominator of $MSTARR_\alpha(X)$, the $MCVaR_\alpha^\triangle(I-X)$ in (15) is always positive.

Following Rockafellar and Uryasev [65], the $\mathrm{CVaR}_{\alpha_k}(I-X)$ can be approximated by the following expression:

$$\beta_k + \frac{1}{1-\alpha_k}\sum_{j=1}^{T} p_j\left(I_j - \sum_{i=1}^{n} x_{ij}w_i - \beta_k\right)^+,$$

for finite T realizations of $I-X$.

The optimization model for the STARR ratio with $MCVaR_\alpha(I-X)$ is the following program:

$$\begin{aligned}
(P1) \qquad & \max_{\tilde\beta,\tilde w} \sum_{i=1}^{n} \mu_i \tilde w_i - \gamma E(I) \\
& \text{subject to} \\
& \sum_{k=1}^{m} \lambda_k CVaR_{\alpha_k}(\gamma I - \tilde X) = 1, \\
& \sum_{i=1}^{n} \tilde w_i = \gamma, \\
& \tilde w_i \geq 0, \quad i = 1, \ldots, n,
\end{aligned}$$

where $\gamma = 1/\sum_{k=1}^{m} \lambda_k CVaR_{\alpha_k}(I-X) > 0$, is a homogenizing variable, and for every other variable ξ, $\tilde\xi = \gamma\xi$, $\tilde\beta = (\tilde\beta_1, \ldots, \tilde\beta_m)$.

The model $(P1)$ gets translated into the following linear program:

$$\begin{aligned}
(MSTARR) \qquad & \max_{\tilde\beta,\tilde w} \sum_{i=1}^{n} \mu_i \tilde w_i - \gamma E(I) \\
& \text{subject to} \\
& \sum_{k=1}^{m} \lambda_k\Big(\beta_k + \frac{1}{(1-\alpha_k)} \sum_{j=1}^{T} p_j \tilde u_{jk}\Big) = 1, \\
& \sum_{i=1}^{n} x_{ij}\tilde w_i + \tilde u_{jk} - \gamma I_j + \beta_k \geq 0, \quad k = 1, \ldots, m;\ j = 1, \ldots, T, \\
& \sum_{i=1}^{n} \tilde w_i = \gamma, \\
& \tilde w_i \geq 0, \quad i = 1, \ldots, n, \\
& \gamma > 0,\ \tilde u_{jk} \geq 0, \quad k = 1, \ldots, m;\ j = 1, \ldots, T.
\end{aligned}$$

In the same spirit, the following linear program gets associated with the STARR ratio with $MCVaR_\alpha^\triangle(I-X)$:

$$
\begin{aligned}
(MSTARR^\triangle) \qquad & \max_{\tilde{\beta},\tilde{w}} \sum_{i=1}^{n} \mu_i \tilde{w}_i - \gamma E(I) \\
& \text{subject to} \\
& \sum_{k=1}^{m} \lambda_k (\beta_k + \frac{1}{1-\alpha_k} \sum_{j=1}^{T} p_j \tilde{u}_{jk}) = 1, \\
& \sum_{i=1}^{n} (x_{ij} - \mu_i)\tilde{w}_i + \tilde{u}_{jk} - \gamma(I_j - E(I)) + \beta_k \geq 0, \\
& \qquad\qquad\qquad\qquad k = 1, \ldots, m;\ j = 1, \ldots, T, \\
& \sum_{i=1}^{n} \tilde{w}_i = \gamma, \\
& \tilde{w}_i \geq 0, \quad i = 1, \ldots, n, \\
& \gamma > 0,\ \tilde{u}_{jk} \geq 0, \quad k = 1, \ldots, m;\ j = 1, \ldots, T,
\end{aligned}
$$

where $\tilde{u}_{jk} = (\sum_{i=1}^{n} \mu_i \tilde{w}_i - \gamma E(I) - (\sum_{i=1}^{n} x_{ij}\tilde{w}_i - \gamma I_j) - \beta_k)^+$, $k = 1, \ldots, m;\ j = 1, \ldots, T$, are mT auxiliary variables, and γ is the homogenizing variable.

The optimal solutions of $(MSTARR)$ and $(MSTARR^\triangle)$ problems are sensitive to the underlying distribution followed by portfolio return X. To design a robust solution strategy, we use the copula representation of X and assume the copula to vary in a set of mixture copulas.

4.3. Robust STARR Ratio Models

For a given uncertainty set U, the most common robust optimization (RO) model is given by

$$
\begin{aligned}
(P) \qquad & \min_{x} f_p(c, x) \\
& \text{subject to} \\
& g_p(A, x) \leq b, \\
& c,\ A,\ b,\ p \in U,
\end{aligned}
$$

where the objective function $f_p(c,x)$ and constraint function $g_p(A,x)$ of the decision variable x involve the unknown parameters c, A, b, and p. The RO framework offers an advantage in the sense that, for many favorite classes of the uncertainty sets U, like, the mixture, boxed, and ellipsoidal sets, model (P) can equivalently be translated into tractable problems. The corresponding worst-case optimization of the problem (P) takes the following form:

$$(WP) \qquad \min_{x} \max_{c,\,A,\,b,\,p\,\in U} f_p(c,x)$$
$$\text{subject to}$$
$$g_p(A,x) \;\leq\; b.$$

Similar to the problem (P), the optimization problems of maximizing STARR ratios also suffer from the estimation error in the probability distribution. In the current article, we wish to find out the robust solutions for both problems ($MSTARR$) and ($MSTARR^{\triangle}$) for the case of mixture uncertainty set.

The mixture uncertainty set $\hat{K}$ of the distribution function is the convex combination of a finite number of priori distribution functions $\{K_1, \ldots, K_l\}$, given by

$$\hat{K} = \{K : K = \sum_{s=1}^{l} \psi_s K_s\,,\ \psi_s \geq 0,\ s = 1, \ldots, l,\ \sum_{s=1}^{l} \psi_s = 1\}.$$

For a fixed portfolio $w \in W$ and a grid of confidence levels α, the worst cases of (14) and (15) with respect to $\hat{K}$ are defined respectively as follows:

$$WMSTARR_K(X) = \min_{K \in \hat{K}} \frac{E(X) - E(I)}{MCVaR_{\alpha}(I - X)}, \tag{16}$$

$$WMSTARR_K^{\triangle}(X) = \min_{K \in \hat{K}} \frac{E(X) - E(I)}{MCVaR_{\alpha}^{\triangle}(I - X)}. \tag{17}$$

We propose to use copula theory to estimate the multiple priors for the distribution functions for mixture uncertainty set. We invoke Sklar's theorem (Theroem 3.1) to replace the joint distributions functions with the corresponding copula functions. Since there is a one to one correspondence between the joint distributions and copulas, corresponding to the mixture distribution set $\hat{K}$, there is an equivalent mixture copula set $\hat{C}$ as the convex combination of a finite number of prior copulas $\{C_1, \ldots, C_l\}$, given by

$$\hat{C} = \{C : C = \sum_{s=1}^{l} \psi_s C_s\,,\ \psi_s \geq 0,\ s = 1, \ldots, l,\ \sum_{s=1}^{l} \psi_s = 1\}.$$

We shall be using the copula in place of probability distribution. The worst case models of (14) and (15), with respect to $\hat{C}$, are respectively defined as follows:

$$WMSTARR_C(X) = \min_{C \in \hat{C}} \frac{E(X) - E(I)}{MCVaR_{\alpha}(I - X)}, \tag{18}$$

$$WMSTARR_C^{\triangle}(X) = \min_{C \in \hat{C}} \frac{E(X) - E(I)}{MCVaR_{\alpha}^{\triangle}(I - X)}. \tag{19}$$

We observe that although the worst case of $\mathrm{MCVaR}_\alpha(I-X)$, that is, $\sup_{C\in\hat{C}} MCVaR_\alpha(I-X)$ is a coherent risk measure [76], the worst case ratio in (18), wherever it is defined, is not necessarily a coherent ratio in the sense defined by Rachev et al. [64]. Similar observation holds for $\mathrm{MCVaR}_\alpha^\triangle(I-X)$ and the ratio in (19).

The corresponding worst case optimization models are as follows:

$$(WMSTARR_C) \quad \max_w \min_{C\in\hat{C}} \frac{E(X)-E(I)}{MCVaR_\alpha(I-X)}, \tag{20}$$

$$(WMSTARR_C^\triangle) \quad \max_w \min_{C\in\hat{c}} \frac{E(X)-E(I)}{MCVaR_\alpha^\triangle(I-X)}. \tag{21}$$

We now aim to formulate equivalent tractable optimization problems for (20) and (21). In this context, we first take a noting of how to formulate the model for the worst case of STARR ratio with MCVaR for the mixture copula.

