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Mean-risk model for uncertain portfolio selection with background risk

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ABSTRACT

When making investment decisions, one will face not only the risk of financial assets within portfolio but also background risk. This paper discusses an uncertain portfolio selection problem in which background risk is considered and the security returns and background assets return are given by experienced experts' evaluations instead of historical data. Regarding the returns of the securities and background assets as uncertain variables, we propose an uncertain mean-risk model with background risk for portfolio selection and the crisp forms of the model are provided when security returns obey different uncertainty distributions. In addition, when everything else is same, it is concluded that the optimal expected return of the mean-risk model with background risk is usually smaller than that without background risk. Moreover, the relationship between the optimal solution of our model and that of the model in Huang's paper "Uncertain Portfolio Selection with Background Risk" is discussed. Finally, numerical examples are presented to illustrate the effectiveness of the model and to show the effect of background risk on investment decision.

1. Introduction

Portfolio selection problem is to consider how to allocate one's capital in different risky securities in order to maximize the return of portfolio with risk control. Markowitz [1] first proposed the mean–variance model in 1952, which is the foundation of modern portfolio theory and has been the most impact-making development in mathematical finance management. However, measuring the risk by the variance of return of a portfolio has some limitations. In view of this, scholars studied other methods to measure investment risk of the portfolio and built a lot of portfolio optimization models, such as, mean–semivariance model [2,3], expected absolute deviation model [4], Value-at-Risk model [5,6], Conditional Value-at-Risk models [7,8], mean–semivariance–CVaR model [9], etc. In this paper, risk curve [10] will be used as the risk measure since it provides information about all the likely losses.

These researches assume that all that investors face only portfolio risk when making portfolio selection decisions. Yet, in reality, investors also face background risk, which arises from various sources, including variations in labor income, investments in real estate, unexpected expenses related to health issues and so on that cannot be traded in the financial market [11,12]. We also refer to the assets that are exposed to background risk as background assets [13]. It is the total risk rather than the sole portfolio risk that is of investors' major concern. Many studies have showed that the presence of background risk can affect investments [14,15]. For example, Heaton and Lucas [16,17] found that labor and entrepreneurial incomes affected portfolio selection. Rosen and Wu [18] showed that investors with bad health were more willing to put

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most of their capital in the low risk assets. The research of Tsanakas [19] revealed that the presence of background risk made risk measurement sensitive to the scale and aggregation of risk. To help investors select portfolio in face of background risk, scholars did a large number of research works and developed a variety of models for portfolio selection with background risk in different situations. Hara et al. [20] found that background risks can increase cautiousness and provided the necessary and sufficient conditions under an individual's expected utility framework. Huang and Wang [21] further studied the portfolio frontier characteristics when risk-free asset is considered. These studies all reveal that investors who consider background risk will become more risk averse and prefer to choose safer assets. Therefore, a portfolio selection model that background risk is considered will be discussed in this paper.

In portfolio theory, the security returns are generally considered as random variables and their characteristics such as expected value and variance are calculated based on the sample of available historical data. It remains valid when there are plenty of data in the developed financial market. However, there may be lack of enough transaction data in some emerging markets. Especially, for background asset returns, there are much subjective impression rather than randomness, then if people still use probability theory to solve this problem, counterintuitive results may occur [22,23]. In these situations, scholars found that the security returns data and background assets return can be estimated by experienced experts and fuzzy set theory appears [24]. In particular, fuzzy theory is also applied to portfolio selection [25,26]. Huang [27,28] established the mean–variance for portfolio selection in fuzzy environment. Subsequently, semivariance of fuzzy variable is employed to measure risk and mean–semivariance model is proposed by Huang [28]. Qin [29] presented fuzzy cross-entropy method for portfolio selection. In 2010, Li [30] further formulated mean–variance–skewness model which considered the skewness to measure the asymmetry of fuzzy portfolio return. Although fuzzy portfolio optimization provided alternatives to estimate security returns when lack of data, fuzzy theory suffers from criticism since a paradox will appear. In order to better describe the subjective imprecise quantity, in 2007, Liu [31] developed uncertainty theory. Based on this framework, Huang [32] introduced uncertainty theory to portfolio selection and produced an uncertain portfolio theory. Later, Huang [33] defined the risk curve in uncertain environment and built an uncertain mean-risk model. In 2012, Huang established a risk index model [34] and a mean–variance model [35] for portfolio selection and the security returns were given by experts' evaluations. For optimal project selection and schedule, Huang [36] presented mean–variance and mean–semivariance model based on uncertain measure. Zhang [37] first applied uncertainty theory to a multinational project selection. Moreover, the mean–variance model for portfolio optimization problem in the simultaneous presence of random and uncertain returns was proposed by Qin [38]. Numerous studies have been done about uncertain portfolio selection but a few papers consider background risk and regard background assets return as uncertain variables. Until 2016, Huang [39] researched the model with background risk for uncertain portfolio selection. In this paper, we will study an uncertain portfolio selection problem with background risk and security returns and background assets return are given by experts' evaluations. We analyze the effect of background risk on portfolio decision.

