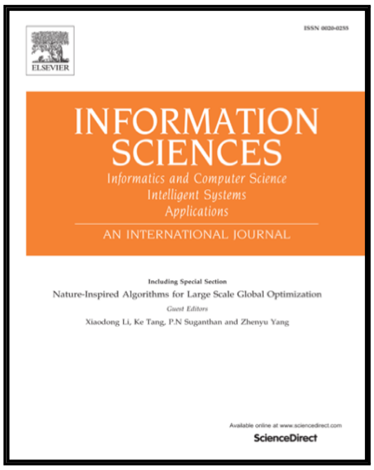
Accepted Manuscript



Hesitant Fuzzy Linguistic Entropy and Cross-Entropy Measures and Alternative Queuing Method for Multiple Criteria Decision Making

Xunjie Gou , Zeshui Xu , Huchang Liao

PII: S0020-0255(16)30813-1
DOI: 10.1016/j.ins.2017.01.033
Reference: INS 12718

To appear in: *Information Sciences*

Received date: 8 September 2016
Revised date: 19 December 2016
Accepted date: 16 January 2017

Please cite this article as: Xunjie Gou , Zeshui Xu , Huchang Liao , Hesitant Fuzzy Linguistic Entropy and Cross-Entropy Measures and Alternative Queuing Method for Multiple Criteria Decision Making, *Information Sciences* (2017), doi: 10.1016/j.ins.2017.01.033

a finite number of decision alternatives having multiple criteria [6]. Lots of methods have been developed to solve the MCDM problems, such as the TOPSIS method [4], the VIKOR method [22], the TODIM method [38], etc. Furthermore, the Granular Computing techniques [1,2,7,8,17,25-27,29,31,33,34,39,43,44,48] can also be used to solve the MCDM problems effectively. In general, MCDM involves two important steps: (1) determining the criteria weights; (2) obtaining a suitable ranking of alternatives. For the first step, when dealing with hesitant fuzzy linguistic MCDM problems, these are few weight-determining methods in the existing literature [11,30]. Farhadinia [11] defined some entropy measures for HFLTSs, which can be used to deal with the MCDM problems, where the information about criteria weights is incomplete. Peng et al. [30] defined the concept of combination weight, and used it to solve the hesitant fuzzy linguistic MCDM problems and overcome the uncertainty caused by subjective weights. For the second step, many aggregation operators and decision making methods have been proposed to deal with the MCDM problems under hesitant fuzzy linguistic information environment, including the hesitant fuzzy linguistic Bonferroni mean (HFLBM) operator and the weighted hesitant fuzzy linguistic Bonferroni mean (WHFLBM) operator [12], the hesitant fuzzy linguistic TOPSIS methods [4,10], the hesitant fuzzy linguistic VIKOR method [22], the hesitant fuzzy linguistic TODIM methods [37,38], the hierarchical hesitant fuzzy linguistic MCDM method [16], and the likelihood-based methods for hesitant fuzzy linguistic MCDM [17,18].

In the existing weight-determining methods and hesitant fuzzy linguistic MCDM methods, there are the following shortcomings:

(1) Lots of information will be lost when we only utilize the entropy measure to determine the weights of criteria because we may neglect the interactive effect of the decision information.

(2) The above aggregation operators and decision making methods are extremely complex or not intuitive.

In order to overcome the above issues, in this paper, we first develop some new hesitant fuzzy linguistic entropy measures and cross-entropy measures. Then, we establish a novel weight-determining model, which considers not only the individual effect of each HFLE, but also the interactive effect between any two HFLEs with respect to each criterion. Furthermore, a hesitant fuzzy linguistic alternative queuing method (HFL-AQM) is proposed to deal with the MCDM problems. This method uses both the graph theory and the precedence relationship matrix skillfully. Especially, the directed graph makes the final ranking results of all alternatives more intuitively to be distinguished.

The rest of the paper is organized as follows: In Section 2, we review some concepts related to HFLTSs. The expectation value and the variance of HFLE are given and a comparison method of HFLEs is established. In Section 3 and Section 4, some hesitant fuzzy linguistic entropy measures and cross-entropy measures are proposed, respectively. In Section 5, we establish a weight-determining model based on the hesitant fuzzy linguistic entropy measures and cross-entropy measures, and then propose the HFL-AQM. In Section 6, a case study concerning the tertiary hospital management is made to verify the weight-determining method and the HFL-AQM. Additionally, a comparison analysis is made to show the advantages of the proposed weight- determining method and the HFL-AQM. Finally, we end the paper with some conclusions in Section 7.

2. Hesitant fuzzy linguistic term set

By combining the hesitant fuzzy set (HFS) [41] with the fuzzy linguistic approach [50], Rodríguez et al. [32] defined the concept of HFLTS as follows:

Definition 2.1 [32]**.** Let $S=\{s_0,\ldots,s_\tau\}$ be a linguistic term set. A hesitant fuzzy linguistic term set (HFLTS), H_S, is an ordered finite subset of the consecutive linguistic terms of S.

Obviously, this definition has some shortcomings [20]: (1) the linguistic term set $S=\{s_0,\ldots,s_\tau\}$ is unreasonable when we use it to do some operations; (2) There is no any mathematical form for the HFLTS. For the first shortcoming, Xu [42] developed a subscript-symmetric additive linguistic term set such as $S=\{s_t | t=-\tau,\ldots,-1,0,1,\ldots,\tau\}$, where the mid-linguistic label s_0 represents an assessment of "indifference," and the rest of them are placed symmetrically around it. $s_{-\tau}$ and s_τ are the lower and upper bounds of linguistic labels, where τ is a positive integer. To understand the HFLTS more clearly, based on the subscript-symmetric additive linguistic term set, Liao et al. [20] redefined the HFLTS with a mathematical form:

Definition 2.2 [20]**.** Let $S=\{s_t | t=-\tau,\ldots,-1,0,1,\ldots,\tau\}$ be a linguistic term set, and $x_i \in X$, $i=1,2,\ldots,N$. Then $H_S=\{<x_i, h_S(x_i)> | x_i \in X\}$ is a HFLTS, where $h_S(x_i)$ is a set of some values in the linguistic term set S and can be expressed as $h_S(x_i)=\{s_{\phi_l}(x_i) | s_{\phi_l}(x_i) \in S, l=1,\ldots,L\}$ with L being the number of linguistic terms in $h_S(x_i)$. $h_S(x_i)$ denotes the possible degree of the linguistic variable x_i to the linguistic term set S. For convenience, $h_S(x_i)$ is called the hesitant fuzzy linguistic element (HFLE), and H_S is the set of all HFLEs.

In addition, Definition 2.2 gives the mathematical form of the extended HFLTS [35]. Wang and Xu [36] also presented the concept of HFLE. As we know, the basic components of a HFS are the hesitant fuzzy elements (HFEs) [41], and there exist some relationships between HFLEs and HFEs. Recently, Gou et al. [12] defined two equivalent transformation functions of HFLEs and HFEs to make the operations among HFLEs much easier.

Definition 2.3 [12]**.** Let $S=\{s_t | t=-\tau,\ldots,-1,0,1,\ldots,\tau\}$ be a finite and totally ordered discrete linguistic term set, $h_S=\{s_t | t\in[-\tau,\tau]\}$ be a HFLE, and $h_\gamma=\{\gamma | \gamma\in[0,1]\}$ be a HFE. Then the linguistic variable s_t that expresses the equivalent information to the membership degree γ is obtained by the following function:

$$g:[-\tau,\tau]\to[0,1],\ g(s_t)=\frac{t}{2\tau}+\frac{1}{2}=\gamma \tag{1}$$

Besides, we can get this function as:

$$g:[-\tau,\tau]\to[0,1],\ g(h_S)=\left\{g(s_t)=\frac{t}{2\tau}+\frac{1}{2}\middle|t\in[-\tau,\tau]\right\}=h_\gamma \tag{2}$$

Additionally, the membership degree γ that expresses the equivalent information to the linguistic variable s_t is obtained by the following function:

$$g^{-1}:[0,1]\to[-\tau,\tau],\ g^{-1}(\gamma)=s_{(2\gamma-1)\tau}=s_t \tag{3}$$

Similar to the analyses above, we get

$$g^{-1}:[0,1]\to[-\tau,\tau],\ g^{-1}(h_\gamma)=\left\{g^{-1}(\gamma)=s_{(2\gamma-1)\tau}\middle|\gamma\in[0,1]\right\}=h_S \tag{4}$$

Motivated by the equivalent transformation function g, a method based on the expectation values and the variances can be developed to compare any two HFLEs:

Definition 2.4. Let $S=\{s_t|t=-\tau,\ldots,-1,0,1,\ldots,\tau\}$ be a finite and totally ordered discrete linguistic term set, $h_S=\{s_{\phi_l}|s_{\phi_l}\in S,l=1,2,\ldots,\#h_S\}$ be a HFLE with $\#h_S$ being the number of linguistic terms in h_S. Then the expectation value of h_S can be defined as $e(h_S)=\frac{1}{\#h_S}\sum_{l=1}^{\#h_S}g(s_{\phi_l})$. Additionally, the variance of h_S can be defined as $\upsilon(h_S)=\frac{1}{\#h_S}\sum_{l=1}^{\#h_S}\left(g(s_{\phi_l})-e(h_S)\right)^2$.

Based on the expectation values and the variances of HFLEs, a method to compare any two HFLEs can be given as follows:

Let $h_{S_1}=\{s^1_{\phi_l}|s^1_{\phi_l}\in S,l=1,2,\ldots,\#h_{S_1}\}$ and $h_{S_2}=\{s^2_{\phi_l}|s^2_{\phi_l}\in S,l=1,2,\ldots,\#h_{S_2}\}$ be two HFLEs, then

(1) If $e(h_{S_1})>e(h_{S_2})$, then h_{S_1} is bigger than h_{S_2}, denoted by $h_{S_1}\succ h_{S_2}$;

(2) If $e(h_{S_1})=e(h_{S_2})$, then

(a) If $\upsilon(h_{S_1}) > \upsilon(h_{S_2})$, then h_{S_1} is smaller than h_{S_2}, denoted by $h_{S_1} \prec h_{S_2}$;

(b) If $\upsilon(h_{S_1}) = \upsilon(h_{S_2})$, then h_{S_1} is equal to h_{S_2}, denoted by $h_{S_1} = h_{S_2}$.

The following example is given to show how this comparison method works:

Example 2.1. Let $S = \{s_t | t = -3, 2, -1, 0, 1, 2, 3\}$ be a linguistic term set, $h_{S_1} = \{s_0\}$ and $h_{S_2} = \{s_{-1}, s_0, s_1\}$ be two HFLEs, then we can calculate the expectation value of them: $e(h_{S_1}) = e(h_{S_2}) = \frac{1}{2}$. Thus, it is necessary to calculate the variances and $\upsilon(h_{S_1}) = 0 < \upsilon(h_{S_1}) = \frac{1}{54}$. So $h_{S_1} \succ h_{S_2}$.

The complementary set of the HFLE was defined by Gou and Xu [14]:

Definition 2.5 [14]. Let $S = \{s_t | t = -\tau, \ldots, -1, 0, 1, \ldots, \tau\}$ be a linguistic term set, and $h_S = \{s_{\phi_l} | s_{\phi_l} \in S, l = 1, 2, \ldots, \# h_S\}$ be a HFLE, then we call $\overline{h}_S$ the complementary set of h_S and $\overline{h}_S = g^{-1}\left\{\bigcup_{l=1}^{\# h_S}\left\{1 - g\left(s_{s_{\phi_l}}\right)\right\}\right\}$.

3. Hesitant fuzzy linguistic entropy and similarity measures

3.1. Hesitant Fuzzy Linguistic Entropy Measures

Considering that the entropy and cross-entropy measures for HFLTSs have not been studied, in this section, we mainly define some entropy and cross-entropy measures for HFLTSs based on the equivalent transformation function g.

Definition 3.1. Let $S = \{s_t | t = -\tau, \ldots, -1, 0, 1, \ldots, \tau\}$ be a linguistic term set, and $h_S = \{s_{\sigma(l)} | l = 1, \ldots, \# h_S\}$, $h_{S_1} = \{s^1_{\sigma(l)} | l = 1, \ldots, \# h_{S_1}\}$ and $h_{S_2} = \{s^2_{\sigma(l)} | l = 1, \ldots, \# h_{S_2}\}$ be three HFLEs ($\# h_S$, $\# h_{S_1}$ and $\# h_{S_2}$ are the numbers of linguistic terms of these three HFLEs, respectively, and $\# h_S = \# h_{S_1} = \# h_{S_2} = L$). Let $\overline{h}_S$ be the complementary set of h_S. We call E an entropy measure for the HFLE h_S if it satisfies:

(1) $0 \le E(h_S) \le 1$;

(2) $E(h_S)=0$ if and only if $g(h_S)=0$ or $g(h_S)=1$;

(3) $E(h_S)=1$ if and only if $g\left(s_{\sigma(l)}\right)+g\left(s_{\sigma(L-l+1)}\right)=1$, for $l=1,2,\ldots,L$;

(4) $E\left(h_{S_1}\right)\le E\left(h_{S_2}\right)$ if $g\left(s^1_{\sigma(l)}\right)\le g\left(s^2_{\sigma(l)}\right)$ for $g\left(s^2_{\sigma(l)}\right)+g\left(s^2_{\sigma(L-l+1)}\right)\le 1$, or if $g\left(s^1_{\sigma(l)}\right)\ge g\left(s^2_{\sigma(l)}\right)$ for $g\left(s^2_{\sigma(l)}\right)+g\left(s^2_{\sigma(L-l+1)}\right)\ge 1$, $l=1,2,\ldots,L$;

(5) $E(h_S)=E\left(\bar{h}_S\right)$.

Remark 1. As descripted in Definition 3.1, we let $\#h_S=\#h_{S_1}=\#h_{S_2}=L$, which means that these three HFLEs have the same number of linguistic elements. However, if they have different numbers of linguistic elements, then we need to make the numbers of all HFLEs be equal by adding the smallest element to each HFLE. For example, for two HFLEs $h_{S_1}=\{s_{-1},s_0,s_1\}$ and $h_{S_2}=\{s_1,s_2\}$, we need to extend $\{s_1,s_2\}$ to $\{s_1,s_1,s_2\}$ by adding the smallest element s_1. In the following section, we also use this method to calculate the cross-entropy between two HFLEs.

Based on the entropy measures for FSs [10,28] and HFSs [46], we define some hesitant fuzzy linguistic entropy measures as:

$$E_1(h_S)=\frac{1}{L\left(\sqrt{2}-1\right)}\sum_{l=1}^{L}\left(\sin\frac{\pi\left(g\left(s_{\sigma(l)}\right)+g\left(s_{\sigma(L-l+1)}\right)\right)}{4}+\sin\frac{\pi\left(2-g\left(s_{\sigma(l)}\right)-g\left(s_{\sigma(L-l+1)}\right)\right)}{4}-1\right) \quad (5)$$

$$E_2(h_S)=\frac{1}{L\left(\sqrt{2}-1\right)}\sum_{l=1}^{L}\left(\cos\frac{\pi\left(g\left(s_{\sigma(l)}\right)+g\left(s_{\sigma(L-l+1)}\right)\right)}{4}+\cos\frac{\pi\left(2-g\left(s_{\sigma(l)}\right)-g\left(s_{\sigma(L-l+1)}\right)\right)}{4}-1\right) \quad (6)$$

$$E_3(h_S)=-\frac{1}{L\ln 2}\sum_{l=1}^{L}\left(\frac{g\left(s_{\sigma(l)}\right)+g\left(s_{\sigma(L-l+1)}\right)}{2}\ln\frac{g\left(s_{\sigma(l)}\right)+g\left(s_{\sigma(L-l+1)}\right)}{2}+\frac{2-g\left(s_{\sigma(l)}\right)-g\left(s_{\sigma(L-l+1)}\right)}{2}\ln\frac{2-g\left(s_{\sigma(l)}\right)-g\left(s_{\sigma(L-l+1)}\right)}{2}\right) \quad (7)$$

$$E_4(h_S)=\frac{1}{L\left(2^{(1-a)b}-1\right)}\sum_{l=1}^{L}\left(\left(\left(\frac{g\left(s_{\sigma(l)}\right)+g\left(s_{\sigma(L-l+1)}\right)}{2}\right)^a+\left(1-\frac{g\left(s_{\sigma(l)}\right)+g\left(s_{\sigma(L-l+1)}\right)}{2}\right)^a\right)^b-1\right)$$

$$b\neq 0,\ a\neq 1,\ a>0 \tag{8}$$

Additionally, we can change the values of the parameters a and b in $E_4(h_S)$:

(1) If $b=1$, then

$$E_4(h_S)=\frac{1}{L\left(2^{(1-a)}-1\right)}\sum_{l=1}^{L}\left(\left(\frac{g\left(s_{\sigma(l)}\right)+g\left(s_{\sigma(L-l+1)}\right)}{2}\right)^a+\left(1-\frac{g\left(s_{\sigma(l)}\right)+g\left(s_{\sigma(L-l+1)}\right)}{2}\right)^a-1\right) \tag{9}$$

(2) If $b=1/a$, then

$$E_4(h_S)=\frac{1}{L\left(2^{(1-a)/a}-1\right)}\sum_{l=1}^{L}\left(\left(\left(\frac{g\left(s_{\sigma(l)}\right)+g\left(s_{\sigma(L-l+1)}\right)}{2}\right)^a+\left(1-\frac{g\left(s_{\sigma(l)}\right)+g\left(s_{\sigma(L-l+1)}\right)}{2}\right)^a\right)^{1/a}-1\right) \tag{10}$$

Let $S=\{s_t \mid t=-3,-2,-1,0,1,2,3\}$ be a linguistic term set, and $h_S=\{s_1,s_2,s_3\}$ be a HFLE. Then we can calculate the hesitant fuzzy linguistic entropy of h_S based on $E_4(h_S)$ by changing the values of the parameters a and b. We take the values of a from 1.01 to 2 with the step 0.01, and take the values of b from 0.01 to 1 with the step 0.01. Furthermore, four kinds of situations can be obtained considering the situations that a and b are increasing or decreasing, respectively (see Fig. 2 to Fig. 5).

