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Adaptive force density method for form-finding problem of tensegrity structures

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Abstract

A numerical method is presented for form-finding of tensegrity structures. Eigenvalue analysis and spectral decomposition are carried out iteratively to find the feasible set of force densities that satisfies the requirement on rank deficiency of the equilibrium matrix with respect to the nodal coordinates. The equilibrium matrix is shown to correspond to the geometrical stiffness matrix in the conventional finite element formulation. A unique and non-degenerate configuration of the structure can then be obtained by specifying an independent set of nodal coordinates. A simple explanation is given for the required rank deficiency of the equilibrium matrix that leads to a non-degenerate structure. Several numerical examples are presented to illustrate the robustness as well as the strong ability of searching new configurations of the proposed method.

Keywords: Tensegrity structure; Form-finding; Force density method; Spectral decomposition

1. Introduction

Tension structures, such as cable nets, membrane structures and tensegric domes, have significant advantages over the conventional structures, such as steel structures, in view of their light-weight characteristics. Since the tension structures can transmit only axial forces, the distribution of axial forces is directly related to the structural shape, and the self-equilibrium shape should be determined by the so-called form-finding analysis that simultaneously finds the feasible set of internal forces and geometry of the structure.

There have been some different definitions and classifications for the tension structures, e.g. some researchers classify them into (1) self-stressed structures, and (2) prestressed structures. Self-stressed structures are free-standing so that they can maintain their self-equilibrium states without any support. The prestressed structures such as tensegric domes, cable nets and membrane structures should be attached to supports to retain equilibrium. However, for the sake of discussion on the availability of the force density method, tension structures

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are classified in view of prestress states of the structural elements in this study as: (1) tensegrity structures, and (2) tensile structures that consist of tensile members (elements) only.

The terminology of *tensegrity*, given by Fuller (1975), refers to a prestressed pin-jointed structure that consists of continuous tensile members (cables) and discontinuous compressional members (struts). However, many of the so-called tensegrity structures developed recently from this basic idea do not satisfy the definition strictly. Detailed description and classification of tensegrity structures are presented by Motro (1996). In this study, we regard *tensegrity structure* as a free-standing prestressed pin-jointed cable-strut system, where contacts are allowed among the struts, e.g. the reciprocal cable prisms proposed by Saitoh et al. (2001) and Wang (1996).

The process of finding an equilibrium configuration, called *form-finding* or *shape finding*, is a key step in the design of a tensegrity structure. Many researchers have made important contributions for this purpose; e.g. Hanaor (1988), Pellegrino and Calladine (1986), Sultan et al. (2002), Motro et al. (1986). The methods extended from the idea of the force density method, which is originally proposed for cable nets by Linkwitz and Schek (1971), are considered to be very effective, because only linear equations need to be solved in the form-finding process. Among them, the analytical technique developed by Vassart and Motro (1999) is found to be particularly suitable for searching new configurations (Tibert and Pellegrino, 2003). They analyzed the equilibrium matrix to obtain the conditions for force densities in symbolic form so that the equilibrium matrix has the required rank deficiency. However, for structures with large number of members, the analytical method may not be efficient enough. In this case, an effective numerical method may be strongly desired, which is the motivation of this study.

The paper is organized as follows: Section 2 introduces the basic idea and formulation of the force density method for the form-finding problem of cable nets, and their extensions to tensegrity structures. A simple explanation on the required rank deficiency of the equilibrium matrix is also presented. Section 3 formulates the linear and geometrical stiffness matrices and shows that the equilibrium matrix with respect to the nodal coordinates is equivalent to the geometrical stiffness matrix. Section 4 formulates the force density vector in terms of the equilibrium matrix and linear constraints on force densities, and presents the *adaptive force density method* based on eigenvalue analysis and spectral decomposition of the equilibrium matrix. Some numerical examples are given in Section 5 to illustrate the robustness, as well as ability of searching new configurations, of the proposed method. Section 6 concludes the study and gives some discussions.

2. Force density method

2.1. Properties of tensegrity structures

The form-finding problem of a tensegrity structure is very similar to that of a cable net, because they use almost the same assumptions described as follows except (d) and (e), which are adopted only for tensegrity structures:

(a) Members are connected by pin joints.
(b) Connectivity between the nodes and members, called *topology*, of the structure is known, and the geometrical configuration of the structure can be described in terms of nodal coordinates only.
(c) No external load is applied and its self-weight is neglected; i.e. the structure is in a self-equilibrium state.
(d) Buckling of the strut is not considered.
(e) The structure is free-standing without any support.

We can learn from the assumptions (a) and (c) that only axial forces, either in compression or tension, are transmitted by the members.

Since the self-equilibrium state of a cable net should be obtained by considering interaction between forces and geometry, there exist some difficulties in the form-finding process, and several methods have been developed to overcome the difficulties. Among these, the force density method that was originally developed by Linkwitz and Schek (1971) is regarded as one of the most powerful approaches. The key feature of the force density method is that the nonlinear equilibrium equations for unknown locations of the nodes are

transformed to a set of linear equations by prescribing the force s_k to length l_k ratio $q_k = s_k/l_k$, called *force density*, for member k. Note that $q_k > 0$ for all members of a cable net. It seems that the formulation for cable nets can be easily applied to the form-finding problem of tensegrity structures. However, some difficulties may arise because tensegrity structures are free-standing and there exist struts with $q_k < 0$.

2.2. Formulation of the equilibrium matrix

We start with the basic formulations of the force density method for cable nets. A typical cable net is attached to supports called *fixed nodes* here because they cannot have any displacement, whereas the nodes that are not constrained are called *free nodes*. For example, the two-dimensional cable net as shown in Fig. 1 has two free nodes 1 and 2, and is attached to fixed nodes 3–8. The structural elements connected by nodes are called *member*s.

For a structure with m members, n free nodes and n^f fixed nodes, its topology can be described by an incidence matrix $\mathbf{C}^s \in \Re^{m\times(n+n^f)}$ defined in the field of graph theory (Kaveh, 2004). If member k connects nodes i and j ($i<j$), then the ith and jth elements of the kth row of $\mathbf{C}^s$ are set to 1 and -1, respectively, as

$$\mathbf{C}^s_{(k,p)} = \begin{cases} 1 & \text{for } p = i \\ -1 & \text{for } p = j \\ 0 & \text{for other cases} \end{cases} \tag{1}$$

The fixed nodes are preceded by the free nodes in the numbering sequence, for convenience, so that $\mathbf{C}^s$ can be partitioned into two parts as

$$\mathbf{C}^s = (\mathbf{C}, \mathbf{C}^f) \tag{2}$$

where $\mathbf{C} \in \Re^{m\times n}$ and $\mathbf{C}^f \in \Re^{m\times n^f}$ describe the connectivities of the members to the free and fixed nodes, respectively.