The rest of the paper is organized as follows. Section 2 in detail describes the mean-risk model with background risk for portfolio selection based on uncertainty theory and some important theorems are proved. In Section 3, we employ the numerical examples to illustrate the validity and significance of the model. In Appendix, we review the necessary knowledge about uncertainty theory.

2. Mean-risk models with background risk

2.1. Uncertain models

Investors' uncertain return includes uncertain portfolio return from financial assets and uncertain return from background assets. When these asset returns are given by experts' estimations, it is better to use uncertain variables to describe them. Since all background assets have same features that are different from financial assets, i.e., non-tradable and unhedgeable, the popular way is to use one parameter r_b to show the returns from all background assets in real life. In addition, we assume that the background asset return rate has zero expected value. The assumption that non-tradable background asset has zero expected return rate, which is in consistent with assumptions of most researches studying portfolio with background risk [15]. The model we proposed is also applicable in the situation where the expected value of the background asset return rate differs from zero.

Firstly, the definitions of risk curve and confidence curve will be introduced as follows:

Definition 1 (*[33]*)**.** Let ξ be an uncertain return rate of a security, and r_f the risk-free interest rate. Then the curve

$$R(r) = M\left\{r_f - \xi \geq r\right\}, \quad \forall r \geq 0$$

is called the risk curve of the security.

Since all investors know that they may lose as well as gain in investment, they will have a maximum tolerance towards occurrence chance of each likely loss level, we call it confidence curve $\alpha(r)$. The common trend of the curve is that the severer the loss, the lower tolerance level the investors have towards the occurrence chance of the loss. Three types of confidence curve are usually employed, for example, the forms of linear function, piecewise function and power function. We can find that a portfolio is safe if its risk curve is below the confidence curve, a portfolio is risky if any part of its risk curve is above the confidence curve.

Let ξ_i be the uncertain return rates of the ith securities, x_i the investment proportions in the ith securities, $i = 1, 2, \ldots, n$. Then the risk curve of the portfolio is denoted as follows:

$$R(x_1, x_2, \ldots, x_n, r_b; r) = M\left\{r_f - (\xi_1 x_1 + \cdots + \xi_n x_n + r_b) \geq r\right\}, \quad \forall r \geq 0,$$

where r_f denote risk-free interest rate, and r_b the return rate of background asset, the expected value of r_b is 0. The background assets are independent with other n independent assets. Let us take $r_p = \sum_{i=1}^{n} \xi_i x_i + r_b$. Since a portfolio is safe if the risk curve is below the confidence curve, so the optimal portfolio should be that its risk curve is below the confidence curve and maximize the expected total return. The uncertain mean-risk model with background risk is expressed as follows:

$$\begin{cases} \max E\left[\xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n + r_b\right] \\ s.t.\, R(x_1, x_2, \ldots, x_n, r_b; r) \leq \alpha(r), \quad \forall r \geq 0, \\ \sum_{i=1}^{n} x_i = 1, \\ x_i \geq 0, \quad i = 1, 2, \ldots, n \end{cases} \tag{1}$$

where E is the expected value operator of uncertain variables and $\alpha(r)$ is the confidence curve.

2.2. Crisp form of mean-risk model with background risk

Theorem 1. *Let Φ_i denote the continuous uncertainty distribution of the ith security return rate ξ_i whose inverse function $\Phi_i^{-1}(\alpha)$ exists and is unique for each $\alpha \in (0, 1)$, $i = 1, 2, \ldots, n$, respectively. Suppose the uncertain return rate of background asset r_b has continuous and strictly increasing uncertainty distribution function Θ. Then the mean-risk model with background risk (1) can be transformed into the following linear model:*

$$\begin{cases} \max x_1 E(\xi_1) + x_2 E(\xi_2) + \cdots + x_n E(\xi_n) \\ s.t. x_1 \Phi_1^{-1}(\alpha(r)) + x_2 \Phi_2^{-1}(\alpha(r)) + \cdots + x_n \Phi_n^{-1}(\alpha(r)) + \Theta^{-1}(\alpha(r)) \geq r_f - r, \quad \forall r \geq 0, \\ \sum_{i=1}^{n} x_i = 1, \quad x_i \geq 0, i = 1, 2, \ldots, n. \end{cases} \tag{2}$$

Proof. It follows from Theorem 7 that the objective function of model (1) can be transformed into the objective function of model (2).