Let $g(w,x):\mathbb{R}^n\to\mathbb{R}$ be the portfolio loss function corresponding to the vector w. Define a function $\hat{g}(w,u):\mathbb{I}^n\to\mathbb{R}$ such that $\hat{g}(w,u)=(goF^{-1})(w,u)=g(w,F^{-1}(u))$, which maps the domain of the loss function from $\mathbb{R}^n$ to $\mathbb{I}^n$. Also, let u follows a continuous distribution with copula density function c. Then $CVaR_\delta(w)$ is given as follows:

$$CVaR_\delta(w)=\min_{\hat{\delta}\in\mathbb{R}}(\hat{\delta}+\frac{1}{1-\delta}\int_{u\in\mathbb{I}^n}(\hat{g}(w,u)-\hat{\delta})^+c(u)\,du).$$

Therefore, for a given grid of confidence levels α,

$$MCVaR_\alpha(w)=\min_\beta\sum_{k=1}^m\lambda_k(\beta_k+\frac{1}{1-\alpha_k}\int_{u\in\mathbb{I}^n}(\hat{g}(w,u)-\beta_k)^+c(u)\,du);\ \beta=(\beta_1,\dots,\beta_m)\in\mathbb{R}^m.$$

And for $C\in\hat{C}$, we have,

$$MCVaR_\alpha(w)=\min_\beta\sum_{k=1}^m\lambda_k(\beta_k+\frac{1}{1-\alpha_k}\sum_{s=1}^l\int_{u\in\mathbb{I}^n}(\hat{g}(w,u)-\beta_k)^+\psi_s c_s(u)du). \tag{22}$$

Similarly, for $C\in\hat{C}$, the expected value of return function $\hat{f}(w,u)$ is given as follows:

$$\sum_{s=1}^l\psi_s\int_{u\in\mathbb{I}^n}\hat{f}(w,u)c_s(u)\,du. \tag{23}$$

We denote

$$G_{\alpha_k}^s(w,\beta_k)=\beta_k+\frac{1}{1-\alpha_k}\int_{u\in\mathbb{I}^n}(\hat{g}(w,u)-\beta_k)^+c_s(u)du,\quad s=1,\dots,l.$$

Using (22) and (23), the mixture copula representation for the general STARR ratio with

MCVaR is as follows:

$$\frac{\sum_{s=1}^{l}\psi_s\int_{u\in\mathbb{I}^n}\hat{f}(w,u)c_s(u)\,du}{\min_\beta\sum_{s=1}^{l}\psi_s[\sum_{k=1}^{m}\lambda_k G_{\alpha_k}^s(w,\beta_k)]} = \max_\beta \frac{\sum_{s=1}^{l}\psi_s\int_{u\in\mathbb{I}^n}\hat{f}(w,u)c_s(u)\,du}{\sum_{s=1}^{l}\psi_s[\sum_{k=1}^{m}\lambda_k G_{\alpha_k}^s(w,\beta_k)]},$$

and the corresponding worst case becomes:

$$\min_{\psi\in\Psi}\max_\beta \frac{\sum_{s=1}^{l}\psi_s\int_{u\in\mathbb{I}^n}\hat{f}(w,u)c_s(u)\,du}{\sum_{s=1}^{l}\psi_s[\sum_{k=1}^{m}\lambda_k G_{\alpha_k}^s(w,\beta_k)]}, \tag{24}$$

where $\Psi = \{\psi : \psi = (\psi_1,\dots,\psi_l),\ \psi_s \geq 0,\ s = 1\dots,l,\ \sum_{s=1}^{l}\psi_s = 1\}$.

We now derive a much simpler form of problem (24) for a general case of expected return $\hat{f}$ and loss function $\hat{g}$. We then give the specific forms of $\hat{f}$ and $\hat{g}$ in terms of simulations to arrive at the solvable models for $(WMSTARR_C)$ and $(WMSTARR_C^{\triangle})$.

Since the ratio in (24) is a quasi concave in β and quasi linear in ψ, we can equivalently write (24) as follows:

$$\max_\beta\min_{\psi\in\Psi} \frac{\sum_{s=1}^{l}\psi_s\int_{u\in\mathbb{I}^n}\hat{f}(w,u)c_s(u)\,du}{\sum_{s=1}^{l}\psi_s[\sum_{k=1}^{m}\lambda_k G_{\alpha_k}^s(w,\beta_k)]}.$$

Thus, the worst case optimization problem is as follows:

$$\max_{w,\beta}\min_{\psi\in\Psi} \frac{\sum_{s=1}^{l}\psi_s\int_{u\in\mathbb{I}^n}\hat{f}(w,u)c_s(u)\,du}{\sum_{s=1}^{l}\psi_s[\sum_{k=1}^{m}\lambda_k G_{\alpha_k}^s(w,\beta_k)]}. \tag{25}$$

Applying Charnes and Cooper transformation [19] with $\gamma > 0$ as a homogenization variable in (25), we obtain,

$$\begin{aligned}
&\max_{\tilde\beta,\tilde w}\min_{\psi}(\sum_{s=1}^{l}\psi_s\int_{u\in\mathbb{I}^n}\hat{f}(\tilde w,u)c_s(u)du)\\
&\text{subject to}\\
&\sum_{s=1}^{l}\psi_s\sum_{k=1}^{m}\lambda_k G_{\alpha_k}^s(\tilde w,\tilde\beta_k) = 1,
\end{aligned} \tag{26}$$

where $\tilde w = w\gamma$, $\tilde\beta = \beta\gamma$, and

$$G_{\alpha_k}^s(\tilde w,\tilde\beta_k) = \tilde\beta_k + \frac{1}{1-\alpha_k}\int_{u\in\mathbb{I}^n}(\hat{g}(\tilde w,u)-\tilde\beta_k)^+ c_s(u)du,\ s = 1,\dots,l.$$

Moreover, for $\psi\in\Psi$, we have following equivalent conditions:

$$\max_{\tilde{\beta},\tilde{w}} \min_{\psi} \sum_{s=1}^{l} \psi_s \int_{u\in\mathbb{I}^n} \hat{f}(\tilde{w},u)c_s(u)du \iff \max_{\tilde{\beta},\tilde{w}} \min_{s=1,\dots,l} \int_{u\in\mathbb{I}^n} \hat{f}(\tilde{w},u)c_s(u)du, \tag{27}$$

$$\sum_{s=1}^{l} \psi_s \sum_{k=1}^{m} \lambda_k G^s_{\alpha_k}(\tilde{w},\tilde{\beta}_k) = 1 \iff \sum_{k=1}^{m} \lambda_k G^s_{\alpha_k}(\tilde{w},\tilde{\beta}_k) = 1, \quad s = 1,\dots,l. \tag{28}$$

Using (27) and (28), along with $\theta = \min_{s=1,\dots,l} \int_{u\in\mathbb{I}^n} \hat{f}(\tilde{w},u)c_s(u)\,du$, we can equivalently express problem (26) as follows:

$$\begin{aligned}
&\max_{\tilde{w},\tilde{\beta}_k} \theta \\
&\text{subject to} \\
&\int_{u\in\mathbb{I}^n} \hat{f}(\tilde{w},u)c_s(u)\,du \geq \theta,\ s = 1,\dots,l, \\
&\sum_{k=1}^{m} \lambda_k G^s_{\alpha_k}(\tilde{w},\tilde{\beta}_k) = 1,\ s = 1,\dots,l.
\end{aligned} \tag{29}$$

Using some sampling technique ([65]), like Monte Carlo simulations, the problem in (29) can easily be approximated to a linear program. Also, it is practically viable to solve the portfolio optimization model for finite scenarios of returns.

Let T^s be the sample size of scenarios generated from the copula C_s and $u^s_{[q]}$ denotes the q^{th} sample. The problem (29) can be written in the following form:

$$\begin{aligned}
&\max_{\tilde{w},\tilde{\beta}_k} \theta \\
&\text{subject to} \\
&\frac{1}{T^s} \sum_{q=1}^{T^s} \hat{f}(\tilde{w},u^s_{[q]}) \geq \theta,\ s = 1,\dots,l, \\
&\sum_{k=1}^{m} \lambda_k \Big(\tilde{\beta}_k + \frac{1}{T^s(1-\alpha_k)} \sum_{q=1}^{T^s} (\hat{g}(\tilde{w},u^s_{[q]}) - \tilde{\beta}_k)^+\Big) = 1,\ s = 1,\dots,l.
\end{aligned} \tag{30}$$

For WMSTARR$_c$ in (20), $\hat{f}(\tilde{w},u^s_{[q]}) = (\tilde{w} - \gamma\widehat{w})' u^s_{[q]}$ is the portfolio return function, and $\hat{g}(\tilde{w},u^s_{[q]}) = (-\tilde{w} + \gamma\widehat{w})' u^s_{[q]}$ is the portfolio loss function in model (30). Also, $\widehat{w}$ is the known weight vector of the constituents applied in formulating the benchmark index.

Finally, for WMSTARR$_C$, model (30) is equivalent to solving the following linear program:

$$
\begin{aligned}
(WSTARR_C) \qquad & \max_{\tilde{\beta},\tilde{w}} \theta \\
& \text{subject to} \\
& \frac{1}{T^s}\sum_{q=1}^{T^s} \tilde{w}' u^s_{[q]} - \gamma \widehat{w}' u^s_{[q]} \geq \theta, \quad s = 1,\ldots,l, \\
& \sum_{k=1}^{m} \lambda_k (\tilde{\beta}_k + \frac{1}{(1-\alpha_k)T^s} \sum_{q=1}^{T^s} \tilde{u}^s_{kq}) = 1, \quad s = 1,\ldots,l, \\
& \tilde{u}^s_{kq} + (\tilde{w}' u^s_{[q]} - \gamma \widehat{w}' u^s_{[q]}) + \tilde{\beta}_k \geq 0, \quad k = 1,\ldots,m; \\
& \qquad\qquad\qquad\qquad\qquad\qquad q = 1,\ldots,T^s;\ s = 1,\ldots,l, \\
& \sum_{i=1}^{n} \tilde{w}_i = \gamma, \\
& \tilde{w}_i \geq 0, \quad i = 1,\ldots,n, \\
& \tilde{u}^s_{kq} \geq 0, \quad k = 1,\ldots,m;\ q = 1,\ldots,T^s;\ s = 1,\ldots,l.
\end{aligned}
$$

For the case of WMSTARR$_C^{\triangle}$ in (21), the portfolio return function $\hat{f}(\tilde{w}, u^s_{[q]}) = (\tilde{w} - \gamma\widehat{w})' u^s_{[q]}$, and the portfolio loss function is $\hat{g}(\tilde{w}, u^s_{[q]}) = (-\tilde{w} + \gamma\widehat{w})' u^s_{[q]} - (-\tilde{w} + \gamma\widehat{w})' u^s_{[q_0]}$ in model (30).

