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Distortion measures and homogeneous financial derivatives

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ABSTRACT

This paper extends the evaluation and allocation of distortion risk measures to apply to arbitrary homogeneous operators ("financial derivatives," e.g. reinsurance recovery) of primitive portfolio elements (e.g. line of business losses). Previous literature argues that the allocation of the portfolio measure to the financial derivative should take the usual special-case form of Aumann–Shapley, being a distortion-weighted "co-measure" expectation. This is taken here as the definition of the "distorted" measure of the derivative "with respect to" the underlying portfolio. Due to homogeneity, the subsequent allocation of the derivative's value to the primitive elements of the portfolio again follows Aumann–Shapley, in the form of the exposure gradient of the distorted measure. However, the gradient in this case is seen to consist of two terms. The first is the familiar distorted expectation of the gradient of the financial derivative with respect to exposure to the element. The second term involves the conditional covariance of the financial derivative with the element. Sufficient conditions for this second term to vanish are provided. A method for estimating the second term in a simulation framework is proposed. Examples are provided.

1. Introduction

This paper discusses the allocation of capital or costs in the particular situation where (1) the capital or costs are computed by a distortion risk measure, (2) that measure is applied to a portfolio of liabilities whose total loss is the sum of the component loss random variables, (3) a nonlinear homogeneous risk transformation (e.g., reinsurance) is contemplated in order to decompose the total loss into a ceded portion that will be paid by another party and the complementary retained portion that will remain in the portfolio, (4) it is desired to evaluate the impact of that risk transformation on capital or costs, and (5) it is desired to allocate that impact back to the original component loss random variables.

Distortion measures are an important class of coherent risk measures. Kusuoka (2001) proved that distortion measures are the only law invariant comonotonic additive coherent risk measures. Distortion measures satisfy numerous desirable properties as pricing principles or capital requirements. See Wirch and Hardy (2001), Goovaerts et al. (2003), or Föllmer and Schied (2011, chapter 4) for elaboration of these properties. Distortion measures are equivalent to spectral measures (Acerbi, 2002). Goovaerts et al. (2010) provides background about the origin of distortion risk measures.

Given a continuous, increasing, concave function g mapping [0,1] onto [0,1] and a random variable U with density f existing everywhere[1] and survival function (complement of cumulative distribution function) $S_U(U)$, the *distortion measure* (Wang, 1996) is defined as:

$$E_H[U] \equiv \int_0^\infty g(S_U(u))\,du - \int_{-\infty}^0 (1 - g(S_U(u)))\,du \\ = \int_{-\infty}^\infty u \cdot g'(S_U(u)) \cdot f(u)\,du = E\left[U \cdot g'(S_U(U))\right]. \tag{1}$$

In keeping with the actuarial perspective, positive U represents losses, so high values are undesirable. We will sometimes drop the subscript U in $S_U()$ when it is obvious. An example of $E_H[U]$ might use $g(s) = \min(s, \alpha)/\alpha$ for some $0 < \alpha < 1$, say 0.01. This corresponds to the well-known tail value at risk (TVaR, sometimes known as CVaR, conditional value at risk) measure. Other well-known examples include the proportional hazards transform $g(s) = s^\alpha$, $0 < \alpha \le 1$, and the Wang transform (Wang, 1996) (a specialization of the Esscher–Girsanov transform (Goovaerts and Laeven, 2008)) $g(s) = \Phi(\Phi^{-1}(s) - \lambda)$ where Φ is the cumulative normal distribution function.

Consider a portfolio whose total loss U is the sum of (possibly correlated) component losses:

$$U = \sum_{c=1}^{C} X_c. \tag{2}$$

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[1] This assumption is made to keep the exposition simple; it can be relaxed at the expense of introducing certain housekeeping details.

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If no further subdivision is available, we refer to the X_c as "primitive" components.

If $E_{\mathrm{H}}[U]$ represents a risk-adjusted premium or a capital requirement, it makes sense to inquire as to how it might be allocated fairly among the components $c = 1, \ldots, C$. Numerous reasonable sets of conditions have been proposed.

Aumann–Shapley allocation (Aumann and Shapley, 1974) has emerged as an important allocation principle. In particular, consider Denault's (2001) motivation for the axioms of "coherent allocation" of capital defined by a coherent risk measure[2]:

> We stress fairness, as all constituents are from the same firm, and none should receive preferential treatment for the purpose of this allocation exercise....
> Upon a component joining the firm (or any subset thereof), the total risk capital increases by no more than the component's own risk capital: in all fairness, that component cannot justifiably be allocated more risk capital than it can possibly have brought to the firm....
> If two components make the same incremental contribution to all subportfolios, then their allocation is the same....
> A riskless component is allocated capital equal to its standalone capital.

Denault (2001) elaborates these principles in an extension to the case where components are able to participate fractionally (i.e. be rescaled by a real number) in the total portfolio. The conclusion is that, if the risk measure is sufficiently differentiable, the Aumann–Shapley value "is the only linear coherent allocation principle". Equivalent axioms and conclusions appear in Aubin (1981), Billera et al. (1981), Tasche (1999), and Tsanakas and Barnett (2003).

Applied to the coherent allocation of a distortion measure $E_{\mathrm{H}}[U]$ to the component X_c, one gets:

$$A_H^c[\mathbf{X}] = \frac{\partial}{\partial \nu_c} E_H[\mathbf{v} \cdot \mathbf{X}] = E\left[X_c \cdot g'\left(S_U(U)\right)\right]. \tag{3}$$

Here, the bold-face notation represents the portfolio components as components of a vector $\mathbf{X} = (X_1, \ldots, X_C)$, and the "$\cdot$" symbol indicates inner product between vectors or multiplication between scalars. The middle expression is a differentiation with respect to exposure, a special case of Aumann–Shapley allocation that obtains when the argument is homogeneous of degree one. The partial derivative is evaluated at $\mathbf{v} = (1, 1, \ldots, 1)$. This operation be elaborated upon below.

The right-hand expression in Eq. (3) is sometimes referred to as a *co-measure* (Kreps, 2005), from an analogy with the way that covariance terms $\mathrm{cov}(X_c, U)$ sum to the variance $\mathrm{cov}(U, U) = \mathrm{var}(U)$.

It should be noted that the Aumann–Shapley allocation principle is also compatible with the optimization approaches to capital allocation in Laeven and Goovaerts (2004). It is also a special case of more general allocation principles treated in Dhaene et al. (2012), also organized around optimization. Zaks and Tsanakas (2014) generalize further to derive an optimal solution for the capital allocation in a hierarchy corporate environment, allowing for conflicting objectives, preferences, and beliefs about risks between board members and line managers.

In all of the papers cited above, when allocation is addressed, the objective is allocation of risk from aggregate loss to the components from which it is summed, e.g., lines of business. Typically, it is assumed that when components grow or shrink, they scale linearly.

[2] Here we have replaced his use of the word "portfolio" with our term "component".

Boonen et al. (2017) is distinguished by addressing the situation of *non-homogeneous* scaling in loss aggregations. The aggregate loss is still defined as the sum of component losses, but the distribution of losses is no longer assumed linear in the exposure parameter. That is, while a doubling of scale may double the mean loss, it will not necessarily quadruple the variance of loss. They show that Aumann–Shapley allocation applied to a linearized version of the problem still provides an appropriate solution. Thus the Aumann–Shapley linear scaling model is still relevant to insurance contexts where linear scaling does not, in fact, apply.

Tsanakas (2004) addresses distortion measure allocation to *nonlinear* portfolio components that sum to the aggregate loss. Consider, for example, an excess-of-loss contract (Strain, 1981) that divides the company-wide original ("gross") losses into reinsurance-paid ("ceded") losses and net-of-reinsurance ("retained") losses according to a nonlinear function of the original loss. Tsanakas (2004) shows that the Aumann–Shapley distortion operator (right hand side of Eq. (3)) is the appropriate way to allocate the capital associated with the company-wide loss down to the ceded and retained portions. This enables one to apply distortion measures to evaluate the efficacy of nonlinear derivative contracts like reinsurance. A restricted version of Tsanakas's conclusion is offered below as Definition 3.