Since

$$\begin{aligned} R(x_1, x_2, \ldots, x_n, r_b; r) &= M\left\{r_f - (\xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n + r_b) \geq r\right\} \\ &= M\left\{\xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n + r_b \leq r_f - r\right\}. \end{aligned}$$

Then according to the monotonicity property of uncertain variable, the first constraint in model (1) can be converted into the following form:

$$x_1 \Phi_1^{-1}(\alpha(r)) + x_2 \Phi_2^{-1}(\alpha(r)) + \cdots + x_n \Phi_n^{-1}(\alpha(r)) + \Theta^{-1}(\alpha(r)) \geq r_f - r.$$

The theorem is completed. □

Next, the crisp forms of model (2) will be introduced when the distribution of security return rates is determined.

Theorem 2. *Suppose the return rates of the ith securities are all normal uncertain variables $\xi_i \sim N(\mu_i, \sigma_i)$, $i = 1, 2, \ldots, n$. The return rate of background asset is r_b and $r_b \sim N(0, \rho)$. Then the model (2) can be transformed into the following form:*

$$\begin{cases} \max x_1 \mu_1 + x_2 \mu_2 + \cdots + x_n \mu_n \\ s.t.\, \sum_{i=1}^{n} \left(\mu_i - \frac{\sqrt{3}\sigma_i}{\pi} \ln \frac{1 - \alpha(r)}{\alpha(r)}\right) \cdot x_i - \frac{\sqrt{3}\rho}{\pi} \ln \frac{1 - \alpha(r)}{\alpha(r)} \geq r_f - r, \quad \forall r \geq 0, \\ \sum_{i=1}^{n} x_i = 1, \\ x_i \geq 0, \quad i = 1, 2, \ldots, n. \end{cases} \tag{3}$$

Theorem 3. *Suppose the return rates of the ith securities are all zigzag uncertain variables $\xi_i \sim z(a_i, b_i, c_i)$, $i = 1, 2, \ldots, n$. The return rate of background asset is r_b and $r_b \sim N(0, \rho)$. Then the model (2) can be transformed into the following form:*

When $\alpha(r) \in \left(0, \frac{1}{2}\right]$,

$$\begin{cases} \max \sum_{i=1}^{n} \frac{a_i + 2b_i + c_i}{4} x_i \\ s.t. \ \sum_{i=1}^{n} 2\alpha(r)(b_i - a_i) x_i + \sum_{i=1}^{n} a_i x_i - \frac{\sqrt{3}\rho}{\pi} \ln \frac{1-\alpha(r)}{\alpha(r)} \geq r_f - r, \quad \forall r \geq 0 \\ \sum_{i=1}^{n} x_i = 1, \\ x_i \geq 0, \quad i = 1, 2, \ldots, n. \end{cases}$$

When $\alpha(r) \in \left[\frac{1}{2}, 1\right)$,

$$\begin{cases} \max \sum_{i=1}^{n} \frac{a_i + 2b_i + c_i}{4} x_i \\ s.t. \ \sum_{i=1}^{n} 2\alpha(r)(c_i - b_i) x_i + \sum_{i=1}^{n} (2b_i - c_i) x_i - \frac{\sqrt{3}\rho}{\pi} \ln \frac{1-\alpha(r)}{\alpha(r)} \geq r_f - r, \quad \forall r \geq 0 \\ \sum_{i=1}^{n} x_i = 1, \\ x_i \geq 0, \quad i = 1, 2, \ldots, n. \end{cases}$$

2.3. Discussion of the optimal solution of the mean-risk model with background risk

Obviously, the model (3) is a linear programming problem. The optimal solution of the model (3) can be obtained by using simplex method. Next, we will discuss the solution of model (3). The standard form of the model (3) is as follows:

$$\begin{cases} \max \mu_1 x_1 + \cdots + \mu_n x_n + 0 \cdot x_{n+1} \\ s.t. \ \sum_{i=1}^{n} \left(\mu_i - \frac{\sqrt{3}\sigma_i}{\pi} \ln \frac{1-\alpha(r)}{\alpha(r)} \right) x_i - x_{n+1} \geq r_f - r + \frac{\sqrt{3}\rho}{\pi} \ln \frac{1-\alpha(r)}{\alpha(r)}, \quad \forall r \geq 0 \\ \sum_{i=1}^{n} x_i = 1, \\ x_i \geq 0, \quad i = 1, 2, \ldots, n+1. \end{cases} \tag{4}$$

Let $\lambda_i = \mu_i, i = 1, 2, \ldots, n, \lambda_{n+1} = 0, h_i = \mu_i - \frac{\sqrt{3}\sigma_i}{\pi} \ln \frac{1-\alpha(r)}{\alpha(r)}, i = 1, 2, \ldots, n, h_{n+1} = -1, d_2 = 1, d_1 = r_f - r + \frac{\sqrt{3}\rho}{\pi} \ln \frac{1-\alpha(r)}{\alpha(r)}$.