The model in (30) yields the following linear programming problem to be solved:

$$
\begin{aligned}
(WSTARR_C^{\triangle}) \qquad & \max_{\tilde{\beta},\tilde{w}} \theta \\
& \text{subject to} \\
& \frac{1}{T^s}\sum_{q=1}^{T^s} \tilde{w}' u^s_{[q]} - \gamma \widehat{w}' u^s_{[q]} \geq \theta, \quad s = 1,\ldots,l, \\
& \sum_{k=1}^{m} \lambda_k (\tilde{\beta}_k + \frac{1}{(1-\alpha_k)T^s} \sum_{q=1}^{T^s} \tilde{u}^s_{kq}) = 1, \quad s = 1,\ldots,l, \\
& \tilde{u}^s_{kq} + (\tilde{w}' u^s_{[q]} - \gamma \widehat{w}' u^s_{[q]}) - (\tilde{w}' u^s_{[q_0]} - \gamma \widehat{w}' u^s_{[q_0]}) + \tilde{\beta}_k \geq 0, \\
& \qquad\qquad\qquad\qquad k = 1,\ldots,m;\ q = 1,\ldots,T^s;\ s = 1,\ldots,l, \\
& \sum_{i=1}^{n} \tilde{w}_i = \gamma, \\
& \tilde{w}_i \geq 0, \quad i = 1,\ldots,n, \\
& \tilde{u}^s_{kq} \geq 0, \quad k = 1,\ldots,m;\ q = 1,\ldots,T^s;\ s = 1,\ldots,l.
\end{aligned}
$$

For the comparison purpose, we solve models ($WSTARR_C$) and ($WSTARR_C^{\triangle}$) for a special case taking $s = 1$, and the single copula C_1 as the Gaussian copula. In that case, we call the two models by ($STARR_G$) and ($STARR_G^{\triangle}$), respectively.

5. Methodology and Data

This section explains the step by step procedure adopted in this paper to carry out the empirical analysis of the proposed robust portfolio optimization models. The section also explains the data used in the present study.

5.1. The Methodology

The step-wise methodology adopted in this paper is described in the following Algorithm 1.

Algorithm 1 A step wise procedure for modeling marginals and joint dependence via copula

---

procedure METHODOLOGY

Transform the price series into log returns using

$$x_{ij} \leftarrow \frac{P_{ij}}{P_{ij-1}}, \quad i = 1, \ldots, n; \quad j = 1, \ldots, T,$$

where P_{ij} and P_{ij-1} are the closing prices of the i-th index on j-th and $(j-1)$-th day/week, respectively, and T is the number of days in the period considered.

Use ARMA-GJR-GARCH process and estimate the marginals $F_i, \ i = 1, \ldots, n$, for each return series. Filter the return series by fitting an ARMA(p, q)-GJR-GARCH(1,1) model, to get the residuals. Standardize the residuals to obtain the standardized residuals.

Using the marginal distribution F_i of each return series, transform the standardized residuals to the uniform random variable $U_i, \ i = 1, \ldots, n$, in the interval $[0, 1]$ so as to fit the copula. Use Kolmogorov-Smirnov (KS) test to ensure that the distribution transformed residuals is uniform $U[0, 1]$.

Estimate the Regular vine tree structure using the next described Algorithm 2.

Using the selected tree and copula parameters, simulate S random samples from the estimated joint probability distribution.

Transform the simulated random samples to the original scales of the log returns using the inverse quantile function of the marginals.

Reintroduce the autocorrelations and heteroscedasticity observed in the original return series using the mean and variance equations of the fitted ARMA(p, q)-GJR-GARCH(1,1) model to get a matrix $[r_1 \quad r_2 \ \cdots \ r_n]_{S\times n}$ of the simulated returns for each associated marginal distribution.

Use these simulated returns data to solve the proposed robust optimization models and obtain the optimal portfolios.

Dissmann et al. [25] suggest the following sequential procedure (Algorithm 2) to identify and estimate an R-vine structure.

Algorithm 2 Sequential procedure of identifying and estimating R-vine tree structure

1: **procedure** VINE TREE STRUCTURE
2: Using Prim's algorithm, the tree with maximum sum of the absolute empirical Kendall's tau correlation coefficients is selected.
3: The pair-copula families associated with the tree specified in the previous step are chosen by minimizing the AIC (Akaike Information Criterion) value.
4: The parameters of the selected copulas are estimated by the maximum likelihood methods.
5: The transformed observations to be used in the next tree are computed using conditional h-function.
6: Using the transformed observations, the above step are repeated in all the remaining trees in regular vine.

5.2. The Data

Three sets of sample data are considered for the present empirical study. All data sets are obtained from EIKON Thomson Reuters data stream. The first two data sets are evidence of bearish markets while the third data set comprises of different market cycles.

The first data set consists of daily closing prices of six indices: S&P EURO (Europe), CNX NIFTY 50 (India), DAX 30 (Germany), Dow Jones (USA), BSE Sensex 30 (India) and Gold ETF. The data belongs to the period April 2007–April 2009. This period is chosen to test the performance of the proposed models in the most talked about financial crisis witnessed in 2008. We took the in-sample period from April 2007 - mid September 2008 (370 realizations), and the out-of-sample period mid September 2008 - April 2009 (160 realizations).

The second data set comprises weekly closing prices of equity indices of five PIIGS countries: PSI-20 (Portugal), FTSE MIB (Italy), ISEQ-20 (Ireland), ATHEX composite (Greece) and IBEX-35 (Spain). A week is taken as Wednesday-to-Tuesday to minimize the weekend effects (see, [67]). The sample period is February 2000 - August 2017 with the in-sample period February 2000- March 2003 (164 realizations), and rest is the out-of-sample period comprising of 753 observations.

The third data set consists of daily closing prices of 4 equity indices namely, AEX (Europe), FTSE (Europe), NIKKEI (Japan) and MSCI (World index) for a period from 21 May 1993–18 May 2018; 6521 observations, and about 25 years. This data is used for carrying out the rolling window analysis.

6. Empirical Analysis

The returns of each index are calculated using $x_{ij} = \ln \frac{P_{ij}}{P_{ij-1}}$, $i = 1, \ldots, n$; $j = 1, \ldots, T$, where P_{ij} and P_{ij-1} are the closing prices of the i-th index on j-th and $(j-1)$-th day/week, respectively.

For all the models, we fix the $1/m$ naive strategy[6] as the benchmark index I to obtain the optimal portfolios in each in-sample period. Furthermore, over the out-of-sample periods, we obtain the performance of all portfolios with respect to the $1/m$ naive strategy portfolio by evaluating the performance measures for the series $X - I$, where I is the return series of the $1/m$ naive portfolio.

For all data sets, we calculate 12 performance measures namely, excess mean return (EMR), Sortino ratio, downside deviation (DD), Treynor ratio, VaR, CVaR, Rachev ratios, and VaR ratios, at 97% and 95% level of confidence for comparative analysis. We provide a brief explanation of these measures in the Appendix at the end of the paper.

We present the analysis of the first data set. A similar analysis is carried out on the second and third data sets.

6.1. Results: First Data Set

Table 1 displays the descriptive statistics along with the results of three statistical tests, Augmented Dickey-Fuller (ADF), Jarque-Bera (JB), and ARCH-LM test to fetch various properties of returns of six indices in first data set over the in-sample period.

Table 1: Summary statistics of six indices along with the outcomes of three statistical tests over the in-sample period; the value 1 indicates rejection of the null hypothesis for a given statistical test, while 0 is used for accepting the same.

	GOLD	CNX	DAX	DJIA	BSE	S&P 500
Mean	0.000264	0.000026	-0.00025	-0.00017	-8E-06	-0.00023
Min	-0.04327	-0.03954	-0.03228	-0.01962	-0.03342	-0.02097
Max	0.031855	0.029187	0.02502	0.015146	0.027835	0.018038
Std. dev	0.005784	0.00856	0.005512	0.005098	0.0085	0.005408
Skewness	-0.80817	-0.24779	-0.49684	-0.22804	-0.15088	-0.22268
Kurtosos	14.86179	5.107722	7.223963	3.821175	4.486202	3.958075
ADF test	1	1	1	1	1	1
JB test	1	1	1	1	1	1
ARCH-LM test	1	1	1	1	1	1

The negative values of skewness and high values of kurtosis suggest non-normality in the return distributions of all the indices. The same is confirmed by JB test. The ADF test rejects the null hypothesis of a unit root in all indices at 3% critical level. The ARCH-LM test indicates presence of conditional heteroscedasticity in returns of all the indices, thereby suggesting the relevance of GARCH model to fit in their distributions.

We use GJR-GARCH$(1, 1)$ to model the conditional volatility and ARMA(p, q) to account for the serial correlation in return series of six indices. We estimate the ordered parameter pair (p, q) in ARMA(p, q) model for the eight pairs in the set

[6]The $1/m$ naive portfolio strategy stems from the allocation of a fraction $1/m$ of a budget to each of the available assets from the corresponding index in each in-sample period.