Let $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ ($\in \Re^n$) and $\mathbf{x}^f$, $\mathbf{y}^f$, $\mathbf{z}^f$ ($\in \Re^{n^f}$) denote the nodal coordinate vectors of the free and fixed nodes, respectively, in x-, y- and z-directions. The force density vector is denoted by $\mathbf{q} = (q_1, q_2, \ldots, q_m)^\top \in \Re^m$. The force density matrix $\mathbf{Q} \in \Re^{m\times m}$ is given as

$$\mathbf{Q} = \mathrm{diag}(\mathbf{q}) \tag{3}$$

The equilibrium equations of the free nodes in each direction of a general pin-jointed structure can be written as follows (Linkwitz and Schek, 1971; Schek, 1974):

$$\begin{aligned} \mathbf{C}^\top\mathbf{Q}\mathbf{C}\mathbf{x} + \mathbf{C}^\top\mathbf{Q}\mathbf{C}^f\mathbf{x}^f &= \mathbf{p}^x \\ \mathbf{C}^\top\mathbf{Q}\mathbf{C}\mathbf{y} + \mathbf{C}^\top\mathbf{Q}\mathbf{C}^f\mathbf{y}^f &= \mathbf{p}^y \\ \mathbf{C}^\top\mathbf{Q}\mathbf{C}\mathbf{z} + \mathbf{C}^\top\mathbf{Q}\mathbf{C}^f\mathbf{z}^f &= \mathbf{p}^z \end{aligned} \tag{4}$$

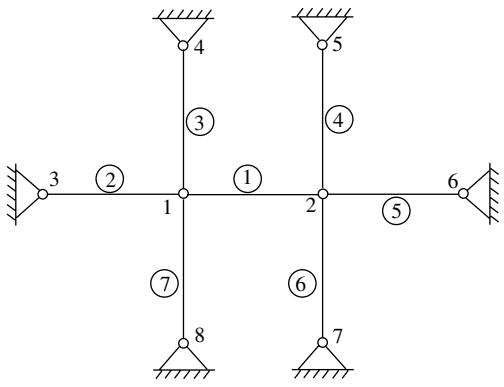

Fig. 1. A two-dimensional cable net.

where $\mathbf{p}^x$, $\mathbf{p}^y$ and $\mathbf{p}^z$ ($\in \Re^n$) are the vectors of external loads applied at the free nodes in x-, y- and z-directions, respectively.

For simplicity, matrices $\mathbf{E} \in \Re^{n\times n}$ and $\mathbf{E}^f \in \Re^{n\times n^f}$ are defined as

$$\begin{aligned} \mathbf{E} &= \mathbf{C}^\top \mathbf{Q}\mathbf{C} \\ \mathbf{E}^f &= \mathbf{C}^\top \mathbf{Q}\mathbf{C}^f \end{aligned} \tag{5}$$

Note that $\mathbf{E}$ and $\mathbf{E}^f$ are constant when the force density matrix $\mathbf{Q}$ is given. From Eqs. (4) and (5), the self-equilibrium equations without external loads can be rewritten as

$$\begin{aligned} \mathbf{E}\mathbf{x} &= -\mathbf{E}^f \mathbf{x}^f \\ \mathbf{E}\mathbf{y} &= -\mathbf{E}^f \mathbf{y}^f \\ \mathbf{E}\mathbf{z} &= -\mathbf{E}^f \mathbf{z}^f \end{aligned} \tag{6}$$

which are *linear* with respect to the unknown coordinates $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ of the free nodes while the coordinates $\mathbf{x}^f$, $\mathbf{y}^f$ and $\mathbf{z}^f$ of the fixed nodes are given. Since $\mathbf{E}$ represents the equilibrium of the free nodes, it is called *equilibrium matrix* in this study.

Instead of using $\mathbf{C}$ and $\mathbf{Q}$ as (5), $\mathbf{E}$ can be written directly from the force densities $\mathbf{q}$ (Connelly and Terrell, 1995; Vassart and Motro, 1999). Let $\mathscr{I}$ denote the set of members connected to free node i. The (i,j)-component $\mathbf{E}_{(i,j)}$ of $\mathbf{E}$ is given as

$$\mathbf{E}_{(i,j)} = \begin{cases} \sum_{k\in\mathscr{I}} q_k & \text{for } i = j \\ -q_k & \text{if free nodes } i \text{ and } j \text{ are connected by member } k \\ 0 & \text{for other cases} \end{cases} \tag{7}$$

For the two-dimensional cable net in Fig. 1, $\mathbf{E}$ can be written directly from (7) as

$$\mathbf{E} = \begin{pmatrix} q_1+q_2+q_3+q_7 & -q_1 \\ -q_1 & q_1+q_4+q_5+q_6 \end{pmatrix}$$

where $\mathbf{E}$ is always square and symmetric, and moreover, *positive-definite* if all members are in tension; i.e. $q_k > 0$ ($k = 1,2,\ldots,m$), without isolated free node (Schek, 1974). Therefore, $\mathbf{E}$ is invertible, and coordinates of the free nodes can be uniquely determined by solving the linear equations (6). This is the original idea of the force density method for the form-finding problem of cable nets.

2.3. Force density method for tensegrity structures

A similar formulation to the force density method for cable nets may be applied to tensegrity structures, because they use almost the same assumptions as listed in Section 2.1. However, for a tensegrity structure, $\mathbf{E}$ always has rank deficiency, therefore it is not invertible as discussed below, because it is free-standing and has compressional members with $q_k < 0$.

Since there exists no fixed node for a tensegrity structure, the self-equilibrium equations in each direction without nodal loads can be written as

$$\mathbf{E}\mathbf{x} = \mathbf{0} \tag{8.1}$$

$$\mathbf{E}\mathbf{y} = \mathbf{0} \tag{8.2}$$

$$\mathbf{E}\mathbf{z} = \mathbf{0} \tag{8.3}$$

For example, the equilibrium matrix $\mathbf{E}$ of the two-dimensional tensegrity structure shown in Fig. 2, where thin and thick lines denote the cables and struts, respectively, can be written as follows from (7):

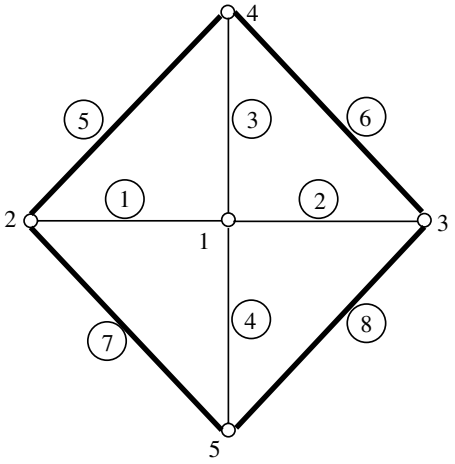

Fig. 2. A two-dimensional tensegrity structure.

$$\mathbf{E}=\begin{pmatrix} q_1+q_2+q_3+q_4 & -q_1 & -q_2 & -q_3 & -q_4 \\ -q_1 & q_1+q_5+q_7 & 0 & -q_5 & -q_7 \\ -q_2 & 0 & q_2+q_6+q_8 & -q_6 & -q_8 \\ -q_3 & -q_5 & -q_6 & q_3+q_5+q_6 & 0 \\ -q_4 & -q_7 & -q_8 & 0 & q_4+q_7+q_8 \end{pmatrix}$$

which is square and symmetric.

Define rank deficiency h of $\mathbf{E}$ as

$$h=n-\operatorname{rank}(\mathbf{E}) \tag{9}$$

It can be seen from Eqs. (8) and (9) that there are h independent components of $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$, respectively, that can be specified arbitrarily.

It may be easily observed from the definition (7) of $\mathbf{E}$ that the sum of the elements of a row or a column of $\mathbf{E}$ is always equal to 0 for a free-standing tensegrity without fixed nodes. Therefore, $\mathbf{E}$ is always *singular* with rank deficiency of at least 1; i.e. $h\geqslant 1$, for any tensegrity structure. Moreover, with the existence of both positive and negative values of the force densities for cables and struts, respectively, h may be larger if the force densities satisfy some specific conditions.