The present paper investigates the *further* allocation of the nonlinear portfolio's capital down to the primitive components. For example, once the gross loss has been transformed to the net loss, how should that new capital amount now be allocated to the lines of business? The retained loss no longer has a natural decomposition into the sum of contributions from the lines of business in the way that the gross loss did.

How one should allocate the capital impact of a reinsurance contract to the lines of business is an important question in the (re)insurance industry. To this author's knowledge, it has not been addressed. In cases where reinsurance can be modeled as a homogeneous financial derivative, this paper provides both a theoretical (closed-form) answer and strategies for computing that answer in a simulation context where closed-form distributional assumptions are not available or applicable.

The remainder of this paper is organized as follows.

In Section 2, the basic mathematical setup is introduced, and financial derivatives of portfolio components are defined as operators on random variables. Examples are provided. The "distorted measure of a financial derivative with respect to the underlying" is defined.

Section 3 addresses the coherent allocation of a homogeneous financial derivative's distorted measure to components and equates it to the sum of two expectations, one a gradient co-measure (analogous to Eq. (3)), and the other involving conditional covariances. Sufficient conditions for the second term to vanish are provided. An example is given where it will not vanish, in general. Sub-allocation is discussed.

Section 4 provides an analytical case study in allocating the distorted expectation of an aggregate excess-of-loss contract to bivariate normal components. The results are given with closed-form formulas.

Section 5 outlines an approach to estimating the allocation in a Monte Carlo simulation setting.

Section 6 provides a numerical case study of the simulation approach. The R code used to implement the example is available from the author.

Section 7 concludes.

2. Financial derivatives of portfolio components

2.1. Financial derivative operators

Definition 1 (*Financial Derivative*). Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a probability space and $\mathscr{X}$ the set of its $\mathcal{P}$-measurable scalar (real-valued) random variables. A financial derivative operator $\mathscr{F}$ is a mapping from $\mathscr{X}^C$ to $\mathscr{X}$ that takes the vector random variable $\mathbf{X} = (X_1, \ldots, X_C)$ to a real-value random variable (the financial derivative) $Z = \mathscr{F}(X_1, \ldots, X_C)$. The variable $\mathbf{X}$ is termed the underlying.

Remark. Denote $\mathscr{U}(\mathbf{X}) = X_1 + \cdots + X_C = U$ as the sum portfolio loss operator.

We say that such an operator is *linear* if $\mathscr{F}(a \cdot \mathbf{X} + b \cdot \mathbf{Y}) = a \cdot \mathscr{F}(\mathbf{X}) + b \cdot \mathscr{F}(\mathbf{Y})$ for any real a, b. We say that such an operator is *homogeneous* (understood to mean degree one) if $\mathscr{F}(t\mathbf{X}) = t\mathscr{F}(\mathbf{X})$ for any real $t > 0$. Clearly, a linear operator is homogeneous, whereas a homogeneous operator might not be linear. Linear combinations of homogeneous operators are homogeneous.

Consider, for example, the following operators on $\mathbf{X} = (X_1, X_2)$:

$$\begin{aligned}
&\mathscr{Z}^{(a)} : \mathbf{X} \to Z = X_1 + X_2 + E\left[X_1 + X_2\right] \\
&\mathscr{Z}^{(b)} : \mathbf{X} \to Z = \max(0, X_1 + X_2 - E\left[X_1 + X_2\right]) \\
&\mathscr{Z}^{(c)} : \mathbf{X} \to Z = X_1 + X_2 + Q_\alpha\left[X_1 + X_2\right] \\
&\mathscr{Z}^{(d)} : \mathbf{X} \to Z = \max(0, \min(X_1 + X_2 - Q_\alpha\left[X_1 + X_2\right], \\
&\quad Q_\beta\left[X_1 + X_2\right] - Q_\alpha\left[X_1 + X_2\right])).
\end{aligned} \tag{4}$$

Here, the notation $Q_\alpha[X]$ signifies the α-quantile (also known as Value at Risk or VaR) of the random variable X.

All the above operators are homogeneous. $\mathscr{Z}^{(a)}$ is linear. $\mathscr{Z}^{(b)}$ is not linear due to the max( ) function. $\mathscr{Z}^{(c)}$ is not linear because quantile operators are not linear in general.

$\mathscr{Z}^{(d)}$ can be recognized as the payout of an excess-of-loss reinsurance contract (equivalently, a call spread) with $1 - \alpha$ probability of attachment (first dollar payment) and $1-\beta$ probability of exhaustion (full payment of the limit, $Q_\beta - Q_\alpha$). It is not linear.

There are two distinct ways in which the argument $\mathbf{X}$ is used in equation block (4). On the one hand, it is used pointwise, in the sense that for any $\omega \in \Omega$, $Z(\omega)$ is a function of $X_1(\omega)$ and $X_2(\omega)$. On the other hand, it is also used holistically, in the sense that $E[X_1{+}X_2]$ or $Q_\alpha[X_1{+}X_2]$ is the same constant value over all choices of ω, but those constants would generally be different for different random variable arguments.

Definition 2 (*Exposure Differentiation*)**.** Let $\mathscr{F}$ be a financial derivative operator. The partial derivative with respect to exposure in component c is the random variable $\mathscr{F}_c[\mathbf{X}]$ whose value at outcome ω is

$$\mathscr{F}_c\left[\mathbf{X}\right](\omega) \equiv \frac{\partial}{\partial \nu_c}\left(\mathscr{F}\left[\mathbf{v} * \mathbf{X}\right](\omega)\right), \tag{5}$$

evaluated at $\mathbf{v} = (1, 1, \ldots, 1)$. Here, the symbol "$*$" signifies Hadamard (elementwise) product of vectors.

Remark. For any particular $\mathbf{v}$, $\mathbf{X} \to \mathscr{F}[\mathbf{v} * \mathbf{X}]$ is a financial derivative. On the other hand, the value of $\mathscr{F}[\mathbf{v} * \mathbf{X}]$ at any particular ω is an ordinary real-valued function of $\mathbf{v}$, which can be differentiated. When extra clarity is needed, we may use the notation $\mathscr{F}_c\left[\mathbf{X}\right] = \frac{\partial}{\partial \nu_c}\mathscr{F}[\mathbf{v} * \mathbf{X}]$.

Theorem 1 (*Euler's Theorem for Homogeneous Financial Derivatives*)**.** *Let $\mathscr{F}$ be a homogeneous financial derivative operator. Then*

$$\mathscr{F}[\mathbf{X}] = \sum_c \mathscr{F}_c\left[\mathbf{X}\right]. \tag{6}$$

Proof. This follows because $\mathscr{F}\left[\mathbf{v} * \mathbf{X}\right](\omega)$ is a homogeneous function of $\mathbf{v}$. ∎

Remark. When applied to $\mathscr{U}[\mathbf{X}]$, this restates Eq. (2).

Second-order exposure differentiation is also possible, with

$$\mathscr{F}_{i,j}\left[\mathbf{X}\right](\omega) \equiv \frac{\partial}{\partial \nu_i}\left(\frac{\partial}{\partial \nu_j}\left(\mathscr{F}\left[\mathbf{v} * \mathbf{X}\right](\omega)\right)\right), \tag{7}$$

again evaluated at $\mathbf{v} = (1, 1, \ldots, 1)$.

2.2. Distortion measures of financial derivatives

Given the example $Z = \mathscr{Z}^{(d)}(\mathbf{X})$ from equation block (4), consider the decomposition of the portfolio total loss $U = X_1 + X_2$ into $U = Y + Z$ where $Y = U - Z$. This splits the *gross loss* U, into the *net loss* Y and *ceded loss* (reinsurance payment), Z.

How should one allocate $E_{\mathrm{H}}[U]$ to Y and Z?