Then model (4) can be transformed into the following form:

$$\begin{cases} \max \sum_{i=1}^{n} \lambda_i x_i \\ s.t. \ \sum_{i=1}^{n+1} h_i x_i = d_1 \\ \sum_{i=1}^{n} x_i = d_2, x_i \geq 0, \quad i = 1, 2, \ldots, n+1. \end{cases} \tag{5}$$

By using simplex method and after a series of iterations, the model (5) can be transformed into the following form:

$$\begin{cases} \max z = z_0 + \sum_{j=3}^{n+1} \lambda'_j x_j \\ s.t. \ x_1 + h'_{1,3} x_3 + \cdots + h'_{1,n+1} x_{n+1} = d'_1 \quad \text{(I)} \\ \quad\ x_2 + h'_{2,3} x_3 + \cdots + h'_{2,n+1} x_{n+1} = d'_2 \quad \text{(II)} \\ \quad\ x_i \geq 0, \quad i = 1, 2, \ldots, n+1 \end{cases}$$

where

$$z_0 = \lambda_1 d'_1 + \lambda_2 d'_2, \qquad \lambda'_j = \lambda_j - h'_{1,j}\lambda_1 + h'_{2,j}\lambda_2, \quad j = 3, \ldots, n+1.$$

Similar to the analysis in Huang's paper [39]. We can obtain the following lemma:

Lemma 1. *Suppose* $X^{(0)} = \left(d_1', d_2', 0, \ldots, 0\right)$ *is a basic feasible solution of the model* (5)*. If* $\lambda_j' \le 0, j = 3, 4, \ldots, n+1$*, then* $X^{(0)}$ *is an optimal solution of the model* (5)*.*

The mean-risk model without background risk is established as follows:

$$\begin{cases} \max x_1 E\left[\xi_1\right] + x_2 E\left[\xi_2\right] + \cdots + x_n E\left[\xi_n\right] \\ s.t.\ x_1 \Phi_1^{-1}\left(\alpha\left(r\right)\right) + x_2 \Phi_2^{-1}\left(\alpha\left(r\right)\right) + \cdots + x_n \Phi_n^{-1}\left(\alpha\left(r\right)\right) \ge r_f - r, \quad \forall r \ge 0, \\ \sum_{i=1}^{n} x_i = 1, \\ x_i \ge 0, \quad i = 1, 2, \ldots, n. \end{cases} \tag{6}$$

Theorem 4. *For any given* r *and confidence curve* $\alpha(r)$ $(0 < \alpha(r) < 0.5)$*, the expected return of the optimal portfolio with background risk is smaller than that without background risk.*

Proof. Let $X = (x_1, x_2, \ldots, x_n)$, then the constraint of the model (2) can be represented as follows

$$f\left(X, \alpha\right) = x_1 \Phi_1^{-1}\left(\alpha\left(r\right)\right) + x_2 \Phi_2^{-1}\left(\alpha\left(r\right)\right) + \cdots + x_n \Phi_n^{-1}\left(\alpha\left(r\right)\right) + \Theta^{-1}\left(\alpha\left(r\right)\right) \ge r_f - r.$$

Let $g\left(X, \alpha\right)$ be the constraint of the model for portfolio selection without background risk, then

$$g\left(X, \alpha\right) = f\left(X, \alpha\right) - \Theta^{-1}\left(\alpha\left(r\right)\right).$$

Suppose X_1 is the optimal solution of the portfolio selection model with background risk, then for any $r \ge 0$, we have $f\left(X_1, \alpha\left(r\right)\right) = r_f - r$. In real life, many investors only care about "$0 < \alpha(r) < 0.5$", because "$\alpha(r) \ge 0.5$" means the investors know that the occurrence chance of each likely loss is too large. When the expected value of r_b is zero and $\alpha(r) < 0.5$, we can obtain $\Theta^{-1}\left(\alpha\left(r\right)\right) < 0$, so $g\left(X, \alpha\right) > r_f - r$. That is to say X_1 is a feasible solution of the model (6).

Similar to the above analysis, suppose X_2 is the optimal solution of the model (6), then

$$g\left(X_2, \alpha\right) = f\left(X_2, \alpha\right) - \Theta^{-1}\left(\alpha\left(r\right)\right) = r_f - r.$$

Hence, it is easy to get $f\left(X_2, \alpha\left(r\right)\right) < r_f - r$. Obviously, X_2 is not the feasible solution of the model with background risk.

For a given r value, the expected return of the optimal portfolio with background risk is smaller than that without background risk. □

In fact, Theorem 4 is true if the uncertainty distribution of background asset return is symmetrical not just the normal uncertainty distribution. Because when background asset return rate r_b is symmetrical, it is clear that in this case the expected value should be the value of $\Theta^{-1}(0.5)$. Thus, Theorem 4 can be proved under the assumption that the expected value is zero in the beginning of the text. On the other hand, when the uncertainty distribution of background asset return is asymmetric, Theorem 4 is also true if the uncertainty distribution function Θ satisfies $\Theta^{-1}\left(\alpha\left(r\right)\right) < 0$ when $\alpha(r) < 0.5$. The upper bound of confidence curve $\alpha(r)$ the investors give is usually less than 0.5 or even smaller, so this theorem is true in most cases.