$\{(0,1),(0,2),(1,0),(2,0),(1,1),(1,2),(2,1),(2,2)\}$ by minimizing Bayes information criterion (BIC). Table 2 reports the analysis. The BIC value corresponding to the ordered pair $(p,q)=(0,1)$ are found to be best for all indices.

Table 2: Bayes information criterion (BIC) values of ARMA(p,q) model for different choices of parameter pair (p,q) on the return series of six indices over the in-sample period.

	(1,0)	(0,1)	(2,0)	(0,2)	(2,1)	(1,2)	(1,1)	(2,2)
GOLD	-7.83396	**-7.834**	-7.81729	-7.81725	-7.8025	-7.80242	-7.81858	-7.8038
CNX	-6.82262	**-6.82262**	-6.81597	-6.81601	-6.80073	-6.80213	-6.80836	-6.80459
DAX	-7.64107	**-7.64146**	-7.62938	-7.62815	-7.61541	-7.61508	-7.63027	-7.60418
DJIA	-7.73529	**-7.73608**	-7.71883	-7.71893	-7.70573	-7.70646	-7.72104	-7.69057
BSE	-6.82418	**-6.82472**	-6.81597	-6.81539	-6.80223	-6.8014	-6.81211	-6.78905
S&P 500	-7.63359	**-7.63659**	-7.62096	-7.62242	-7.6117	-7.61164	-7.62834	-7.5957

Table 3 presents the values of the parameters for the best fitted ARMA$(0,1)$-GJR-GARCH$(1,1)$ model (1) – (3).

Table 3: The estimated parameters for the best fit ARMA$(0,1)$-GJR-GARCH$(1,1)$ model calculated over the in-sample data where the residuals follow skewed student's t distribution with parameters τ and ν, the skewness and shape parameters, respectively.

	GOLD ETF	CNX	DAX	DJIA	BSE	S&P 500
μ	0.000256	0.000379	-0.000144	-0.000178	0.000387	-0.000286
θ_1	-0.000234	-0.003210	-0.067705	-0.128432	0.028155	-0.168949
ω	0.000000	0.000004	0.000001	0.000000	0.000003	0.000000
a	0.081789	0.032964	0.000001	0.000000	0.027254	0.000000
b	0.976821	0.768784	0.874894	0.928342	0.794079	0.924413
d	-0.115537	0.334411	0.150771	0.147402	0.319940	0.156270
τ	1.079301	0.959575	0.842914	0.819655	0.970291	0.797416
ν	5.418194	5.071561	10.003472	6.753333	5.454994	7.366613

From the skewed student's t distribution, the asymmetric parameter τ is less than one for five return series indicating left skewness while it is more than one for GOLD ETF. The positive values of b indicate presence of time dependency in conditional volatility in all indices. The positive values of d for all except GOLD ETF indicating existence of leverage effects i. e., the volatility increases with the negative sentiments for all indices except GOLD ETF. Moreover, from the values of shape parameter ν, we observe that the degree of freedom is less than 15 for all indices which favors application of skewed student's t distribution in fitting the residual terms.

To further confirm the adequacy of ARMA$(0,1)$-GJR-GARCH$(1,1)$ model for estimating the marginal distributions, in addition, we apply Ljung-Box (LB) test to check serial dependence in residuals for all indices followed by Kolmogorov-Smirnov (KS) test on the $U[0,1]$ transformed residual variables using the specified distribution. The summary from the two tests is reported in Table 4 along with the values of the Pearson's correlation coefficients.

The alternative hypothesis of both the tests is rejected at 1% significance level thereby confirming that the residuals are independent and identically distributed uniform variates

Table 4: The Pearson's correlation coefficients and the outcomes of KS and LB tests obtained from the in-sample data; the value 0 indicates rejection of the alternative hypothesis for a given statistical test, while 1 is used otherwise.

	GOLD ETF	CNX	DAX	DJIA	BSE	S&P 500
GOLD ETF	1	-0.07935	-0.09627	-0.09021	-0.09889	-0.07741
CNX	-0.07935	1	0.39608	0.17702	0.97743	0.19193
DAX	-0.09627	0.39608	1	0.51961	0.40643	0.54019
DJIA	-0.09021	0.17702	0.51961	1	0.19760	0.96852
BSE	-0.09889	0.97743	0.40643	0.19760	1	0.21407
S&P 500	-0.07741	0.19193	0.54019	0.96852	0.21407	1
KS test	0	0	0	0	0	0
LB test	0	0	0	0	0	0

for all six indices. It also proves that the return series are correctly modeled by model (1)–(3).

Pearson's correlation coefficients between GOLD ETF and other indices are small and negative indicating different price movements in GOLD ETF vis a vis equity indices, which is somewhat an expected phenomena.

We next follow Algorithm 2 to identify the tree structure and best fit pair-copula. We use five copula families viz., Gumbel, Frank, Clayton, Guassian and Joe, in our present analysis.[7]. A brief description of the five copulas is jotted down in the Appendix.

Thereafter, we simulate 1000 paths, each of 500 scenarios, for all six indices. Each simulated path is used as the in-sample data to solve four models, $(WSTARR_C)$, $(STARR_G)$, $(WSTARR_C^{\triangle})$ and $(STARR_G^{\triangle})$. The optimal portfolios are named $(WS_C(X))$, $(S_G(X))$, $(WS_C^{\triangle}(X))$, and $(S_G^{\triangle}(X))$, respectively. In this way, we obtain 1000 optimal portfolios for each model.

We compute the out-of-sample returns of optimal portfolios to obtain 1000 out-of-sample return series for each of the four models. We then take the average of these 1000 out-of-sample series to obtain one out-of-sample return series for each model.

Table 5 records the out-of-sample performance of the portfolios from the four models on excess mean return (EMR), Sortino ratio, downside deviation (DD), Treynor ratio, VaR, CVaR, Rachev ratios, and VaR ratios at 97% and 95% level of confidence.

We observe that the portfolios, $(WS_C(X))$ and $(WS_C^{\triangle}(X))$, improve their respective counterparts $(S_G(X))$ and $(S_G^{\triangle}(X))$, in all performance measures.

The results designate that, to carry out worst-case analysis of robust STARR ratio optimization models, it is more efficacious to utilize several antecedently selected copulas to capture the changing dynamics of the market than a single Gaussian copula.

[7]We first select a copula family among these five. We determine the best pair copula at each node amongst the various possible rotations of the chosen copula family that are available in the R package VineCopula. We follow this procedure with each of the five copulas.

Table 5: The out-of-sample values of the performance measures of four portfolios $(WS_C(X))$, $(WS_C^{\triangle}(X))$, $(S_G(X))$ and $(S_G^{\triangle}(X))$ in the out-of-sample period Sept 2008 - April 2009 of first data set.

	$(WS_C(X))$	$(S_G(X))$	$(WS_C^{\triangle}(X))$	$(S_G^{\triangle}(X))$
EMR	**0.000652**	0.000559	**0.000647**	0.000578
Sortino ratio	**0.144156**	0.135112	**0.143852**	0.137492
DD	**0.004563**	0.005053	**0.004582**	0.00494
Treynor Ratio	**0.029412**	0.005876	**0.028034**	0.010202
$\text{CVaR}_{95\%}$	**0.015045**	0.015924	**0.015066**	0.015633
$\text{CVaR}_{97\%}$	**0.01705**	0.018589	**0.017101**	0.018248
$\text{VaR}_{95\%}$	**0.010059**	0.010546	**0.010053**	0.010194
$\text{VaR}_{97\%}$	**0.011875**	0.012594	**0.011893**	0.012438
$\text{Rachev}_{95\%}$	**0.00022**	0.000177	**0.000218**	0.000186
$\text{Rachev}_{97\%}$	**0.000274**	0.000221	**0.00027**	0.00023
$\text{VaR}_{95\%}$ Ratio	**0.00012**	0.000105	**0.000119**	0.000108
$\text{VaR}_{97\%}$ Ratio	**0.000162**	0.000122	**0.00016**	0.000129

6.2. Results: Second Data Set

Table 6 presents the final values of the performance measures from the four models in the out-of-sample period of the second data set.

From Table 6, the findings are very similar to those from Table 5. The $(WS_C(X))$ and $(WS_C^{\triangle}(X))$ exhibit superior performance than their respective counterparts $(S_G(X))$ and $(S_G^{\triangle}(X))$, in almost all performance measures.

Table 6: The out-of-sample values of the performance measures of four portfolios $(WS_C(X))$, $(WS_C^{\triangle}(X))$, $(S_G(X))$ and $(S_G^{\triangle}(X))$ on the data of PIIGS countries in the out-of-sample period April 2003 - August 2017.

	$(WS_C(X))$	$(S_G(X))$	$(WS_C^{\triangle}(X))$	$(S_G^{\triangle}(X))$
EMR	**0.000091**	0.000055	**0.000088**	0.000052
Sortino ratio	**0.028965**	0.018265	**0.028218**	0.017471
DD	0.003136	**0.003003**	0.003120	**0.003002**
Treynor Ratio	**0.003891**	0.000705	**0.003643**	0.000493
$\text{CVaR}_{95\%}$	0.032422	**0.032392**	0.032416	**0.032400**
$\text{CVaR}_{97\%}$	0.037496	**0.037143**	0.037452	**0.037155**
$\text{VaR}_{95\%}$	**0.022318**	0.023035	**0.022288**	0.022824
$\text{VaR}_{97\%}$	0.027506	**0.027085**	0.027520	**0.027015**
$\text{Rachev}_{95\%}$	**0.000109**	0.000102	**0.000107**	0.000102
$\text{Rachev}_{97\%}$	**0.000141**	0.000133	**0.000140**	0.000133
$\text{VaR}_{95\%}$ Ratio	**0.000057**	0.000053	**0.000057**	0.000052
$\text{VaR}_{97\%}$ Ratio	**0.000080**	0.000078	**0.000080**	0.000078

The empirical findings demonstrate that the robust optimization with mixture copulas captures the underlying dependency exceedingly well than a single copula. The worst-case copula approach in the MCVaR framework provides immunization against the worst loss in all possible realizations of uncertainty during periods of crisis.