Billera et al. (1981) set out numerous desirable properties that a cost allocation procedure should possess, and found Aumann–Shapley to uniquely satisfy them. Their property 3, additivity, was described as follows:

> Stated more formally, suppose that the cost of a process $f(x_1, x_2)$ is decomposed into two costs, $k(x_1, x_2)$ and $h(x_1, x_2)$: $f(x_1, x_2) = k(x_1, x_2) + h(x_1, x_2)$. We say a cost allocation procedure is additive if it yields an identical per unit charge regardless of whether the procedure is applied directly to the process $f(a_1, a_2)$, or indirectly to each of the different cost components of the process, $k(a_1, a_2)$, and $h(a_1, a_2)$, and then summed.

Here we envision the portfolio loss distortion measure "cost" (as a function of exposure $\mathbf{v}$) taking the role of f, being decomposed into the cost of the ceded loss (reinsurance payout) k and that of the retained (net of reinsurance) loss h.

Tsanakas (2004), as previously noted, found that when nonlinear components sum to the aggregate, the appropriate allocation to them is still Aumann–Shapley.

This motivates the following:

Definition 3 (*Distorted Measure of a Financial Derivative with Respect to the Underlying*)**.** Let $\mathscr{F}$ be a financial derivative operator taking a vector random variable $\mathbf{X} = (X_1, \ldots, X_C)$ to a scalar random variable Z. The distorted measure of $\mathscr{F}$ with respect to the underlying portfolio of primitive components $\mathbf{X}$ is defined as:

$$E_{H,\mathscr{F}}[\mathbf{X}] \equiv E\left[\mathscr{F}(\mathbf{X}) \cdot g'\left(S_U\left(U\right)\right)\right] \tag{8}$$

where $U = X_1 + \cdots + X_C$.

Remark. Regarding the previous discussion, clearly we will have $f = E_{\mathrm{H}}[U] =$ the sum of $E_{\mathrm{H},\mathscr{F}}[\mathbf{X}] = k + h$.

Remark. The terminology adopted here distinguishes between "distor*tion* measure" and "distor*ted* measure". If the derivative $\mathscr{F}(\mathbf{X})$ were considered in isolation, its distortion measure would be $E_{\mathrm{H}}[\mathscr{F}(\mathbf{X})]$, using $g'(S_{\mathscr{F}(\mathbf{X})}(\mathscr{F}(\mathbf{X})))$ instead of $g'(S_U(U))$. This measure would not, in general, be equal to the "distorted" $E_{\mathrm{H},\mathscr{F}}[\mathbf{X}]$. Furthermore, the sum of the measures of complementary loss random variables, $E_{\mathrm{H}}[\mathscr{F}(\mathbf{X})] + E_{\mathrm{H}}[U - \mathscr{F}(\mathbf{X})]$, would not in general equal that of the total losses, $E_{\mathrm{H}}[U]$.

Next we consider allocation down from $\mathscr{F}(\mathbf{X})$ to the primitive components.

3. Allocating homogeneous financial derivatives

If $\mathscr{F}$ is a homogeneous financial derivative operator, then it is easily verified that the distorted measure $E_{\mathrm{H}\mathscr{F}}[\mathbf{X}]$ is also homogeneous in the sense that

$$E_{H,\mathscr{F}}[\mathbf{v} * \mathbf{X}] \equiv E\left[\mathscr{F}(\mathbf{v} * \mathbf{X}) \cdot g'\left(S_{\mathbf{v} \cdot \mathbf{X}}\left(\mathbf{v} \cdot \mathbf{X}\right)\right)\right], \tag{9}$$

as a real-valued function of $\mathbf{v}$, is homogeneous.

Theorem 2 (*Coherent Allocation of Financial Derivatives*)**.** *If $\mathcal{F}$ is a homogeneous financial derivative operator, then the coherent allocation of $E_{\mathrm{H},\mathcal{F}}[\mathbf{X}]$ to the components X_c of $\mathbf{X}$ is given by*

$$A^c_{H,\mathcal{F}}[\mathbf{X}] \equiv \frac{\partial}{\partial \nu_c} E\left[\mathcal{F}(\mathbf{v} * \mathbf{X}) \cdot g'\left(S_{\mathbf{v}\cdot\mathbf{X}}(\mathbf{v}\cdot\mathbf{X})\right)\right]\bigg|_{\mathbf{v}=(1,\ldots,1)}. \tag{10}$$

Proof. This follows from Aumann–Shapley allocation applied to a homogeneous function of $\mathbf{v}$. ■

Theorem 3 (*Coherent Allocation is the Sum of Two Expectations*)**.** *If the distribution of $\boldsymbol{X}$ and the smoothness of $\mathcal{F}$ are such that differentiation on $\mathbf{v}$ can be exchanged with expectation, then the following is equivalent to* Eq. (10)*:*

$$A^c_{H,\mathcal{F}}[\mathbf{X}] = E\left[\frac{\partial}{\partial \nu_c}\mathcal{F}(\mathbf{v} * \mathbf{X})\bigg|_{\mathbf{v}=(1,\ldots,1)} \cdot g'\left(S\left(U\right)\right)\right] + E_2 \tag{11}$$

where

$$E_2 = -E_U\left[f_U\left(U\right) \cdot \mathrm{cov}\left[\mathcal{F}(\mathbf{X}), X_c | U\right] \cdot g''\left(S\left(U\right)\right)\right], \tag{12}$$

and $f_U(U)$ is the marginal density of U.

Proof. See Appendix A. ■

Remark. There are two sources of complication here. First, the exposure differentiation of $\mathcal{F}(\mathbf{v}\cdot\mathbf{X})$ appearing in Eq. (11) may not resolve nicely if $\mathcal{F}$ is not linear. Second, there is the E_2 term, involving a weighted expectation of the conditional covariance of the financial derivative with the underlying cth component.

Remark. Since g is assumed concave, g'' will be everywhere non-positive. If the conditional covariance between $\mathcal{F}$ and X_c is consistently of one sign in some region of U values where $f \cdot g'' \neq 0$, and the product of the covariance and $f \cdot g''$ is zero elsewhere, then E_2 will take on the same sign. If the conditional covariance takes on both signs, then the sign of E_2 will depend on the relative weightings provided by $f^2 \cdot g''$.

From now on, in any discussion of allocation, we will assume that the premises of Theorem 3 hold.

3.1. Example exposure gradient of $\mathcal{F}$

Consider the example $Z = \mathcal{Z}^{(\mathrm{d})}[\mathbf{X}]$ of equation block (4), differentiating on the first exposure component. We need to express the new random variable:

$$\begin{aligned} Z_1 &\equiv \frac{\partial}{\partial \nu_1}\mathcal{Z}^{(d)}\left(\nu_1 \cdot X_1, X_2\right) \\ &= \frac{\partial}{\partial \nu_1}\max(0, \min(\nu_1 \cdot X_1 + X_2 - Q_\alpha\left[\nu_1 \cdot X_1 + X_2\right], \\ &\quad Q_\beta\left[\nu_1 \cdot X_1 + X_2\right] - Q_\alpha\left[\nu_1 \cdot X_1 + X_2\right])). \end{aligned} \tag{13}$$

The partial derivative when the contract has not attached (payoff has not started) is zero. When it has exhausted (reached its maximum), the partial derivative is

$$\begin{aligned} Z_1 &= \frac{\partial}{\partial \nu_1}\left(Q_\beta\left[\nu_1 \cdot X_1 + X_2\right] - Q_\alpha\left[\nu_1 \cdot X_1 + X_2\right]\right) \\ &= E\left[X_1 | U = Q_\beta\right] - E\left[X_1 | U = Q_\alpha\right] \end{aligned} \tag{14}$$

where on the right hand side the notation has been simplified in an obvious manner. See Tasche (2001) for the gradient of a quantile.

The derivative in the interior range is given by:

$$\begin{aligned} Z_1 &= \frac{\partial}{\partial \nu_1}\left(\nu_1 \cdot X_1 + X_2 - Q_\alpha\left[\nu_1 \cdot X_1 + X_2\right]\right) \\ &= X_1 - E\left[X_1 | U = Q_\alpha\right]. \end{aligned} \tag{15}$$

Putting the pieces together,

$$Z_1 = \begin{cases} 0, & U < Q_\alpha \\ X_1 - E\left[X_1 | U = Q_\alpha\right], & Q_\alpha \le U < Q_\beta \\ E\left[X_1 | U = Q_\beta\right] - E\left[X_1 | U = Q_\alpha\right], & Q_\beta \le U. \end{cases} \tag{16}$$

It can easily be verified that this is an allocation: $Z_1 + Z_2 = Z$.