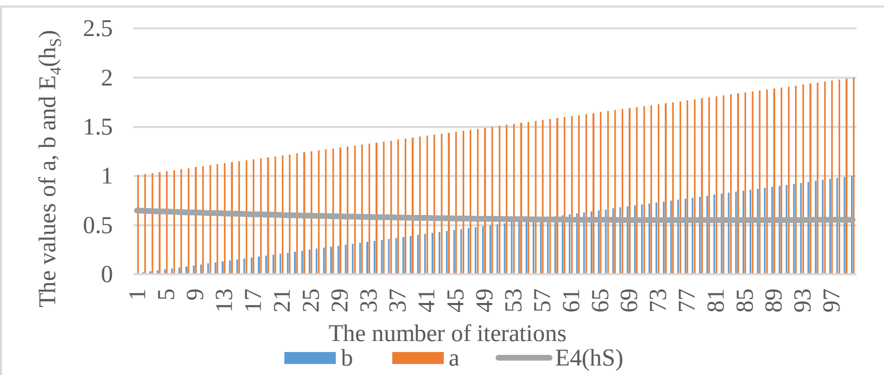


Fig. 2. The change trend of $E_4(h_S)$ when both a and b are increasing.

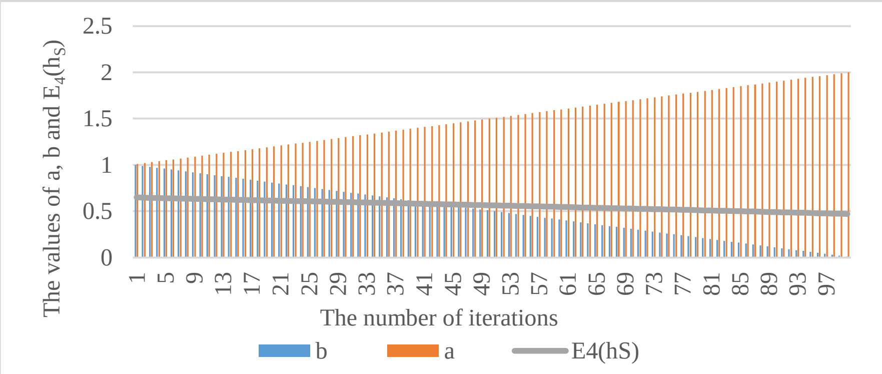


Fig. 3. The change trend of $E_4\left(h_{S_2}\right)$ when a is increasing and b is decreasing.

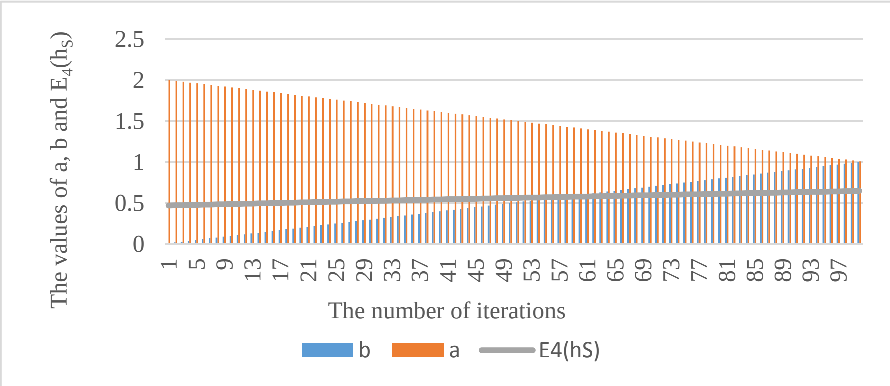


Fig. 4. The change trend of $E_4\left(h_{S_2}\right)$ when a is decreasing and b is increasing.

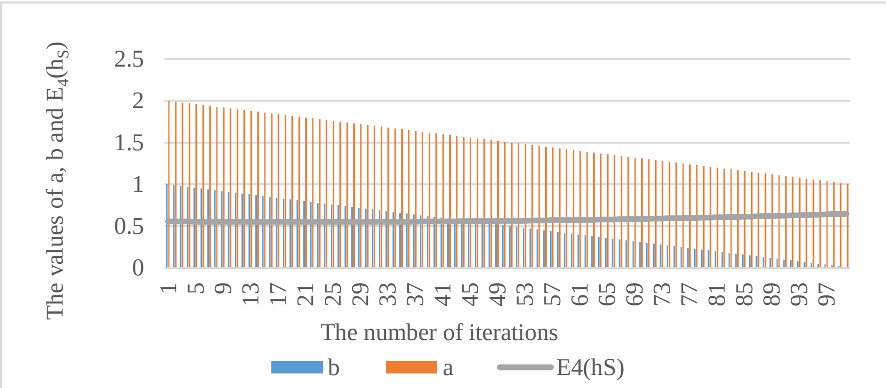


Fig. 5. The change trend of $E_4\left(h_{S_2}\right)$ when both a and b are decreasing.

Remark 2. From Fig. 2 to Fig. 5, we can see that the hesitant fuzzy linguistic entropy $E_4\left(h_S\right)$ is a decreasing function when both a and b are increasing or a is increasing and b is decreasing. On the contrary, the hesitant fuzzy linguistic entropy $E_4\left(h_S\right)$ is an increasing function when both a and b are decreasing or a is decreasing and b is increasing.

In order to understand the hesitant fuzzy linguistic entropy measures more clearly, we give the following example:

Example 3.1. Let $S=\{s_t|t=-3,-2,-1,0,1,2,3\}$ be a linguistic term set, and $h_S=\{s_{-1},s_0,s_1\}$ be a HFLE. Then the hesitant fuzzy linguistic entropy of h_S can be obtained as:

(1) $E_1(h_S)=\frac{1}{3(\sqrt{2}-1)}\left(\sin\frac{\pi}{4}+\sin\frac{\pi}{4}+\sin\frac{\pi}{4}+\sin\frac{\pi}{4}+\sin\frac{\pi}{4}+\sin\frac{\pi}{4}-3\right)=1;$

(2) $E_2(h_S)=\frac{1}{3(\sqrt{2}-1)}\left(\cos\frac{\pi}{4}+\cos\frac{\pi}{4}+\cos\frac{\pi}{4}+\cos\frac{\pi}{4}+\cos\frac{\pi}{4}+\cos\frac{\pi}{4}-3\right)=1;$

(3) $E_3(h_S)=-\frac{1}{3\ln 2}\left(\frac{1}{2}\ln\frac{1}{2}+\frac{1}{2}\ln\frac{1}{2}+\frac{1}{2}\ln\frac{1}{2}+\frac{1}{2}\ln\frac{1}{2}+\frac{1}{2}\ln\frac{1}{2}+\frac{1}{2}\ln\frac{1}{2}\right)=1;$

(4) $E_4(h_S)=\frac{1}{3(2^1-1)}\left(3\left(\left(\frac{1}{2}\right)^{\frac{1}{2}}+\left(\frac{1}{2}\right)^{\frac{1}{2}}\right)^2-3\right)=1$, if $b=2, a=\frac{1}{2}$.

3.2. The Relationships between Hesitant Fuzzy Linguistic Entropy Measures and Similarity Measures

As we know, there exist some relationships between the entropy measures and the similarity measures [11,46]. Liao et al. [19] proposed some hesitant fuzzy linguistic similarity measures. However, on the basis of the equivalent transformation function g, in the following, we put forward some new hesitant fuzzy linguistic similarity measures, and then discuss the relationships between them and the hesitant fuzzy linguistic entropy measures:

Definition 3.2. Let $S=\{s_t|t=-\tau,\ldots,-1,0,1,\ldots,\tau\}$ be a linguistic term set, and $h_{S_i}=\{s^i_{\sigma(l)}|l=1,\ldots,L\}$ $(i=1,2,3)$ be three HFLEs. We call ρ the similarity measure between two HFLTSs if it satisfies:

(1) $\rho(h_{S_1},h_{S_2})=0$ if and only if $g(h_{S_1})=0$, $g(h_{S_2})=1$ or $g(h_{S_1})=1$, $g(h_{S_2})=0$;

(2) $\rho(h_{S_1},h_{S_2})=1$ if and only if $g(s^1_{\sigma(l)})=g(s^2_{\sigma(l)})$, $l=1,2,\ldots,L$;

(3) $\rho(h_{S_1},h_{S_3})\le\rho(h_{S_1},h_{S_2})$, $\rho(h_{S_1},h_{S_3})\le\rho(h_{S_2},h_{S_3})$, if and only if $g(s^1_{\sigma(l)})\le g(s^2_{\sigma(l)})\le g(s^3_{\sigma(l)})$ or

$g\left(s_{\sigma(l)}^{1}\right) \geq g\left(s_{\sigma(l)}^{2}\right) \geq g\left(s_{\sigma(l)}^{3}\right)$, $l = 1,2,\ldots,L$;

(4) $\rho\left(h_{S_1}, h_{S_2}\right) = \rho\left(h_{S_2}, h_{S_1}\right)$.

Based on Definition 3.2, some hesitant fuzzy linguistic similarity measures between h_{S_1} and h_{S_2} can be defined as follows:

(1) The Hamming similarity measure:

$$\rho_1\left(h_{S_1}, h_{S_2}\right) = 1 - \frac{1}{L}\sum_{l=1}^{L}\left|g\left(s_{\sigma(l)}^{1}\right) - g\left(s_{\sigma(l)}^{2}\right)\right| \tag{11}$$

(2) The Euclidean similarity measure:

$$\rho_2\left(h_{S_1}, h_{S_2}\right) = 1 - \left(\frac{1}{L}\sum_{l=1}^{L}\left(\left|g\left(s_{\sigma(l)}^{1}\right) - g\left(s_{\sigma(l)}^{2}\right)\right|\right)^2\right)^{1/2} \tag{12}$$

(3) The generalized similarity measure:

$$\rho_3\left(h_{S_1}, h_{S_2}\right) = 1 - \left(\frac{1}{L}\sum_{l=1}^{L}\left(\left|g\left(s_{\sigma(l)}^{1}\right) - g\left(s_{\sigma(l)}^{2}\right)\right|\right)^\lambda\right)^{1/\lambda} \tag{13}$$

(4) The generalized Hausdorff similarity measure:

$$\rho_4\left(h_{S_1}, h_{S_2}\right) = 1 - \left(\max_{l=1,\ldots,L}\left(\left|g\left(s_{\sigma(l)}^{1}\right) - g\left(s_{\sigma(l)}^{2}\right)\right|\right)^\lambda\right)^{1/\lambda} \tag{14}$$

In particular, if $\lambda = 1$, then the generalized Hausdorff similarity measure reduces to the Hamming-Hausdorff similarity measure:

$$\rho_5\left(h_{S_1}, h_{S_2}\right) = 1 - \max_{l=1,\ldots,L}\left|g\left(s_{\sigma(l)}^{1}\right) - g\left(s_{\sigma(l)}^{2}\right)\right| \tag{15}$$

If $\lambda = 2$, then the generalized Hausdorff similarity measure reduces to the Euclidean-Hausdorff similarity measure:

$$\rho_6\left(h_{S_1}, h_{S_2}\right) = 1 - \left(\max_{l=1,\ldots,L}\left(\left|g\left(s_{\sigma(l)}^{1}\right) - g\left(s_{\sigma(l)}^{2}\right)\right|\right)^2\right)^{1/2} \tag{16}$$

Additionally, we can give some hybrid similarity measures between h_{S_1} and h_{S_2} by combining the above similarity measures:

(5) The hybrid Hamming similarity measure:

$$\rho_7\left(h_{S_1},h_{S_2}\right)=1-\frac{1}{2}\left(\frac{1}{L}\sum_{l=1}^{L}\left|g\left(s_{\sigma(l)}^1\right)-g\left(s_{\sigma(l)}^2\right)\right|+\max_{l=1,\dots,L}\left|g\left(s_{\sigma(l)}^1\right)-g\left(s_{\sigma(l)}^2\right)\right|\right) \tag{17}$$

(6) The hybrid Euclidean similarity measure:

$$\rho_8\left(h_{S_1},h_{S_2}\right)=1-\left(\frac{1}{2}\left(\frac{1}{L}\sum_{l=1}^{L}\left(\left|g\left(s_{\sigma(l)}^1\right)-g\left(s_{\sigma(l)}^2\right)\right|\right)^2+\max_{l=1,\dots,L}\left(\left|g\left(s_{\sigma(l)}^1\right)-g\left(s_{\sigma(l)}^2\right)\right|\right)^2\right)\right)^{1/2} \tag{18}$$

(7) The generalized hybrid similarity measure:

$$\rho_9\left(h_{S_1},h_{S_2}\right)=1-\left(\frac{1}{2}\left(\frac{1}{L}\sum_{l=1}^{L}\left(\left|g\left(s_{\sigma(l)}^1\right)-g\left(s_{\sigma(l)}^2\right)\right|\right)^\lambda+\max_{l=1,\dots,L}\left(\left|g\left(s_{\sigma(l)}^1\right)-g\left(s_{\sigma(l)}^2\right)\right|\right)^\lambda\right)\right)^{1/\lambda} \tag{19}$$

In what follows, we study the relationships between the proposed entropy measures and similarity measures:

Theorem 3.1. Let $S=\left\{s_t\,|\,t=-\tau,\dots,-1,0,1,\dots,\tau\right\}$ be a linguistic term set, $h_S=\left\{s_{\sigma(l)}\,|\,l=1,\dots,L\right\}$, $h_{S_i}=\left\{s_{\sigma(l)}^i\,|\,l=1,\dots,L\right\}\left(i=1,2,3\right)$ be four HFLEs, and $\bar{h}_S$ be the complementary set of h_S. Then $\rho\left(h_S,\bar{h}_S\right)$ is an entropy of h_S.

Proof. (1) $\rho\left(h_S,\bar{h}_S\right)=0\Leftrightarrow g\left(h_S\right)=0$, $g\left(\bar{h}_S\right)=1$ or $g\left(h_S\right)=1$, $g\left(\bar{h}_S\right)=0$.

(2) $\rho\left(h_S,\bar{h}_S\right)=1\Leftrightarrow h_S=\bar{h}_S\Leftrightarrow g\left(s_{\sigma(l)}\right)+g\left(s_{\sigma(L-l+1)}\right)=1$, for all $l=1,2,\dots,L$.

