Concepts of digital topology

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Abstract

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In an earlier paper written for a different readership [*Computers and Graphics* 13(2) (1989) 159-166] the first author defined a *digital fundamental group*—an analog, for binary digital pictures, of the fundamental group. In general the definition of the digital fundamental group involves continuous deformation. But an alternative, discrete, definition of the digital fundamental group was proposed for the *strongly normal digital picture spaces* defined in the same paper. The above-mentioned paper also defined a "continuous analog" $C(\mathscr{P})$ for each binary digital picture $\mathscr{P}$ on such a DPS (DPS = digital picture space). $C(\mathscr{P})$ is a polyhedron constructed by "filling in the gaps" between black points (1's) of the binary digital picture $\mathscr{P}$ in a specific way. Other kinds of continuous analog had previously been used by the first two authors.

In seeking the simplest and most efficient algorithms for performing image processing operations, researchers have considered many different combinations of grids and adjacency relations. Almost all of those combinations are isomorphic to special cases of the concept of a strongly normal DPS.

The main contribution of the present paper is a proof that the digital fundamental groups of binary digital pictures on a strongly normal DPS are naturally isomorphic to the fundamental groups of the digital pictures' continuous analogs. We use this result to establish that on a strongly normal DPS the discrete and continuous definitions of the digital fundamental group are equivalent, up to a natural group isomorphism. We also show that many topological results which hold in the Euclidean plane or Euclidean 3-space have analogs that hold in every strongly normal DPS. Our results suggest that a strongly normal DPS is a suitable domain for studying topology-related image processing operations such as thinning, border tracking and contour filling.

The definitions of digital fundamental group, strongly normal DPS and continuous analog are included in this paper so as to make it self-contained.

Keywords: Strongly normal digital picture space, digital fundamental group, continuous analog, polyhedral analog, digital topology, binary digital picture, binary digital image, border, adjacency tree, connectedness, component, hole, tunnel, Euler characteristic, Jordan curve.

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1. Introduction

This paper is about the digital fundamental group, strongly normal digital picture spaces, and the continuous analog $C(\mathcal{P})$ of a binary digital picture $\mathcal{P}$ on a strongly normal digital picture space. These concepts of digital topology were introduced in [11], a paper written for a different readership.[1]

The digital fundamental group is an analog for binary digital pictures[2] of the fundamental group, just as digital connectedness (e.g., 4- or 8-connectedness) is an analog for binary digital pictures of the topological notion of connectedness. The importance of the fundamental group in 3-d polyhedral topology suggests that the digital fundamental group will be a useful concept of 3-d digital topology. In fact the digital fundamental group has an immediate application to the theory of 3-d image thinning algorithms. For in order to preserve "tunnels" a 3-d thinning algorithm must preserve the digital fundamental groups of the input binary digital picture. See [11, Section 2.3; 18, Section 10; 14] for further discussion of this topic.

In seeking the simplest and most efficient algorithms for performing image processing operations such as thinning and border tracking, researchers have considered many different combinations of grids and adjacency relations. Almost all of those combinations are isomorphic to special cases of the concept of a strongly normal digital picture space.

The continuous analog $C(\mathcal{P})$ of a binary digital picture $\mathcal{P}$ on a strongly normal digital picture space is a polyhedron constructed by "filling in the gaps" between black points of $\mathcal{P}$ in accordance with a number of fairly natural rules. The principal result of our paper, Theorem 6.1.1 in Section 6, shows that the fundamental groups of $C(\mathcal{P})$ and its complement are naturally isomorphic to the digital fundamental groups of $\mathcal{P}$ and the complementary binary digital picture $\bar{\mathcal{P}}$. This is evidence that the digital fundamental group has been appropriately defined. Theorem 6.1.1 also shows that $C(\mathcal{P})$ is analogous to $\mathcal{P}$ in other useful ways.

In Section 7 we make extensive use of Theorem 6.1.1 to establish that a strongly normal digital picture space has the good "topological" properties which make it a suitable domain for studying topology-related image processing operations. Also, it follows from Proposition 7.9.1 that for strongly normal digital picture spaces the "continuous" definition of the digital fundamental group [11, Definition 3.3.4] is equivalent[3] to the discrete definition of the group in [11, Section 4.4].

To make this paper self-contained, many of the definitions given in [11] are included in Sections 3-5.