We next present the empirical study on the third data set which embraces all types of market scenarios.

6.3. Third Data Set: Rolling Window Analysis

The findings of the previous two empirical analysis, favoring the proposed models over their Gaussian copula counterparts especially when the markets are in stress, encouraged us to check the performance of the proposed scheme and models on the third data set which embraces all types of market scenarios. We use the rolling window approach to perform the empirical analysis on the third data set.

Moreover, we extend the scope of our comparative analysis by including three more models viz, Markowitz model [53], and two multivariate GARCH models - DCC GARCH [28] and aDCC GARCH ([17]).

For d_1 and d_2 be the respective time lengths (in days) for the in-sample and out-of-sample periods, we solve our models for five different pairs of (d_1, d_2) values, specifically: (252, 126), (252, 252), (504, 252), (1260, 252) and (1260, 504).

We follow the rolling window approach of in-sample length d_1 and out-of-sample length d_2 on each of these period. That is, the first in-sample period is $1, \ldots, d_1$ with $d_1 + 1, \ldots, d_1 + d_2$ as the out-of-sample period; the next in-sample period gets forward by d_2 to $1 + d_2, \ldots, d_1 + d_2$ with next $d_1 + d_2 + 1, \ldots, d_1 + d_2 + d_2$ out-of-sample period, and so on.

The following table 7 reports the number of out-of-sample periods for each pair of (d_1, d_2).

Table 7: The in-sample and out-of-sample periods of analysis (in days) along with total number of out-of-sample rolling windows.

d_1	d_2	Total no. of out-of-sample windows
252	126	49
252	252	24
504	252	22
1260	252	20
1260	504	10

Similar to the case of copulas, we also solve (MSTARR) and (MSTARR$^{\triangle}$) models for the 1000 simulated paths, each having 500 scenarios, where the paths are obtained from the multivariate GARCH DCC and aDCC models. We name so formed optimal portfolios as ($SDCC(X)$) and ($SaDCC(X)$) from (MSTARR) model with simulations from GARCH DCC and GARCH aDCC model, respectively. ($SDCC^{\triangle}(X)$) and ($SaDCC^{\triangle}(X)$) from (MSTARR$^{\triangle}$) model with simulations from GARCH DCC model and GARCH aDCC model, respectively.

For each out-of-sample period, the out-of-sample returns series are obtained for 1000 optimal portfolios. These series are then converted into a single out-of-sample return series by taking the average returns. This analysis is performed for each value of (m, n) listed in Table 7 over all out-of-sample windows.

In Markowitz mean-variance model, we set the mean in the constraint to be the in-sample average return of the corresponding naive portfolio. The optimal portfolio obtained from the Markowitz model is named ($MW(X)$). Note that we solve the Markowitz model using original historical data for all periods of (d_1, d_2) with the same rolling window scheme as for the other models.

Since the complete analysis is enormous to present here, we report only one instance of it when ($d_1 = 1260$, $d_2 = 504$). The out-of-sample performance of the portfolios ($WS_C(X)$), ($S_G(X)$), ($WS_C^{\triangle}(X)$), ($S_G^{\triangle}(X)$), ($MW(X)$), ($SDCC(X)$), ($SaDCC(X)$), ($SDCC^{\triangle}(X)$) and ($SaDCC^{\triangle}(X)$), obtained from the nine models are recorded in Tables 8 and 9.

The portfolios ($WS_C(X)$) and ($WS_C^{\triangle}(X)$) improve the other models on EMR and Treynor ratio in a total of 9 out of 10 out-of-sample windows. The proposed portfolios also dominate the other portfolios in terms of Sortino ratio in all the windows. Although, Markowitz portfolios achieve least values for VaR, CVaR (at both 95% and 97%), and downside deviation 80%, but the proposed portfolios outperform the Markowitz portfolio by achieving higher values of Rachev ratio and VaR ratios, at both 95% and 97%, in 8 out of 10 windows. The proposed portfolios thus help to achieve our aim of maximizing the reward-risk ratios thereby maintaining a better trade-off between risk and return.

To sum up our analysis, we concatenate the out-of-sample returns series of each out-of-sample window corresponding to the model. These return series are then used for evaluating the performance measures as reported in Table 10.

It is worth noting that the proposed worst case portfolios have higher EMR, Sortino ratio and Treynor ratio.

7. Conclusions

Portfolio return is a multivariate random variable whose distribution depends on the underline dependence structure among its constitutes. This dependency needs to be capture carefully for correct investment decisions. Copula theory is a widely accepted mechanism to capture the dependency structure by accounting for most stylized features present in the financial data, like asymmetry, fat-tails, and nonlinearity. However, the ever-changing market dynamics is more complicated to be adequately represented by a single copula function. We propose to use mixture copulas, a linear combination of various copulas, to fit the dependence structure in portfolio returns.

The present study provides a robust optimization tool to obtain portfolios with enhanced return-reward trade-off by conducting the worst-case analysis of two variants of STARR ratio models with the mixed conditional value-at-risk (MCVaR) and the deviation MCVaR (DMCVaR). We proposed two models to maximize the minimum (worst) value of two variants of STARR ratio calculated for many feasible copulas. We use the regular vine copula to capture the dependence structure among the assets after fitting the marginal distribution of each asset by the GJR-GARCH model. We take Gumbel, Frank, Clayton, Gaussian and Joe copulas in our empirical analysis.

We perform an empirical analysis on three datasets, the first data consisting of six indices including Gold ETF, the second data is of Eurozone crisis period of the PIIGS countries, and the third data of a relatively long period is comprising of 4 global indices. We

	$(WS_C(X))$	$(S_G(X))$	$(WS_C^{\triangle}(X))$	$(S_G^{\triangle}(X))$	(MW(X)	(SDCC(X))	(SaDCC(X))	$(SDCC^{\triangle}$(X))	$(SaDCC^{\triangle}$(X))
	window 1								
EMR	**0.000126**	0.000036	4.000000	-0.000056	-0.000044	0.000073	0.000067	0.000028	0.000073
Sortino Ratio	**0.166315**	0.065952	0.093474	-0.054180	-0.046303	0.027309	0.025117	0.008155	0.027309
Treynor Ratio	**0.045457**	0.027716	0.033832	0.006536	0.010780	0.026193	0.025167	0.029576	0.026193
DD	0.000757	**0.000551**	0.000795	0.001042	0.000952	0.002686	0.002686	0.003462	0.002686
$\text{CVaR}_{95\%}$	0.002274	**0.001628**	0.002366	0.003018	0.002843	0.008819	0.008819	0.009994	0.008819
$\text{CVaR}_{97\%}$	0.002477	**0.001784**	0.002567	0.003281	0.003076	0.010242	0.010242	0.010964	0.010242
$\text{VaR}_{95\%}$	0.001811	**0.001330**	0.001927	0.002422	0.002353	0.005865	0.005865	0.007999	0.005865
$\text{VaR}_{97\%}$	0.002119	**0.001459**	0.002172	0.002769	0.002603	0.007241	0.007241	0.008747	0.007241
$\text{Rachev}_{95\%}$	**0.000099**	0.000004	0.000009	0.000008	0.000008	0.000066	0.000066	0.000012	0.000066
$\text{Rachev}_{97\%}$	**0.000095**	0.000006	0.000012	0.000010	0.000009	0.000089	0.000089	0.000015	0.000089
$\text{VaR}_{95\%}$ Ratio	**0.000082**	0.000002	0.000003	0.000005	0.000005	0.000031	0.000031	0.000075	0.000031
$\text{VaR}_{97\%}$ Ratio	**0.000046**	0.000002	0.000005	0.000007	0.000006	0.000044	0.000044	0.000039	0.000044
	window 3								
EMR	**0.000152**	0.000120	0.000132	0.000133	0.000048	0.000021	-0.000209	-0.000209	-0.000209
Sortino Ratio	**0.097546**	0.048545	0.044344	0.046323	0.053222	0.015527	-0.063356	-0.063356	-0.063356
Treynor Ratio	**0.006199**	-0.000828	0.001907	0.002303	-0.014550	-0.018941	-0.041014	-0.041014	-0.041014
DD	0.001553	0.002463	0.000983	0.002877	**0.000899**	0.001352	0.003303	0.003303	0.003303
$\text{CVaR}_{95\%}$	0.004922	0.007667	0.009231	0.008924	**0.002852**	0.004028	0.009892	0.009892	0.009892
$\text{CVaR}_{97\%}$	0.005431	0.008548	0.010318	0.009965	**0.003157**	0.004486	0.010911	0.010911	0.010911
$\text{VaR}_{95\%}$	0.003731	0.005910	0.006881	0.006831	**0.002164**	0.003073	0.007757	0.007757	0.007757
$\text{VaR}_{97\%}$	0.004443	0.006583	0.008224	0.007956	**0.002652**	0.003705	0.008843	0.008843	0.008843
$\text{Rachev}_{95\%}$	0.000026	0.000056	**0.000807**	0.000075	0.000008	0.000017	0.000103	0.000103	0.000103
$\text{Rachev}_{97\%}$	0.000033	0.000068	**0.000986**	0.000092	0.000010	0.000021	0.000126	0.000126	0.000126
$\text{VaR}_{95\%}$ Ratio	0.000014	0.000035	**0.000482**	0.000048	0.000005	0.000010	0.000062	0.000062	0.000062
$\text{VaR}_{97\%}$ Ratio	0.000018	0.000042	**0.000644**	0.000060	0.000007	0.000014	0.000081	0.000081	0.000081
	window 5								
EMR	**0.000089**	0.000051	0.000130	0.000011	0.000011	-0.000030	-0.000054	0.000041	0.000041
Sortino Ratio	**0.051383**	0.016337	0.046965	0.004271	0.050300	-0.036890	-0.018306	0.034867	0.034867
Treynor Ratio	0.051705	0.040418	**0.066707**	0.026216	0.025539	0.013819	0.005275	0.032421	0.032421
DD	0.001730	0.003109	0.002772	0.002525	**0.000225**	0.000821	0.002936	0.001170	0.001170
$\text{CVaR}_{95\%}$	0.005265	0.008413	0.008474	0.007698	**0.000699**	0.002437	0.008926	0.003554	0.003554
$\text{CVaR}_{97\%}$	0.005949	0.010334	0.009539	0.008670	**0.000791**	0.002695	0.010022	0.003978	0.003978
$\text{VaR}_{95\%}$	0.003860	0.004988	0.006228	0.005701	**0.000525**	0.001915	0.006534	0.002641	0.002641
$\text{VaR}_{97\%}$	0.004438	0.005680	0.007162	0.006500	**0.000623**	0.002118	0.007636	0.003137	0.003137
$\text{Rachev}_{95\%}$	0.000034	0.000074	**0.000079**	0.000060	0.000001	0.000006	0.000076	0.000013	0.000013
$\text{Rachev}_{97\%}$	0.000046	0.000113	**0.000101**	0.000078	0.000001	0.000007	0.000097	0.000017	0.000017
$\text{VaR}_{95\%}$ Ratio	0.000015	0.000025	**0.000041**	0.000030	0.000000	0.000003	0.000040	0.000007	0.000007
$\text{VaR}_{97\%}$ Ratio	0.000021	0.000034	**0.000057**	0.000043	0.000000	0.000004	0.000057	0.000010	0.000010