(3) For two HFLEs h_{S_1} and h_{S_2}, suppose that $g\left(s_{\sigma(l)}^1\right)\le g\left(s_{\sigma(l)}^2\right)$, for $g\left(s_{\sigma(l)}^2\right)+g\left(s_{\sigma(L-l+1)}^2\right)\le 1$, then $g\left(s_{\sigma(l)}^1\right)\le g\left(s_{\sigma(l)}^2\right)\le 1-g\left(s_{\sigma(L-l+1)}^2\right)\le 1-g\left(s_{\sigma(L-l+1)}^1\right)$. Based on the similarity measure of HFLTSs, we can obtain $\rho\left(h_{S_1},\bar{h}_{S_1}\right)\le\rho\left(h_{S_2},\bar{h}_{S_1}\right)\le\rho\left(h_{S_2},\bar{h}_{S_2}\right)$. Similarly, we can also prove the case where $g\left(s_{\sigma(l)}^1\right)\ge g\left(s_{\sigma(l)}^2\right)$, for $g\left(s_{\sigma(l)}^2\right)+g\left(s_{\sigma(L-l+1)}^2\right)\ge 1$.

(4) $\rho\left(h_S,\bar{h}_S\right)=\rho\left(\bar{h}_S,h_S\right)$.

Example 3.2. Motivated by Theorem 3.1 and based on the nine similarity measures ρ_m $(m=1,2,\cdots,9)$, we can get some special hesitant fuzzy linguistic entropy measures:

$$\rho_1'\left(h_S,\overline{h}_S\right)=1-\frac{1}{L}\sum_{l=1}^{L}\left|g\left(s_{\sigma(l)}\right)+g\left(s_{\sigma(L-l+1)}\right)-1\right|$$

$$\rho_2'\left(h_S,\overline{h}_S\right)=1-\left(\frac{1}{L}\sum_{l=1}^{L}\left(\left|g\left(s_{\sigma(l)}\right)+g\left(s_{\sigma(L-l+1)}\right)-1\right|\right)^2\right)^{1/2}$$

$$\rho_3'\left(h_S,\overline{h}_S\right)=1-\left(\frac{1}{L}\sum_{l=1}^{L}\left(\left|g\left(s_{\sigma(l)}\right)+g\left(s_{\sigma(L-l+1)}\right)-1\right|\right)^\lambda\right)^{1/\lambda}$$

$$\rho_4'\left(h_S,\overline{h}_S\right)=1-\left(\max_{l=1,\dots,L}\left(\left|g\left(s_{\sigma(l)}\right)+g\left(s_{\sigma(L-l+1)}\right)-1\right|\right)^\lambda\right)^{1/\lambda}$$

$$\rho_5'\left(h_S,\overline{h}_S\right)=1-\max_{l=1,\dots,L}\left|g\left(s_{\sigma_l}\right)+g\left(s_{\sigma(L-l+1)}\right)-1\right|$$

$$\rho_6'\left(h_S,\overline{h}_S\right)=1-\left(\max_{l=1,\dots,L}\left(\left|g\left(s_{\sigma(l)}\right)+g\left(s_{\sigma(L-l+1)}\right)-1\right|\right)^2\right)^{1/2}$$

$$\rho_7'\left(h_S,\overline{h}_S\right)=1-\frac{1}{2}\left(\frac{1}{L}\sum_{l=1}^{L}\left|g\left(s_{\sigma(l)}\right)+g\left(s_{\sigma(L-l+1)}\right)-1\right|+\max_{l=1,\dots,L}\left|g\left(s_{\sigma(l)}\right)+g\left(s_{\sigma(L-l+1)}\right)-1\right|\right)$$

$$\rho_8'\left(h_S,\overline{h}_S\right)=1-\left(\frac{1}{2}\left(\frac{1}{L}\sum_{l=1}^{L}\left(\left|g\left(s_{\sigma(l)}\right)+g\left(s_{\sigma(L-l+1)}\right)-1\right|\right)^2+\max_{l=1,\dots,L}\left(\left|g\left(s_{\sigma(l)}\right)+g\left(s_{\sigma(L-l+1)}\right)-1\right|\right)^2\right)\right)^{1/2}$$

$$\rho_9'\left(h_S,\overline{h}_S\right)=1-\left(\frac{1}{2}\left(\frac{1}{L}\sum_{l=1}^{L}\left(\left|g\left(s_{\sigma(l)}\right)+g\left(s_{\sigma(L-l+1)}\right)-1\right|\right)^\lambda+\max_{l=1,\dots,L}\left(\left|g\left(s_{\sigma(l)}\right)+g\left(s_{\sigma(L-l+1)}\right)-1\right|\right)^\lambda\right)\right)^{1/\lambda}$$

Theorem 3.2. Let $S=\left\{s_t\,\middle|\,t=-\tau,\dots,-1,0,1,\dots,\tau\right\}$ be a linguistic term set, $h_S=\left\{s_{\sigma(l)}\,\middle|\,l=1,2,\dots,L\right\}$ be a HFLE, and let $h_{S_1}=\left\{s_{\sigma_1},s_{\sigma_2},\dots,s_{\sigma\left(\overline{[L/2]}\right)}\right\}$ and $h_{S_2}=\left\{1-s_{\sigma_L},1-s_{\sigma_{L-1}},\dots,1-s_{\sigma([L/2])}\right\}$. Then $\rho\left(h_{S_1},h_{S_2}\right)$ is an entropy of h_S, where $\overline{[L/2]}$ denotes the largest integer no bigger than $1/2$, and $[L/2]$ denotes the smallest integer no smaller than $1/2$.

Proof. (1) $\rho\left(h_{S_1},h_{S_2}\right)=0 \Leftrightarrow g\left(h_{S_1}\right)=0$, $g\left(h_{S_2}\right)=1$ or $g\left(h_{S_1}\right)=1$, $g\left(h_{S_2}\right)=0$.

(2) $\rho\left(h_{S_1},h_{S_2}\right)=1 \Leftrightarrow h_{S_1}=h_{S_2} \Leftrightarrow g\left(s_{\sigma(l)}\right)=1-g\left(s_{\sigma(L-l+1)}\right)$, for all $l=1,2,\dots,L$.

(3) For two HFLEs $h_S=\left\{s_{\sigma(l)}\,\middle|\,l=1,2,\dots,L\right\}$ and $h_S'=\left\{s'_{\sigma(l)}\,\middle|\,l=1,2,\dots,L\right\}$. Suppose that $s_{\sigma(l)}\le s'_{\sigma(l)}$, for $s'_{\sigma(l)}+s'_{\sigma(L-l+1)}\le 1, l=1,2,\dots,L$, then $s_{\sigma(l)}\le s'_{\sigma(l)}\le 1-s'_{\sigma(L-l+1)}\le 1-s_{\sigma(L-l+1)}$. Therefore, based on the

similarity measure of HFLTSs, we can obtain $\rho\left(h_{S_1}, h_{S_2}\right) \le \rho\left(h'_{S_1}, h_{S_2}\right) \le \rho\left(h'_{S_2}, h'_{S_2}\right)$. Similarly, we can prove the case where $s_{\sigma(l)} \ge s'_{\sigma(l)}$, for $s'_{\sigma(l)} + s'_{\sigma(L-l+1)} \ge 1, l = 1, 2, \ldots, L$.

(4) $\rho\left(h_{S_1}, h_{S_2}\right) = \rho\left(h_{S_2}, h_{S_1}\right)$.

Example 3.3. Motivated by Theorem 3.2 and based on the nine similarity measures ρ_m $(m = 1, 2, \cdots, 9)$, some hesitant fuzzy linguistic entropy measures can also be obtained as follows:

$$\rho''_1\left(h_{S_1}, h_{S_2}\right) = 1 - \frac{1}{\lceil L/2 \rceil} \sum_{l=1}^{\lceil L/2 \rceil} \left| g\left(s_{\sigma(l)}\right) + g\left(s_{\sigma(L-l+1)}\right) - 1 \right|$$

$$\rho''_2\left(h_{S_1}, h_{S_2}\right) = 1 - \left(\frac{1}{\lceil L/2 \rceil} \sum_{l=1}^{\lceil L/2 \rceil} \left(\left| g\left(s_{\sigma(l)}\right) + g\left(s_{\sigma(L-l+1)}\right) - 1 \right| \right)^2 \right)^{1/2}$$

$$\rho''_3\left(h_{S_1}, h_{S_2}\right) = 1 - \left(\frac{1}{\lceil L/2 \rceil} \sum_{l=1}^{\lceil L/2 \rceil} \left(\left| g\left(s_{\sigma(l)}\right) + g\left(s_{\sigma(L-l+1)}\right) - 1 \right| \right)^\lambda \right)^{1/\lambda}$$

$$\rho''_4\left(h_{S_1}, h_{S_2}\right) = 1 - \left(\max_{l=1,\ldots,L} \left(\left| g\left(s_{\sigma(l)}\right) + g\left(s_{\sigma(L-l+1)}\right) - 1 \right| \right)^\lambda \right)^{1/\lambda}$$

$$\rho''_5\left(h_{S_1}, h_{S_2}\right) = 1 - \max_{l=1,\ldots,L} \left| g\left(s_{\sigma(l)}\right) + g\left(s_{\sigma(L-l+1)}\right) - 1 \right|$$

$$\rho''_6\left(h_{S_1}, h_{S_2}\right) = 1 - \left(\max_{l=1,\ldots,L} \left(\left| g\left(s_{\sigma(l)}\right) + g\left(s_{\sigma(L-l+1)}\right) - 1 \right| \right)^2 \right)^{1/2}$$

$$\rho''_7\left(h_{S_1}, h_{S_2}\right) = 1 - \frac{1}{2} \left(\frac{1}{\lceil L/2 \rceil} \sum_{l=1}^{\lceil L/2 \rceil} \left| g\left(s_{\sigma(l)}\right) + g\left(s_{\sigma(L-l+1)}\right) - 1 \right| + \max_{l=1,\ldots,L} \left| g\left(s_{\sigma(l)}\right) + g\left(s_{\sigma(L-l+1)}\right) - 1 \right| \right)$$

$$\rho''_8\left(h_{S_1}, h_{S_2}\right) = 1 - \left(\frac{1}{2} \left(\frac{1}{\lceil L/2 \rceil} \sum_{l=1}^{\lceil L/2 \rceil} \left(\left| g\left(s_{\sigma(l)}\right) + g\left(s_{\sigma(L-l+1)}\right) - 1 \right| \right)^2 + \max_{l=1,\ldots,L} \left(\left| g\left(s_{\sigma(l)}\right) + g\left(s_{\sigma(L-l+1)}\right) - 1 \right| \right)^2 \right) \right)^{1/2}$$

$$\rho''_9\left(h_{S_1}, h_{S_2}\right) = 1 - \left(\frac{1}{2} \left(\frac{1}{\lceil L/2 \rceil} \sum_{l=1}^{\lceil L/2 \rceil} \left(\left| g\left(s_{\sigma(l)}\right) + g\left(s_{\sigma(L-l+1)}\right) - 1 \right| \right)^\lambda + \max_{l=1,\ldots,L} \left(\left| g\left(s_{\sigma(l)}\right) + g\left(s_{\sigma(L-l+1)}\right) - 1 \right| \right)^\lambda \right) \right)^{1/\lambda}$$

Theorem 3.3. Let $S = \left\{ s_t \middle| t = -\tau, \ldots, -1, 0, 1, \ldots, \tau \right\}$ be a linguistic term set, and $h_{S_i} = \left\{ s^i_{\sigma(l)} \middle| l = 1, 2, \ldots, L \right\} (i = 1, 2)$ be two HFLEs. Assume that $\left| g\left(s^1_{\sigma(l)}\right) - g\left(s^2_{\sigma(l)}\right) \right| < \left| g\left(s^1_{\sigma(l+1)}\right) - g\left(s^2_{\sigma(l+1)}\right) \right|$ $l = 1, 2, \ldots, L$, and

$$F\left(h_{S_1},h_{S_2}\right)=\left(\frac{\left|g\left(s^1_{\sigma(1)}\right)-g\left(s^2_{\sigma(1)}\right)\right|+1}{2},\frac{\left|g\left(s^1_{\sigma(2)}\right)-g\left(s^2_{\sigma(2)}\right)\right|+1}{2},\ldots,\frac{\left|g\left(s^1_{\sigma(L)}\right)-g\left(s^2_{\sigma(L)}\right)\right|+1}{2}\right) \tag{20}$$

then $E\left(F\left(h_{S_1},h_{S_2}\right)\right)$ is a similarity measure between h_{S_1} and h_{S_2}.

Proof. (1) $E\left(F\left(h_{S_1},h_{S_2}\right)\right)=0 \Leftrightarrow F\left(h_{S_1},h_{S_2}\right)=1$ or $F\left(h_{S_1},h_{S_2}\right)=0 \Leftrightarrow g\left(h_{S_1}\right)=0$, $g\left(h_{S_2}\right)=1$ or $g\left(h_{S_1}\right)=1$, $g\left(h_{S_2}\right)=0$.

(2) $E\left(F\left(h_{S_1},h_{S_2}\right)\right)=1 \Leftrightarrow \frac{\left|g\left(s^1_{\sigma(l)}\right)-g\left(s^2_{\sigma(l)}\right)\right|+1}{2}+\frac{\left|g\left(s^1_{\sigma(L-l+1)}\right)-g\left(s^2_{\sigma(L-l+1)}\right)\right|+1}{2}=1 \Leftrightarrow \left|g\left(s^1_{\sigma(l)}\right)-g\left(s^2_{\sigma(l)}\right)\right|$ $+\left|g\left(s^1_{\sigma(L-l+1)}\right)-g\left(s^2_{\sigma(L-l+1)}\right)\right|=0 \Leftrightarrow h_{S_1}=h_{S_2}$.

(3) Suppose that there are three HFLEs $h_{S_i}=\left\{s^i_{\sigma(l)}\middle| l=1,2,\ldots,L\right\}\ (i=1,2,3)$ and $g\left(s^1_{\sigma(l)}\right)\le g\left(s^2_{\sigma(l)}\right)$ $\le g\left(s^3_{\sigma(l)}\right)\ (l=1,2,\ldots,L)$. Then $\frac{\left|g\left(s^1_{\sigma(l)}\right)-g\left(s^3_{\sigma(l)}\right)\right|+1}{2}\ge\frac{\left|g\left(s^1_{\sigma(l)}\right)-g\left(s^2_{\sigma(l)}\right)\right|+1}{2}$, $l=1,2,\ldots,L$, and $F\left(h_{S_1},h_{S_3}\right)\ge F\left(h_{S_1},h_{S_2}\right)$. From the definition of $F\left(h_{S_1},h_{S_2}\right)$, we have $F\left(h_{S_1},h_{S_2}\right)_{\sigma(l)}+F\left(h_{S_1},h_{S_2}\right)_{\sigma(L-l+1)}\ge 1$ $(l=1,2,\ldots,L)$. Based on the hesitant fuzzy linguistic entropy measure, we get $E\left(F\left(h_{S_1},h_{S_3}\right)\right)\ge E\left(F\left(h_{S_1},h_{S_2}\right)\right)$. Similarly, we can prove the case where $g\left(s^1_{\sigma(l)}\right)\ge g\left(s^2_{\sigma(l)}\right)\ge g\left(s^3_{\sigma(l)}\right)$ $(l=1,2,\ldots,L)$.

(4) $E\left(F\left(h_{S_1},h_{S_2}\right)\right)=E\left(F\left(h_{S_2},h_{S_1}\right)\right)$.