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[1] Digital fundamental groups and continuous analogs of binary digital pictures were also considered in [14, 19]. Continuous analogs of binary images had previously been used by the first two authors in [15, 16], by the first author and Khalimsky in [12] and by Kopperman, Meyer and Wilson in [20].

[2] A binary digital picture is a binary image equipped with adjacency relations that are used to define connectedness for sets of 1's and sets of 0's. A more precise definition will be given in Section 3. In this paper a 1 in a binary image is called a *black* point and a 0 is called a *white* point. (Some authors use the opposite convention, calling 1's white and 0's black.)

[3] Up to a natural group isomorphism.

2. Standard grids and adjacency relations

A number of different grids and adjacency relations have been considered by researchers seeking the best algorithms for performing image processing operations. Most of the adjacency relations considered have been based on the *Voronoi neighborhoods* of the grid points (c.f. [1]). The Voronoi neighborhood of a grid point p in a 2-d grid (3-d grid) is the set of all points in the Euclidean plane (in Euclidean 3-space) that are at least as close to p as to any other grid point. It is always a closed convex polygon or polyhedron. The *Voronoi adjacency relations* are the adjacency relations in which two grid points are adjacent if their Voronoi neighborhoods (i) share a vertex, (ii) share an edge, and (in the 3-d case) (iii) share a face. However, these adjacency relations may not all be distinct. In the case of the body-centered cubic grid defined below all three of them are the same, because the Voronoi neighborhoods of two grid points share a vertex only if they share a face.

A Voronoi adjacency relation in which each grid point is adjacent to just n other grid points is referred to as the *n-adjacency relation* on the grid; two adjacent points are then said to be *n-adjacent* to each other and each is called an *n-neighbor* of the other. We call a straight line segment joining two n-neighbors an *n-adjacency*.

We now describe most of the grids that have been considered in the image processing literature, and their Voronoi adjacency relations:

● *2-d square grid*: The grid points are the points (x, y) with integer coordinates. This is of course the most commonly used 2-d grid. The Voronoi neighborhood of each grid point is a unit square and the Voronoi adjacency relations are the standard 4- and 8-adjacency relations.

● *2-d isometric hexagonal grid*: The grid points are the centers of the hexagons in a tiling of the plane by regular hexagons. These hexagons are the Voronoi neighborhoods of the grid points. There is only one Voronoi adjacency relation: the 6-adjacency relation. This grid has been used by many authors (e.g., [4]).

● *2-d triangular grid*: The grid points are the centroids of the triangles in a tiling of the plane by equilateral triangles. These triangles are the Voronoi neighborhoods of the grid points. There are two Voronoi adjacency relations: the 3- and the 12-adjacency relations. This grid is discussed in [3].

● *3-d cubic grid*: The grid points are the points (x, y, z) with integer coordinates. This is the most obvious and commonly used 3-d grid. The Voronoi neighborhood of each grid point is a unit cube and the Voronoi adjacencies are the standard 6-, 18- and 26-adjacency relations.

● *3-d face-centered cubic grid*: The grid points are the points with coordinates (x, y, z), where x, y and z are integers such that $x + y + z$ is even. The Voronoi neighborhood of each grid point is a rhombic dodecahedron. There are just two Voronoi adjacency relations: the 12- and the 18-adjacency relations. The 12-neighbors of a grid point p are the grid points at a distance of $\sqrt{2}$ from p. The 18-neighbors of p are the 12-neighbors of p plus the six grid points at a distance of 2 from p. This grid is considered in [5, 25].

• 3-d *body-centered cubic grid*: The grid points are the points with coordinates (x, y, z), where x, y and z are integers such that $x \equiv y \equiv z \pmod 2$. The Voronoi neighborhood of each grid point is a truncated octahedron. There is only one Voronoi adjacency relation: the 14-adjacency relation. The 14-neighbors of a grid point p are the eight grid points at a distance of $\sqrt{3}$ from p and the six grid points at a distance of 2 from p. This grid is used in [21].

Khalimsky's adjacency relations. Khalimsky has introduced a different approach to digital topology in which images are represented by locally finite T_0 topological spaces [6-10]. Kovalevsky has independently developed a similar theory from a more practical standpoint (see [22]). For an introduction to this "topological" approach to digital topology see [13, 19].