2.4. Non-degeneracy condition of tensegrity structures

Let $\mathbf{x}_0$, $\mathbf{y}_0$ and $\mathbf{z}_0$ be defined as

$$\mathbf{x}_0=\alpha_0^x\bar{\mathbf{I}} \tag{10.1}$$

$$\mathbf{y}_0=\alpha_0^y\bar{\mathbf{I}} \tag{10.2}$$

$$\mathbf{z}_0=\alpha_0^z\bar{\mathbf{I}} \tag{10.3}$$

where all the elements of the vector $\bar{\mathbf{I}}\in\Re^n$ are equal to 1, and the coefficients α_0^z, α_0^y and α_0^z can have arbitrary values. Since the sum of the elements of any row of $\mathbf{E}$ is always equal to zero for a tensegrity structure, it is obvious that $\mathbf{x}_0$, $\mathbf{y}_0$ and $\mathbf{z}_0$ are the solutions of Eqs. (8.1)–(8.3), respectively. Accordingly, the solutions of Eqs. (8.1)–(8.3) can be combined to a general form as

$$\begin{pmatrix}\mathbf{x}\\ \mathbf{y}\\ \mathbf{z}\end{pmatrix}=\begin{pmatrix}\mathbf{x}_0\\ \mathbf{y}_0\\ \mathbf{z}_0\end{pmatrix}+\sum_{i=1}^{h-1}\begin{pmatrix}\alpha_i^x & 0 & 0\\ 0 & \alpha_i^y & 0\\ 0 & 0 & \alpha_i^z\end{pmatrix}\begin{pmatrix}\boldsymbol{\sigma}_i\\ \boldsymbol{\sigma}_i\\ \boldsymbol{\sigma}_i\end{pmatrix} \tag{11}$$

where $\boldsymbol{\sigma}_i$ is in the null-space of $\mathbf{E}$ as $\mathbf{E}\boldsymbol{\sigma}_i = \mathbf{0}$.

From (11), the following properties hold:

(1) If $h = 1$, all nodes degenerate into one node $(\alpha_0^x, \alpha_0^y, \alpha_0^z)$, which is called *base node* here.
(2) If $h = 2$, (11) defines a line that passes through the base node.
(3) Eq. (11) forms a two-dimensional space (plane) in the case of $h = 3$, and a three-dimensional space if $h = 4$. Both of these solution spaces contain the base node.

Therefore, to our problem, which is to obtain a non-degenerate d-dimensional ($d = 2$ or 3) tensegrity structure, the rank deficiency h of $\mathbf{E}$ should satisfy the following condition:

$$h = h^\star = d + 1 \tag{12}$$

Hence, the problem of finding a feasible shape for a tensegrity structure turns out to be that of finding a set of force densities satisfying $h = h^\star$.

Vassart and Motro (1999) suggested three practical techniques for finding a feasible set of force densities to achieve the required rank deficiency: (1) intuitive, (2) iterative, and (3) analytical. The intuitive method is suitable for structures with only a few members. The iterative method seems to be based on trial-and-error experiments, or more refined iterative search for a set of force densities that yields the required deficiency. However, the details of the method were not presented. The analytical method is thought to be most effective among these methods: the equilibrium matrix $\mathbf{E}$ is analyzed based on Gaussian elimination in symbolic form to find the relation among the force densities to satisfy the non-degeneracy condition (12).

3. Stiffness of tensegrity structures

The tangent stiffness matrix of a general prestressed pin-jointed structure is formulated in this section to show that the equilibrium matrix $\mathbf{E}$ corresponds to the geometrical stiffness matrix in general finite element formulation.

Let e_k and A_k denote Young's modulus and cross-sectional area of member k, respectively. The lengths of member k in the prestressed and initial unstressed states are denoted by l_k and l_k^0, respectively. Assuming that struts and cables consist of linear elastic materials, so the force density q_k can be written as

$$q_k = \frac{1}{l_k}\left(e_k A_k \frac{l_k - l_k^0}{l_k^0}\right) = e_k A_k \left(\frac{1}{l_k^0} - \frac{1}{l_k}\right) \tag{13}$$

Let $\mathbf{L}_0$, $\mathbf{L}$ and $\bar{\mathbf{K}}$ denote the diagonal matrices of which the kth diagonal elements are l_k^0, l_k and $e_k A_k$, respectively. $\bar{\mathbf{K}}$ can be considered to be constant since the members are assumed to be linear elastic so that e_k is constant and the changes of cross-sectional areas A_k can be neglected while the strains are very small. The force density matrix $\mathbf{Q}$ can be written as

$$\mathbf{Q} = \bar{\mathbf{K}}(\mathbf{L}_0^{-1} - \mathbf{L}^{-1}) \tag{14}$$

The equivalent nodal load vectors in x-, y- and z-directions, which are compatible to the deformation of the structure, are denoted by $\mathbf{f}^x$, $\mathbf{f}^y$ and $\mathbf{f}^z$, respectively. The following relations can be derived from the equilibrium equations:

$$\begin{aligned} \mathbf{f}^x &= \mathbf{E}\mathbf{x} + \mathbf{E}^f\mathbf{x}^f \\ \mathbf{f}^y &= \mathbf{E}\mathbf{y} + \mathbf{E}^f\mathbf{y}^f \\ \mathbf{f}^z &= \mathbf{E}\mathbf{z} + \mathbf{E}^f\mathbf{z}^f \end{aligned} \tag{15}$$

Partial differentiation of (15) with respect $\mathbf{x}$ results in

$$\frac{\partial \mathbf{f}^x}{\partial \mathbf{x}} = \left(\frac{\partial \mathbf{E}}{\partial x_1}\mathbf{x} + \frac{\partial \mathbf{E}^f}{\partial x_1}\mathbf{x}^f, \frac{\partial \mathbf{E}}{\partial x_2}\mathbf{x} + \frac{\partial \mathbf{E}^f}{\partial x_2}\mathbf{x}^f, \ldots, \frac{\partial \mathbf{E}}{\partial x_n}\mathbf{x} + \frac{\partial \mathbf{E}^f}{\partial x_n}\mathbf{x}^f\right) + \mathbf{E} \tag{16.1}$$

$$\frac{\partial \mathbf{f}^y}{\partial \mathbf{x}} = \left(\frac{\partial \mathbf{E}}{\partial x_1}\mathbf{y} + \frac{\partial \mathbf{E}^f}{\partial x_1}\mathbf{y}^f, \frac{\partial \mathbf{E}}{\partial x_2}\mathbf{y} + \frac{\partial \mathbf{E}^f}{\partial x_2}\mathbf{y}^f, \ldots, \frac{\partial \mathbf{E}}{\partial x_n}\mathbf{y} + \frac{\partial \mathbf{E}^f}{\partial x_n}\mathbf{y}^f\right) \tag{16.2}$$

$$\frac{\partial \mathbf{f}^z}{\partial \mathbf{x}} = \left(\frac{\partial \mathbf{E}}{\partial x_1}\mathbf{z} + \frac{\partial \mathbf{E}^f}{\partial x_1}\mathbf{z}^f, \frac{\partial \mathbf{E}}{\partial x_2}\mathbf{z} + \frac{\partial \mathbf{E}^f}{\partial x_2}\mathbf{z}^f, \ldots, \frac{\partial \mathbf{E}}{\partial x_n}\mathbf{z} + \frac{\partial \mathbf{E}^f}{\partial x_n}\mathbf{z}^f\right) \tag{16.3}$$

where x_i denotes the x-coordinate of free node i. By incorporating the definitions $\mathbf{E} = \mathbf{C}^\top\mathbf{Q}\mathbf{C}$ and $\mathbf{E}^f = \mathbf{C}^\top\mathbf{Q}\mathbf{C}^f$ in (5), where $\mathbf{C}$ and $\mathbf{C}^f$ are constant, we obtain the following relations:

$$\begin{aligned} \frac{\partial \mathbf{E}}{\partial x_i} &= \mathbf{C}^\top \frac{\partial \mathbf{Q}}{\partial x_i}\mathbf{C}, \\ \frac{\partial \mathbf{E}^f}{\partial x_i} &= \mathbf{C}^\top \frac{\partial \mathbf{Q}}{\partial x_i}\mathbf{C}^f \end{aligned} \tag{17}$$