Example 3.4. For two HFLEs $h_{S_i}=\left\{s^i_{\sigma(l)}\middle| l=1,2,\ldots,L\right\}\ (i=1,2)$, we have

$$E_1'\left(F\left(h_{S_1},h_{S_2}\right)\right)=\frac{1}{L\left(\sqrt{2}-1\right)}\sum_{l=1}^{L}\left(\sin\frac{\pi\left(2+\left|g\left(s^1_{\sigma(l)}\right)-g\left(s^2_{\sigma(l)}\right)\right|+\left|g\left(s^1_{\sigma(L-l+1)}\right)-g\left(s^2_{\sigma(L-l+1)}\right)\right|\right)}{8}+\sin\frac{\pi\left(2-\left|g\left(s^1_{\sigma(l)}\right)-g\left(s^2_{\sigma(l)}\right)\right|-\left|g\left(s^1_{\sigma(L-l+1)}\right)-g\left(s^2_{\sigma(L-l+1)}\right)\right|\right)}{8}-1\right)$$

$$E_2'\left(F\left(h_{S_1},h_{S_2}\right)\right)=\frac{1}{L\left(\sqrt{2}-1\right)}\sum_{l=1}^{L}\left(\cos\frac{\pi\left(2+\left|g\left(s^1_{\sigma(l)}\right)-g\left(s^2_{\sigma(l)}\right)\right|+\left|g\left(s^1_{\sigma(L-l+1)}\right)-g\left(s^2_{\sigma(L-l+1)}\right)\right|\right)}{8}+\right.$$

$$\left.\cos\frac{\pi\left(2-\left|g\left(s^1_{\sigma(l)}\right)-g\left(s^2_{\sigma(l)}\right)\right|-\left|g\left(s^1_{\sigma(L-l+1)}\right)-g\left(s^2_{\sigma(L-l+1)}\right)\right|\right)}{8}-1\right)$$

$$E_3'\left(F\left(h_{S_1},h_{S_2}\right)\right)=-\frac{1}{L\ln 2}\sum_{l=1}^{L}\left(\frac{2+\left|g\left(s^1_{\sigma(l)}\right)-g\left(s^2_{\sigma(l)}\right)\right|+\left|g\left(s^1_{\sigma(L-l+1)}\right)-g\left(s^2_{\sigma(L-l+1)}\right)\right|}{4}\right.$$

$$\left.\times\ln\frac{2+\left|g\left(s^1_{\sigma(l)}\right)-g\left(s^2_{\sigma(l)}\right)\right|+\left|g\left(s^1_{\sigma(L-l+1)}\right)-g\left(s^2_{\sigma(L-l+1)}\right)\right|}{4}-\frac{2-\left|g\left(s^1_{\sigma(l)}\right)-g\left(s^2_{\sigma(l)}\right)\right|-\left|g\left(s^1_{\sigma(L-l+1)}\right)-g\left(s^2_{\sigma(L-l+1)}\right)\right|}{4}\right)$$

$$=-\frac{1}{L\ln 2}\sum_{l=1}^{L}\left(\frac{2+\left|g\left(s^1_{\sigma(l)}\right)-g\left(s^2_{\sigma(l)}\right)\right|+\left|g\left(s^1_{\sigma(L-l+1)}\right)-g\left(s^2_{\sigma(L-l+1)}\right)\right|}{4}\times\right.$$

$$\left.\ln\frac{2-\left|g\left(s^1_{\sigma(l)}\right)-g\left(s^2_{\sigma(l)}\right)\right|-\left|g\left(s^1_{\sigma(L-l+1)}\right)-g\left(s^2_{\sigma(L-l+1)}\right)\right|}{4}\right)$$

$$E_4'\left(F\left(h_{S_1},h_{S_2}\right)\right)=\frac{1}{L\left(2^{(1-a)b}-1\right)}\sum_{l=1}^{L}\left(\left(\left(\frac{2+\left|g\left(s^1_{\sigma(l)}\right)-g\left(s^2_{\sigma(l)}\right)\right|+\left|g\left(s^1_{\sigma(L-l+1)}\right)-g\left(s^2_{\sigma(L-l+1)}\right)\right|}{4}\right)^a\right.\right.$$

$$\left.\left.+\left(\frac{2+\left|g\left(s^1_{\sigma(l)}\right)-g\left(s^2_{\sigma(l)}\right)\right|+\left|g\left(s^1_{\sigma(L-l+1)}\right)-g\left(s^2_{\sigma(L-l+1)}\right)\right|}{4}\right)^a\right)^b-1\right)$$

Theorem 3.4. Let $S=\left\{s_t\left|t=-\tau,\ldots,-1,0,1,\ldots,\tau\right.\right\}$ be a linguistic term set, $h_{S_i}=\left\{s^i_{\sigma(l)}\left|l=1,2,\ldots,L\right.\right\}\left(i=1,2\right)$ be two HFLEs, and $\overline{F\left(h_{S_1},h_{S_2}\right)}$ be the complementary set of $F\left(h_{S_1},h_{S_2}\right)$. Then we call $E\left(\overline{F\left(h_{S_1},h_{S_2}\right)}\right)$ the similarity measure between h_{S_1} and h_{S_2}.

Theorem 3.5. Let $h_{S_1}=\left\{s^1_{\sigma(l)}\left|l=1,2,\ldots,L\right.\right\}$ and $h_{S_2}=\left\{s^2_{\sigma(l)}\left|l=1,2,\ldots,L\right.\right\}$ be two HFLEs, and $\left|g\left(s^1_{\sigma(l)}\right)-g\left(s^2_{\sigma(l)}\right)\right|<\left|g\left(s^1_{\sigma(l+1)}\right)-g\left(s^2_{\sigma(l+1)}\right)\right|, l=1,2,\ldots,L$. We call $E\left(G\left(h_{S_1},h_{S_2}\right)\right)$ a similarity measure between h_{S_1} and h_{S_2}, where

$$G\left(h_{S_1},h_{S_2}\right)=\left(\frac{\left|g\left(s_{\sigma(1)}^1\right)-g\left(s_{\sigma(1)}^2\right)\right|^P+1}{2},\frac{\left|g\left(s_{\sigma(2)}^1\right)-g\left(s_{\sigma(2)}^2\right)\right|^P+1}{2},\ldots,\frac{\left|g\left(s_{\sigma(L)}^1\right)-g\left(s_{\sigma(L)}^2\right)\right|^P+1}{2}\right), P>0 \quad (21)$$

Based on the proposed entropy measures and similarity measures, Theorem 3.4 and Theorem 3.5 are obvious, and we omit the proofs of them here.

Theorem 3.6. Let $S=\left\{s_t \middle| t=-\tau,\ldots,-1,0,1,\ldots,\tau\right\}$ be a linguistic term set, $h_S=\left\{s_{\sigma(l)} \middle| l=1,2,\ldots,L\right\}$ be a HFLE, and suppose that $\left|g\left(s_{\sigma(l)}\right)+g\left(s_{\sigma(L-l+1)}\right)-1\right|<\left|g\left(s_{\sigma(l+1)}\right)+g\left(s_{\sigma(L-l)}\right)-1\right|, l=1,2,\ldots,[L/2]$. Then we call $\rho\left(m\left(h_S\right),n\left(h_S\right)\right)$ the entropy of h_S, where

$$m\left(h_S\right)=\left(\frac{1+\left|g\left(s_{\sigma(1)}\right)+g\left(s_{\sigma(L)}\right)-1\right|}{2},\frac{1+\left|g\left(s_{\sigma(2)}\right)+g\left(s_{\sigma(L-1)}\right)-1\right|}{2},\ldots,\frac{1+\left|g\left(s_{\sigma\left(\overline{[L/2]}\right)}\right)+g\left(s_{\sigma\left(L-\overline{[L/2]}+1\right)}\right)-1\right|}{2}\right) \quad (22)$$

$$n\left(h_S\right)=\left(\frac{1-\left|g\left(s_{\sigma(1)}\right)+g\left(s_{\sigma(L)}\right)-1\right|}{2},\frac{1-\left|g\left(s_{\sigma(2)}\right)+g\left(s_{\sigma(L-1)}\right)-1\right|}{2},\ldots,\frac{1-\left|g\left(s_{\sigma\left(\overline{[L/2]}\right)}\right)+g\left(s_{\sigma\left(L-\overline{[L/2]}+1\right)}\right)-1\right|}{2}\right) \quad (23)$$

Proof. (1) $\rho\left(m\left(h_S\right),n\left(h_S\right)\right)=0 \Leftrightarrow m\left(h_S\right)=1$, $n\left(h_S\right)=0$ or $m\left(h_S\right)=0$, $n\left(h_S\right)=1 \Leftrightarrow g\left(h_S\right)=1$, $g\left(h_S\right)=0$.

(2) $\rho\left(m\left(h_S\right),n\left(h_S\right)\right)=1 \Leftrightarrow m\left(h_S\right)=n\left(h_S\right) \Leftrightarrow g\left(s_{\sigma(l)}\right)+g\left(s_{\sigma(L-l+1)}\right)=1$, for all $l=1,2,\ldots,\overline{[L/2]} \Leftrightarrow$ $g\left(s_{\sigma(l)}\right)+g\left(s_{\sigma(L-l+1)}\right)=1$, for all $l=1,2,\ldots,L$.

(3) For two HFLEs $h_S=\left\{s_{\sigma(l)} \middle| l=1,2,\ldots,L\right\}$ and $h'_S=\left\{s'_{\sigma(l)} \middle| l=1,2,\ldots,L\right\}$. Suppose that $s_{\sigma(l)} \le s'_{\sigma(l)}$, for $s'_{\sigma(l)}+s'_{\sigma(L-l+1)} \le 1, l=1,2,\ldots,L$, then $s_{\sigma(l)} \le s'_{\sigma(l)} \le 1-s'_{\sigma(L-l+1)} \le 1-s_{\sigma(L-l+1)}$. Thus, $\left|g\left(s_{\sigma(l)}\right)+g\left(s_{\sigma(L-l+1)}\right)-1\right| \ge \left|g\left(s'_{\sigma(l)}\right)+g\left(s'_{\sigma(L-l+1)}\right)-1\right|, l=1,2,\ldots,L$. Based on the similarity measure of HFLTSs, we obtain $\rho\left(m\left(h_S\right),n\left(h_S\right)\right) \le \rho\left(m\left(h'_S\right),n\left(h_S\right)\right) \le \rho\left(m\left(h'_S\right),n\left(h'_S\right)\right)$. Similarly, if $s_{\sigma(l)} \ge s'_{\sigma(l)}$,

for $s'_{\sigma(l)} + s'_{\sigma(L-l+1)} \geq 1,\ \ l = 1, 2, \ldots, L$, then we can also get $\rho\left(m\left(h_S\right), n\left(h_S\right)\right) \leq \rho\left(m\left(h'_S\right), n\left(h'_S\right)\right)$.

(4) $\rho\left(m\left(h_S\right), n\left(h_S\right)\right) = \rho\left(m\left(\bar{h}_S\right), n\left(\bar{h}_S\right)\right)$.

4. Hesitant fuzzy linguistic cross-entropy measures

In this section, we mainly discuss two hesitant fuzzy linguistic cross-entropy measures. The definition of hesitant fuzzy linguistic cross-entropy measure can be given as follows:

Definition 4.1. Let $S = \left\{s_t \middle| t = -\tau, \ldots, -1, 0, 1, \ldots, \tau\right\}$ be a linguistic term set, $h_{S_1} = \left\{s^1_{\sigma(l)} \middle| l = 1, 2, \ldots, L\right\}$ and $h_{S_2} = \left\{s^2_{\sigma(l)} \middle| l = 1, 2, \ldots, L\right\}$ be two HFLEs. Then we call $CE\left(h_{S_1}, h_{S_2}\right)$ the hesitant fuzzy linguistic cross-entropy measure between h_{S_1} and h_{S_2} if it satisfies:

(1) $CE\left(h_{S_1}, h_{S_2}\right) \geq 0$;

(2) $CE\left(h_{S_1}, h_{S_2}\right) = 0$ if and only if $g\left(s^1_{\sigma(l)}\right) = g\left(s^2_{\sigma(l)}\right),\ l = 1, 2, \ldots, L$.

Here we give a hesitant fuzzy linguistic cross-entropy measure formula between two HFLEs h_{S_1} and h_{S_2}:

$$
\begin{aligned}
CE_Q\left(h_{S_1}, h_{S_2}\right) = \frac{1}{L\aleph}\sum_{l=1}^{L}\Bigg(& \frac{\left(1+qg\left(s^1_{\sigma(l)}\right)\right)\ln\left(1+qg\left(s^1_{\sigma(l)}\right)\right)+\left(1+qg\left(s^2_{\sigma(l)}\right)\right)\ln\left(1+qg\left(s^2_{\sigma(l)}\right)\right)}{2} \\
& -\frac{2+qg\left(s^1_{\sigma(l)}\right)+qg\left(s^2_{\sigma(l)}\right)}{2}\ln\frac{2+qg\left(s^1_{\sigma(l)}\right)+qg\left(s^2_{\sigma(l)}\right)}{2} \\
& +\frac{\left(1+q\left(1-g\left(s^1_{\sigma(L-l+1)}\right)\right)\right)\ln\left(1+q\left(1-g\left(s^1_{\sigma(L-l+1)}\right)\right)\right)+\left(1+q\left(1-g\left(s^2_{\sigma(L-l+1)}\right)\right)\right)\ln\left(1+q\left(1-g\left(s^2_{\sigma(L-l+1)}\right)\right)\right)}{2} \\
& -\frac{2+q\left(1-g\left(s^1_{\sigma(L-l+1)}\right)+1-g\left(s^2_{\sigma(L-l+1)}\right)\right)}{2}\ln\frac{2+q\left(1-g\left(s^1_{\sigma(L-l+1)}\right)+1-g\left(s^2_{\sigma(L-l+1)}\right)\right)}{2}\Bigg),\ q>0 \qquad (24)
\end{aligned}
$$

where $\aleph = (1+q)\ln(1+q) - (2+q)\left(\ln(2+q) - \ln 2\right),\ q > 0$.

Remark 3. For $CE_Q\left(h_{S_1},h_{S_2}\right)$, since $\frac{d\aleph}{dq}=1+\ln(1+q)-1-\left(\ln(2+q)-\ln 2\right)=\ln\frac{2+2q}{2+q}>\ln 1=0$, then $\aleph(q)$ is an increasing function. Also since the minimal value of $\aleph(q)$ is 0 when $q=0$, then $\aleph(q)>0$. For the function $f(x)=(1+qx)\ln(1+qx), 0\le x\le 1$, we can get $f'(x)=q\ln(1+qx)+q\ge 0$ and $f''(x)=\frac{q^2}{(1+qx)}>0$. Then, $f(x)$ is a concave-up function of x, and thus, $CE_Q\left(h_{S_1},h_{S_2}\right)\ge 0$ and $CE_Q\left(h_{S_1},h_{S_2}\right)=0$ if and only if $g\left(s^1_{\sigma(l)}\right)=g\left(s^2_{\sigma(l)}\right), l=1,2,\ldots,L$. Based on Definition 4.1, $CE_1\left(h_{S_Q},h_{S_2}\right)$ is a hesitant fuzzy linguistic cross-entropy measure between h_{S_1} and h_{S_2}.

The other hesitant fuzzy linguistic cross-entropy measure formula can be shown as follows:

$$CE_P\left(h_{S_1},h_{S_2}\right)=\frac{1}{L\left(1-2^{1-p}\right)}\sum_{l=1}^{L}\left(\frac{\left(g\left(s^1_{\sigma(l)}\right)\right)^p+\left(g\left(s^2_{\sigma(l)}\right)\right)^p}{2}+\frac{\left(1-g\left(s^1_{\sigma(L-l+1)}\right)\right)^p+\left(1-g\left(s^2_{\sigma(L-l+1)}\right)\right)^p}{2}\right.$$
$$\left.-\left(\frac{g\left(s^1_{\sigma(l)}\right)+g\left(s^2_{\sigma(l)}\right)}{2}\right)^p-\left(\frac{1-g\left(s^1_{\sigma(L-l+1)}\right)+1-g\left(s^2_{\sigma(L-l+1)}\right)}{2}\right)^p\right), p>1 \quad (25)$$

Remark 4. Considering the function $f(x)=x^p$, $0\le x\le 1$ and $p>1$, we get $f'(x)=px^{p-1}>0$ and $f''(x)=p(p-1)x^{p-2}>0$. Therefore, $f(x)$ is a concave-up function of x, and then $CE_P\left(h_{S_1},h_{S_2}\right)\ge 0$ and $CE_P\left(h_{S_1},h_{S_2}\right)=0$ if and only if $g\left(s^1_{\sigma(l)}\right)=g\left(s^2_{\sigma(l)}\right), l=1,2,\ldots,L$. According to Definition 4.1, $CE_P\left(h_{S_1},h_{S_2}\right)$ is a hesitant fuzzy linguistic cross-entropy measure between h_{S_1} and h_{S_2}.

Theorem 4.1. Let $S=\left\{s_t \middle| t=-\tau,\ldots,-1,0,1,\ldots,\tau\right\}$ be a linguistic term set, and $h_S=\left\{s_{\sigma(l)} \middle| l=1,2,\ldots,L\right\}$ be a HFLE. Then both $E_Q\left(h_S\right)=1-CE_Q\left(h_S,\bar{h}_S\right)$ and $E_P\left(h_S\right)=1-CE_P\left(h_S,\bar{h}_S\right)$ are the hesitant fuzzy linguistic entropy measures.