3.2. The E_2 term

In general, the E_2 term is rather complex. There are a few things that can be said, however.

Theorem 4 (E_2 *is Zero Sum*)**.** *The sum of the E_2 allocation components is zero.*

Proof. Summing Eq. (12) across components c, we see that in collecting the covariance terms, we obtain $\mathrm{cov}[\mathcal{F}, U|U]$ which equals zero because U is constant given U. ■

This implies that the first term of Eq. (11), across c, does indeed form an allocation, but it only happens to be the Aumann–Shapley allocation if the corresponding E_2 terms are all zero.

Here are some situations where the E_2 term will vanish.

Theorem 5 (*Sufficient Conditions for $E_2 = 0$*)**.** *The following are sufficient conditions for $E_2 = 0$:*

(a) *$\mathcal{F} = \mathcal{H} \circ \mathcal{U}$, that is, $\mathcal{F}(\boldsymbol{X})$ can be expressed as $\mathcal{H}(U)$ operating on $U = X_1 + \cdots + X_C$ alone.*
(b) *The conditional distribution of each X_c on U consists of a degenerate distribution (i.e., X_c is a deterministic function of U).*
(c) *Each X_c is conditionally (on U) uncorrelated with $\mathcal{F}(\boldsymbol{X})$. (This generalizes item b.)*

Proof. Referring to Eq. (12), in case (a), $\mathcal{F}(\mathbf{X}) = \mathcal{H}(U)$ conditional on U is constant, and therefore the covariance term is zero. In case (b) it is X_c in the covariance term which is constant. Case (c) directly implies the covariance is zero. ■

Case (a) is likely to occur when $\mathcal{F}$ represents aggregate reinsurance. Case (b) is unlikely to occur in "natural" settings. Case (c) might occur with a component X_c when $\mathcal{F}$ does not involve X_c and X_c is independent of the other components; however, it is not likely to occur over all components simultaneously.

It should be noted that case (b) occurs with high probability in a Monte Carlo setting where the distribution of U has a density everywhere (no mass points) except possibly at $U = 0$ (where all X_c must also be zero). In this situation, the probability of two realizations of U taking on the same nonzero value is effectively zero,[3] and so the conditional distribution of X_c on U is represented by one realization (or realizations of zero). This is not a fact to be celebrated; it is to be lamented. Without some extra work, a naïve Monte Carlo simulation will not provide a consistent estimate of the coherent allocation of $\mathcal{F}$. This is the subject of the next section.

For all the examples of equation block (4), $E_2 = 0$ because case (a) applies.

An example where E_2 is not zero is given by:

$$\begin{aligned} \mathcal{Z}^e[\mathbf{X}] = \max&\left(0, \min\left(X_1 + X_2 - Q_\alpha\left[X_1 + X_2\right],\right.\right. \\ &\left.\left.\min\left(X_1, Q_\beta\left[X_1 + X_2\right] - Q_\alpha\left[X_1 + X_2\right]\right)\right)\right). \end{aligned} \tag{17}$$

This differs from $\mathcal{Z}^{(\mathrm{d})}$ of equation block (4) in that the limit has been replaced by the minimum of X_1 and the previous limit term $Q_\beta - Q_\alpha$. The payoff is still based on the excess of $X_1 + X_2$ over a threshold, but it is limited to not exceed the loss of X_1 by itself.

[3] Assuming sufficient machine precision.

Clearly, it is homogeneous. For a given $U = X_1 + X_2 > Q_\beta$, there are two possibilities. If $X_1 > Q_\beta - Q_\alpha$, then Z is constantly $Q_\beta - Q_\alpha$; otherwise $Z = X_1$. This gives Z a nonzero (and nontrivial) conditional covariance with X_1 for those values of U. These conditional covariances will not, in general, integrate to zero over the entire expectation of Eq. (12).

3.3. Sub-allocation

We can look at $a_1 = A_H^1[\mathbf{X}] = E\left[X_1 \cdot g'(S(U))\right]$ in two ways. On the one hand, it is the component-one allocation of the distortion measure $E_H[U]$. On the other hand, we can consider it the distorted measure $E_{H,\mathcal{P}}[\mathbf{X}]$ of the projection derivative $\mathcal{P}(\mathbf{X}) = X_1$. In the second interpretation, we can ask, per Theorems 2 and 3, how do we allocate $E_{H,\mathcal{P}}[\mathbf{X}]$? This is equivalent to asking, in the first interpretation, how do we *further* allocate, or *sub-allocate*, a_1?

Because a_1 is homogeneous, it can be allocated using Theorem 3. In general, however, one will find that $E_2 \neq 0$, and so, for example, the allocation of a_1 to the first component does not equal a_1. This might seem paradoxical. Should not all of a_1, which, after all, belongs to the first component, go to the first component upon suballocation? Not necessarily. Suballocation, following Aumann–Shapley logic, considers the marginal impact each component has on a_1. Since the distortion measure $E_H[]$ is affected by all components through their influence on $S_U(U)$, it is to be expected that they all influence a_1.

Fortunately, this does not result in an inconsistent appraisal of the first component's share of $E_H[U]$:

Theorem 6 (*Sub-allocations Add up from the First Level*)**.** *Let $\mathcal{F}$ be a homogeneous financial derivative operator whose distorted measure allocations $A^c_{H,\mathcal{F}}[\mathbf{X}]$ all have $E_2 = 0$. Assume that the partial derivative operators $\mathcal{F}_c[\mathbf{X}]$ are themselves homogeneous and that all $\mathcal{F}_{i,j}[\mathbf{X}]$ are continuous. Then*

$$A^c_{H,\mathcal{F}}[\mathbf{X}] = \sum_i A^c_{H,\mathcal{F}_i}[\mathbf{X}]. \tag{18}$$

Proof. See Appendix B. ■

Remark. While the first-level allocation to component c, $A^c_{H,\mathcal{F}}[\mathbf{X}]$, is, on the second level, sub-allocated to other components, the other first-level components are also being sub-allocated, in part, to component c. The sum of those second-level allocations to component c equals the original first-level allocation. Fig. 1 illustrates this.

4. Analytical case study

This section presents an example of a distorted measure applied to a homogeneous financial derivative and its allocation to the underlying components. Formulas are suitable for implementation in a spreadsheet.

4.1. The underlying

Consider the underlying random variable $\mathbf{X} = (X_1, X_2)$ distributed as a bivariate normal with mean vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$, its p.d.f. being

$$\phi(\mathbf{X}, \boldsymbol{\mu}, \boldsymbol{\Sigma}) = \frac{1}{2 \cdot \pi \cdot \sqrt{\det(\boldsymbol{\Sigma})}} \cdot \exp\left(-\frac{1}{2} \cdot (\mathbf{X} - \boldsymbol{\mu})' \cdot \boldsymbol{\Sigma}^{-1} \cdot (\mathbf{X} - \boldsymbol{\mu})'\right). \tag{19}$$

The aggregate loss $U = X_1 + X_2$ is therefore distributed as a normal with mean $\mu_U = \mu_1 + \mu_2$ and variance $\sigma_U^2 = \sigma_1^2 + 2 \cdot \rho \cdot \sigma_1 \cdot \sigma_2 + \sigma_2^2$. Define $S(U) = 1 - F_U(U)$.