Since $\mathbf{L}_0$ in (14) is constant,

$$\frac{\partial \mathbf{Q}}{\partial x_i} = \bar{\mathbf{K}}(\mathbf{L}^{-1})^2 \frac{\partial \mathbf{L}}{\partial x_i} \tag{18}$$

holds. Let $\mathbf{U}$, $\mathbf{V}$ and $\mathbf{W}$ denote diagonal matrices, of which the kth diagonal elements are the coordinate differences of member k in x-, y- and z-directions, respectively; i. e.

$$\begin{aligned} \mathbf{U} &= \mathrm{diag}(\mathbf{C}\mathbf{x} + \mathbf{C}^f\mathbf{x}^f) \\ \mathbf{V} &= \mathrm{diag}(\mathbf{C}\mathbf{y} + \mathbf{C}^f\mathbf{y}^f) \\ \mathbf{W} &= \mathrm{diag}(\mathbf{C}\mathbf{z} + \mathbf{C}^f\mathbf{z}^f) \end{aligned} \tag{19}$$

$\mathbf{L}$ in prestressed state satisfies

$$\mathbf{L}^2 = \mathbf{U}^2 + \mathbf{V}^2 + \mathbf{W}^2 \tag{20}$$

Partial differentiation of (20) with respect to x_i leads to

$$\frac{\partial \mathbf{L}}{\partial x_i} = \mathbf{L}^{-1}\left(\mathbf{U}\frac{\partial \mathbf{U}}{\partial x_i} + \mathbf{V}\frac{\partial \mathbf{V}}{\partial x_i} + \mathbf{W}\frac{\partial \mathbf{W}}{\partial x_i}\right) \tag{21}$$

From (19),

$$\frac{\partial \mathbf{U}}{\partial x_i} = \mathrm{diag}(\mathbf{C}_i) \tag{22.1}$$

$$\frac{\partial \mathbf{V}}{\partial x_i} = \mathbf{0} \tag{22.2}$$

$$\frac{\partial \mathbf{W}}{\partial x_i} = \mathbf{0} \tag{22.3}$$

where $\mathbf{C}_i$ is the ith column of $\mathbf{C}$. Hence, from Eqs. (17)–(19), (21), and (22.1)–(22.3), we obtain

$$\begin{aligned} \frac{\partial \mathbf{E}}{\partial x_i}\mathbf{x} + \frac{\partial \mathbf{E}^f}{\partial x_i}\mathbf{x}^f &= \mathbf{C}^\top\bar{\mathbf{K}}(\mathbf{L}^{-1})^3\mathbf{U}\,\mathrm{diag}(\mathbf{C}_i)(\mathbf{C}\mathbf{x} + \mathbf{C}^f\mathbf{x}^f) = \mathbf{C}^\top\bar{\mathbf{K}}(\mathbf{L}^{-1})^3\mathbf{U}\,\mathrm{diag}(\mathbf{C}\mathbf{x} + \mathbf{C}^f\mathbf{x}^f)\mathbf{C}_i \\ &= \mathbf{C}^\top\bar{\mathbf{K}}(\mathbf{L}^{-1})^3\mathbf{U}^2\mathbf{C}_i \end{aligned} \tag{23}$$

Using (23) and letting $\mathbf{D}^x = \mathbf{C}^\top\mathbf{U}\mathbf{L}^{-1}$, (16.1) can be written as

$$\frac{\partial \mathbf{f}^x}{\partial \mathbf{x}} = \mathbf{C}^\top\bar{\mathbf{K}}(\mathbf{L}^{-1})^3\mathbf{U}^2\mathbf{C} + \mathbf{E} = \mathbf{D}^x\bar{\mathbf{K}}\mathbf{L}^{-1}\mathbf{D}^{x\top} + \mathbf{E} \tag{24}$$

Similarly, Eqs. (16.2) and (16.3) can be written as

$$\frac{\partial \mathbf{f}^y}{\partial \mathbf{x}} = \mathbf{D}^y\bar{\mathbf{K}}\mathbf{L}^{-1}\mathbf{D}^{x\top} \tag{25}$$

$$\frac{\partial \mathbf{f}^z}{\partial \mathbf{x}} = \mathbf{D}^z\bar{\mathbf{K}}\mathbf{L}^{-1}\mathbf{D}^{x\top} \tag{26}$$

where $\mathbf{D}^y = \mathbf{C}^\top\mathbf{V}\mathbf{L}^{-1}$ and $\mathbf{D}^z = \mathbf{C}^\top\mathbf{W}\mathbf{L}^{-1}$.

The tangent stiffness matrix $\mathbf{K} \in \Re^{3n\times 3n}$ of a structure is defined by partial differentiation of the equivalent nodal load vector $\mathbf{F} = (\mathbf{f}^{x\top}, \mathbf{f}^{y\top}, \mathbf{f}^{z\top})^\top \in \Re^{3n}$ with respect to nodal coordinate vector $\mathbf{X} = (\mathbf{x}^\top, \mathbf{y}^\top, \mathbf{z}^\top)^\top \in \Re^{3n}$, which can be written as

$$\mathbf{K} = \frac{\partial \mathbf{F}}{\partial \mathbf{X}} = \begin{pmatrix} \frac{\partial \mathbf{f}^x}{\partial \mathbf{x}} & \frac{\partial \mathbf{f}^x}{\partial \mathbf{y}} & \frac{\partial \mathbf{f}^x}{\partial \mathbf{z}} \\ \frac{\partial \mathbf{f}^y}{\partial \mathbf{x}} & \frac{\partial \mathbf{f}^y}{\partial \mathbf{y}} & \frac{\partial \mathbf{f}^y}{\partial \mathbf{z}} \\ \frac{\partial \mathbf{f}^z}{\partial \mathbf{x}} & \frac{\partial \mathbf{f}^z}{\partial \mathbf{y}} & \frac{\partial \mathbf{f}^z}{\partial \mathbf{z}} \end{pmatrix} \tag{27}$$

Let $\mathbf{I} \in \Re^{3\times 3}$ denote an identity matrix, and $\mathbf{D}^\top = (\mathbf{D}^{x\top}, \mathbf{D}^{y\top}, \mathbf{D}^{z\top})$. From Eqs. (24)–(26) and the similar equations for partial differentiation with respect to $\mathbf{y}$ and $\mathbf{z}$, $\mathbf{K}$ is written as

$$\mathbf{K} = \mathbf{K}^E + \mathbf{K}^G \tag{28}$$

where

$$\begin{aligned} \mathbf{K}^E &= \mathbf{D}\bar{\mathbf{K}}\mathbf{L}^{-1}\mathbf{D}^\top \\ \mathbf{K}^G &= \mathbf{I} \otimes \mathbf{E} \end{aligned} \tag{29}$$

$\mathbf{K}^E$ is the linear stiffness matrix, and $\mathbf{K}^G$ is the geometrical stiffness matrix corresponding to the prestress, where the stressed equilibrium state is considered as the reference state. It can be easily observed that $\mathbf{K}^E$, $\mathbf{K}^G$ and $\mathbf{K}$ are all symmetric, because $\bar{\mathbf{K}}$, $\mathbf{L}^{-1}$ and $\mathbf{E}$ are symmetric.