Proof. (1) It follows from Eq. (24) that

$$E_Q\left(h_S\right)=1-CE_Q\left(h_S,\bar{h}_S\right)$$

$$=1-\frac{2}{L\aleph}\sum_{l=1}^{L}\left(\frac{\left(1+qg\left(s_{\sigma(l)}\right)\right)\ln\left(1+qg\left(s_{\sigma(l)}\right)\right)+\left(1+q\left(1-g\left(s_{\sigma(L-l+1)}\right)\right)\right)\ln\left(1+q\left(1-g\left(s_{\sigma(L-l+1)}\right)\right)\right)}{2}\right.$$

$$\left.-\frac{2+qg\left(s_{\sigma(l)}\right)+q\left(1-g\left(s_{\sigma(L-l+1)}\right)\right)}{2}\ln\frac{2+qg\left(s_{\sigma(l)}\right)+q\left(1-g\left(s_{\sigma(L-l+1)}\right)\right)}{2}\right), q>0 \qquad (26)$$

where $\aleph=(1+q)\ln(1+q)-(2+q)\left(\ln(2+q)-\ln 2\right), q>0$.

For two HFLEs $h_S=\left\{s_{\sigma(l)} \middle| l=1,2,\ldots,L\right\}$ and $h'_S=\left\{s'_{\sigma(l)} \middle| l=1,2,\ldots,L\right\}$, if $s_{\sigma(l)}\leq s'_{\sigma(l)}$, for $s'_{\sigma(l)}+s'_{\sigma(L-l+1)}\leq 1$, $l=1,2,\ldots,L$, then we have $s_{\sigma(l)}\leq s'_{\sigma(l)}\leq 1-s'_{\sigma(L-l+1)}\leq 1-s_{\sigma(L-l+1)}$. Thus, $\left|s_{\sigma(l)}+s_{\sigma(L-l+1)}-1\right|\geq\left|s'_{\sigma(l)}+s'_{\sigma(L-l+1)}-1\right|$. Let $0\leq x,y\leq 1$ and $\xi=|x-y|$, then

$$f(x,y)=\frac{(1+qx)\ln(1+qx)+(1+qy)\ln(1+qy)}{2}-\frac{1+qx+1+qy}{2}\ln\frac{1+qx+1+qy}{2}, q>0 \qquad (27)$$

Assume that $x\geq y$, then $x=t+y$. Thus, Eq. (27) becomes

$$f(\xi,y)=\frac{(1+q(\xi+y))\ln(1+q(\xi+y))+(1+qy)\ln(1+qy)}{2}-\frac{1+q(\xi+y)+1+qy}{2}\ln\frac{1+q(\xi+y)+1+qy}{2}, q>0$$

(28)

and then

$$\frac{df(\xi,y)}{d\xi}=\frac{q+q\ln\left(1+q(y+\xi)\right)+q+q\ln(1+qy)}{2}-\frac{q}{2}-\frac{q}{2}\ln\frac{1+q(y+\xi)+1+qy}{2}\geq 0, q>0 \qquad (29)$$

Therefore, we can get that the function $f(x,y)$ is a non-decreasing function of $\xi=|x-y|$, for $x\geq y$. Similarly, we can prove the other case where $x\leq y$. Therefore, we have $E_Q(h_S)\leq E_Q(h'_S)$, and get the maximal value 1 of $E_Q(h_S)$ when $h_S=\bar{h}_S$, as well as get the minimal value 0 of $E_Q(h_S)$ when $g(h_S)=0$ or $g(h_S)=1$.

(2) It follows from Eq. (25) that

$$E_P(h_S)=1-CE_P\left(h_S,\bar{h}_S\right)$$

$$=1-\frac{2}{L\left(1-2^{1-p}\right)}\sum_{i=1}^{L}\left(\frac{\left(g\left(s_{\sigma(l)}\right)\right)^{p}+\left(1-g\left(s_{\sigma(L-l+1)}\right)\right)^{p}}{2}-\left(\frac{g\left(s_{\sigma(l)}\right)+1-g\left(s_{\sigma(L-l+1)}\right)}{2}\right)^{p}\right),\ p>1 \tag{30}$$

Similarly, if $\zeta=|x-y|$, then

$$\varphi(x,y)=\frac{x^{p}+y^{p}}{2}-\left(\frac{x+y}{2}\right)^{p},\ 0\le x,y\le 1,\ p>1 \tag{31}$$

and if $x\ge y$, then $x=\zeta+y$. In this case, Eq. (31) becomes

$$\varphi(\zeta,y)=\frac{(\zeta+y)^{p}+y^{p}}{2}-\left(\frac{\zeta+2y}{2}\right)^{p},\ p>1 \tag{32}$$

and

$$\frac{d\varphi(\zeta,y)}{d\zeta}=\frac{p}{2}\left((\zeta+y)^{p-1}-\left(\frac{\zeta+2y}{2}\right)^{p-1}\right)\ge 0,\ p>1 \tag{33}$$

Based on Eq. (33), the function $\varphi(x,y)$ is a non-decreasing function. In a similar way, we can prove the other case where $x\le y$. Thus, we have $E_Q(h_S)\le E_Q(h_S')$, and obtain the maximal value 1 of $E_Q(h_S)$ when $h_S=\bar{h}_S$, as well as get the minimal value 0 of $E_Q(h_S)$ when $g(h_S)=0$ or $g(h_S)=1$.

Theorem 4.2. Let $S=\{s_t \mid t=-\tau,\ldots,-1,0,1,\ldots,\tau\}$ be a linguistic term set, and $h_{S_i}=\{s^{i}_{\sigma(l)} \mid l=1,2,\ldots,L\}\ (i=1,2)$ be two HFLEs. Suppose that there is only one element in h_{S_1} and $h_{S_1}=\{s_0\}$. Then $CE_Q\left(h_{S_1},h_{S_2}\right)=CE_Q\left(h_{S_1},\bar{h}_{S_2}\right)$ and $CE_P\left(h_{S_1},h_{S_2}\right)=CE_P\left(h_{S_1},\bar{h}_{S_2}\right)$.

Proof. By extending h_{S_1}, we have $s^{1}_{\sigma(1)}=s^{1}_{\sigma(2)}=\cdots=s^{1}_{\sigma(L)}=s_0$ and $1-g\left(s^{1}_{\sigma(l)}\right)=g\left(s^{1}_{\sigma(l)}\right)=g\left(s_0\right)$ $(l=1,2,\ldots,L)$. Suppose that $\bar{h}_{S_2}=\left(\bar{s}^{2}_{\sigma(l)} \mid l=1,2,\ldots,L\right)$, and $g\left(s^{2}_{\sigma(l)}\right)+g\left(\bar{s}^{2}_{\sigma(L-l+1)}\right)=1$, then

(1) Eq. (24) can be transformed into

$$CE_Q\left(h_{S_1},\bar{h}_{S_2}\right)=\frac{1}{L\aleph}\sum_{l=1}^{L}\left(\frac{\left(1+qg\left(s^{1}_{\sigma(l)}\right)\right)\ln\left(1+qg\left(s^{1}_{\sigma(l)}\right)\right)+\left(1+q\left(1-g\left(s^{2}_{\sigma(L-l+1)}\right)\right)\right)\ln\left(1+q\left(1-g\left(s^{2}_{\sigma(L-l+1)}\right)\right)\right)}{2}\right.$$

$$-\frac{2+q\left(1-g\left(s^{1}_{\sigma(L-l+1)}\right)\right)+q\left(1-g\left(s^{2}_{\sigma(L-l+1)}\right)\right)}{2}\ln\frac{2+q\left(1-g\left(s^{1}_{\sigma(L-l+1)}\right)\right)+q\left(1-g\left(s^{2}_{\sigma(L-l+1)}\right)\right)}{2}$$

$$+\frac{\left(1+q\left(1-g\left(s^{1}_{\sigma(L-l+1)}\right)\right)\right)\ln\left(1+q\left(1-g\left(s^{1}_{\sigma(L-l+1)}\right)\right)\right)+\left(1+qg\left(s^{2}_{\sigma(l)}\right)\right)\ln\left(1+qg\left(s^{2}_{\sigma(l)}\right)\right)}{2}$$

$$\left.-\frac{2+q\left(g\left(s^{1}_{\sigma(l)}\right)+g\left(s^{2}_{\sigma(l)}\right)\right)}{2}\ln\frac{2+q\left(g\left(s^{1}_{\sigma(l)}\right)+g\left(s^{2}_{\sigma(l)}\right)\right)}{2}\right)$$

$$=\frac{1}{L\aleph}\sum_{l=1}^{L}\left(\frac{\left(1+qg\left(s^{1}_{\sigma(l)}\right)\right)\ln\left(1+qg\left(s^{1}_{\sigma(l)}\right)\right)+\left(1+qg\left(s^{2}_{\sigma(l)}\right)\right)\ln\left(1+qg\left(s^{2}_{\sigma(l)}\right)\right)}{2}\right.$$

$$-\frac{2+q\left(g\left(s^{1}_{\sigma(l)}\right)+g\left(s^{2}_{\sigma(l)}\right)\right)}{2}\ln\frac{2+q\left(g\left(s^{1}_{\sigma(l)}\right)+g\left(s^{2}_{\sigma(l)}\right)\right)}{2}$$

$$+\frac{\left(1+q\left(1-g\left(s^{1}_{\sigma(L-l+1)}\right)\right)\right)\ln\left(1+q\left(1-g\left(s^{1}_{\sigma(L-l+1)}\right)\right)\right)+\left(1+q\left(1-g\left(s^{2}_{\sigma(L-l+1)}\right)\right)\right)\ln\left(1+q\left(1-g\left(s^{2}_{\sigma(L-l+1)}\right)\right)\right)}{2}$$

$$\left.-\frac{2+q\left(1-g\left(s^{1}_{\sigma(L-l+1)}\right)\right)+q\left(1-g\left(s^{2}_{\sigma(L-l+1)}\right)\right)}{2}\ln\frac{2+q\left(1-g\left(s^{1}_{\sigma(L-l+1)}\right)\right)+q\left(1-g\left(s^{2}_{\sigma(L-l+1)}\right)\right)}{2}\right)=CE_Q\left(h_{S_1},h_{S_2}\right)$$

Similarly, Eq. (25) can be transformed into

$$CE_P\left(h_{S_1},\bar{h}_{S_2}\right)=\frac{1}{L\left(1-2^{1-p}\right)}\sum_{l=1}^{L}\left(\frac{\left(g\left(s^{1}_{\sigma(l)}\right)\right)^p+\left(1-g\left(s^{2}_{\sigma(L-l+1)}\right)\right)^p}{2}+\frac{\left(1-g\left(s^{1}_{\sigma(L-l+1)}\right)\right)^p+\left(g\left(s^{2}_{\sigma(l)}\right)\right)^p}{2}\right.$$

$$\left.-\left(\frac{1-g\left(s^{1}_{\sigma(L-l+1)}\right)+1-g\left(s^{2}_{\sigma(L-l+1)}\right)}{2}\right)^p-\left(\frac{g\left(s^{1}_{\sigma(l)}\right)+g\left(s^{2}_{\sigma(l)}\right)}{2}\right)^p\right)$$

$$=\frac{1}{L\left(1-2^{1-p}\right)}\sum_{l=1}^{L}\left(\frac{\left(g\left(s^{1}_{\sigma(l)}\right)\right)^p+\left(g\left(s^{2}_{\sigma(l)}\right)\right)^p}{2}+\frac{\left(1-g\left(s^{1}_{\sigma(L-l+1)}\right)\right)^p+\left(1-g\left(s^{2}_{\sigma(L-l+1)}\right)\right)^p}{2}\right.$$

$$\left.-\left(\frac{g\left(s^{1}_{\sigma(l)}\right)+g\left(s^{2}_{\sigma(l)}\right)}{2}\right)^p-\left(\frac{1-g\left(s^{1}_{\sigma(L-l+1)}\right)+1-g\left(s^{2}_{\sigma(L-l+1)}\right)}{2}\right)^p\right)=CE_P\left(h_{S_1},h_{S_2}\right)$$

Suppose that there are two HFLEs $h_{S_1}=\left(s_{-1},s_0,s_1\right)$ and $h_{S_2}=\left(s_{-2},s_{-1}\right)$, we can describe the change trends of the values of the hesitant fuzzy linguistic cross-entropy measures $CE_Q\left(h_{S_1},h_{S_2}\right)$ and $CE_P\left(h_{S_1},h_{S_2}\right)$ by changing the values of the parameters q and p. We take the values of p from 1.01 to

2 with the step size 0.03, and take the values of q from 0.01 to 1 with the same step size 0.03. By using the MATLAB software, the results can be shown in Fig. 6.

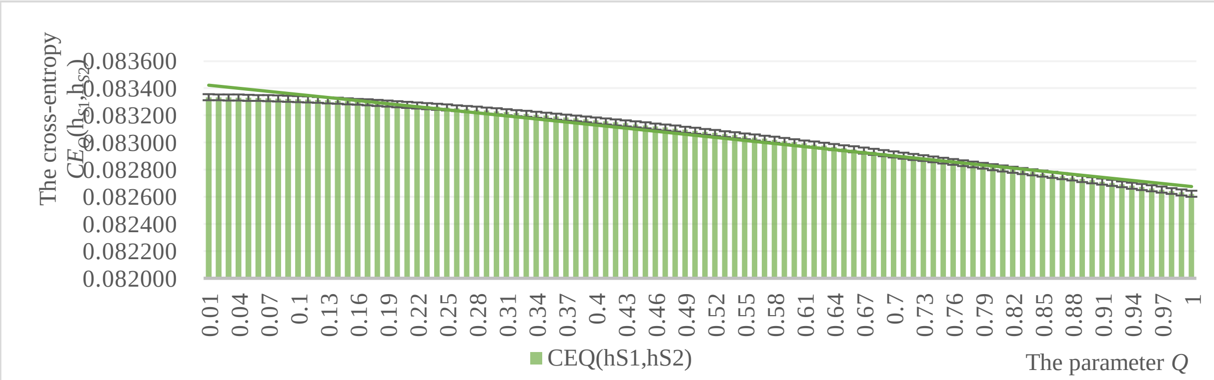


Fig. 6. The change trend of $CE_Q\left(h_{S_1},h_{S_2}\right)$ based on the parameter q.

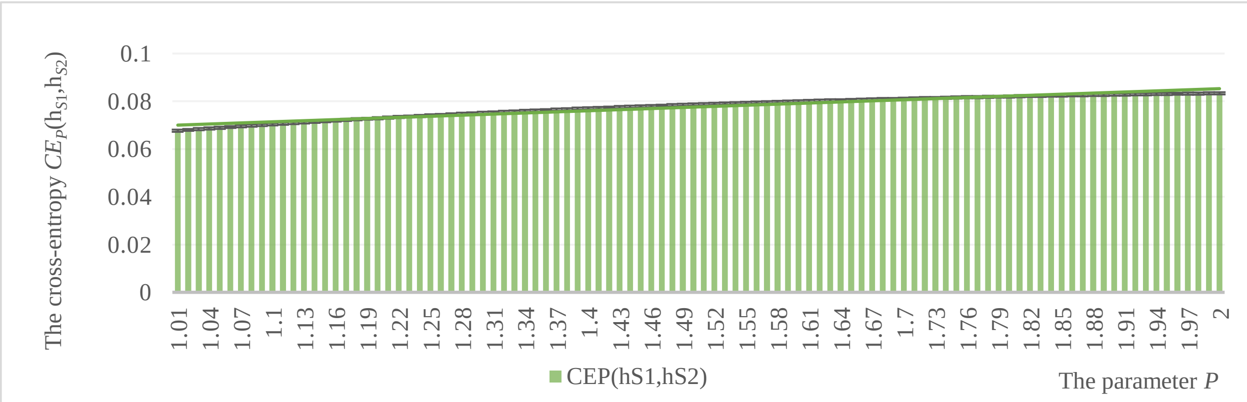


Fig. 7. The change trend of $CE_P\left(h_{S_1},h_{S_2}\right)$ based on the parameter p.

Remark 5. From Fig. 6 and Fig. 7, we can obtain that the cross-entropy $CE_Q\left(h_{S_1},h_{S_2}\right)$ is decreasing when increasing the values of q. On the contrary, the cross-entropy $CE_P\left(h_{S_1},h_{S_2}\right)$ is also increasing when increasing the values of p.