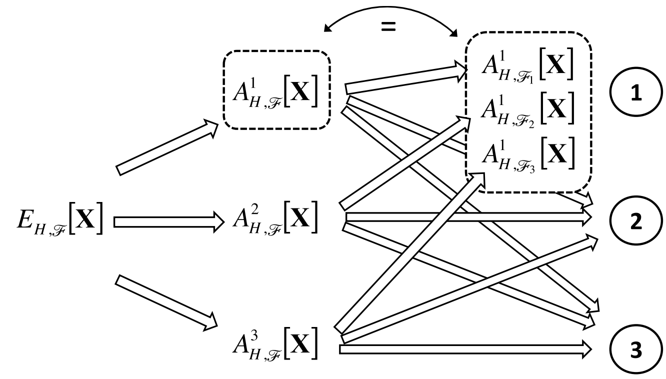

Fig. 1. Sub-allocations into component 1 sum up to equal the first-level allocation.

4.2. The distortion

Consider the Wang transform distortion measure. This transforms an arbitrary cumulative distribution function F into H:

$$H(U) = \Phi\left(\Phi^{-1}(F(U)) - \lambda\right). \tag{20}$$

In the case of the normal F_U, the c.d.f. H is also normal, with the same variance σ_U^2 and shifted mean $\mu_U + \Delta_U$ where $\Delta_U = \lambda \cdot \sigma_U$. This leads to

$$g'(S(U)) = \exp\left(\frac{\Delta_U}{\sigma_U^2} \cdot \left(U - \mu_U - \frac{\Delta_U}{2}\right)\right). \tag{21}$$

It can be shown that

$$\phi(\mathbf{X}, \boldsymbol{\mu}, \boldsymbol{\Sigma}) \cdot g'(S(X_1 + X_2)) = \phi(\mathbf{X}, \boldsymbol{\eta}, \boldsymbol{\Sigma}) \tag{22}$$

where $\boldsymbol{\eta} = \boldsymbol{\mu} + \boldsymbol{\beta} \cdot \Delta_U$ and $\boldsymbol{\beta}$ are the regression coefficients of $\mathbf{X}$ on U:

$$\begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \frac{1}{\sigma_U^2} \cdot \begin{pmatrix} (\sigma_1 + \rho \cdot \sigma_2) \cdot \sigma_1 \\ (\rho \cdot \sigma_1 + \sigma_2) \cdot \sigma_2 \end{pmatrix}. \tag{23}$$

This means that with the distribution of $\mathbf{X}$, distorted expectations of the form $E[\Theta \cdot g'(S(U))]$ can be obtained as $E_H[\Theta]$, the expectation after a change of measure, shifting the mean by $\boldsymbol{\beta} \cdot \Delta_U$.

4.3. The financial derivative

Consider the aggregate excess-of-loss contract on U defined by the attachment and exhaustion parameters Att and Exh (themselves defined as particular quantiles of U):

$$AX(U) = \max(0, \min(U - Att, Exh - Att)). \tag{24}$$

This is an instance of $\mathcal{L}^{(d)}$ from equation block 4. The distorted expectation of AX is given by

$$\begin{aligned} E_H[AX(U)] = &\left(\mu' - Att\right) \cdot (\Phi(e) - \Phi(a)) \\ &- \sigma_U \cdot (\phi(e) - \phi(a)) \\ &+ (Exh - Att) \cdot (1 - \Phi(e)) \end{aligned} \tag{25}$$

where $\mu' = \mu_U + \Delta_U$, $a = (Att - \mu')/\sigma_U$, and $e = (Exh - \mu')/\sigma_U$. This follows from the well-known properties of the truncated normal distribution.

4.4. Coherent allocation

Per Theorem 3 there are two terms to the allocation of $E_H[AX]$ to X_1, but per Theorem 5 part (a) the E_2 term is zero:

$$A^1_{H,AX}[\mathbf{X}] = E_H\left[\left.\frac{\partial}{\partial \nu_1} AX(\boldsymbol{\nu} * \mathbf{X})\right|_{\boldsymbol{\nu}=(1,\ldots,1)}\right]. \tag{26}$$

The first gradient component of AX was presented in Eq. (16) as Z_1, so:

$$A^1_{H,AX}[\mathbf{X}] = \iint_{Att<U<Exh} Z_1 \cdot \phi(\mathbf{x}, \boldsymbol{\eta}, \boldsymbol{\Sigma})\, dx_1 dx_2 + \iint_{Exh<U} Z_1 \cdot \phi(\mathbf{x}, \boldsymbol{\eta}, \boldsymbol{\Sigma})\, dx_1 dx_2. \tag{27}$$

Change variables to (x_1, U) and factor the joint distribution into conditional and marginal:

$$A^1_{H,AX}[\mathbf{X}] = \int_{Att}^{Exh} f_U(U) \cdot \int_{-\infty}^{\infty} Z_1 \cdot f_{X_1|U}(x_1, U)\, dx_1 dU + (E[X_1|U = Exh] - E[X_1|U = Att]) \cdot (1 - \Phi(e)). \tag{28}$$

The conditional expectations of $\mathbf{X}$ given U for this bivariate normal example are given by the regression:

$$E[\mathbf{X}|U] = \boldsymbol{\mu} + \boldsymbol{\beta} \cdot (U - \mu_U). \tag{29}$$

Substituting into Z_1, evaluating conditional expectations, and simplifying, we arrive at

$$A^1_{H,AX}[\mathbf{X}] = \beta_1 \cdot \int_{Att}^{Exh} (U - Att) \cdot f_U(U)\, dU + \beta_1 \cdot (Exh - Att) \cdot (1 - \Phi(e)). \tag{30}$$

Finally,

$$A^1_{H,AX}[\mathbf{X}] = \beta_1 \cdot E_H[AX(U)]. \tag{31}$$

4.5. Numerical instantiation

If we take $\boldsymbol{\mu} = (7, 11)$, $\boldsymbol{\sigma} = (1, 1.5)$, $\rho = -0.4$, then $\mu_U = 18$, $\sigma_U = 1.432$, and $\boldsymbol{\beta} = (0.195, 0.805)$. If we further set $Att = Q_{0.5} = 18$, $Exh = Q_{0.9} = 19.835$, and $\Delta_U = 1$, then the following results are obtained: (undistorted) $\mathrm{E}[AX] = 0.503$, and (distorted) $E_{\mathrm{H}}[AX] = 0.957$. The allocation vector of $\mathrm{E}[AX]$ to the two components is $(0.098, 0.405)$ and the allocation of $E_{\mathrm{H}}[AX]$ is $(0.187, 0.770)$.

5. Evaluating and allocating distortion measures by Monte Carlo

This section presumes the existence of a sample of N realizations $\mathbf{X}(\omega)$, $(\omega = 1, \ldots, N)$ of the joint distribution of the portfolio component losses $\mathbf{X} = (X_1, \ldots, X_C)$, each with associated probability $p(\omega)$ summing to 1. As before, the aggregate loss $U = \mathscr{U}(\mathbf{X})$ is the sum of the component losses. The realizations are assumed sorted so that $U(\omega) < U(\omega + 1)$. Analytical formulas for the distortion function $g()$ and its second derivative $g''()$ are assumed as given.

Define the survival function:

$$S(\omega) = \sum_{i=\omega}^{N} p(i). \tag{32}$$

Let $Z(\omega)$ be the realization of $Z = \mathscr{F}(\mathbf{X})$ at ω and let $Z_c(\omega)$ be the realization of $\mathscr{F}_c[\mathbf{X}] = \frac{\partial}{\partial \nu_c} \mathscr{F}[\mathbf{v} * \mathbf{X}]$ at ω. The following are offered without proof. Interested readers are referred to Hammersley and Handscomb (1964) or Rubinstein and Kroese (2016).

A consistent estimator for $E_{\mathrm{H},\mathscr{F}}[\mathbf{X}]$ is:

$$\widehat{E}_{H,\mathscr{F}}[\mathbf{X}] = \sum_{\omega=1}^{N} Z(\omega) \cdot (g(S(\omega)) - g(S(\omega + 1))) \tag{33}$$

where $S(m)$ for $m > N$ is understood to be zero.