In the topological approach one gives the integers $\mathbb{Z}$ the (locally finite T_0) topology with basis

$$\{\{2k+1\} \mid k \in \mathbb{Z}\} \cup \{\{2k-1, 2k, 2k+1\} \mid k \in \mathbb{Z}\}.$$

Then $\mathbb{Z}^n$ is topologized as a product of n copies of this space. We shall refer to the product space as *Khalimsky n-space*.

Say that two distinct points p and q in $\mathbb{Z}^n$ are *Khalimsky-adjacent* if the two-point set $\{p, q\}$ is connected in Khalimsky n-space. Call a closed straight line segment in Euclidean n-space whose endpoints are Khalimsky-adjacent points in $\mathbb{Z}^n$ a *Khalimsky adjacency*. Then a set $S \subseteq \mathbb{Z}^n$ is connected in Khalimsky n-space if and only if the union of S and all the Khalimsky adjacencies that join two points in S is connected in Euclidean n-space.

Following Khalimsky, call a grid point a *pure point* if its coordinates are all even or all odd, and a *mixed point* otherwise. Then when $n = 2$ or 3 the Khalimsky adjacencies are the 8-adjacencies ($n = 2$) or 26-adjacencies ($n = 3$) in which at least one endpoint is a pure point, together with all 6-adjacencies that join two mixed points when $n = 3$. Thus the Khalimsky adjacency relation is an example of a non-Voronoi adjacency relation on the 2-d square and 3-d cubic grids.

In this section we have described many different combinations of grids and adjacency relations. All of these combinations are special cases of the general concept of a *binary digital picture space* which we now introduce.

3. Binary digital picture spaces and binary digital pictures

3.1. Choice of representation

In [16] the first two authors presented a general theory of binary digital pictures. In that theory a binary digital picture was represented by an ordered pair (A, S), where S was the set of "black" grid points on the 2-d square grid or 3-d cubic grid,

and A was an adjacency relation, on the set of all grid points, that satisfied certain regularity conditions. It was shown that well-behaved binary digital pictures had "continuous analogs". However, the continuous analogs constructed in [16] are not consistent with any reasonable theory of digital fundamental groups.

The (A, S) representation of binary digital pictures is not a convenient notation for discussing image processing operations such as thinning or digital rotation. The reason is that these operations would normally alter the A part of a binary digital picture (A, S) as well as the black point set S. Instead of the A part of the (A, S) representation, it is better to have something that is invariant under conventional image processing operations. For this reason we use a different representation of binary digital pictures that was introduced in [11].

3.2. Binary digital picture spaces

A *binary digital picture space* is a triple (V, β, ω), where V is the set of grid points in a 2-d or 3-d grid and each of β and ω is a set of closed straight line segments joining pairs of points in V. By "V is the set of grid points in a 2-d or 3-d grid" we mean that V is an infinite set of points in E^2 (2-d case) or E^3 (3-d case), V has no accumulation points, and there exists a positive constant D such that every point in E^2 or respectively E^3 is within distance D of a point in V. (We write E^2 for the Euclidean plane and E^3 for Euclidean 3-space.) In this paper, as in [11], we refer to a binary digital picture space simply as a *digital picture space*; and we will often abbreviate this to *DPS*.

We call the members of V the *points* of (or in) the DPS (V, β, ω). Most often we take $V = \mathbb{Z}^2$ or $\mathbb{Z}^3$, corresponding to the square or cubic grid. (We write $\mathbb{Z}^2$ for the set of points with integer coordinates in E^2, and $\mathbb{Z}^3$ for the set of points with integer coordinates in E^3.) Some other possibilities for V were described in the previous section.

An important notion of digital topology is that of *adjacency* between points. Typically, different adjacency relations are used for the black and the white points. On a DPS (V, β, ω) these adjacency relations are defined by the line segments in the sets β and ω. The set β contains all straight line segments joining points in V that will be considered adjacent to each other if they are both black. Similarly, the set ω contains all straight line segments joining points in V that will be considered adjacent to each other if they are both white. Neither β nor ω need have the same symmetries as V. (More precisely, neither $\bigcup\beta$ nor $\bigcup\omega$ need be invariant under isometries of E^2 or E^3 that map V onto itself.)