In the following, we give an example to illustrate these two hesitant fuzzy linguistic cross-entropy measures:

Example 4.1. Let $S=\left\{s_t \middle| t=-3,-2,-1,0,1,2,3\right\}$ be a linguistic term set, $h_{S_1}=\left\{s_0,s_1\right\}$ and $h_{S_2}=\left\{s_{-1},s_0\right\}$ be two HFLEs. The hesitant fuzzy linguistic cross-entropy between h_{S_1} and h_{S_2} can be obtained as:

$$CE_Q\left(h_{S_1},h_{S_2}\right)=\frac{1}{2\left(5\ln 5-3\ln 3\right)}\left(\frac{3}{2}\ln\frac{3}{2}+\frac{5}{3}\ln\frac{5}{3}-\frac{19}{6}\ln\frac{19}{12}+\frac{3}{2}\ln\frac{3}{2}+\frac{4}{3}\ln\frac{4}{3}-\frac{17}{6}\ln\frac{17}{12}\right)=0.006,\text{ if } q=1.$$

$$CE_P\left(h_{S_1},h_{S_2}\right)=\frac{1}{2\left(1-2^{1-2}\right)}\left(\frac{1}{9}+\frac{1}{4}-\frac{25}{72}+\frac{4}{9}+\frac{1}{4}-\frac{49}{72}\right)=0.028,\text{ if } p=2.$$

5. Weight-determining method and hesitant fuzzy linguistic alternative queuing method

5.1. Hesitant fuzzy linguistic entropy and cross-entropy-based weight-determining method

As we know, a hesitant fuzzy linguistic MCDM problem can be described as follows: Suppose that $A=\{A_1,A_2,\ldots,A_m\}$ is a set of alternatives, $C=\{C_1,C_2,\ldots,C_n\}$ is a set of criteria, and $w=(w_1,w_2,\ldots,w_n)^T$ is the weight vector of all criteria, where $w_j\geq 0$, $j=1,2,\ldots,n$, and $\sum_{j=1}^{n}w_j=1$. Let $H=\left(h_{S_{ij}}\right)_{m\times n}$ $(i=1,2,\ldots,m;\ j=1,2,\ldots,n)$ be the hesitant fuzzy linguistic decision matrix, where $h_{S_{ij}}$ is a HFLE for the alternative A_i with respect to the criterion C_j. The decision matrix can be shown as:

$$H=\begin{bmatrix} h_{S_{11}} & h_{S_{12}} & \cdots & h_{S_{1n}} \\ h_{S_{21}} & h_{S_{22}} & \cdots & h_{S_{2n}} \\ \vdots & \vdots & \ddots & \vdots \\ h_{S_{m1}} & h_{S_{m2}} & \cdots & h_{S_{mn}} \end{bmatrix}$$

For most hesitant fuzzy linguistic MCDM problems, the weights of criteria are completely unknown or incompletely known due to the time pressure, the lack of knowledge or data, and the experts' limited expertise about the considered problems. Therefore, the first step of solving the MCDM problems is to determine the criteria weights, and a number of methods can be used to determine the criteria weights under hesitant fuzzy linguistic environment such as the entropy measure method [11]. However, if we use the entropy method, then it can only consider the influence of individual HFLEs, but the interrelationships between the evaluation values of alternatives with respect to each criterion are not taken into account. Therefore, it is necessary to develop a more reasonable method to determine the criteria weights based on the hesitant fuzzy linguistic entropy and cross-entropy measures.

For any criterion C_j, we can define the average cross-entropy of the alternative A_i to all the other alternatives, which can be shown as $\frac{1}{m-1}\sum_{\delta=1,i\neq\delta}^{m} CE\left(h_{S_{ij}}, h_{S_{\delta j}}\right)$. Then, the overall cross-entropy for the criterion C_j, i.e., the divergence degree of all alternatives corresponding to the criterion C_j is given as $\sum_{i=1}^{m}\left(\frac{1}{m-1}\sum_{\delta=1,i\neq\delta}^{m} CE\left(h_{S_{ij}}, h_{S_{\delta j}}\right)\right)$. It is natural that if the performance value of each alternative has little difference under a criterion, then it implies that this criterion plays a slightly important role in the priority procedure, so this criterion should be given a small weight. On the contrary, if there exist obvious differences about the performance values under a criterion, then such a criterion plays an important role in the MCDM problems and thus, it should be assigned a bigger weight.

Similarly, by considering the hesitant fuzzy linguistic entropy, the overall entropy of the criterion C_j can be shown as $\sum_{i=1}^{m} E\left(h_{S_{ij}}\right)$. Based on the entropy theory, the experts should assign the criterion a bigger weight if the associated entropy value of the criterion is smaller considering that it can provide more useful information to the experts.

By combining the hesitant fuzzy linguistic cross-entropy and entropy measures, we get

$$\sum_{i=1}^{m}\left(\frac{1}{m-1}\sum_{\delta=1,i\neq\delta}^{m} CE\left(h_{S_{ij}}, h_{S_{\delta j}}\right)+\left(1-E\left(h_{S_{ij}}\right)\right)\right) \tag{34}$$

Therefore, we can draw a conclusion that the criterion C_j can be assigned a bigger weight when Eq. (34) has a bigger value. Thus, we can give a formula to determine the weight of each criterion:

$$w_j=\frac{\sum_{i=1}^{m}\left(\frac{1}{m-1}\sum_{\delta=1,i\neq\delta}^{m} CE\left(h_{S_{ij}}, h_{S_{\delta j}}\right)+\left(1-E\left(h_{S_{ij}}\right)\right)\right)}{\sum_{j=1}^{n}\sum_{i=1}^{m}\left(\frac{1}{m-1}\sum_{\delta=1,i\neq\delta}^{m} CE\left(h_{S_{ij}}, h_{S_{\delta j}}\right)+\left(1-E\left(h_{S_{ij}}\right)\right)\right)}, \quad j=1,2,\ldots,n \tag{35}$$

In fact, the experts may not be able to judge the correct values for the criteria weights because they may lack the specialized knowledge or the practical decision situations are complicated. Sometimes, the experts can provide partial knowledge about the criteria weights. For convenience, we let Φ be the set of all information about the criteria weights. Then, Model 1 can be established to obtain the optimal weight vector:

Model 1

$$Max\ E_w = \sum_{j=1}^{n}\sum_{i=1}^{m}\left(\frac{1}{m-1}\sum_{\delta=1,i\neq\delta}^{m} CE\left(h_{S_{ij}}, h_{S_{\delta j}}\right) + \left(1 - E\left(h_{S_{ij}}\right)\right)\right)\cdot w_j$$
$$s.t.\begin{cases} w = \left(w_1, w_2, \ldots, w_n\right)^T \in \Phi \\ \sum_{j=1}^{n} w_j = 1,\ w_j \geq 0,\ j = 1, 2, \ldots, n \end{cases}$$

Remark 6. In Model 1, there exists the incomplete weight information Φ, and the criterion C_j can be assigned a bigger weight when Eq. (34) has a bigger value. Thus, we combine Eq. (34) and the weight vector $w = \left(w_1, w_2, \ldots, w_n\right)^T$ as the objective function E_w, and then calculate the maximum of E_w.

5.2. Hesitant fuzzy linguistic alternative queuing method

For the classical decision making methods, we need to aggregate the criteria values for each alternative and get the ranking of all alternatives. However, considering that the decision information can be disposed from the angle of criterion, in other words, we can deal with all the alternatives under each criterion, Gou et al. [13] established the AQM, which can be used to deal with hybrid fuzzy information, ranking information, and both of them by combining the directed graph and 0-1 precedence relationship matrix. Obviously, we can also develop a HFL-AQM to deal with the MCDM problems with hesitant fuzzy linguistic information. Firstly, we need to show the directed graph and the 0-1 precedence relationship matrix.

(1) Directed graph

As shown in Fig. 8, there are five alternatives $A_1, A_2, \ldots, A_5$. The directed arc $A_1 \rightarrow A_3$ expresses $A_1 \succ A_3$ (A_1 is superior to A_3). If two alternatives are equal ($A_1 = A_4$), then we need to draw two directed arcs $A_1 \to A_4$ and $A_4 \to A_1$. Specially, we do not need to draw any directed arc if two alternatives cannot be compared (A_2 and A_5).

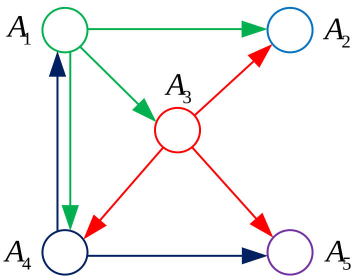


Fig. 8. A directed graph of alternatives.

(2) 0-1 precedence relationship matrix

The 0-1 precedence relationship matrix, denoted by $P = (p_{ik})_{m\times m} (i, k = 1, 2, \ldots, m)$, can be used to express the precedence relationships among the alternatives A_i $(i = 1, 2, ..., m)$. We denote $p_{ik} = 1$ and $p_{ki} = 0$ if $A_i \succ A_k$, denote $p_{ik} = p_{ki} = 1$ if $A_i = A_k$, and denote $p_{ik} = p_{ki} = 0$ if A_i and A_k cannot be compared.

In fact, the directed graph and the 0-1 precedence relationship matrix can be equivalently transformed to each other. Therefore, Fig. 8 can be transformed into the 0-1 precedence relationship matrix M:

$$M = \begin{matrix} & \begin{matrix} A_1 & A_2 & A_3 & A_4 & A_5 \end{matrix} \\ \begin{matrix} A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \end{matrix} & \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{matrix}$$

When dealing with the MCDM problem, the final results can be shown in a directed graph or a 0-1 precedence relationship matrix. Then we can derive the ranking of all alternatives. In the directed graph, let κ_i be the number of directed arcs which start from A_i, and ι_i be the number of directed arcs which point to A_i. Then we have $\Delta_i = \kappa_i - \iota_i$. Obviously, the bigger the value of Δ_i is, the more optimal the alternative A_i should be. Finally, we can get the ranking of all alternatives based on the values of all Δ_i $(i = 1, 2, ..., m)$.

In the following, by combining the hesitant fuzzy linguistic entropy and cross-entropy measures with the HFL-AQM, we give an algorithm for dealing with the MCDM problems with incomplete weight information, which mainly includes two parts: (1) calculating the weight vector of criteria; and (2) deriving the ranking of alternatives. The algorithm can be shown as follows:

Step 1. Calculate the weight vector $w = (w_1, w_2, \ldots, w_n)^T$ of the criteria by utilizing the hesitant fuzzy linguistic entropy and cross-entropy measures.

Step 2. Make pairwise comparisons among all alternatives with respect to each criterion by utilizing the expectation values and the variances of HFLEs. Then the directed graph or the 0-1 precedence relationship matrix with respect to each criterion can be established. For each alternative pair (A_i, A_k) corresponding to the criterion C_j, we denote $(A_i \succ A_k)_j$ if the alternative A_i is superior to A_k; On the contrary, we denote $(A_i \prec A_k)_j$ if the alternative A_k is superior to A_i; In particular, we denote $(A_i = A_k)_j$ if there is no difference between A_i and A_k.

Step 3. Based on the considered three cases in Step 2, the overall pros weights $w(A_i \succ A_k)$ $(i, k = 1, 2, \ldots, m)$, the overall cons weights $w(A_i \prec A_k)$ $(i, k = 1, 2, \ldots, m)$ and the overall indifference weights $w(A_i = A_k)$ of all alternative pairs (A_i, A_k) $(i, k = 1, 2, \ldots, m)$ can be calculated. Then, summing all the weights of $(A_i \succ A_k)_j$ over the criteria C_j $(j = 1, 2, \ldots, n)$, we obtain $w(A_i \succ A_k) = \sum_{j \in (A_i \succ A_k)_j} w_j$.

Analogously, we can also calculate $w(A_i = A_k)$ and $w(A_i \prec A_k)$, respectively.

Step 4. Calculate the overall pros and cons indicated value about the alternative pair (A_i, A_k):

$$O_{\Im}(A_i, A_k) = \frac{w(A_i \succ A_k) + \Im w(A_i = A_k)}{w(A_i \prec A_k) + \Im w(A_i = A_k)} \tag{36}$$

where $0 \le \Im \le 1$. The parameter $\Im$ indicates the important degree of $\left(A_i = A_k\right)$.

Step 5. Given the threshold value $\aleph > 1$, we get the relationships among all alternatives:

$$\begin{cases} A_i \succ A_k, & O_{\Im}\left(A_i, A_k\right) \ge \aleph \\ A_i = A_k, & 1/\aleph < O_{\Im}\left(A_i, A_k\right) < \aleph \\ A_i \prec A_k, & 0 < O_{\Im}\left(A_i, A_k\right) \le 1/\aleph \end{cases} \tag{37}$$

Then, the ultima directed graph or the 0-1 precedence relationship matrix can be constructed.

Step 6. Based on the ultima directed graph or the 0-1 precedence relationship matrix, we calculate the ranking value of each alternative A_i: $\Delta_i = \kappa_i - \iota_i$, $i = 1, 2, \ldots, m$.

Step 7. Rank all alternatives $A_i\left(i = 1, 2, \ldots, m\right)$ by comparing the values $\Delta_i\left(i = 1, 2, \ldots, m\right)$. The larger Δ_i, the better the alternative A_i, and thus, we get the optimal alternative with the largest of $\Delta_i\left(i = 1, 2, \ldots, m\right)$.

To illustrate the calculation process intuitively, a flow chart can be drawn as shown in Fig. 9.

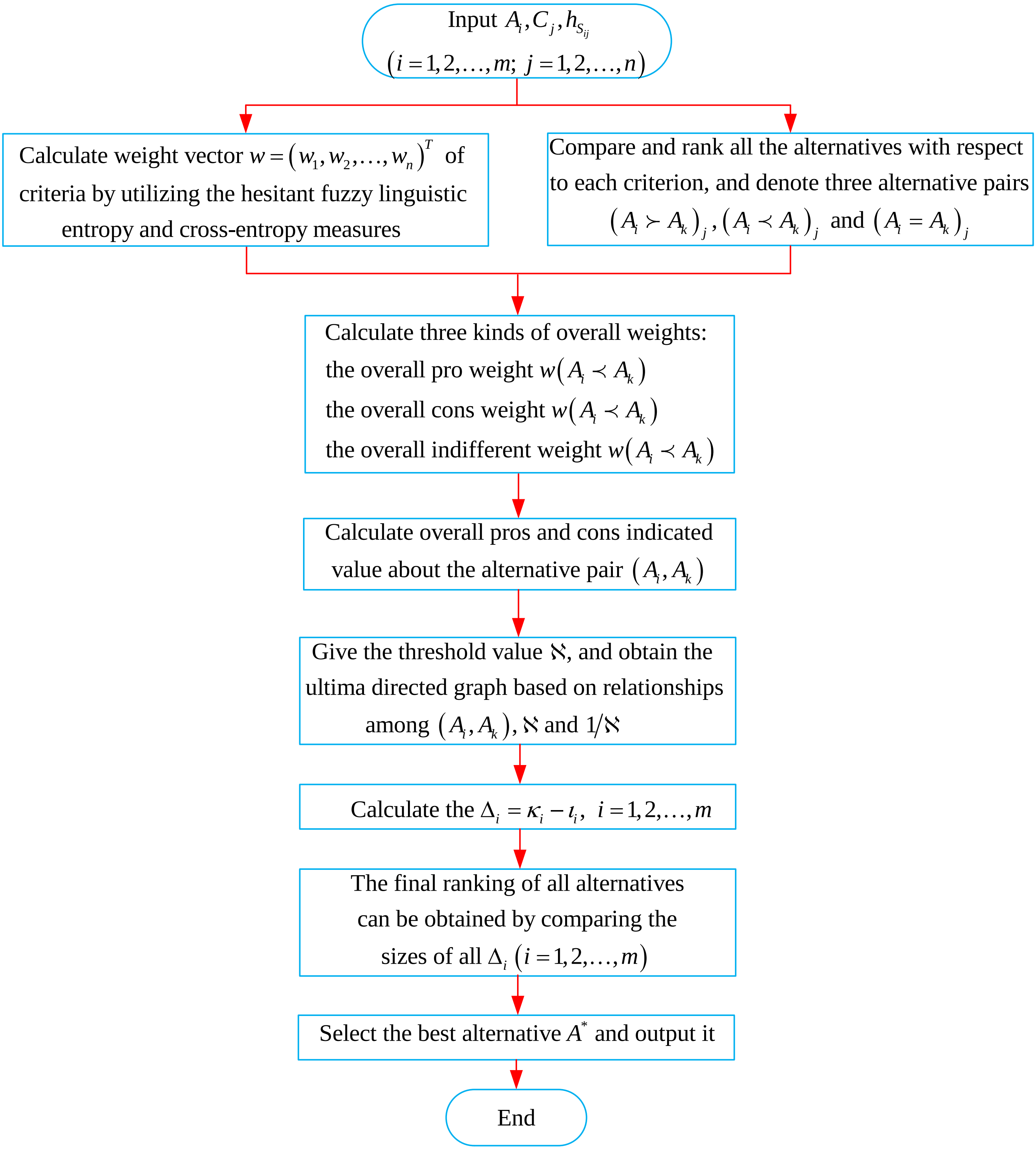


Fig. 9. The algorithm based on hesitant fuzzy linguistic entropy and cross-entropy measures and HFL-AQM.