A consistent estimator for the allocation of $E_{\mathrm{H},\mathscr{F}}[\mathbf{X}]$ to the cth component (per Theorem 3) is:

$$\hat{A}^c_{H,\mathscr{F}}[\mathbf{X}] = \sum_{\omega=1}^{N} Z_c(\omega) \cdot (g(S(\omega)) - g(S(\omega + 1))) + \widehat{E}_2 \tag{34}$$

where

$$\hat{E}_2 = -\sum_{\omega=1}^{N} \left(Z(\omega) - \widehat{E}(Z_c|U(\omega))\right) \cdot \left(X_c(\omega) - \widehat{E}(X_c|U(\omega))\right) \cdot \widehat{f}_U(U(\omega)) \cdot g''(S(\omega)) \cdot p(\omega). \tag{35}$$

In Eq. (35), the three "hatted" terms need to be defined. Inside the first two factors (the covariance product) are conditional expectation operators $\widehat{E}(\bullet|U(\omega))$. These may be estimated by locally weighted regression (loess) (Loader, 1999) or splines (Green and Silverman, 1994). The probability density f_U may be estimated by kernel smoothing methods (Wand and Jones, 1994). The example in Section 5 discusses implementation in the R programming language.

It should be noted that in some cases the realizations $Z_c(\omega)$ may not be known in closed-form. This is the case with excess-of-loss financial derivatives whose gradients involve conditional expectations where the underlying distribution is only known through simulation. In such a case, locally weighted regression or splines might be used to provide a consistent estimator.

6. Numerical example

In this section we address the numerical estimation of $E_{\mathrm{H}\mathscr{F}}[\mathrm{X}]$ for four homogeneous financial derivatives and the allocation thereof to primitive portfolio components.

6.1. Preliminaries

The underlying consists of (X_1, X_2) distributed as bivariate normal with means of 7 and 11, respectively, standard deviations of 1 and 1.5, respectively, and correlation coefficient -0.4.

The four financial derivatives are described in Table 1.

In terms of previous discussion in the paper, "U" is $\mathscr{U}(\mathbf{X})$ from the remark after Definition 1, "AX" is an instance of $\mathscr{Z}^{(d)}$ from equation block (4), "SX" is the sum of two such instances, and "VX" is an instance of $\mathscr{Z}^{(e)}$ from Eq. (17).

The method is straightforward Monte Carlo simulation. The R code implementing these calculations is available from the author. One million samples of (X_1, X_2) are drawn. The sample statistics are given in Table 2.

Despite the fact that quantiles of a normal distribution are readily calculated, functions were defined to compute quantiles from the sample via smoothing splines, using R's built-in `smooth.spline( )` function. These were used to compute the quantiles specified in Table 1. The sample (U, X1, X2) vectors were then augmented with the associated values of AX, SX, and VX.

Equal-probability sampling was used, so the weight $p = 10^{-6}$ was associated with each draw. After sorting the samples in descending value of U, the exceedance probability S (Eq. (32)) was computed as the cumulative sum of p.

6.2. Distortion measure

For this example, the distortion function $g(s) = \sqrt{s}$ was used. Values of $E_{\mathrm{H},\mathscr{F}}[\mathrm{X}]$ were computed according to Eq. (33). The results are in Table 3.

As expected, the distorted measures of the payouts are larger than their mean values.

Table 1
Target financial derivatives.

Symbol	Name	Description
U	Portfolio gross loss	$U = X_1 + X_2$
AX	Aggregate XOL	Payoff of U in excess of $Q_{50\%}$(U); exhausts at $Q_{90\%}$(U)
SX	Silo XOLs	Sum of individual payoffs: X_1 in excess of $Q_{50\%}$ up to $Q_{90\%}$, plus X_2 in excess of $Q_{80\%}$ up to $Q_{95\%}$.
VX	Variable limit XOL	Payoff of U in excess of $Q_{0.1\%}$(U); exhausts at $Q_{99.37\%}$(U), but payoff limited to X_1.

Table 2
Summary statistics of $N = 1{,}000{,}000$ bivariate normal draws.

Variable	Mean	Standard deviation
U	18.0028	1.4304
X1	7.0010	1.0002
X2	11.0018	1.4992

Correlation coefficient: −0.4008.

Table 3
Distorted measures.

Quantity	U	AX	SX	VX
Mean	18.003	0.50283	0.48761	4.4965
E_H	19.009	0.93645	0.76571	5.3605

Table 4
Aumann–Shapley allocations by definition.

Quantity	U	AX	SX	VX
E_H	19.009	0.93645	0.76571	5.3605
A_H^1	7.196	0.18412	0.51364	2.0765
A_H^2	11.813	0.75242	0.25217	3.2844
E_H - $(A^1 + A^2)$	0.00002	−0.00009	−0.00010	−0.0004
Approx σ(A)	0.002	0.01	0.002	0.02

Table 5
Allocation E_1 shortfalls.

Quantity	U	AX	SX	VX
A_H^1	7.196	0.18412	0.51364	2.0765
A_H^2	11.813	0.75242	0.25217	3.2844
E_1^1	7.196	0.18273	0.43698	1.9721
E_1^2	11.813	0.75371	0.32873	3.3883
Shortfall^1	−1.6E−06	0.00139	0.07666	0.1043
Shortfall^2	−2.1E−05	−0.00130	−0.07656	−0.1039
Approx σ(sf)	3E−5	0.006	0.002	0.01

Table 6
E_2 vs. E_1 shortfall.

Quantity	U	AX	SX	VX
Shortfall^1	−1.6E−06	0.00139	0.07666	0.1043
Shortfall^2	−2.1E−05	−0.00130	−0.07656	−0.1039
E_2^1	−5.1E−11	−2.0E−08	0.08038	0.1235
E_2^2	5.1E−11	2.0E−08	−0.08038	−0.1235
Approx $\sigma(E_2)$	7E−12	1E−7	0.0045	0.0046
E_2^1 - Shortfall^1	1.6E−06	−0.00139	0.0037	0.0192
E_2^2 - Shortfall^2	2.1E−05	0.00130	−0.0038	−0.0196
Approx $\sigma(E_2$-sf)	3E−05	0.006	0.005	0.009

6.3. Allocation by definition

Because all of these financial derivatives are homogeneous, their Aumann–Shapley allocations are defined by Theorem 2 (Eq. (10)). This was implemented by a secant approximation to the gradient. For example, consider the allocation to the component $c = 1$. Replace all X1 values by 1.01 · X1, and recalculate all of the other variables, including S, in a new sort order. Compute the E_H values corresponding to Table 3. Repeat using 0.99 · X1. Take the difference between the two E_H values and divide by (2 · 0.01). The result is the estimate of the allocation defined in Eq. (10).

The results are shown in Table 4. While the two allocation components should sum to the original measure, there is a slight numerical discrepancy which is also shown. Being sample-based estimates, these figures are themselves subject to sampling error. A separate Monte Carlo study suggested the standard deviations of the A figures posted in the last row.

Clearly, the estimators for A_H^1 and A_H^2 are highly negatively correlated.

6.4. Allocation by formula, without E_2

The first term of Eq. (11) was computed for the four financial derivatives. This required expressions for the gradients (with respect to component exposures) of the cash flows themselves. These were programmed following the logic laid out in Section 3.1. The "E_1" terms were calculated according to the first term of Eq. (34). "Shortfall" is defined as $A^c - E^c$. A separate Monte Carlo study estimated the sampling variability of the shortfalls. The results are presented in Table 5.

In Table 5, the shortfalls represent an estimate of E_2 (Eq. (12)). By Theorem 5 part (a), the shortfalls for U and AX should be zero. The shortfalls in SX and VX should not be expected to be zero. These expectations are confirmed by the following facts: The measured shortfalls for U and AX are less than 1% of their smaller allocations and fall within 0.7 standard deviations of zero; those of SX and VX are greater than 3% of their larger allocations and are more than 10 standard deviations away from zero.

6.5. E_2

This section shows the results of calculating the E_2 term (Eqs. (12) and (35)). The R function `density()` was used to construct a 512-element estimate of the marginal density of U and then `smooth.spline()` was used to create a function that would interpolate the density at any U value. Conditional expectations were computed using `smooth.spline()`. Results are shown in Table 6.