It can also be easily observed that the linear stiffness matrix is independent of the initial lengths l_k^0 of the members but dependent on the current lengths l_k after deformation by prestress. In the case that the structure has no prestress, the geometrical stiffness matrix will vanish, and we will have $l_k = l_k^0$ since no prestress is introduced so that there is no extension in any member.

The tangent stiffness matrix presented above can be used for any pin-jointed structure, including prestressed structures. Note that the rigid-body motions should be appropriately constrained for analysis of tensegrity structures without fixed nodes. The final form of the tangent stiffness matrix derived above is equivalent to those by Guest (2006), Masic et al. (2005) and Murakami (2001).

We can also see that if all the eigenvalues of the geometrical stiffness matrix are non-negative, which may lead to *super stability* described by Connelly (1999), it will surely increase the possibility of achieving a super stable structure. This is the basic idea of the adaptive force density method presented in the next section, which sets the negative eigenvalues of the equilibrium matrix to 0 for satisfying the requirement on rank deficiency.

4. Adaptive force density method

The analytical approach presented by Vassart and Motro (1999) may not be effective enough for a structure with moderately large number of members, because the force densities are analyzed in symbolic form. This motivates us to propose a new numerical method to achieve the required rank deficiency of the equilibrium matrix $\mathbf{E}$ with less human effort. The proposed method is called *adaptive force density method* because (a) it is an extension of the basic formulation and initial idea of the force density method proposed for the form-finding problem of cable nets, and (b) the method is based on eigenvalue analysis of $\mathbf{E}$, and can automatically adjust the values of the force densities to adapt to the requirement on rank deficiency.

4.1. Formulation of force density vector

Let $\mathscr{I}$ denote the set of members connected to node i. From the direct definition of $\mathbf{E}$ in (7), the ith column $\mathbf{E}_i$ of $\mathbf{E}$ can be written in terms of the force density vector $\mathbf{q}$ by a matrix $\mathbf{B}^i \in \Re^{n\times m}$ as

$$\mathbf{B}^i\mathbf{q} = \mathbf{E}_i \tag{30}$$

where (j,k)-component $\mathbf{B}^i_{(j,k)}$ $(k = 1,2,\ldots,m)$ of $\mathbf{B}^i$ is defined as

$$\mathbf{B}^i_{(j,k)} = \begin{cases} 1 & \text{if } i = j \text{ and } k \in \mathscr{I} \\ -1 & \text{if nodes } i \text{ and } j \text{ are connected by member } k \\ 0 & \text{for other cases} \end{cases} \tag{31}$$

For the tensegrity structure shown in Fig. 2, e.g., the matrix $\mathbf{B}^1$ can be written as

$$\mathbf{B}^1 = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Letting $\mathbf{B}^\top = (\mathbf{B}^{1\top}, \ldots, \mathbf{B}^{i\top}, \ldots, \mathbf{B}^{n\top})$ and $\mathbf{g}^\top = (\mathbf{E}_1^\top, \ldots, \mathbf{E}_i^\top, \ldots, \mathbf{E}_n^\top)$, the following relation holds for $\mathbf{q}$:

$$\mathbf{B}\mathbf{q} = \mathbf{g} \tag{32}$$

From the definition of $\mathbf{B}$, we can see that there exists a row of which the kth component ($k = 1, 2, \ldots, m$) is -1 while the others are zero. So the rank of $\mathbf{B}$ is m; i.e. $\mathbf{B}$ is full-rank.

Linear constraints on some specific force densities, e.g. relation between two values, and direct assignment of the components, can be formulated using constant matrix $\mathbf{B}^e$ and vector $\mathbf{g}^e$ as

$$\mathbf{B}^e\mathbf{q} = \mathbf{g}^e \tag{33}$$

Letting $\bar{\mathbf{B}}^\top = (\mathbf{B}^\top, \mathbf{B}^{e\top})$ and $\bar{\mathbf{g}}^\top = (\mathbf{g}^\top, \mathbf{g}^{e\top})$, Eqs. (32) and (33) are combined as

$$\bar{\mathbf{B}}\mathbf{q} = \bar{\mathbf{g}} \tag{34}$$

where $\bar{\mathbf{B}}$ is constant and full-rank. The least square solution of (34) is obtained as (Borse, 1997)

$$\mathbf{q} = \bar{\mathbf{B}}^-\bar{\mathbf{g}} \tag{35}$$

where $\bar{\mathbf{B}}^-$ is the generalized inverse matrix of $\bar{\mathbf{B}}$.

4.2. Eigenvalue analysis and spectral decomposition of the equilibrium matrix

A symmetric matrix $\mathbf{E} \in \Re^{n\times n}$ can be written as follows by applying spectral decomposition:

$$\mathbf{E} = \mathbf{\Phi}\mathbf{\Lambda}\mathbf{\Phi}^\top \tag{36}$$

where diagonal elements $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ of the diagonal matrix $\mathbf{\Lambda}$ are the eigenvalues of $\mathbf{E}$, and they are numbered in non-decreasing order as

$$\lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_n \tag{37}$$

The ith column $\mathbf{\Phi}_i$ of $\mathbf{\Phi}$ is the eigenvector corresponding to λ_i. It is clear that the number of non-zero eigenvalues of $\mathbf{E}$ is equal to its rank. Let r denote the number of non-positive eigenvalues of $\mathbf{E}$, we then have the following two cases:

Case 1: $r \leqslant h^\bigstar$.
Case 2: $r > h^\bigstar$.

For Case 1, we can simply assign 0 to the first $h^\bigstar$ eigenvalues of $\mathbf{E}$ to zero as

$$\lambda_i = 0, \quad (i = 1, 2, \ldots, h^\bigstar) \tag{38}$$

to obtain $\bar{\mathbf{\Lambda}}$ with modified eigenvalues. The equilibrium matrix is modified using $\bar{\mathbf{\Lambda}}$ as

$$\bar{\mathbf{E}} = \mathbf{\Phi}\bar{\mathbf{\Lambda}}\mathbf{\Phi}^\top \tag{39}$$

This way, $\bar{\mathbf{E}}$ will have the required rank deficiency $h^\bigstar$ without any negative eigenvalue.

However, for Case 2 where $r > h^{\star}$, the rank deficiency will be larger than required, if the same operation as Case 1 is applied. For this case, we may have several alternatives; e.g., (a) assign positive values to some of the negative eigenvalues, or (b) specify more than $h^{\star}$ independent coordinates in the form-finding process presented in the next section, etc. Since arbitrary chosen initial force densities usually result in $r \leqslant h^{\star}$, we will focus on only Case 1 in this study.

Instead of assigning 0 to $h^{\star}$ smallest eigenvalues zero, we can also assign 0 to eigenvalues with the $h^{\star}$ smallest absolute values. In some cases, the latter strategy may show stronger ability of searching new self-equilibrium configurations, which will be shown in the last example in Section 5.

4.3. Form-finding process

The adaptive force density method is first presented to obtain a feasible set of force densities, by carrying out eigenvalue analysis of $\mathbf{E}$ and assigning 0 to its $h^{\star}$ smallest eigenvalues so as to obtain $\bar{\mathbf{E}}$ from (39). An independent set of nodal coordinates is then specified to determine a unique and non-degenerate geometrical configuration of the structure.