The model in Huang's paper can be converted into the following form:

$$\begin{cases} \max E\left[\xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n + r_b\right] \\ s.t.\ x_1 \Phi_1^{-1}\left(\alpha\right) + x_2 \Phi_2^{-1}\left(\alpha\right) + \cdots + x_n \Phi_n^{-1}\left(\alpha\right) + \Theta^{-1}\left(\alpha\right) \ge H, \\ \sum_{i=1}^{n} x_i = 1, \\ x_i \ge 0, \quad i = 1, 2, \ldots, n. \end{cases} \tag{7}$$

Theorem 5. *The optimal solution of the model* (1) *is also the optimal solution of model* (7)*, but not vice versa.*

Proof. If r is a deterministic value, the model (1) can degenerate the model (7). It is obvious that the optimal solution of model (7) in Huang's paper is only one solution of model (1) for a given r, but is not always the optimal solution of model (1).

3. Numerical examples

In this section, numerical examples are presented to illustrate the proposed mean-risk model with background risk for uncertain portfolio selection and to show the effect of the background risk on the portfolio selection decision. The following results are programmed in Matlab2009a.

Table 1
Normal uncertain return rates of 10 securities.

Security i	Uncertain return rate ξ_i	Security i	Uncertain return rate ξ_i
1	$N(0.027, 0.14)$	6	$N(0.028, 0.15)$
2	$N(0.033, 0.19)$	7	$N(0.030, 0.08)$
3	$N(0.032, 0.16)$	8	$N(0.032, 0.18)$
4	$N(0.039, 0.20)$	9	$N(0.025, 0.10)$
5	$N(0.031, 0.15)$	10	$N(0.026, 0.06)$

Table 2
Optimal portfolios at different risk level (%).

r	0	0.03	0.06	0.09	0.12	0.15	0.18	0.21	0.24	0.27	0.3
Return rate 1	3.71	3.61	3.54	3.44	3.28	3.40	3.48	3.53	3.54	3.50	3.36
Return rate 2	3.25	3.31	3.27	3.18	3.03	3.15	3.23	3.28	3.29	3.25	3.12

Remark: Return rate 1 denotes the return rate without background risk and Return rate 2 the return rate with background risk.

Table 3
Zigzag uncertain return rates of 10 securities.

Security i	Uncertain return rate ξ_i	Security i	Uncertain return rate ξ_i
1	$Z(-0.16, -0.01, 0.14)$	6	$Z(-0.17, 0.00, 0.33)$
2	$Z(-0.13, 0.01, 0.19)$	7	$Z(-0.24, 0.02, 0.45)$
3	$Z(-0.13, 0.03, 0.20)$	8	$Z(-0.14, 0.03, 0.21)$
4	$Z(-0.18, 0.02, 0.16)$	9	$Z(-0.21, -0.01, 0.18)$
5	$Z(-0.14, 0.00, 0.18)$	10	$Z(-0.10, 0.03, 0.20)$

Example 1. We select ten stocks. The data we select is from the paper [33]. The estimation of the security return rates is given in Table 1.

Suppose the monthly risk-free interest rate is 0.01. The experts believe that the background asset return rate has uncertainty distribution $r_b \sim N(0, 0.01)$, and the investor gives the confidence curve as follows:

$$\alpha(r) = \begin{cases} -2.75r + 0.43, & 0 \le r \le 0.12, \\ -0.5r + 0.16, & 0.12 \le r \le 0.3, \\ 0.01, & r \ge 0.3. \end{cases}$$

We discuss the optimal portfolio when security return rates are normal uncertain variables and calculate allocation proportions of ten securities to maximize corresponding expected return rate. Set $r \in [0, 0.3]$ in the model and let $r = 0, 0.03, 0.06, \ldots, 0.3$, respectively. The optimal portfolios with and without background risk at different risk level can be shown in detail in Table 2.

Several conclusions can be obtained from above numerical experiment. For any determined r value, the expected optimal portfolio with background risk is smaller than that without background risk. When security returns are normal uncertain variables, the expected return of the optimal portfolio with background risk is 3.27% which is smaller than the expected return of the optimal portfolio without background risk 3.28%. Considering background risk, investors should invest security 4 and security 7 and the allocation proportion is 29.64% and 70.36%, respectively. In addition, we have checked that the risk curve of optimal portfolio with background risk is below the confidence curve, i.e. corresponding portfolio is in safe area. In Table 2, higher expected return rates than 3.27% can be found when background risk is considered, but part of their risk curve is above the confidence curve, i.e. corresponding portfolio is in risky area.

The relationship between risk curve and confidence curve when security returns are normal uncertain variables is shown in Fig. 1. In this figure, the red broken line represents confidence curve and the blue curve represents risk curve. It can be seen that risk curve is below the confidence curve.