Theoretically, the E_2 values should equal the E_1 allocation shortfalls; for U and AX they should all be zero and for SX and AX they should be material. The computed E_2 values for U and AX are closer to zero than the measured shortfalls, which were already statistically indistinguishable from zero. The fact that they appear to be statistically significant is not material. The computed E_2 values for SX and VX are within 5% and 19%, respectively, of the measured shortfalls. These are approximately 1 and 2 standard deviations, respectively, away from equality.

7. Conclusion

This paper reviewed the basic mathematics of distortion risk measures – the only law-invariant comonotonic additive coherent risk measures – and the coherent allocation of such to portfolio components. It argued that since financial derivatives must have the portfolio risk measure allocated to them in the same way that portfolio components have – via "co-measure" distorted expectation – then that expectation is also a suitable definition for the "distorted" measure of a financial derivative "with respect to" (i.e., in the context of evaluating) an underlying portfolio. Examples of nonlinear homogeneous financial derivatives, motivated by reinsurance, were given.

The coherent allocation of a homogeneous financial derivative's distorted risk measure to portfolio components, via Aumann–Shapley, was found to include an additional term beyond the gradient co-measure distorted expectation, involving conditional covariance. Sufficient conditions for this additional term to vanish were provided. A method for numerical evaluation of the distorted measure and its allocation in a Monte Carlo simulation setting was proposed and exhibited.

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Appendix A

This section derives Theorem 3, Eqs. (11) and (12) for the case $c = 1$ and $C = 2$. This is sufficient to prove the general case of $C \geq 2$ because, with Aumann–Shapley allocations, whether one decomposes $U = X_1 + \cdots + X_C$ or $U = X_1 + Y$ where $Y = X_2 + \cdots + X_C$, the same allocation for X_1 will be obtained. The labeling of $c = 1$ is, of course, arbitrary and without loss of generality.

Start with $E_{H,\mathscr{F}}[\mathbf{X}]$ per Definition 3, Eq. (8):

$$E_{H,\mathscr{F}}[\mathbf{X}] \equiv \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathscr{F}(x_1, x_2) \cdot g'\left(S_U\left(x_1 + x_2\right)\right) \cdot f\left(x_1, x_2\right) dx_1 dx_2 \tag{A.1}$$

where f is the joint distribution function of (X_1, X_2). Perturbing by the factor $\mathbf{v}$ produces:

$$E_{H,\mathscr{F}}[\mathbf{v} * \mathbf{X}] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathscr{F}(\nu_1 \cdot x_1, \nu_2 \cdot x_2) \cdot g'\left(S_{\mathbf{v}\cdot\mathbf{X}}\left(\nu_1 \cdot x_1, \nu_2 \cdot x_2\right)\right) \cdot f\left(x_1, x_2\right) dx_1 dx_2. \tag{A.2}$$

The definition of Aumann–Shapley allocation in the special case of homogeneous functions is the partial derivative of the function (of $\mathbf{v}$): $A^1_{H,\mathscr{F}}[\mathbf{X}] = \frac{\partial}{\partial \nu_1} E_{H,\mathscr{F}}[\mathbf{v} * \mathbf{X}]$. Taking the first partial derivative, moving the derivative inside the integrals, and applying the product rule, we get:

$$\begin{aligned} A^1_{H,\mathscr{F}}[\mathbf{X}] = & \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(\frac{\partial}{\partial \nu_1}\mathscr{F}(\nu_1 \cdot x_1, \nu_2 \cdot x_2)\right) \\ & \cdot g'\left(S\left(\nu_1 \cdot x_1 + \nu_2 \cdot x_2\right)\right) \cdot f\left(x_1, x_2\right) dx_1 dx_2 \\ & + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathscr{F}(\nu_1 \cdot x_1, \nu_2 \cdot x_2) \\ & \cdot \left(\frac{\partial}{\partial \nu_1} g'\left(S\left(\nu_1 \cdot x_1 + \nu_2 \cdot x_2\right)\right)\right) \cdot f\left(x_1, x_2\right) dx_1 dx_2. \end{aligned} \tag{A.3}$$

Calling the second term E_2, and evaluating the first term at $\mathbf{v} = (1, 1)$ we obtain Eq. (11).

Applying the chain rule, we get another version of E_2:

$$\begin{aligned} E_2 = & \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathscr{F}(x_1, x_2) \\ & \cdot g''\left(S\left(x_1, x_2\right)\right) \cdot \left(\frac{\partial}{\partial \nu_1} S\left(\nu_1 \cdot x_1 + \nu_2 \cdot x_2\right)\right) \\ & \cdot f\left(x_1, x_2\right) dx_1 dx_2. \end{aligned} \tag{A.4}$$

Examining the derivative term, note that:

$$\begin{aligned} S\left(\nu_1 \cdot x_1 + \nu_2 \cdot x_2\right) \equiv & \iint_{\substack{\nu_1 \cdot z_1 + \nu_2 \cdot z_2 \geq \\ \nu_1 \cdot x_1 + \nu_2 \cdot x_2}} f\left(z_1, z_2\right) dz_1 dz_2 \\ = & \int_{-\infty}^{\infty}\int_{\frac{\nu_1 \cdot x_1 + \nu_2 \cdot x_2 - \nu_2 \cdot z_2}{\nu_1}}^{\infty} f\left(z_1, z_2\right) dz_1 dz_2. \end{aligned} \tag{A.5}$$

Fig. A.1 illustrates the situation.

$$\begin{aligned} & \frac{\partial}{\partial \nu_1}\int_{-\infty}^{\infty}\int_{\frac{\nu_1 \cdot x_1 + \nu_2 \cdot x_2 - \nu_1 \cdot z_1}{\nu_2}}^{\infty} f\left(z_1, z_2\right) dz_2 dz_1 \\ & = \int_{-\infty}^{\infty}\left(\frac{\partial}{\partial \nu_1}\int_{z_2 = \frac{\nu_1 \cdot x_1 + \nu_2 \cdot x_2 - \nu_1 \cdot z_1}{\nu_2}}^{\infty} f\left(z_1, z_2\right) dz_2\right) dz_1 \\ & = \int_{-\infty}^{\infty}\left(-f\left(z_1, \frac{\nu_1 \cdot x_1 + \nu_2 \cdot x_2 - \nu_1 \cdot z_1}{\nu_2}\right) \cdot \frac{\partial}{\partial \nu_1}\frac{\nu_1 \cdot x_1 + \nu_2 \cdot x_2 - \nu_1 \cdot z_1}{\nu_2}\right) dz_1 \\ & = \int_{-\infty}^{\infty}\left(-f\left(z_1, x_2 + \frac{\nu_1}{\nu_2} \cdot \left(x_1 - z_1\right)\right) \cdot \frac{\partial}{\partial \nu_1}\left(x_2 + \frac{\nu_1}{\nu_2} \cdot \left(x_1 - z_1\right)\right)\right) dz_1 \\ & = \frac{1}{\nu_2} \cdot \int_{-\infty}^{\infty}\left(z_1 - x_1\right) \cdot f\left(z_1, x_2 - \frac{\nu_1}{\nu_2} \cdot \left(z_1 - x_1\right)\right) dz_1 \\ & = \int_{-\infty}^{\infty}\left(z_1 - x_1\right) \cdot f\left(z_1, x_1 + x_2 - z_1\right) dz_1. \end{aligned}$$

In the last step, the expression is evaluated at $\mathbf{v} = (1, 1)$. Fig. A.2 illustrates the result.