	$(WS_C(X))$	$(S_G(X))$	$(WS_C^{\triangle}(X))$	$(S_G^{\triangle}(X))$	(MW(X))	(SDCC(X))	(SaDCC(X))	$()SDCC^{\triangle}$(X))	$()SaDCC^{\triangle}$(X))
	window 2								
EMR	0.000297	-0.000217	**0.000467**	-0.000209	0.000072	0.000070	0.000070	-0.000132	0.000173
Sortino Ratio	0.101101	-0.069842	**0.120505**	-0.057248	0.094844	0.029635	0.029635	-0.031269	0.101827
Treynor Ratio	-0.000986	-0.128334	**0.046560**	-0.130006	-0.049378	-0.044655	-0.044655	-0.117551	-0.026393
DD	0.002936	0.003110	0.003872	0.003659	**0.000762**	0.002364	0.002364	0.004227	0.001697
$\text{CVaR}_{95\%}$	0.009174	0.009617	0.012238	0.011278	**0.002355**	0.007468	0.007468	0.012998	0.005335
$\text{CVaR}_{97\%}$	0.010676	0.011180	0.014498	0.013206	**0.002673**	0.008585	0.008585	0.015257	0.005986
$\text{VaR}_{95\%}$	0.006420	0.006746	0.008135	0.007801	**0.001738**	0.005033	0.005033	0.008934	0.003902
$\text{VaR}_{97\%}$	0.007495	0.007861	0.009191	0.008917	**0.001976**	0.006136	0.006136	0.010325	0.004679
$\text{Rachev}_{95\%}$	0.000116	0.000093	**0.000195**	0.000129	0.000006	0.000053	0.000053	0.000178	0.000033
$\text{Rachev}_{97\%}$	0.000155	0.000126	**0.000270**	0.000176	0.000007	0.000068	0.000068	0.000243	0.000043
$\text{VaR}_{95\%}$ Ratio	0.000054	0.000041	**0.000085**	0.000058	0.000003	0.000026	0.000026	0.000083	0.000016
$\text{VaR}_{97\%}$ Ratio	0.000081	0.000062	**0.000118**	0.000083	0.000004	0.000037	0.000037	0.000113	0.000022
	window 4								
EMR	0.000089	0.000085	**0.000100**	0.000090	0.000015	-0.000029	-0.000029	0.000051	0.000068
Sortino Ratio	**0.055315**	0.047838	0.047066	0.047642	0.028510	-0.035717	-0.035717	0.047073	0.031657
Treynor Ratio	0.109226	0.104969	**0.109571**	0.106142	0.091080	0.085355	0.085355	0.102446	0.098096
DD	0.001617	0.001781	0.002133	0.001899	**0.000531**	0.000815	0.000815	0.001082	0.002149
$\text{CVaR}_{95\%}$	0.005162	0.005665	0.006801	0.006042	**0.001630**	0.002366	0.002366	0.003449	0.006822
$\text{CVaR}_{97\%}$	0.005785	0.006330	0.007625	0.006753	**0.001863**	0.002578	0.002578	0.003891	0.007625
$\text{VaR}_{95\%}$	0.003779	0.004149	0.004979	0.004424	**0.001193**	0.001935	0.001935	0.002572	0.005091
$\text{VaR}_{97\%}$	0.004807	0.005268	0.006311	0.005614	**0.001380**	0.002097	0.002097	0.003052	0.006311
$\text{Rachev}_{95\%}$	0.000026	0.000031	**0.000045**	0.000035	0.000003	0.000006	0.000006	0.000012	0.000045
$\text{Rachev}_{97\%}$	0.000032	0.000038	**0.000055**	0.000043	0.000003	0.000007	0.000007	0.000014	0.000055
$\text{VaR}_{95\%}$ Ratio	0.000016	0.000019	**0.000027**	0.000021	0.000002	0.000004	0.000004	0.000007	0.000026
$\text{VaR}_{97\%}$ Ratio	0.000022	0.000026	**0.000038**	0.000029	0.000002	0.000005	0.000005	0.000009	0.000037
	window 6								
EMR	0.000265	0.000011	**0.000266**	-0.000084	0.000046	-0.000075	-0.000075	-0.000075	-0.000122
Sortino Ratio	0.057847	0.005689	**0.089976**	-0.020870	0.063774	-0.013503	-0.013503	-0.013503	-0.041240
Treynor Ratio	-0.006542	-0.034262	**-0.004325**	-0.053897	-0.031877	-0.057796	-0.057796	-0.057796	-0.047892
DD	0.004579	0.001847	0.002961	0.004036	**0.000716**	0.005537	0.005537	0.005537	0.002961
$\text{CVaR}_{95\%}$	0.010476	0.005812	0.009659	0.013051	**0.002304**	0.017971	0.017971	0.017971	0.009849
$\text{CVaR}_{97\%}$	0.014196	0.006589	0.011069	0.014881	**0.002609**	0.020634	0.020634	0.020634	0.011494
$\text{VaR}_{95\%}$	0.004147	0.004165	0.006776	0.009005	**0.001665**	0.011834	0.011834	0.011834	0.006350
$\text{VaR}_{97\%}$	0.004960	0.005001	0.008528	0.011231	**0.001944**	0.015391	0.015391	0.015391	0.008033
$\text{Rachev}_{95\%}$	0.000156	0.000035	0.000118	0.000160	0.000006	**0.000311**	**0.000311**	**0.000311**	0.000087
$\text{Rachev}_{97\%}$	0.000299	0.000046	0.000164	0.000205	0.000009	**0.000404**	**0.000404**	**0.000404**	0.000119
$\text{VaR}_{95\%}$ Ratio	0.000019	0.000017	0.000052	0.000085	0.000003	**0.000156**	**0.000156**	**0.000156**	0.000037
$\text{VaR}_{97\%}$ Ratio	0.000029	0.000026	0.000075	0.000118	0.000005	**0.000228**	**0.000228**	**0.000228**	0.000061

Table 8: The out-of-sample performance of optimal portfolios from all the models with $d_1 = 1026$ days in-sample and $d_2 = 504$ days out-of-sample period in each window. The best values amongst all models are highlighted in **bold.**