4.3.1. Feasible force densities

From Eqs. (38) and (39), we can easily obtain $\bar{\mathbf{E}}$ that has the required rank deficiency. A new set of force densities is then found from (35) with the updated $\mathbf{E}$ $(= \bar{\mathbf{E}})$. However, the final feasible set of force densities has to be obtained by iterative process, because (35) gives an approximate solution of $\mathbf{q}$. Let $\hat{\mathbf{q}}$ denote the feasible force density vector that ensures the required rank deficiency of $\mathbf{E}$. The algorithm can be summarized as follows, where the superscript indicates the iteration counter:

Algorithm 1. Feasible force densities

Step 0: Specify an initial force density vector $\mathbf{q}^0$ to obtain $\mathbf{E}^0$ by (5). Formulate $\bar{\mathbf{B}}$ and $\bar{\mathbf{g}}^0$ with the specified linear constraints. Set $i := 0$.
Step 1: Assign 0 to the $h^{\star}$ smallest (absolute) eigenvalues of $\mathbf{E}^i$ to reconstruct $\bar{\mathbf{E}}^i$ by (39).
Step 2: Obtain $\bar{\mathbf{g}}^{i+1}$, calculate $\mathbf{q}^{i+1}$ from (35) and update $\mathbf{E}^{i+1}$ by (5).
Step 3: If $n - \mathrm{rank}(\mathbf{E}^{i+1}) = h^{\star}$ holds, then let $\hat{\mathbf{q}} = \mathbf{q}^{i+1}$ and terminate the process; otherwise, set $i \leftarrow i + 1$ and return to Step 1.

This way, we can find the feasible force density vector $\hat{\mathbf{q}}$ and its corresponding equilibrium matrix $\hat{\mathbf{E}}$ which has the required rank deficiency $h^{\star}$.

4.3.2. Determination of nodal coordinates

Let $\mathbf{H} \in \Re^{dn \times dn}$ ($d = 2$ or 3) denote the tensor product of the identity matrix $\mathbf{I} \in \Re^{d \times d}$ and $\mathbf{E} = \hat{\mathbf{E}}$ as

$$\mathbf{H} = \mathbf{I} \otimes \mathbf{E} \tag{40}$$

The equilibrium equations in all directions can be combined to

$$\mathbf{H}\mathbf{X} = \mathbf{0} \tag{41}$$

It should be noticed that there are $h^{\star}$ components of nodal coordinates in each direction that can be specified, because the rank deficiency of $\mathbf{E}$ is equal to $h^{\star}$. Therefore, the rank deficiency of $\mathbf{H}$ is $dh^{\star}$, and $dh^{\star}$ independent nodal coordinates can be specified. Hence, the solution of (41) can be written as

$$\mathbf{X} = \mathbf{G}\boldsymbol{\beta} \tag{42}$$

where $\mathbf{G} \in \Re^{dn \times dh^{\star}}$ satisfies $\mathbf{H}\mathbf{G} = \mathbf{0}$, and $\boldsymbol{\beta} \in \Re^{dh^{\star}}$ is the coefficient vector. If we specify an independent set of nodal coordinates $\bar{\mathbf{X}} \in \Re^{dh^{\star}}$ and obtain the corresponding components $\bar{\mathbf{G}} \in \Re^{dh^{\star} \times dh^{\star}}$ in $\mathbf{G}$, where $\mathrm{rank}(\bar{\mathbf{G}}) = dh^{\star}$, the configuration of the structure in terms of nodal coordinates $\mathbf{X}$ can be determined by

$$\mathbf{X} = \mathbf{G}\bar{\mathbf{G}}^{-1}\bar{\mathbf{X}} \tag{43}$$

$\bar{\mathbf{G}}$ can be obtained by using the algorithm presented by Zhang et al. (2006), where the Reduced Row-Echelon Form (RREF) of $\mathbf{G}^{\top}$ is extensively used to specify the independent set of nodal coordinates consecutively.

As a simple example for demonstrating how to determine the independent set of nodal coordinates to be specified based on the RREF of the transpose form $\mathbf{G}^{\top}$ of $\mathbf{G}$, the two-dimensional tensegrity structure consisting of ($n=$) 5 nodes and ($m=$) 8 members as shown in Fig. 2 is considered. The force densities of members 1–4 and 5–8 can be 1.0 and −0.5, respectively, so that the structure is in a state of self-equilibrium. The equilibrium matrix $\mathbf{E}$ is written as follows by using (7):

$$\mathbf{E}=\begin{pmatrix} 4.0 & -1.0 & -1.0 & -1.0 & -1.0 \\ -1.0 & 0.0 & 0.0 & 0.5 & 0.5 \\ -1.0 & 0.0 & 0.0 & 0.5 & 0.5 \\ -1.0 & 0.5 & 0.5 & 0.0 & 0.0 \\ -1.0 & 0.5 & 0.5 & 0.0 & 0.0 \end{pmatrix} \tag{44}$$

Since $\mathbf{H}$ is defined by $\mathbf{E}$ as (40), it is sufficient to investigate only the null-space $\mathbf{G}$ of $\mathbf{E}$, which can be written as

$$\mathbf{G}=\begin{pmatrix} 0.0 & 1.0 & 0.0 \\ 1.0 & 1.0 & 0.0 \\ -1.0 & 1.0 & 0.0 \\ 0.0 & 1.0 & -1.0 \\ 0.0 & 1.0 & 1.0 \end{pmatrix} \tag{45}$$

It can be seen from (45) that the rank of $\mathbf{G}$ is 3. Therefore, the rank deficiency h of $\mathbf{E}$ is 3, which satisfies the non-degeneracy condition (12) for a two-dimensional structure. The RREF of $\mathbf{G}^{\top}$ is

$$\mathrm{RREF}(\mathbf{G}^{\top})=\begin{pmatrix} 1.0 & 0.0 & 2.0 & 0.0 & 2.0 \\ 0.0 & 1.0 & -1.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 1.0 & -1.0 \end{pmatrix} \tag{46}$$

It can be observed from (46) that the columns corresponding to the node groups $\{1,2,4\}$, $\{1,2,5\}$, $\{1,3,4\}$ and $\{1,3,5\}$ are linearly independent, respectively. Therefore, we can specify the coordinates of three nodes in one of these four node groups to obtain a unique and non-degenerate configuration.

Since tensegrity structure should satisfy the self-equilibrium conditions, the vector of unbalanced loads $\boldsymbol{\epsilon}\in\Re^{dn}$ defined as follows can be used for evaluating the accuracy of the results:

$$\boldsymbol{\epsilon}=\mathbf{HX} \tag{47}$$

The Euclidean norm of $\boldsymbol{\epsilon}$ is used to define the design error ξ as

$$\xi=\sqrt{\boldsymbol{\epsilon}^{\top}\boldsymbol{\epsilon}} \tag{48}$$

The form-finding procedure by the proposed adaptive force density method can be summarized as follows:

Algorithm 2. Adaptive Force Density Method

Step 0: Define topology of the structure.
Step 1: Specify linear constraints and an initial set of force densities.
Step 2: Obtain a feasible set of force densities by implementation of Algorithm 1.
Step 3: Specify an independent set of nodal coordinates to obtain a unique and non-degenerate configuration of the structure.

5. Numerical examples

In order to investigate the robustness and ability of searching new configurations of the proposed adaptive force density method, numerical examples are presented for several tensegrity structures using MATLAB Ver. 6.5.1 (Borse, 1997).

5.1. Two-stage tensegrity structures

The proposed adaptive force density method is first applied to a two-stage tensegrity structure as shown in Fig. 3. The structure is composed of 12 nodes and 30 members; i.e. $n = 12$, $m = 30$. Its six struts are divided into two groups: (1) struts of the upper stage, and (2) struts of the lower stage. The 24 cables are divided into: (3) top and bottom bases, (4) saddle, (5) vertical, and (6) diagonal (Sultan et al., 2002), as indicated in Fig. 3(c).