Rewrite the joint distribution as the product of conditional and marginal:

$$f\left(z_1, x_1 + x_2 - z_1\right) = f_U\left(x_1 + x_2\right) \cdot f_{X_1|U}\left(z_1 | x_1 + x_2\right). \tag{A.6}$$

Substitute into Eq. (A.4):

$$\begin{aligned} E_2 = & \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathscr{F}(x_1, x_2) \cdot g''\left(S\left(x_1 + x_2\right)\right) \\ & \cdot \left(\int_{-\infty}^{\infty}\left(z_1 - x_1\right) \cdot f_U\left(x_1 + x_2\right) \cdot f_{X_1|U}\left(z_1 | x_1 + x_2\right) dz_1\right) \\ & \cdot f\left(x_1, x_2\right) dx_1 dx_2. \end{aligned} \tag{A.7}$$

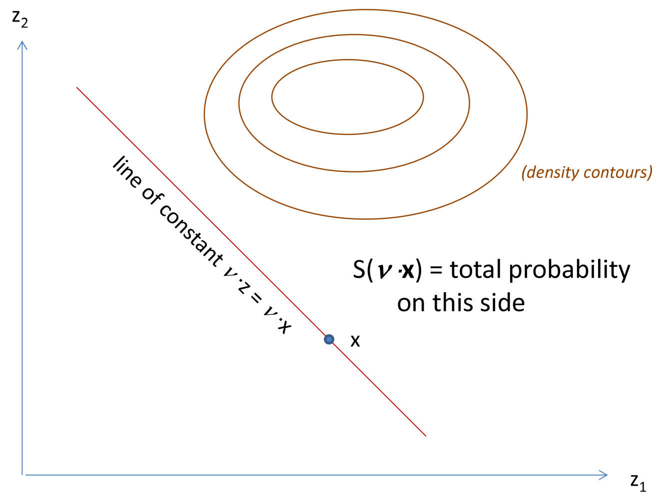

Fig. A.1. Visual interpretation of $S(\mathbf{v}\cdot\mathbf{x})$.

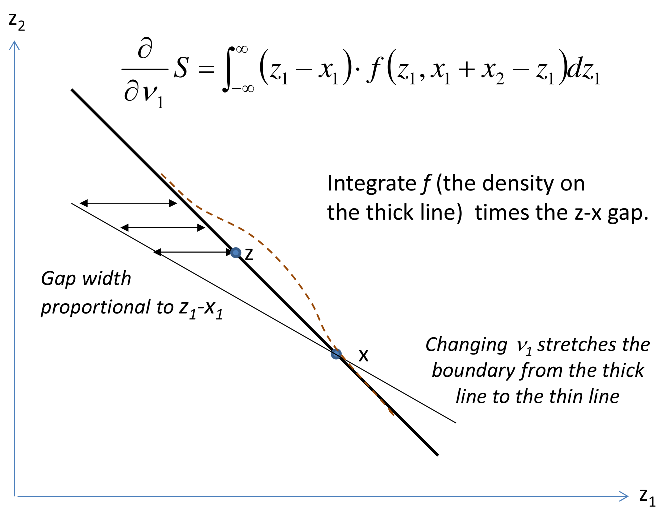

Fig. A.2. Visual interpretation of $\partial S/\partial v_1$.

Since the marginal density term is independent of z_1, it can be factored out of the inner integral:

$$E_2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathcal{F}(x_1,x_2)\cdot g''(S(x_1+x_2))\cdot f_U(x_1+x_2) \cdot\left(\int_{-\infty}^{\infty}(z_1-x_1)\cdot f_{X_1|U}(z_1|\,x_1+x_2)\,dz_1\right) \cdot f(x_1,x_2)\,dx_1dx_2. \tag{A.8}$$

Reinterpreting the inner integral as an expectation:

$$E_2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_U(x_1+x_2) \cdot\{\mathcal{F}(x_1,x_2)\cdot E[X_1-x_1|\,U=x_1+x_2]\} \cdot g''(S(x_1+x_2))\cdot f(x_1,x_2)\,dx_1dx_2. \tag{A.9}$$

Reinterpreting the outer integral as an expectation:

$$E_2 = E_{\mathbf{X}}\left[f_U(U)\cdot\{\mathcal{F}(\mathbf{X})\cdot(E[X_1|\,U]-X_1)\}\cdot g''(S(U))\right]. \tag{A.10}$$

Rewriting $E_{\mathbf{X}}[\cdot]$ as $E_U[E_{X1}[\cdot|U]]$ (a change of variables with Jacobian determinant of one):

$$E_2 = E_U\left[f_U(U)\cdot\left\{E_{X_1}[\mathcal{F}(\mathbf{X})\cdot(E[X_1|\,U]-X_1)|\,U]\right\} \cdot g''(S(U))\right]. \tag{A.11}$$

Noting that $E_{X_1}[E[X_1|\,U]-X_1|\,U]=0$, we can recognize the term in braces as being equal to the covariance of $\mathcal{F}$ and $-X_1$, resulting in:

$$E_2 = -E_U\left[f_U(U)\cdot\mathrm{cov}[\mathcal{F}(\mathbf{X}),X_1|\,U]\cdot g''(S(U))\right]. \tag{A.12}$$

This is the $c=1$ equivalent of Eq. (12). ■

Appendix B

This section derives Theorem 6, Eq. (18). Let $\mathcal{F}$ be a homogeneous financial derivative operator whose distorted measure allocations $A^c_{H,\mathcal{F}}[\mathbf{X}]$ all have $E_2=0$. Assume that the random variables $\mathcal{F}_c[\mathbf{X}]$ are themselves homogeneous and that all $\mathcal{F}_{i,j}[\mathbf{X}]$ are continuous.

From Theorem 3, we have

$$A^c_{H,\mathcal{F}_i}[\mathbf{X}] = E\left[\frac{\partial}{\partial v_c}\mathcal{F}_i(\mathbf{v}*\mathbf{X})\cdot g'(S(U))\right] - E_U\left[f_U(U)\cdot\mathrm{cov}[\mathcal{F}_i(\mathbf{X}),X_c|\,U]\cdot g''(S(U))\right]. \tag{B.1}$$

Using the notation for second-order exposure differentiation,

$$A^c_{H,\mathcal{F}_i}[\mathbf{X}] = E\left[\mathcal{F}_{c,i}(\mathbf{X})\cdot g'(S(U))\right] - E_U\left[f_U(U)\cdot\mathrm{cov}[\mathcal{F}_i(\mathbf{X}),X_c|\,U]\cdot g''(S(U))\right]. \tag{B.2}$$

A pointwise application of the Clairaut–Schwarz theorem shows that $\mathcal{F}_{i,j}[\mathbf{X}]=\mathcal{F}_{j,i}[\mathbf{X}]$ for all i,j. Applying this to the first term,

$$A^c_{H,\mathcal{F}_i}[\mathbf{X}] = E\left[\mathcal{F}_{i,c}(\mathbf{X})\cdot g'(S(U))\right] - E_U\left[f_U(U)\cdot\mathrm{cov}[\mathcal{F}_i(\mathbf{X}),X_c|\,U]\cdot g''(S(U))\right]. \tag{B.3}$$

Therefore,

$$\sum_i A^c_{H,\mathcal{F}_i}[\mathbf{X}] = E\left[\sum_i\mathcal{F}_{i,c}(\mathbf{X})\cdot g'(S(U))\right] - E_U\left[f_U(U)\cdot\mathrm{cov}\left[\sum_i\mathcal{F}_i(\mathbf{X}),X_c\middle|\,U\right] \cdot g''(S(U))\right] \equiv T_1+T_2. \tag{B.4}$$

Consider the T_2 term first. Because $\mathcal{F}$ is homogeneous, Theorem 1 applies and $\sum_i\mathcal{F}_i[\mathbf{X}]=\mathcal{F}[\mathbf{X}]$. This makes T_2 equal to the E_2 term of $A^c_{H,\mathcal{F}}[\mathbf{X}]$ which is zero by assumption. So $T_2=0$.

Expanding the T_1 term, we have

$$\sum_i A^c_{H,\mathcal{F}_i}[\mathbf{X}] = E\left[\sum_i\frac{\partial}{\partial v_i}\mathcal{F}_c(\mathbf{v}*\mathbf{X})\cdot g'(S(U))\right]. \tag{B.5}$$

Because $\mathcal{F}_c$ is assumed homogeneous, Theorem 1 applies and so

$$\sum_i A^c_{H,\mathcal{F}_i}[\mathbf{X}] = E\left[\mathcal{F}_c(\mathbf{X})\cdot g'(S(U))\right] = A^c_{H,\mathcal{F}}[\mathbf{X}] \tag{B.6}$$

which is Eq. (18). ■

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