Example 2. Using the data in Table 1, we can compute the optimal solution of model (7) when security returns are normal uncertain variables. In this model, let $\alpha = 0.15$, $H = -0.08$, $\rho = 0.01$. Corresponding risk curve and risk point of this model are shown in Fig. 2. The point is below confidence curve but part of risk curve is above the confidence curve, thus the numerical example demonstrates that the optimal solution of the model (7) is not the optimal solution of model (1) in this paper.

Example 3. Suppose an investor plans to invest his money among 10 securities. Without loss of generality, assume the future returns of the securities are zigzag uncertain variables denoted by $\xi_i = Z(a_i, b_i, c_i)$ for $i = 1, 2, \ldots, 10$. The return rates of 10 securities are given in Table 3.

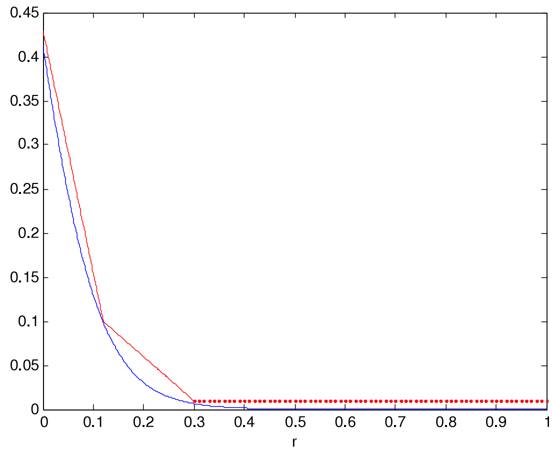

Fig. 1. Confidence curve and risk curve when security returns are normal uncertain variables. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

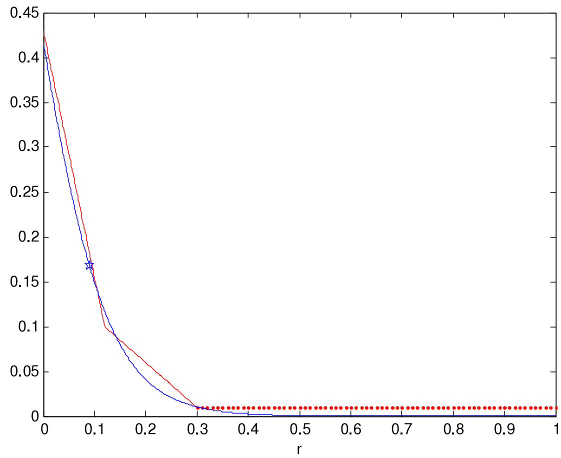

Fig. 2. The risk curve of model (7).

Set the risk-free interest rate is 0.01, the standard deviation of background assets return rate is $\rho = 0.01$, and the investor gives his/her confidence curve as follows:

$$\alpha(r) = \begin{cases} -2.90r + 0.448, & 0 \le r \le 0.12, \\ -0.5r + 0.16, & 0.12 \le r \le 0.3, \\ 0.01, & r \ge 0.3. \end{cases}$$

It is clear that $\alpha(r) \in (0, 0.5)$. We discuss the optimal portfolio when security return rates are zigzag uncertain variables and calculate allocation proportions of ten securities to maximize corresponding expected return rate. When considering background risk, we can calculate optimal portfolio according to the model in Theorem 3. When background risk is not considered, the model in Theorem 3 becomes the following form:

$$\begin{cases} \max \sum_{i=1}^{n} \frac{a_i + 2b_i + c_i}{4} x_i \\ s.t. \sum_{i=1}^{n} 2\alpha(r)(b_i - a_i) x_i + \sum_{i=1}^{n} a_i x_i \ge r_f - r, \quad \forall r \ge 0, \\ \sum_{i=1}^{n} x_i = 1, x_i \ge 0, \quad i = 1, 2, \ldots, n. \end{cases} \tag{8}$$

Table 4
Optimal portfolios with background risk and without background risk (%).

Security i	1	2	3	4	5	6	7	8	9	10
Allocation proportion 1	0	0	0	0	0	0	28.04	0	0	71.96
Allocation proportion 2	0	0	0	0	0	0	30.04	0	0	69.96

Remark: Allocation proportion 1 denotes the optimal portfolio solved by the model with background risk and Allocation proportion 2 the optimal portfolio solved by the model without background risk.

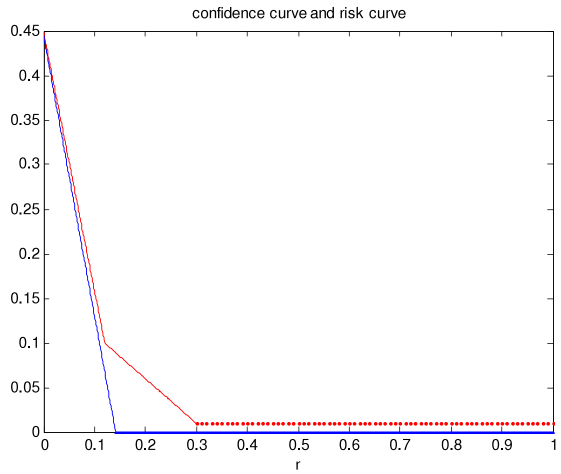

Fig. 3. Confidence curve and risk curve when security returns are zigzag uncertain variables.