6. Case study

6.1. A practical application of the weight-determining method and the HFL-AQM

In China, the hierarchical medical (HM) is the most important work of healthcare reform in 2016. The tertiary hospitals play a key role in medical science, technological innovation and talent cultivation. We can retrospect the fountainhead of the HM at the beginning of hospital hierarchy partition. Even though the function of the HM can be judged from the hospital hierarchy partition, considering the medical technology and medical facility are uneven in different levels of hospitals, the high quality resources are concentrated in tertiary hospitals and the shunt effect of medical insurance payment system is not obvious. Thus, the above shortcomings lead to a series of problems such as the serious phenomenon of disorder hospitalizing, the medical resource allocation increasingly imbalance, intensifying the doctor-patient conflicts, etc.

Because of this, China vigorously carries out the HM by taking the top-down pushing approach, such as giving policy funding to support the development of basic medical institution and limiting the expansion of tertiary hospitals. In spite that the tertiary hospitals still dominate the absolute advantage positions in medical system, at the same time, some practical difficulties appear on the road of the HM implementation. Some aspects of the resultant force have caused a huge impact to tertiary hospitals and the hospitals' administrators have to face these problems: the outpatient service volume reduction, the revenue reduction, the development limitation, the medical personnel increased mobility, the management of personnel increased difficulty, etc.

In order to deal with these problems, the managers come up with some alternative methods:

A_1: Creating a relatively close regional medical association;

A_2: Optimizing the structure of hospital business;

A_3: Building some key departments;

A_4: Deepening the cooperation with county-level hospitals;

A_5: Deepening the cooperation with social capitals.

This is a MCDM problem for selecting the optimal alternative methods to deal with the influences of the HM. Meanwhile, for the problem we discussed above, some criteria can be set, such as C_1: the outpatient service volume, C_2: the revenue, C_3: the development, C_4: the medical personnel mobility, C_5: the management of personnel. For this MCDM problem, the weights of the criteria are incompletely known and can be given as $\Phi = \{w_1 \neq w_2 \neq w_3 \neq w_4, 0.1 \leq w_3 \leq 0.25, w_2 \leq 4w_1 \leq 2w_4, w_4 \leq w_5, w_2 \leq 0.4\}$. The experts give their evaluations about each alternative with respect to each criterion. The evaluation values are expressed as HFLEs and they establish a decision matrix $R = \left(h_{S_{ij}}\right)_{5\times5}$ as shown in Table 1.

Table 1.
The decision matrix of evaluation values.

	C_1	C_2	C_3	C_4	C_5
A_1	$\{s_0, s_1, s_2\}$	$\{s_0, s_1, s_2\}$	$\{s_0\}$	$\{s_1, s_2\}$	$\{s_1, s_2, s_3\}$
A_2	$\{s_{-1}, s_0, s_1\}$	$\{s_2\}$	$\{s_0, s_1\}$	$\{s_0, s_1, s_2\}$	$\{s_1, s_2\}$
A_3	$\{s_{-3}, s_{-2}, s_{-1}\}$	$\{s_1, s_2\}$	$\{s_{-1}, s_0, s_1\}$	$\{s_1, s_2, s_3\}$	$\{s_2\}$
A_4	$\{s_0\}$	$\{s_{-2}, s_{-1}\}$	$\{s_1\}$	$\{s_{-2}, s_{-1}\}$	$\{s_0, s_1\}$
A_5	$\{s_1, s_2, s_3\}$	$\{s_{-3}\}$	$\{s_1, s_2\}$	$\{s_0\}$	$\{s_2\}$

We can utilize the hesitant fuzzy linguistic entropy and cross-entropy based weight-determining method and the HFL-AQM to deal with this MCDM problem.

Step 1. According to Model 1, we can calculate the weight vector of criteria by the following model:

Model 2

$$Max\ E_w = \sum_{j=1}^{n}\sum_{i=1}^{m}\omega_\varsigma\left(\frac{1}{m-1}\sum_{\delta=1, i\neq\delta}^{m} CE\left(h_{S_{ij}}, h_{S_{\delta j}}\right) + \left(1 - E\left(h_{S_{ij}}\right)\right)\right)\cdot w_j$$

$$s.t.\begin{cases} w_1 \neq w_2 \neq w_3 \neq w_4 \neq w_5 \\ 0.1 \leq w_3 \leq 0.25, w_2 \leq 4w_1 \leq 2w_4, w_4 \leq w_5, w_2 \leq 0.4 \\ \sum_{j=1}^{n} w_j = 1, w_j \geq 0, \ j = 1, 2, \ldots, n \end{cases}$$

In Section 3 and Section 4, we define four hesitant fuzzy linguistic entropy measures and two hesitant fuzzy linguistic cross-entropy measures. We can use them to obtain eight weight vectors of criteria and the results can be shown in Table 2. In the following step, we only take the first combination as an example.

Table 2.
The weight vector of the criteria based on different combinations.

The entropy and cross-entropy Combination	The weight vector w
Combining CE_Q and E_1	$w^1=(0.094,0.379,0.100,0.198,0.229)^T$
Combining CE_Q and E_2	$w^2=(0.094,0.379,0.100,0.198,0.229)^T$
Combining CE_Q and E_3	$w^3=(0.099,0.398,0.100,0.200,0.203)^T$
Combining CE_Q and E_4	$w^4=(0.099,0.398,0.100,0.201,0.202)^T$
Combining CE_P and E_1	$w^5=(0.094,0.378,0.100,0.190,0.238)^T$
Combining CE_P and E_2	$w^6=(0.094,0.378,0.100,0.190,0.238)^T$
Combining CE_P and E_3	$w^7=(0.099,0.395,0.100,0.199,0.207)^T$
Combining CE_P and E_4	$w^8=(0.099,0.399,0.100,0.200,0.202)^T$

Step 2. Obtain the ranking of all alternatives with respect to each criterion by utilizing the expectation value and the variance of HFLEs. For example, $e\left(h_{S_{11}}\right)=\frac{1}{3}\left(\frac{1}{2}+\frac{2}{3}+\frac{5}{6}\right)=\frac{2}{3}>e\left(h_{S_{21}}\right)=\frac{1}{3}\left(\frac{1}{3}+\frac{1}{2}+\frac{2}{3}\right)=\frac{1}{2}$, then we denote 1 in the precedence relationship matrix of C_1. Similarly, the five 0-1 precedence relationship matrixes with respect to $C_i\ (i=1,2,\ldots,5)$ can be established, shown as $M_\zeta\ (\zeta=1,2,\ldots,5)$, respectively:

$$M_1=\begin{matrix} & A_1 & A_2 & A_3 & A_4 & A_5 \\ A_1 & 1 & 1 & 1 & 1 & 0 \\ A_2 & 0 & 1 & 1 & 0 & 0 \\ A_3 & 0 & 0 & 1 & 0 & 0 \\ A_4 & 0 & 1 & 1 & 1 & 0 \\ A_5 & 1 & 1 & 1 & 1 & 1 \end{matrix},\quad M_2=\begin{matrix} & A_1 & A_2 & A_3 & A_4 & A_5 \\ A_1 & 1 & 0 & 0 & 1 & 1 \\ A_2 & 1 & 1 & 1 & 1 & 1 \\ A_3 & 1 & 0 & 1 & 1 & 1 \\ A_4 & 0 & 0 & 0 & 1 & 1 \\ A_5 & 0 & 0 & 0 & 0 & 1 \end{matrix},\quad M_3=\begin{matrix} & A_1 & A_2 & A_3 & A_4 & A_5 \\ A_1 & 1 & 0 & 1 & 0 & 0 \\ A_2 & 1 & 1 & 1 & 0 & 0 \\ A_3 & 0 & 0 & 1 & 0 & 0 \\ A_4 & 1 & 1 & 1 & 1 & 0 \\ A_5 & 1 & 1 & 1 & 1 & 1 \end{matrix},$$

$$M_4=\begin{matrix} & A_1 & A_2 & A_3 & A_4 & A_5 \\ A_1 & 1 & 1 & 0 & 1 & 1 \\ A_2 & 0 & 1 & 0 & 1 & 1 \\ A_3 & 1 & 1 & 1 & 1 & 1 \\ A_4 & 0 & 0 & 0 & 1 & 0 \\ A_5 & 0 & 0 & 0 & 1 & 1 \end{matrix},\quad M_5=\begin{matrix} & A_1 & A_2 & A_3 & A_4 & A_5 \\ A_1 & 1 & 1 & 0 & 1 & 0 \\ A_2 & 0 & 1 & 0 & 1 & 0 \\ A_3 & 1 & 1 & 1 & 1 & 1 \\ A_4 & 0 & 0 & 0 & 1 & 0 \\ A_5 & 1 & 1 & 1 & 1 & 1 \end{matrix}$$

Step 3. Calculate the overall pros weights $w\left(A_i \succ A_k\right)(i,k=1,2,\cdots,5)$, the overall cons weights

$w(A_i \prec A_k)$ ($i,k=1,2,\cdots,5$) and the overall indifference weights $w(A_i = A_k)$ ($i,k=1,2,\cdots,5$) of all alternative pairs (A_i, A_k) ($i,k=1,2,\cdots,5$). Based on $w(A_i \succ A_k) = \sum_{j\in(A_i \succ A_k)_j} w_j$ and the weight vector $w^1 = (0.094, 0.379, 0.100, 0.198, 0.229)^T$, the weights of all alternative pairs can be calculated. For example, we can find that $A_1 \succ A_2$ in M_1, M_4 and M_5, so $w(A_1 \succ A_2) = 0.094 + 0.198 + 0.229 = 0.521$. Similarly, the weights of all alternative pairs under the combination of CE_Q and E_1 can be shown in Table 3.

Table 3.

The weights of all alternative pairs under the combination of CE_Q and E_1.

Overall pros weight	Weight value	Overall cons weight	Weight value	Overall indifference weight	Weight value
$w(A_1 \succ A_2)$	0.521	$w(A_1 \prec A_2)$	0.479	$w(A_1 = A_2)$	0
$w(A_1 \succ A_3)$	0.199	$w(A_1 \prec A_3)$	0.801	$w(A_1 = A_3)$	0
$w(A_1 \succ A_4)$	0.521	$w(A_1 \prec A_4)$	0.479	$w(A_1 = A_4)$	0
$w(A_1 \succ A_5)$	0.806	$w(A_1 \prec A_5)$	0.194	$w(A_1 = A_5)$	0
$w(A_2 \succ A_3)$	0.573	$w(A_2 \prec A_3)$	0.427	$w(A_2 = A_3)$	0
$w(A_2 \succ A_4)$	0.806	$w(A_2 \prec A_4)$	0.194	$w(A_2 = A_4)$	0
$w(A_2 \succ A_5)$	0.577	$w(A_2 \prec A_5)$	0.423	$w(A_2 = A_5)$	0
$w(A_3 \succ A_4)$	0.806	$w(A_3 \prec A_4)$	0.194	$w(A_3 = A_4)$	0
$w(A_3 \succ A_5)$	0.577	$w(A_3 \prec A_5)$	0.194	$w(A_3 = A_5)$	0.229
$w(A_4 \succ A_5)$	0379	$w(A_4 \prec A_5)$	0.621	$w(A_4 = A_5)$	0

Step 4-Step 5. Utilize Eq. (34) and Eq. (35) (let $\Im = 0.8$ and $\aleph = 1.1$) to construct the ultima precedence relationship matrix and the directed graph as shown in M^* and Fig. 10.

$$M^* = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \end{array} \begin{array}{c} \begin{array}{ccccc} A_1 & A_2 & A_3 & A_4 & A_5 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \end{array}$$

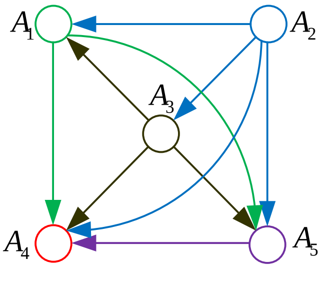


Fig. 10. A directed graph of all alternatives.

Step 6. Based on M^* and Fig. 10, we can calculate the ranking value of each alternative:

$$\Delta_1 = 3-2=1,\ \Delta_2 = 5-0=5,\ \Delta_3 = 4-1=3,\ \Delta_4 = 1-4=-3, \text{ and } \Delta_5 = 2-3=-1$$

Step 7. The final ranking of all alternatives can be obtained by comparing all $\Delta_i\left(i=1,2,\ldots,5\right)$: $A_2 \succ A_3 \succ A_1 \succ A_5 \succ A_4$, and thus, the most optimal alternative is A_2.

Additionally, based on the remaining combinations, the rankings of alternatives can be shown in Table 4.

Table 4.
The ranking orders of alternatives based on the remaining combinations.

The entropy and cross-entropy combination	The ranking orders of alternatives	The optimal alternative
Combining CE_Q and E_2	$A_2 \succ A_3 \succ A_1 \succ A_5 \succ A_4$	A_2
Combining CE_Q and E_3	$A_2 \succ A_3 \succ A_1 \succ A_5 \succ A_4$	A_2
Combining CE_Q and E_4	$A_2 \succ A_3 \succ A_1 \succ A_5 \succ A_4$	A_2
Combining CE_P and E_1	$A_2 \succ A_3 \succ A_1 \succ A_5 \succ A_4$	A_2
Combining CE_P and E_2	$A_2 \succ A_3 \succ A_1 \succ A_5 \succ A_4$	A_2
Combining CE_P and E_3	$A_2 \succ A_3 \succ A_1 \succ A_5 \succ A_4$	A_2
Combining CE_P and E_4	$A_2 \succ A_3 \succ A_1 \succ A_5 \succ A_4$	A_2

From the calculation process above, we can make some analyses as follows:

(1) Based on the hesitant fuzzy linguistic entropy and cross-entropy measures, we have established a model to calculate the weight vector of criteria. This model considers not only the individual effect of each HFLE (by the hesitant fuzzy linguistic entropy measures), but also the interaction effect between any two HFLEs (by the hesitant fuzzy linguistic cross-entropy measures). Therefore, this weight-determining method is reasonable considering that it is sufficient to deal with the decision information.

(2) The HFL-AQM has been developed to deal with the hesitant fuzzy linguistic information. This method uses both the graph theory and the precedence relationship matrix, especially, the directed graph makes the final ranking results of all alternatives more intuitively to be distinguished.

(3) In this case study, we have listed eight combinations of the hesitant fuzzy linguistic entropy and

cross-entropy measures and obtained eight different weight vectors of criteria (See Table 2). From the final results (See Table 4), we can know that these eight combinations get the same decision making result. Thus, in the future work, we can only select one combination to calculate the weight vector of criteria for the MCDM problems.

6.2. Comparative analyses

We can make a comparative analysis between our weight-determining method and the satisfaction degree-based method [23]. Considering the relationships between the HFSs and the HFLTSs, in the following, we first use the satisfaction degree-based method to determine the weights of criteria under hesitant fuzzy linguistic environment.

Step 1. Calculate the expectation values of HFLEs in Table 2, and they are shown in Table 5.

Table 5.
The expectation values of the HFLEs.