	window 7								
	$(WS_C(X))$	$(S_G(X))$	$(WS_C^{\triangle}(X))$	$(S_G^{\triangle}(X))$	(MW(X))	(SDCC(X))	(SaDCC(X))	$(SDCC^{\triangle}$(X))	$(SaDCC^{\triangle}$(X))
EMR	**0.000201**	0.000019	0.000144	0.000072	-0.000006	-0.000021	-0.000033	0.000075	-0.000056
Sortino Ratio	**0.367295**	0.005367	0.215105	0.092642	-0.009475	-0.006060	-0.021635	0.064121	-0.037526
Treynor Ratio	**0.030082**	-0.019458	0.014867	-0.001047	-0.018685	-0.030420	-0.022102	-0.000733	-0.026646
DD	**0.000547**	0.003566	0.000672	0.000781	0.000595	0.003531	0.001503	0.001163	0.001505
$CVaR_{95\%}$	**0.001603**	0.010933	0.002071	0.002382	0.001837	0.010782	0.004525	0.003502	0.004525
$CVaR_{97\%}$	**0.001763**	0.012639	0.002337	0.002633	0.002106	0.012494	0.005034	0.003966	0.005034
$VaR_{95\%}$	**0.001278**	0.007570	0.001485	0.001849	0.001309	0.007342	0.003410	0.002587	0.003410
$VaR_{97\%}$	**0.001486**	0.009058	0.001799	0.002193	0.001553	0.008944	0.004077	0.002930	0.004077
$Rachev_{95\%}$	0.000008	**0.000129**	0.000009	0.000007	0.000003	0.000119	0.000024	0.000013	0.000023
$Rachev_{97\%}$	0.000012	**0.000173**	0.000014	0.000010	0.000004	0.000160	0.000031	0.000018	0.000029
$VaR_{95\%}$ Ratio	0.000002	**0.000061**	0.000002	0.000003	0.000002	0.000054	0.000013	0.000007	0.000013
$VaR_{97\%}$ Ratio	0.000002	**0.000084**	0.000004	0.000004	0.000002	0.000081	0.000018	0.000009	0.000018
	window 9								
EMR	**0.000168**	-0.000110	0.000088	0.000110	-0.000008	**0.000168**	0.000036	0.000108	0.000129
Sortino Ratio	0.061563	-0.111605	**0.106555**	0.054574	-0.030524	0.061562	0.032004	0.038291	0.046883
Treynor Ratio	**0.108787**	0.022205	0.093447	0.093069	0.054385	**0.108787**	0.071199	0.091512	0.096130
DD	0.002727	0.000988	0.000827	0.002024	**0.000270**	0.002727	0.001134	0.002821	0.002745
$CVaR_{95\%}$	0.009011	0.002732	0.002523	0.006625	**0.000798**	0.009011	0.003583	0.009390	0.009011
$CVaR_{97\%}$	0.010368	0.002991	0.002918	0.007645	**0.000895**	0.010368	0.004177	0.010700	0.010368
$VaR_{95\%}$	0.005855	0.002231	0.001723	0.004288	**0.000610**	0.005855	0.002335	0.006081	0.005866
$VaR_{97\%}$	0.008153	0.002552	0.002102	0.005946	**0.000694**	0.008153	0.003087	0.008492	0.008153
$Rachev_{95\%}$	**0.000075**	0.000009	0.000009	0.000041	0.000001	**0.000075**	0.000012	0.000078	**0.000075**
$Rachev_{97\%}$	**0.000097**	0.000011	0.000014	0.000053	0.000001	**0.000097**	0.000016	0.000101	**0.000097**
$VaR_{95\%}$ Ratio	**0.000035**	0.000005	0.000003	0.000019	0.000000	**0.000035**	0.000006	0.000037	**0.000035**
$VaR_{97\%}$ Ratio	**0.000058**	0.000008	0.000005	0.000031	0.000000	**0.000058**	0.000009	0.000060	**0.000058**

	window 8								
	$(WS_C(X))$	$(S_G(X))$	$(WS_C^{\triangle}(X))$	$(S_G^{\triangle}(X))$	(MW(X))	(SDCC(X))	(SaDCC(X))	$(SDCC^{\triangle}$(X))	$(SaDCC^{\triangle}$(X))
EMR	0.000099	0.000024	0.000060	-0.000024	0.000031	**0.000113**	-0.000053	-0.000053	0.000075
Sortino Ratio	0.060865	0.014494	**0.061091**	-0.014470	0.042156	0.032162	-0.045665	-0.045665	0.021161
Treynor Ratio	0.090653	0.069354	0.081608	0.056504	0.080155	**0.106614**	0.055104	0.055104	0.095659
DD	0.001633	0.001634	0.000983	0.001650	**0.000746**	0.003522	0.001169	0.001169	0.003566
$CVaR_{95\%}$	0.004763	0.004763	0.002956	0.004810	**0.002264**	0.010740	0.003290	0.003290	0.010927
$CVaR_{97\%}$	0.005224	0.005224	0.003322	0.005272	**0.002586**	0.012441	0.003636	0.003636	0.012683
$VaR_{95\%}$	0.003832	0.003832	0.002232	0.003849	**0.001636**	0.007572	0.002637	0.002637	0.007724
$VaR_{97\%}$	0.004298	0.004298	0.002547	0.004429	**0.001869**	0.008565	0.002885	0.002885	0.008855
$Rachev_{95\%}$	0.000031	0.000027	0.000012	0.000025	0.000005	**0.000107**	0.000012	0.000012	**0.000107**
$Rachev_{97\%}$	0.000040	0.000034	0.000016	0.000031	0.000006	**0.000139**	0.000015	0.000015	**0.000139**
$VaR_{95\%}$ Ratio	0.000016	0.000015	0.000005	0.000015	0.000003	**0.000059**	0.000007	0.000007	**0.000060**
$VaR_{97\%}$ Ratio	0.000023	0.000021	0.000007	0.000021	0.000004	**0.000075**	0.000009	0.000009	**0.000075**
	window 10								
EMR	**0.000049**	-0.000036	0.000051	0.000037	0.000006	-0.000049	-0.000070	-0.000070	-0.000070
Sortino Ratio	**0.057249**	-0.022408	0.041161	0.030054	0.012413	-0.024999	-0.020256	-0.020256	-0.020256
Treynor Ratio	**0.021585**	-0.001441	0.020798	0.017319	0.009992	-0.004816	-0.011693	-0.011693	-0.011693
DD	0.000855	0.001606	0.001235	0.001235	**0.000481**	0.001967	0.003475	0.003475	0.003475
$CVaR_{95\%}$	0.002880	0.005492	0.004049	0.004049	**0.001517**	0.006689	0.011895	0.011895	0.011895
$CVaR_{97\%}$	0.003428	0.006598	0.004727	0.004727	**0.001689**	0.008024	0.014262	0.014262	0.014262
$VaR_{95\%}$	0.001791	0.003328	0.002611	0.002611	**0.001173**	0.003914	0.007115	0.007115	0.007115
$VaR_{97\%}$	0.002253	0.004590	0.003374	0.003374	**0.001320**	0.005631	0.009786	0.009786	0.009786
$Rachev_{95\%}$	**0.000081**	0.000028	0.000018	0.000018	0.000002	0.000042	0.000013	0.000013	0.000013
$Rachev_{97\%}$	**0.000077**	0.000040	0.000025	0.000025	0.000003	0.000059	0.000018	0.000018	0.000018
$VaR_{95\%}$ Ratio	**0.000033**	0.000011	0.000008	0.000007	0.000001	0.000016	0.000005	0.000005	0.000005
$VaR_{97\%}$ Ratio	**0.000045**	0.000019	0.000012	0.000012	0.000002	0.000028	0.000009	0.000009	0.000009

Table 9: The out-of-sample performance of optimal portfolios from all the models with $d_1 = 1026$ days in-sample and $d_2 = 504$ days out-of-sample period in each window. The best values amongst all models are highlighted in **bold.**