Example 1. By specifying an initial set of force densities as $\{-1.5, -1.5, 1.0, 2.0, 1.0, 1.0\}$ for the six groups, Algorithm 1 finds the feasible set of force densities $\{-1.8376, -1.8376, 0.9281, 1.9918, 1.1737, 0.9958\}$ with 158 iterations.

The relative error of force density vector at each iteration, defined as the Euclidean norm of the difference of $\mathbf{q}^i$ from the final value $\hat{\mathbf{q}}$, is plotted in Fig. 4. Termination condition of Algorithm 1 is that the equilibrium matrix $\hat{\mathbf{E}}$ has the required rank deficiency $h^\star$ $(= n - \mathrm{rank}(\hat{\mathbf{E}}) = 4)$ where $|\lambda_{h^\star}| < 10^{-5}$ and $|\lambda_{h^\star+1}| > 10^{-5}$. A very good convergence of Algorithm 1 can be seen from Fig. 4, where the relative error comes very close to zero with only 20 iterations.

If we specify the coordinates of nodes a, b and c, which are defined in Fig. 3(a), as $\{(-2.6667, 0.0, 0.0), (1.3333, -2.3094, 0.0), (1.3334, 2.3094, 0.0)\}$ to make the bottom base located on the xy-plane, and node d in

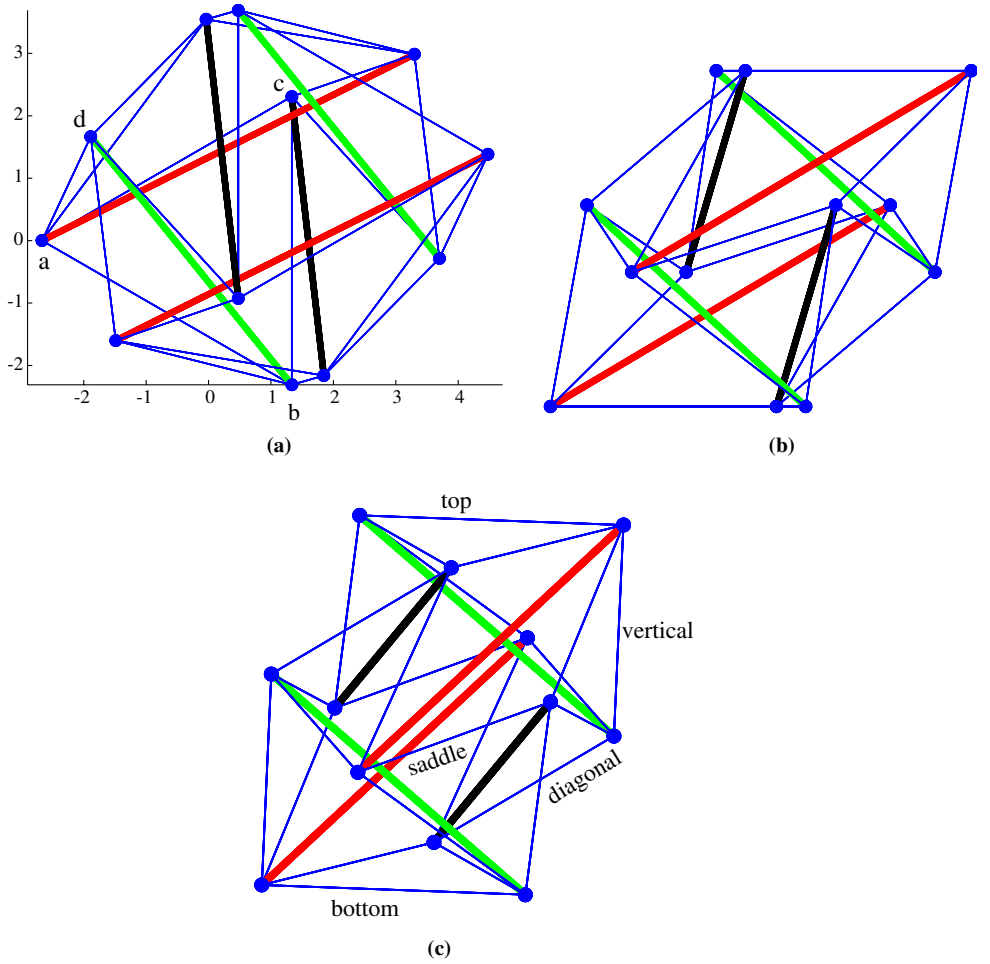

Fig. 3. Example 1: A two-stage tensegrity structure (type 1). (a) Top view, (b) side view, and (c) perspective view.

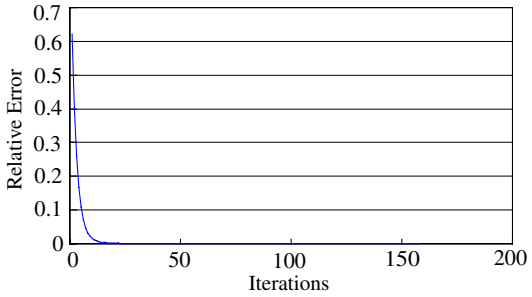

Fig. 4. Convergence of the algorithm for feasible force densities.

the lower stage as $(-1.8867, 1.6666, 3.3333)$, we can then achieve the final configuration of the structure as shown in Fig. 3.

Example 2. Since the initial force densities and independent nodal coordinates can be arbitrarily given by the designers, we can have some control over the geometrical and mechanical properties of the structure. Furthermore, new configurations can be easily found by changing the values of initial force densities and nodal coordinates.

If x-coordinate of node d is modified to -2.8867 from -1.8867 in Example 2 without changing any other parameter, a new configuration of the two-stage tensegrity structure as shown in Fig. 5 is obtained. We can also change the initial force densities at the first step of Algorithm 1 to search for new configurations.

The design errors ξ defined in (48) are less than 10^{-13} for Examples 1 and 2. Both the structures obtained in Examples 1 and 2 have only one infinitesimal mechanism and one self-stress mode; i.e. they are kinematically and statically indeterminate (Pellegrino and Calladine, 1986). In the meantime, the geometrical stiffness matrices for both cases are positive semi-definite, and the structures are super stable. So, it is clear in these examples that introduction of prestress stiffens the infinitesimal mechanism to make the structures stable.

5.2. Three-stage tensegrity structure

The adaptive force density method is applicable to the structures with much more members. A rather complicated three-stage structure, which has upper, center and lower stages, is considered. The struts are classified into two groups: (1) six struts in the upper and lower stages, and (2) three struts in the center stage. The cables are classified into the same groups as the two-stage tensegrity structure. Therefore, there are six groups in total.

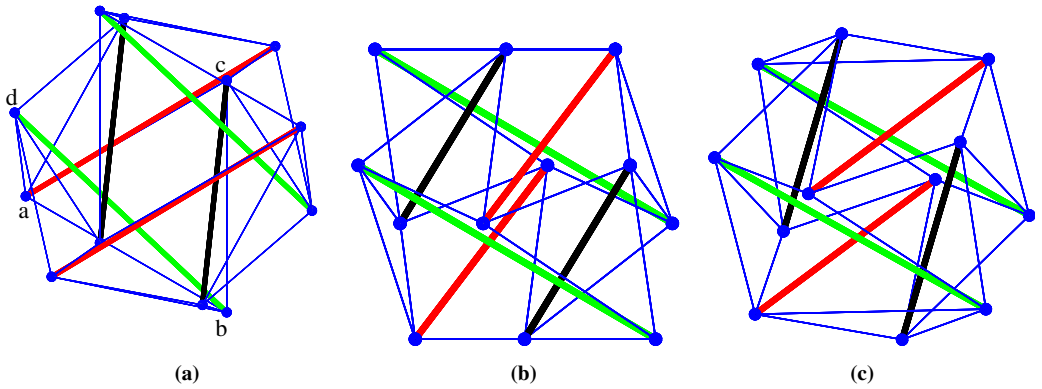

Fig. 5. Example 2: A two-stage tensegrity structure (type 2). (a) Top view, (b) side view and (c) perspective view.