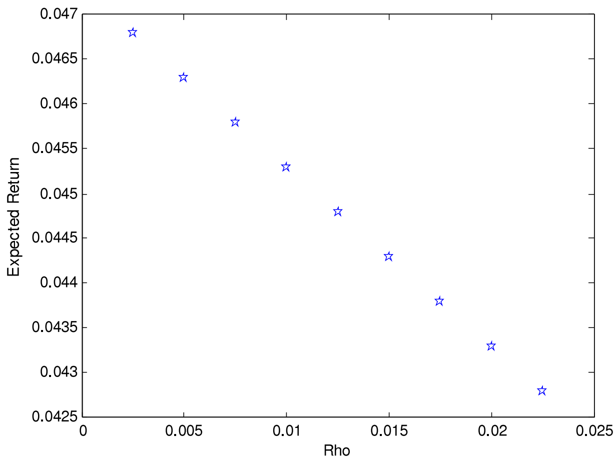

Fig. 4. The expected return rate of optimal portfolios with different standard deviation (ρ) values of the background asset return rate.

By solving the linear programming problems, we can obtain the optimal portfolios with background risk and without background risk when security return rates are zigzag uncertain variables. The results are provided in Table 4.

For any determined r, we can obtain that the optimal expected return with background risk is smaller than that without background risk. Furthermore, the optimal expected return rate with background risk is 4.63% and that without background risk is 4.68%. It can be seen that $0.0463 < 0.0468$, which means that the expected value of the optimal portfolio with background risk is smaller than that without background risk. Fig. 3 reveals the risk curve is below the confidence curve when security return rates are zigzag uncertain variables.

In order to further show the impact of the background risk on the investment decision, we change the background risk levels (i.e., change the standard deviation (ρ) values of the background asset return rates) and select the portfolios. The change of the values of expected return rate and standard deviation of the optimal portfolios with the change of background risk level (ρ) are obtained and shown in Figs. 4 and 5, respectively. From the detailed computational results, we find that the relative errors among expected values do not exceed 1%. It also can be seen from Figs. 4 and 5 that the bigger the standard

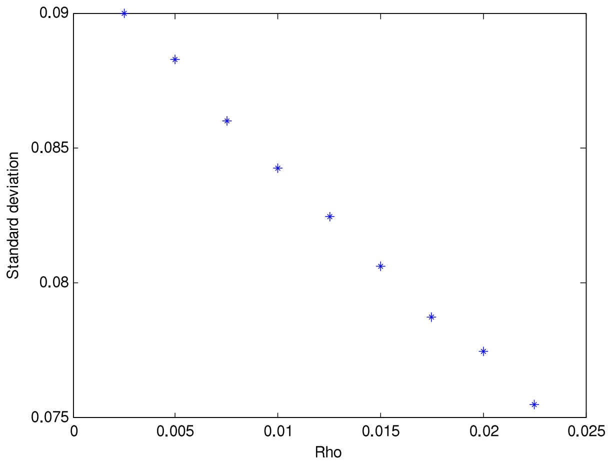

Fig. 5. The standard deviation values of optimal portfolios with different standard deviation (ρ) values of the background asset return rate.

deviation of the return rate of background asset, the smaller the expected return and standard deviation of the selected optimal portfolio. This is because the greater the standard deviation of the return rate of background asset, the greater the background risk that investors are facing. Hence, when investors face greater background risk, they prefer to put more of their wealth in the low risk financial assets rather than risky assets. Obviously, background risk has a great effect on the optimal portfolio selection decision.

4. Conclusions

In this paper, the effect of the background risk on investments has been discussed. In the complex financial and social environment, there are situations where background assets return and the security returns have to be evaluated by experienced experts due to the unexpected things and the lack of historical data. This paper has discussed that risk level can be measured by risk curve and proposed uncertain mean-risk model with background risk for portfolio selection. The crisp equivalents of the model are provided. In addition, for any given r, we have found that the expected return of the optimal portfolio with background risk is smaller than that without background risk. Different from Huang's model, our model can consider all the possible risks that investors could tolerate.

Finally, both the numerical examples and the analysis show that the mean-risk model with background risk is effective and background risk has a great effect on the investment decision.

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Appendix. Uncertainty theory

In this section, uncertain measure, uncertain variable and uncertainty distribution will be introduced for easy understanding of the paper. In 2007, Liu [31] proposed uncertainty theory and got a wide range of applications. Firstly, we introduce the definition of uncertain measure.