	252-126					252-252				
	$(WS_C(X))$	$(S_G(X))$	$(WS_C^{\triangle}(X))$	$(S_G^{\triangle}(X))$	(MW(X))	$(WS_C(X))$	$(S_G(X))$	$(WS_C^{\triangle}(X))$	$(S_G^{\triangle}(X))$	(MW(X))
EMR	0.001539	0.001393	**0.001865**	0.001283	0.001347	**0.000190**	0.000126	0.000178	0.000141	0.000007
Sortino Ratio	**0.661000**	0.584259	0.044990	0.560818	0.025108	**0.087129**	0.053243	0.052573	0.040667	0.011912
Treynor Ratio	0.129061	0.053455	**0.135010**	0.001790	0.032546	**0.058445**	0.041300	0.042536	0.034984	0.011580
DD	0.002329	0.002385	0.041453	0.002287	**0.000537**	0.002178	0.002367	0.003380	0.003463	**0.000626**
$CVaR_{95\%}$	0.008266	0.008552	0.075056	0.008025	**0.001875**	0.007548	0.008154	0.011260	0.011489	**0.002195**
$CVaR_{97\%}$	0.009874	0.010189	0.120626	0.009427	**0.002337**	0.009110	0.009709	0.014652	0.014700	**0.002629**
$VaR_{95\%}$	0.005139	0.005327	0.005938	0.005167	**0.001046**	0.004618	0.005196	0.005274	0.005764	**0.001364**
$VaR_{97\%}$	0.006711	0.007067	0.008163	0.006644	**0.001356**	0.005940	0.006649	0.007314	0.007713	**0.001787**
$Rachev_{95\%}$	0.000271	0.000273	**0.007908**	0.000232	0.000052	0.000070	0.000077	0.000142	**0.000146**	0.000005
$Rachev_{97\%}$	0.000448	0.000464	**0.018926**	0.000390	0.000104	0.000105	0.000111	0.000239	**0.000240**	0.000007
$VaR_{95\%}$ Ratio	0.000050	0.000043	**0.000073**	0.000040	0.000002	0.000025	0.000030	0.000032	**0.000037**	0.000002
$VaR_{97\%}$ Ratio	0.000162	0.000162	**0.000335**	0.000114	0.000018	0.000041	0.000048	0.000063	**0.000066**	0.000003
	504-252					1260-252				
EMR	0.000389	0.000081	**0.000425**	0.000097	0.000007	**0.000354**	0.000035	0.000174	0.000073	0.000017
Sortino Ratio	0.155814	0.029264	**0.181615**	0.035890	0.008985	**0.141202**	0.014264	0.102130	0.029070	0.025179
Treynor Ratio	0.092465	0.037647	**0.095848**	0.040441	0.024004	**0.088567**	0.008380	0.042390	0.017205	0.005387
DD	0.002496	0.002779	0.002340	0.002690	**0.000789**	0.002507	0.002434	0.001707	0.002506	**0.000694**
$CVaR_{95\%}$	0.008394	0.009254	0.008073	0.009053	**0.002387**	0.008430	0.008139	0.005841	0.008541	**0.002304**
$CVaR_{97\%}$	0.010220	0.011102	0.009576	0.010923	**0.002925**	0.010178	0.009725	0.007058	0.010095	**0.002679**
$VaR_{95\%}$	0.005025	0.005759	0.005229	0.005562	**0.001394**	0.005187	0.005110	0.003570	0.005512	**0.001559**
$VaR_{97\%}$	0.006502	0.007382	0.006581	0.007198	**0.001845**	0.006602	0.006455	0.004669	0.007063	**0.001918**
$Rachev_{95\%}$	0.000097	0.000089	**0.000106**	0.000088	0.000006	**0.000086**	0.000066	0.000046	0.000076	0.000005
$Rachev_{97\%}$	0.000152	0.000130	**0.000166**	0.000130	0.000009	**0.000130**	0.000094	0.000071	0.000107	0.000007
$VaR_{95\%}$ Ratio	0.000030	**0.000034**	0.000032	0.000031	0.000002	0.000029	0.000027	0.000014	**0.000032**	0.000002
$VaR_{97\%}$ Ratio	0.000048	**0.000054**	0.000052	0.000052	0.000003	0.000049	0.000041	0.000025	**0.000050**	0.000004
	1260-504									
	$(WS_C(X))$	$(S_G(X))$	$(WS_C^{\triangle}(X))$	$(S_G^{\triangle}(X))$	(MW(X))	(SDCC(X))	(SaDCC(X))	$(SDCC^{\triangle}$(X))	$(SaDCC^{\triangle}$(X))	
EMR	**0.000153**	-0.000002	0.000151	0.000008	0.000017	0.000024	-0.000035	-0.000024	0.000010	
Sortino Ratio	**0.069104**	-0.000802	0.068275	0.003339	0.026013	0.008368	-0.012303	-0.007636	0.003804	
Treynor Ratio	0.034624	0.000087	**0.036826**	0.002322	0.005297	0.005070	-0.007652	-0.005884	0.001028	
DD	0.002221	0.002264	0.002217	0.002408	**0.000660**	0.002878	0.002840	0.003110	0.002653	
$CVaR_{95\%}$	0.006430	0.007288	0.007644	0.008079	**0.002204**	0.009807	0.009674	0.010607	0.008825	
$CVaR_{97\%}$	0.007806	0.008599	0.009048	0.009515	**0.002542**	0.011775	0.011658	0.012538	0.010343	
$VaR_{95\%}$	0.003949	0.004733	0.004848	0.005267	**0.001522**	0.006139	0.005939	0.006906	0.005866	
$VaR_{97\%}$	0.004822	0.005846	0.006245	0.006552	**0.001878**	0.007662	0.007523	0.008630	0.007285	
$Rachev_{95\%}$	0.000053	0.000055	**0.000169**	0.000065	0.000005	0.000094	0.000092	0.000115	0.000075	
$Rachev_{97\%}$	0.000082	0.000076	**0.000199**	0.000089	0.000007	0.000134	0.000133	0.000161	0.000102	
$VaR_{95\%}$ Ratio	0.000017	0.000023	**0.000057**	0.000028	0.000002	0.000038	0.000035	0.000048	0.000034	
$VaR_{97\%}$ Ratio	0.000027	0.000035	**0.000085**	0.000043	0.000003	0.000058	0.000057	0.000077	0.000052	

Table 10: The average out-of-sample performance analysis of the returns data series obtained by serially concatenating the out-of-sample returns series from the out-of-sample windows of the optimal portfolios of the corresponding model for all the in-sample and out-of-sample periods considered. The best values among all models, for each in-sample and out-of-sample period, are highlighted in **bold.**

conduct a comparative analysis of the proposed worst-case strategy with the case when only Gaussian copula is used to capture dependency. Furthermore, we used the Markowitz model, naive portfolio and two multivariate GARCH models in our comparative analysis.

The extensive empirical analysis of several indices from different markets and across different time periods exhibits the superior performance of the proposed robust reward-risk models on EMR, Sortino ratio, Rachev ratio, VaR ratio, and Treynor ratio.

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Appendix

Following is the description of the performance measures considered for the out-of-sample analysis:

- Excess Mean Return (EMR): It is defined as the average of the difference between the portfolio return and the benchmark index return

$$EMR = \frac{\sum_{j=1}^{T}(X_j - I_j)}{T},$$

  where X_j and I_j are respectively the j-th realization, $j = 1, \ldots, T$, of the tracking portfolio and the benchmark index. It measures the improvement of the portfolio over the benchmark index in terms of the return value.

- Sortino ratio: It is the ratio of EMR to the under achievement from the benchmark index as the measure of risk

$$\frac{\frac{1}{T}\sum_{j=1}^{T}(X_j - I_j)}{\sqrt{\sum_{j=1}^{T}\frac{(\min\{X_j - I_j, 0\})^2}{T}}}.$$

Higher values of this ratio are desirable.

- Rachev Ratio (RR): It measures the right tail reward potential relative to the left tail risk in a non-Gaussian setting. Intuitively, it allows a trade off between the potential of extreme positive returns to the risk of extreme losses. The ratio is defined as follows:

$$RR_{\alpha_1,\alpha_2} = \frac{CVaR_{\alpha_1}(X-I)}{CVaR_{\alpha_2}(I-X)}, \quad \alpha_1, \alpha_2 \in (0,1).$$

  Larger values are worthwhile.

- VaR ratio: It the ratio of the best value to the worst value of the portfolio with respect to the benchmark index measured using VaR

$$VaR_{\alpha_1,\alpha_2} = \frac{VaR_{\alpha_1}(X-I)}{VaR_{\alpha_2}(I-X)}, \quad \alpha_1, \alpha_2 \in (0,1).$$

- Downside deviation (DD): It depicts the under achievement of portfolio from the index. It is given by

$$\sqrt{\sum_{j=1}^{T} \frac{((I_j - X_j)^+)^2}{T}}.$$

- Treynor ratio (TR): The Treynor ratio uses beta of the portfolio as the volatility measure and is defined as

$$TR = \frac{\frac{1}{T}\sum_{j=1}^{T}(X_j - I_j)}{\beta}.$$

Archimedean Copulas

Let ϕ be a continuous, strictly decreasing function from $\mathbb{I} = [0,1]$, to $[0,\infty]$ such that $\phi(1) = 0$. The pseudo-inverse of ϕ is the function $\phi^{[-1]}$, with $\mathrm{Dom}\phi^{[-1]} = [0,\infty]$ and $\mathrm{Ran}\phi^{[-1]} = \mathbb{I}$, is given by

$$\phi^{[-1]}(t) = \begin{cases} \phi^{-1}(t), & 0 \le t \le \phi(0) \\ 0, & \text{otherwise.} \end{cases}$$

Further, let ϕ be convex, and $C : \mathbb{I}^2 \to \mathbb{I}$ be given by

$$C(u,v) = \phi^{[-1]}(\phi(u) + \phi(v)).$$

Then, C is called an Archimedean copula generated by the generator function ϕ.

In the following, we present a brief description of aforementioned copulas.

Let $u, v \in \mathbb{I}$.

(1) **Frank copula** is defined as follows:

$$C(u,v) = \frac{-1}{\theta} \ln \left(1 + \frac{(e^{-\theta u} - 1)(e^{-\theta v} - 1)}{e^{-\theta} - 1}\right), \quad \theta \in (-\infty, \infty) \setminus \{0\},$$

with generator function

$$\phi(t) = -\ln\left(\tfrac{\exp(-\theta t) - 1}{\exp(-\theta) - 1}\right).$$

This copula is radially symmetric and allows for both positive dependence (when θ is positive) and negative dependence (when θ is negative). The larger is the value of θ the greater is the dependence between the variables. This copula can not capture the tail dependence.

(2) **Gumbel copula** is defined as follows:

$$C(u,v) = \exp\left(((-\ln u)^{\theta} + (-\ln v)^{\theta})^{\frac{1}{\theta}}\right), \quad \theta \in [1, \infty),$$

with generator function

$$\phi(t) = (-\ln t)^{\theta}.$$

This copula is asymmetric. It allows for positive dependence and captures upper tail dependence; the larger values of θ indicate greater dependence between the variables.

(3) **Clayton copula** is defined as follows:

$$C(u,v) = (u^{-\theta} + v^{-\theta} - 1)^{\frac{-1}{\theta}}, \quad \theta \in (0, \infty),$$

with generator function

$$\phi(t) = \frac{1}{t^{\theta}} - 1.$$

It is specially used to capture the lower tail dependence. The larger is the value of the parameter θ, the greater is the dependence between the variables.

(4) **Gaussian copula** is defined as follow:

$$C(u,v) = \int_{-\infty}^{\Phi^{-1}(u)} \int_{-\infty}^{\Phi^{-1}(v)} \frac{1}{2\pi(1-\theta^2)^{1/2}} \exp\{\frac{-x^2 - 2\theta xy + y^2}{2(1-\theta^2)}\} dxdy,$$

where $\Phi^{-1}(.)$ is the inverse of the standard univariate Gaussian distribution function. This copula is symmetric and shows no tail dependence.

(5) **Joe copula** is defined as

$$C(u,v) = 1 - \left[(1-u)^{\theta} + (1-v)^{\theta} - (1-u)^{\theta}(1-v)^{\theta}\right]^{1/\theta}, \quad \theta \in [1, \infty),$$

with generator function

$$\phi(t) = -\log\left(1 - (1-t)^{\theta}\right).$$

It is used to capture upper tail dependence.