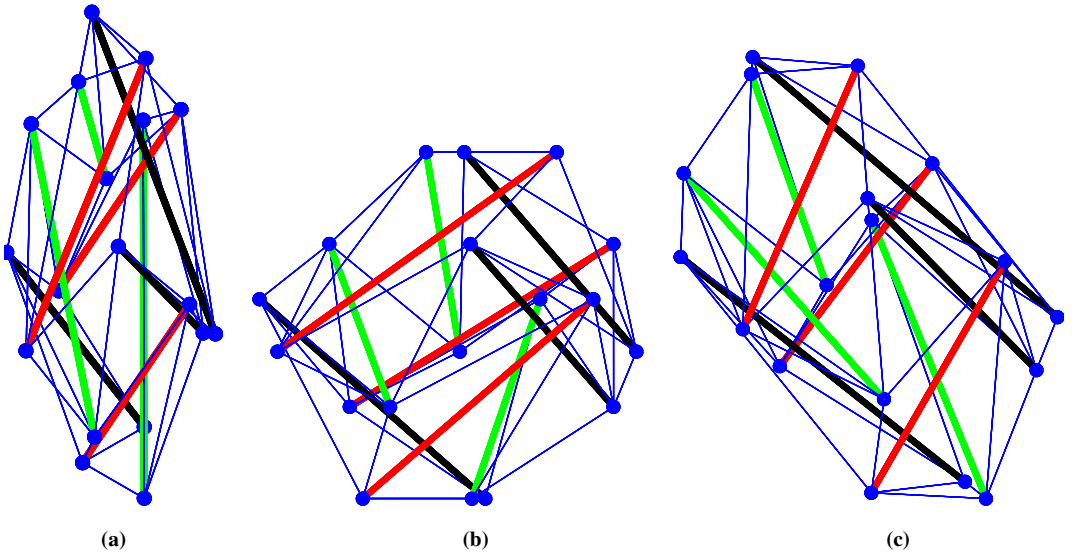

Fig. 6. Example 3: A three-stage tensegrity structure (type 1). (a) Top view, (b) side view and (c) perspective view.

Example 3. By specifying an initial set of force densities for these six groups as $\{-0.8, -1.0, 0.871, 0.3, 0.7244, 0.5\}$, and specifying the coordinates of the nodes in the bottom base and a lower node in the center stage as $\{(-2.6667, 0.0, 0.0), (1.3333, -2.3094, 0.0), (1.3334, 2.3094, 0.0)\}$ and $(-1.8867, 1.6666, 3.3333)$, respectively, we obtain the configuration of the three-stage tensegrity structure as shown in Fig. 6. It can be observed that the structure is asymmetric although we have specified symmetric force densities for the members by dividing them into six groups.

Example 4. Instead of assigning 0 to the $h^\star$ smallest eigenvalues of $\mathbf{E}$, we can also assign 0 to its $h^\star$ smallest absolute values. The initial force densities and independent nodal coordinates are same as in Example 3. The configuration of the structure in this case is obtained as shown in Fig. 7.

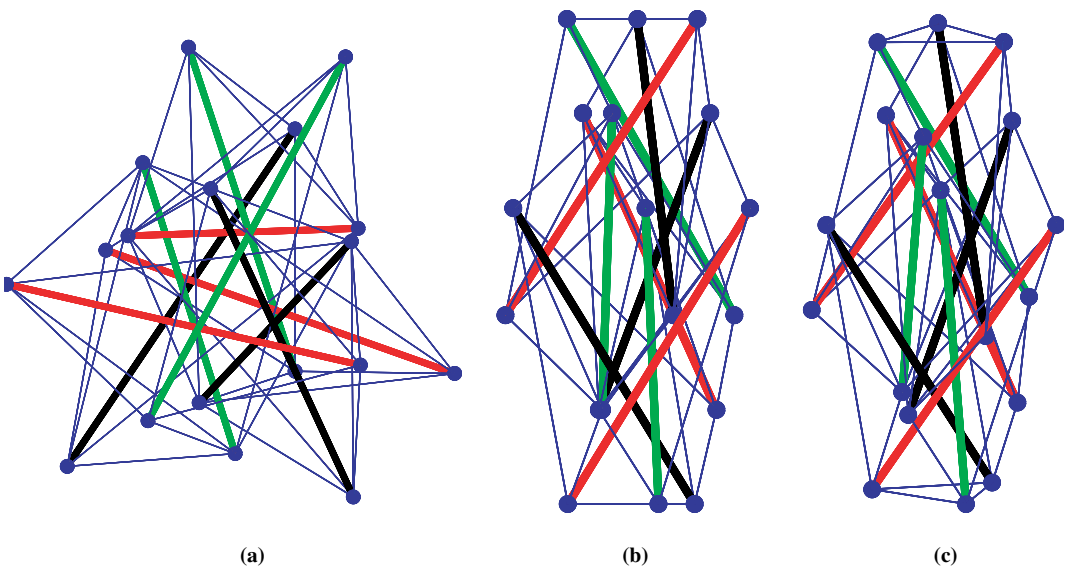

Fig. 7. Example 4: A three-stage tensegrity structure (type 2). (a) Top view, (b) side view and (c) perspective view.

Both of the three-stage tensegrity structures in Examples 3 and 4 have one infinitesimal mechanism and one self-stress mode. Therefore, they are kinematically and statically indeterminate.

Note that extensive numerical experiments have been done to confirm that new configurations can be systematically found by changing the initial force densities and nodal coordinates, although not all of the results have been shown here.

6. Discussions and conclusions

The adaptive force density method has been presented for form-finding problem of tensegrity structures based on eigenvalue analysis and spectral decomposition of the equilibrium matrix with respect to the nodal coordinates. Linear constraints on some specific force densities can be included in the formulation, and the prestress state of each member can be specified as expected; i.e. tension for cables and compression for struts. The force density vector is updated as a least-square solution of the equation defined by the equilibrium matrix and the given linear constraints. After obtaining the feasible set of force densities of which the corresponding equilibrium matrix has the required rank deficiency, an independent set of nodal coordinates can be specified to generate a unique and non-degenerate configuration of the structure. The requirement on rank deficiency of the equilibrium matrix for the purpose of obtaining a non-degenerate tensegrity structure has been discussed.

Formulations of the tangent, linear and geometrical stiffness matrices have been given. These formulations can be easily applied to any kind of pin-jointed structures. The equilibrium matrix is shown to correspond to the geometrical stiffness matrix in the conventional finite element formulation.

In the numerical examples, a very good convergence of the proposed algorithm has been demonstrated. It has also been shown that the proposed approach has a very strong ability of searching new and non-degenerate configurations by changing the values of the initial set of force densities at the first step and the nodal coordinates at the last step of the form-finding procedure.

However, the proposed method cannot have direct and exact control over the geometrical and mechanical properties of the structure, because the parameters in the method are neither forces nor lengths but the force-to-length ratios. As discussed by Tibert and Pellegrino (2003), asymmetric configurations may be found for a given set of symmetric force densities by the proposed method, because the member lengths cannot be described explicitly and linearly in the formulation.

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