In the special case $V = \mathbb{Z}^2$ it is most usual for one of β and ω to be the set of all 4-adjacencies of $\mathbb{Z}^2$ and the other to be the set of all 8-adjacencies of $\mathbb{Z}^2$. In the special case $V = \mathbb{Z}^3$ it is most usual for one of β and ω to be the set of all 6-adjacencies of $\mathbb{Z}^3$ and the other to be either the set of all 18-adjacencies or the set of all 26-adjacencies of $\mathbb{Z}^3$. In general, if β happens to be the set of all m-adjacencies of V for some integer m, and ω happens to be the set of all n-adjacencies for some

integer n, then we may denote the DPS (V, β, ω) by (V, m, n), as in $(\mathbb{Z}^2, 8, 4)$ or $(\mathbb{Z}^3, 6, 26)$.

A line segment in β is called a *β-adjacency*. Similarly, a line segment in ω is called an ω-adjacency. If p and q are the endpoints of a β-adjacency (ω-adjacency) we say p is *β-adjacent* (*ω-adjacent*) to q.

An *isomorphism* of a DPS $\mathcal{S}_1 = (V_1, \beta_1, \omega_1)$ to a DPS $\mathcal{S}_2 = (V_2, \beta_2, \omega_2)$ is a homeomorphism h of the Euclidean plane (2-d case) or Euclidean 3-space (3-d case) to itself such that h maps V_1 onto V_2, each β_1-adjacency onto a β_2-adjacency and each ω_1-adjacency onto an ω_2-adjacency, and h^{-1} maps each β_2-adjacency onto a β_1-adjacency and each ω_2-adjacency onto an ω_1-adjacency.

Thus the DPS $(V, 6, 6)$ where V is the set of grid points in a 2-d isometric hexagonal grid is isomorphic to the DPS (V, β, ω) in which $V = \mathbb{Z}^2$ and $\beta = \omega = $ the 4-adjacencies and the south-west–north-east diagonals of unit lattice squares[4]. The latter DPS is an example of a DPS (V, β, ω) in which β and ω do not have the same symmetries as V.

If S is any set of points in the DPS $\mathcal{S} = (V, \beta, \omega)$ then the *complement* of S (with respect to $\mathcal{S}$), written $\bar{S}$, is the set $V - S$. The *complement* of a DPS $\mathcal{S} = (V, \beta, \omega)$, written $\bar{\mathcal{S}}$, is the DPS (V, ω, β). For example, if $\mathcal{S} = (\mathbb{Z}^2, 8, 4)$, then $\bar{\mathcal{S}} = (\mathbb{Z}^2, 4, 8)$.

3.3. Binary digital pictures

A *binary digital picture* is a quadruple (V, β, ω, B), where (V, β, ω) is a DPS and B is a subset of V. In this paper, as in [11], we refer to a binary digital picture simply as a *digital picture*; and often we just call it a *picture*. We say (V, β, ω, B) is a picture *on* the DPS (V, β, ω), and points of the DPS (i.e., points in V) are also referred to as points of the picture. Points in B are called *black* points of the picture; each black point represents a pixel or voxel that has value 1. Points in $\bar{B}$ correspond to pixels or voxels with value 0 and are called *white* points of the picture. The general effect of image processing operations such as shrinking, thinning, border finding and digital rotation is to transform a digital picture to another digital picture on the same digital picture space.

An *isomorphism of a picture* $\mathcal{P}_1 = (V_1, \beta_1, \omega_1, B_1)$ to a picture $\mathcal{P}_2 = (V_2, \beta_2, \omega_2, B_2)$ is an isomorphism of the DPS (V_1, β_1, ω_1) to the DPS (V_2, β_2, ω_2) that maps B_1 onto B_2.

Two black points of the picture $\mathcal{P} = (V, \beta, \omega, B)$ are said to be *$\mathcal{P}$-adjacent* if they are β-adjacent. Two white points or a white point and a black point are said to be *$\mathcal{P}$-adjacent* if they are ω-adjacent.

A β-adjacency that joins two black points of $\mathcal{P} = (V, \beta, \omega, B)$ is called a *black adjacency*; an ω-adjacency that joins two white points of $\mathcal{P}$ is called a *white adjacency*.

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[4] On a 2-d square grid or 3-d cubic grid a *unit lattice square* is a unit square whose corners are all lattice points. Similarly, on a 3-d cubic grid a *unit lattice cube* is a unit cube whose corners are all lattice points.

A black (white) adjacency that joins two points in a subset S of V is called a black (white) adjacency of S.