Definition 2 (*[31]*)**.** Let L be a σ-algebra over a nonempty set Ω. A set function $M : L \to [0, 1]$ is called an uncertain measure if it satisfies the following four axioms:

(1) (Normality) $M\{\Omega\} = 1$;

(2) (Self-duality) $M\{\Gamma\} + M\{\Gamma^c\} = 1$;

(3) (Countable subadditivity) For every countable sequence of events $\{\Gamma_i\}$, we have

$$M\left\{\bigcup_{i=1}^{\infty} \Gamma_i\right\} \leq \sum_{i=1}^{\infty} M\{\Gamma_i\}.$$

The triplet (Ω, L, M) is called an uncertainty space

(4) (Product measure [40]) For uncertainty spaces (Ω_i, L_i, M_i), $i = 1, 2, \ldots,$ the product uncertain measure is

$$M\left\{\prod_{i=1}^{\infty}\Gamma_i\right\} = \bigwedge_{i=1}^{\infty} M_i\left\{\Gamma_i\right\}$$

where Γ_i are arbitrary chosen events from L_i for $i = 1, 2, \ldots$, respectively.

It is easy to prove that any uncertain measure M is increasing. That is,

$$M\left\{\Gamma_1\right\} \le M\left\{\Gamma_2\right\}$$

for any events $\Gamma_1 \subset \Gamma_2$.

Definition 3 (*[31]*). An uncertain variable is a function $\xi : (\Omega, L, M) \to R$, i.e. for any Borel set B of real numbers, the set

$$\{\xi \in B\} = \{\chi \in \Omega \,|\xi(\chi) \in B\}$$

is an event.

Definition 4 (*[31]*). For any $x \in R$, the uncertainty distribution of an uncertain variable ξ is defined as $\Phi(x) = M\{\xi \le x\}$. It is said to be regular if it is a continuous and strictly increasing function with respect to x at which $0 < \Phi(x) < 1$, and

$$\lim_{x\to-\infty}\Phi(x) = 0, \qquad \lim_{x\to+\infty}\Phi(x) = 1.$$

The inverse function $\Phi^{-1}(\alpha)$ is called the inverse uncertainty distribution of ξ if it exists and is unique for each $\alpha \in (0, 1)$. Inverse uncertainty distribution plays a crucial role in the operations of independent uncertain variables.

Next we introduce some commonly used uncertainty distributions. An uncertain variable is called zigzag if it has a zigzag uncertainty distribution

$$\Phi(x) = \begin{cases} 0, & \text{if } x \le a, \\ (x-a)/2(b-a), & \text{if } a \le x \le b, \\ (x+c-2b)/2(c-b), & \text{if } b \le x \le c, \\ 1, & \text{if } x \ge c, \end{cases}$$

denoted by $Z(a, b, c)$ where a, b, c are real numbers with $a < b < c$. An uncertain variable is called normal if it has a normal uncertainty distribution

$$\Phi(x) = \left(1 + \exp\left(\frac{\pi(e-x)}{\sqrt{3}\sigma}\right)\right)^{-1}, \quad x \in R,$$

denoted by $N(e, \sigma)$ where e and σ are real numbers with $\sigma > 0$.

Theorem 6 (*[31]*). *Let $\xi_1, \xi_2, \ldots, \xi_n, \xi_{n+1}, \ldots, \xi_{n+m}$ be independent uncertain variables with continuous and strictly increasing uncertainty distribution functions $\Psi_1, \Psi_2, \ldots, \Psi_n, \Psi_{n+1}, \ldots, \Psi_{n+m}$, respectively. Let $f(r_1, r_2, \ldots, r_n, r_{n+1}, \ldots, r_{n+m})$ be strictly increasing with respect to $r_1, r_2, \ldots, r_n$ and strictly decreasing with respect to $r_{n+1}, r_{n+2}, \ldots, r_{n+m}$. Then*

$$\xi = f(\xi_1, \xi_2, \ldots, \xi_n, \xi_{n+1}, \ldots, \xi_{n+m})$$

is an uncertain variable whose inverse uncertainty distribution function is

$$\Phi^{-1}(\alpha) = f\left(\Psi_1^{-1}(\alpha), \Psi_2^{-1}(\alpha), \ldots, \Psi_n^{-1}(\alpha), \Psi_{n+1}^{-1}(1-\alpha), \ldots, \Psi_{n+m}^{-1}(1-\alpha)\right), \quad 0 < \alpha < 1.$$

Definition 5 (*[31]*). The expected value of an uncertain variable ξ is defined by

$$E[\xi] = \int_0^{+\infty} M\{\xi \ge x\}\,dx - \int_{-\infty}^{0} M\{\xi \le x\}\,dx$$

provided that at least one of the two integrals exists.

Theorem 7 (*[31]*). *Let ξ_1 and ξ_2 be independent uncertain variables with finite expected values. Then for any real numbers a_1 and a_2, we have*

$$E[a_1\xi_1 + a_2\xi_2] = a_1E[\xi_1] + a_2E[\xi_2].$$

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