	C_1	C_2	C_3	C_4	C_5
A_1	2/3	2/3	1/2	3/4	5/6
A_2	1/2	5/6	3/5	2/3	3/4
A_3	1/6	3/4	1/4	5/6	5/6
A_4	1/2	1/4	2/3	1/4	3/5
A_5	5/6	0	3/4	1/2	5/6

Step 2. Let the parameter $\theta = 0.4$, then we establish Model 3 to calculate the weight information:

Model 3

$$\max \frac{0.6\left(2/3w_1+2/3w_2+1/2w_3+3/4w_4+5/6w_5\right)}{0.4\left(1/3w_1+1/3w_2+1/2w_3+1/4w_4+1/6w_5\right)+0.6\left(2/3w_1+2/3w_2+1/2w_3+3/4w_4+5/6w_5\right)}$$
$$+\frac{0.6\left(1/2w_1+5/6w_2+3/5w_3+2/3w_4+3/4w_5\right)}{0.4\left(1/2w_1+1/6w_2+2/5w_3+1/3w_4+1/4w_5\right)+0.6\left(1/2w_1+5/6w_2+3/5w_3+2/3w_4+3/4w_5\right)}$$
$$+\frac{0.6\left(1/6w_1+3/4w_2+1/2w_3+5/6w_4+5/6w_5\right)}{0.4\left(5/6w_1+2/4w_2+1/2w_3+1/6w_4+1/6w_5\right)+0.6\left(1/6w_1+3/4w_2+1/2w_3+5/6w_4+5/6w_5\right)}$$
$$+\frac{0.6\left(1/2w_1+1/4w_2+2/3w_3+1/4w_4+3/5w_5\right)}{0.4\left(1/2w_1+3/4w_2+1/3w_3+3/4w_4+2/5w_5\right)+0.6\left(1/2w_1+1/4w_2+2/3w_3+1/4w_4+3/5w_5\right)}$$
$$+\frac{0.6\left(5/6w_1+0w_2+3/4w_3+1/2w_4+5/6w_5\right)}{0.4\left(1/6w_1+w_2+1/4w_3+1/2w_4+1/6w_5\right)+0.6\left(5/6w_1+0w_2+3/4w_3+1/2w_4+5/6w_5\right)}$$

$$s.t.\begin{cases} w_1 \neq w_2 \neq w_3 \neq w_4 \neq w_5 \\ 0.1 \leq w_3 \leq 0.25 \\ w_2 \leq 4w_1 \leq 2w_4 \\ w_4 \leq w_5 \\ w_2 \leq 0.4 \\ \sum_{j=1}^{n} w_j = 1,\ w_j \geq 0,\ j = 1,2,\ldots,n \end{cases}$$

Solving Model 3 by the MATLAB software, we get the weight vector $w = (0.099, 0.39, 0.1, 0.198, 0.213)^T$. Based on the HFL-AQM, the final ranking of all alternatives is $A_2 \succ A_3 \succ A_1 \succ A_5 \succ A_4$, and thus, the most optimal alternative is A_2.

Moreover, we can make a comparative analysis between the HFL-AQM and the weighted hesitant fuzzy linguistic Bonferroni mean (WHFLBM) operator [7]. The WHFLBM operator is shown as:

$$WHFLB_{\omega}^{p,q}\left(h_{S_1}, h_{S_2}, \ldots, h_{S_n}\right) = g^{-1}\left(\left(\frac{1}{n(n-1)}\left(\underset{\substack{i,j=1 \\ i \neq j}}{\overset{n}{\oplus}}\left(\left(g\left(\omega_i h_{S_i}\right)\right)^p \otimes \left(g\left(\omega_j h_{S_j}\right)\right)^q\right)\right)\right)^{\frac{1}{p+q}}\right) \tag{38}$$

which can be used to aggregate the evaluation information of each alternative. Based on the proposed weight-determining method and the WHFLBM operator, the final ranking of all alternatives is $A_2 \succ A_3 \succ A_1 \succ A_5 \succ A_4$, and thus, the most optimal alternative is A_2.

In what follows, we make some comparative analyses of the above numerical results:

(1) The hesitant fuzzy linguistic entropy and cross-entropy measures mainly consider the direction of thermodynamics. By combining the entropy and cross-entropy measures, the decision information can be considered more adequately. By transforming the decision information into the expectation values, the satisfaction degree-based method utilizes a model to derive the weight information. Even though these two methods obtain almost the same criteria weights, the hesitant fuzzy linguistic entropy and cross-entropy measure-based method is more reasonable because it not only considers the decision information more adequately, but also does not lose any information.

(2) The HFL-AQM mainly uses both the graph theory and the precedence relationship matrix. Furthermore, the directed graph makes the final ranking result of all alternatives more intuitively to be distinguished. From the direction of information aggregation, the WHFLBM operator can not only consider the significances of the weights of all criteria, but also reflect the interrelationships among the criteria. However, the main shortcoming of the WHFLBM operator is that the calculation complexity is very high when we need to do the addition and multiplication over HFLEs. On the contrary, the HFL-AQM is very simple and easy to get the optimal result.

(3) From the decision making results above, both of these two methods get $A_2 \succ A_3 \succ A_1 \succ A_5 \succ A_4$ and the optimal alternative is A_2. However, by considering the information integrity and the calculation integrity, the hesitant fuzzy linguistic entropy and cross-entropy measure-based weight-determining method and the HFL-AQM are reasonable to deal with the hesitant fuzzy linguistic MCDM problems.

7. Conclusions

In this paper, we have introduced some hesitant fuzzy linguistic entropy and cross-entropy measures, and discussed their properties. By combining them, a weight-determining model has been established. Additionally, we have developed a HFL-AQM to deal with the MCDM problems based on the directed graph and the precedence relationship matrix. Finally, a case study concerning the tertiary hospital management has been made to verify the weight-determining method and the HFL-AQM, in which the optimal method has been selected to deal with the problems of tertiary hospitals related to hierarchical medical (HM). Some comparisons of our weight-determining method and the HFL-AQM with the satisfaction degree-based method and the WHFLBM operator have been made so as to validate the efficiency and effectiveness of our method.

Acknowledgments

The authors would like to thank the editors and the anonymous referees for their insightful and constructive comments and suggestions that have led to this improved version of the paper. The work was supported in part by the National Natural Science Foundation of China (Nos. 71502235, and 71532007), the China Postdoctoral Science Foundation (No. 2016T90863), and the Central University Basic Scientific Research Business Expenses Project (Nos. skgt201501, skqy201649).

References

[1] S.S.S. Ahmad, W. Pedrycz, The development of granular rule-based systems: A study in structural model compression, Granul. Comput. (2016), doi: 10.1007/s41066-016-0022-5.

[2] B. Apolloni, S. Bassis, J. Rota, G.L. Galliani, M. Gioia, L. Ferrari, A neurofuzzy algorithm for learning from complex granules, Granul. Comput. (2016), doi: 10.1007/s41066-016-0018-1.

[3] K.T. Atanassov, Intuitionistic fuzzy set, Fuzzy Sets Syst. 20 (1986) 87-96.

[4] I. Beg, T. Rashid, TOPSIS for hesitant fuzzy linguistic term sets, Int. J. Intell. Syst. 28 (2013) 1162-1171.

[5] S.M. Chen, J.A. Hong, Multicriteria linguistic decision making based on hesitant fuzzy linguistic term sets and the aggregation of fuzzy sets, Inf. Sci. 286 (2014) 63-74.

[6] N. Chen, Z.S. Xu. Hesitant fuzzy ELECTRE II approach: A new way to handle multi-criteria decision making problems, Inf. Sci. 292 (2015) 175-197.

[7] D. Ciucci, Orthopairs and granular computing, Granul. Comput. 1(3) (2016) 159-170.

[8] D. Dubois, H. Prade, Bridging gaps between several forms of granular computing, Granul. Comput. 1(2) (2016) 115-126.

[9] F.J. Estrella, R.M. Rodriguez, L. Martinez, A hesitant linguistic fuzzy TOPSIS approach integrated into FLINTSTONES. In: 16th World Congress of the International-Fuzzy-Systems-Association (IFSA)/9th Conference of the European Society for Fuzzy Logic and Technology (EUSFLAT), Gijon, SPAIN 89 (2015) 799-806.

[10] J.L. Fan, Some new fuzzy entropy formulas, Fuzzy Sets Syst. 128 (2002) 277-284.

[11] B. Farhadinia, Multiple criteria decision-making methods with completely unknown weights in hesitant fuzzy linguistic term setting, Knowl. Based Syst. 93 (2016) 135-144.

[12] X.J. Gou, Z.S. Xu, H.C. Liao, Multi-criteria decision making based on Bonferroni means with hesitant fuzzy linguistic information, Soft Comput. (2016), doi: 10.1007/s00500-016-2211-1.

[13] X.J. Gou, Z.S. Xu, H.C. Liao, Alternative queuing method for multiple criteria decision making with hybrid fuzzy and ranking order information, Inf. Sci. 357 (2016) 144-160.

[14] X.J. Gou, Z.S. Xu, Novel basic operational laws for linguistic terms, hesitant fuzzy linguistic term sets and probabilistic linguistic term sets, Inf. Sci. 372 (2016) 407-427.

[15] G. Hesamian, M. Shams, Measuring similarity and ordering based on hesitant fuzzy linguistic term sets, J. Intell. Fuzzy Syst. 28(2) (2015) 983-990.

[16] C. Kahraman, B. Oztaysi, S.C. Onar, A multicriteria supplier selection model using hesitant fuzzy linguistic term sets, J. Mult.-Valued Log. Soft Comput. 26(3-5) (2016) 315-333.

[17] L.W. Lee, S.M. Chen, Fuzzy decision making based on likelihood-based comparison relations of hesitant fuzzy linguistic term sets and hesitant fuzzy linguistic operators, Inf. Sci. 294 (2015) 513-529.

[18] L. W. Lee, S. M. Chen, Fuzzy decision making and fuzzy group decision making based on likelihood-based comparison relations of hesitant fuzzy linguistic term sets, J. Intell. Fuzzy Syst. 29(3) (2015) 1119-1137.

[19] H.C. Liao, Z.S. Xu, X. J. Zeng, Distance and similarity measures for hesitant fuzzy linguistic term sets and their application in multi-criteria decision making, Inf. Sci. 271 (2014) 125-142.

[20] H.C. Liao, Z.S. Xu, X.J. Zeng, J. M. Merigó, Qualitative decision making with correlation coefficients of hesitant fuzzy linguistic term sets, Knowl. Based Syst. 76 (2015) 127-138.

[21] H.C. Liao, Z.S. Xu, Approaches to manage hesitant fuzzy linguistic information based on the cosine distance and similarity measures for HFLTSs and their application in qualitative decision making, Expert Syst. Appl. 42 (2015) 5328-5336

[22] H.C. Liao, Z.S. Xu, X.J. Zeng, Hesitant fuzzy linguistic VIKOR method and its application in qualitative multiple criteria decision making, IEEE Trans. Fuzzy Syst. 23(5) (2015) 1343-1355 .

[23] H.C. Liao, Z.S. Xu, Satisfaction degree based interactive decision making under hesitant fuzzy environment with incomplete weights, Int. J. Uncertainty Fuzziness Knowl.-Based Syst. 22(4) (2014) 553-572.

[24] H.B. Liu, J.F. Cai, L. Jiang, On improving the additive consistency of the fuzzy preference relations based on comparative linguistic expressions, Int. J. Int. Syst. 29 (2014) 544-559.

[25] L. Livi, A. Sadeghian, Granular computing, computational intelligence, and the analysis of non-geometric input spaces, Granul. Comput. 1(1) (2016) 13-20.

[26] V. Loia, G. D'Aniello, A. Gaeta, F. Orciuoli, Enforcing situation awareness with granular computing: A systematic overview and new perspectives, Granul. Comput. 1(2) (2016) 127-143.

[27] L. Maciel, R. Ballini, F. Gomide, Evolving granular analytics for interval time series forecasting, Granul. Comput. (2016), doi: 10.1007/s41066-016-0016-3.

[28] O. Parkash, P.K. Sharma, R. Mahajan, Newmeasures of weighted fuzzy entropy and their applications for the study of maximum weighted fuzzy entropy principle, Inf. Sci. 178 (2008) 2389-2395.

[29] W. Pedrycz, S.M. Chen, Granular Computing and Decision-Making: Interactive and Iterative Approaches, Springer, Heidelberg, Germany, 2015.

[30] X.D. Peng, Y. Yang, J.P. Song, Y. Jiang, Hesitant fuzzy linguistic decision method based on combination weight, Comput. Eng. 41(9) (2015) 190-198.

[31] G. Peters, R. Weber, DCC: A framework for dynamic granular clustering, Granul. Comput. 1(1) (2016) 1-11.

[32] R.M. Rodríguez, L. Martínez, F. Herrera, Hesitant fuzzy linguistic terms sets for decision making, IEEE Trans. Fuzzy Syst. 20 (2012) 109-119.

[33] A. Skowron, A. Jankowski, S. Dutta, Interactive granular computing, Granul. Comput. 1(2) (2016) 95-113.

[34] M. Song, Y. Wang, A study of granular computing in the agenda of growth of artificial neural networks, Granul. Comput. (2016), doi: 10.1007/s41066-016-0020-7.

[35] H. Wang, Extended hesitant fuzzy linguistic term sets and their aggregation in group decision making, Int. J. Comput. Intell. Syst. 8 (2015) 14-33.

[36] H. Wang, Z.S. Xu, Some consistency measures of extended hesitant fuzzy linguistic preference relations, Inf. Sci. 297 (2015) 316-331.

[37] J.Q. Wang, J.T. Wu, J. Wang, H.Y. Zhang, X.H. Chen, Multi-criteria decision-making methods based on the Hausdorff distance of hesitant fuzzy linguistic numbers, Soft Comput. 20(4) (2016) 1621-1633.

[38] C.P. Wei, Z.L. Ren, R.M. Rodriguez, A hesitant fuzzy linguistic TODIM method based on a score function, Int. J. Comput. Intell. Syst. 8(4) (2015) 701-712.

[39] G. Wilke, E. Portmann, Granular computing as a basis of human-data interaction: A cognitive cities use case, Granul. Comput. 1(3) (2016) 181-197.

[40] M.M. Xia, Z.S. Xu, H.C. Liao, Preference relations based on intuitionistic multiplicative information, IEEE Trans. Fuzzy Syst. 21(1) (2013) 113-133.

[41] M.M. Xia, Z.S. Xu, Hesitant fuzzy information aggregation in decision making, Int. J. Approx. Reason. 52 (2011) 395-407.

[42] Z.S. Xu, Deviation measures of linguistic preference relations in group decision making, Omega 33 (2005) 249-254.

[43] Z.S. Xu, X.J. Gou, An overview of interval-valued intuitionistic fuzzy information aggregations and applications, Granul. Comput. (2016), doi: 10.1007/s41066-016-0023-4.

[44] Z.S. Xu, H. Wang, Managing multi-granularity linguistic information in qualitative group decision making: An overview, Granul. Comput. 1(1) (2016) 21-35.

[45] Z.S. Xu, H. Wang, On the syntax and semantics of virtual linguistic terms for information fusion in decision making, Inf. Fusion. 34 (2017) 43-48.

[46] Z.S. Xu, M.M. Xia, Hesitant fuzzy entropy and cross-entropy and their use in multiattribute decision-making, Int. J. Intell. Syst. 27 (2012) 799-822.

[47] R.R. Yager, A.M. Abbasov, Pythagorean membership grades, complex numbers, and decision making, Int. J. Intell. Syst. 28(5) (2013) 436-452.

[48] Y. Yao, A triarchic theory of granular computing, Granul. Comput. 1(2) (2016) 145-157.

[49] L.A. Zadeh, Fuzzy sets, Inf. Control. 8 (1965) 338-356.

[50] L.A. Zadeh, The concept of a linguistic variable and its application to approximate reasoning-I, Inf. Sci. 8(3) (1975) 199-249.

[51] J.L. Zhang, X.W. Qi, Research on Multiple attribute decision making under hesitant fuzzy linguistic environment with application to production strategy decision making, Adv. Mat. Res. (753-755) (2013) 2829-2836.

[52] Z.M. Zhang, C. Wu, On the use of multiplicative consistency in hesitant fuzzy linguistic preference relations, Knowl. Based Syst. 72 (2014) 13-27.

[53] B. Zhu, Z.S. Xu, Consistency measures for hesitant fuzzy linguistic preference relations, IEEE Trans. Fuzzy Syst. 22(1) (2014) 35-45.