The *complement* of a picture $\mathcal{P} = (V, \beta, \omega, B)$, written $\bar{\mathcal{P}}$, is the picture $(V, \omega, \beta, \bar{B})$. In other words, the picture $\bar{\mathcal{P}}$ is the same as the picture $\mathcal{P}$ but with the black and white points and their associated adjacency relations interchanged. Thus $\mathcal{P}$-adjacent black points of $\mathcal{P}$ are $\bar{\mathcal{P}}$-adjacent white points of $\bar{\mathcal{P}}$. However, a black point and a white point that are $\mathcal{P}$-adjacent are not necessarily $\bar{\mathcal{P}}$-adjacent.

3.4. Connectedness. Components. Paths. Simple closed curves

Let $\mathcal{P} = (V, \beta, \omega, B)$ be any picture and let $\cdot$ be $\mathcal{P}, \beta, \omega$, or a positive integer (e.g., 4 or 8 when $V = \mathbb{Z}^2$). In this section "point" will mean "point of $\mathcal{P}$".

If a point p is $\cdot$-adjacent to a point q then we say p is a $\cdot$-*neighbor* of q. A point p is said to be $\cdot$-*adjacent* to a set of points S if p is $\cdot$-adjacent to some point in S. Two sets of points S and T are said to be $\cdot$-*adjacent* to each other if some point in S is $\cdot$-adjacent to some point in T.

A set of points is $\cdot$-*connected* if it is not a union of two disjoint nonempty sets which are not $\cdot$-adjacent to each other. A $\cdot$-*component* of a nonempty set of points S is a maximal $\cdot$-connected subset of S. Thus a $\cdot$-component of S is a nonempty $\cdot$-connected subset of S that is not $\cdot$-adjacent to any other point in S.

A $\mathcal{P}$-component of B (or, equivalently, a β-component of B) is called a *black component* of $\mathcal{P}$. A $\mathcal{P}$-component of $\bar{B} = V - B$ (or, equivalently, an ω-component of $\bar{B}$) is called a *white component* of $\mathcal{P}$.

A $\cdot$-*path* of $\mathcal{P}$ is a sequence $p_1, p_2, \ldots, p_n$ of $n \geqslant 1$ points in which each point p_i is $\cdot$-adjacent to p_{i-1} ($1 < i \leqslant n$). A $\cdot$-*path from p to q* is a $\cdot$-path whose initial and final points are respectively p and q. It is easy to show that two points p and q lie in the same $\cdot$-component of a set of points S if and only if there is a $\cdot$-path in S from p to q.

A *simple closed $\cdot$-curve* of $\mathcal{P}$ is a finite $\cdot$-connected set of points in which each point is $\cdot$-adjacent to exactly two other points in the set.

A $\cdot$-path or simple closed $\cdot$-curve of $\mathcal{P}$ is said to be *black* (*white*) if all its points are black (white). Notice that a black $\mathcal{P}$-path of $\mathcal{P}$ is a β-path in B and a white $\mathcal{P}$-path of $\mathcal{P}$ is an ω-path in $\bar{B}$. Analogous remarks apply to black and white simple closed $\mathcal{P}$-curves of $\mathcal{P}$.

3.5. Borders. Surrounding. Holes and cavities. The background

In the rest of this paper we use the terms *adjacent, neighbor, connected, component, path* and *simple closed curve* to mean (respectively) $\mathcal{P}$-adjacent, $\mathcal{P}$-neighbor, $\mathcal{P}$-connected, $\mathcal{P}$-component, $\mathcal{P}$-path, and simple closed $\mathcal{P}$-curve, where $\mathcal{P}$ is whatever picture is currently being discussed.

Given a black component C in a picture $\mathcal{P} = (V, \beta, \omega, B)$, a point in C that is adjacent to a white point of $\mathcal{P}$ is called a *border point* of C in $\mathcal{P}$. (Note that here "adjacent" means "ω-adjacent".) The set of all border points of C in $\mathcal{P}$ is called

the *border* of C in $\mathscr{P}$. If D is a white component of $\mathscr{P}$, then the *border of C with respect to D* in $\mathscr{P}$ is the set of all points in C that are adjacent to D.

We say that a DPS $\mathscr{S} = (V, \beta, \omega)$ is *connected* if V is connected in every picture on $\mathscr{S}$. All the digital picture spaces described in Section 2 are connected.

In a picture (V, β, ω, B) on a connected DPS a nonempty set $X \subseteq V$ is adjacent to each component of $\bar{X}$. For if C is a component of $\bar{X}$, then by the connectedness of V there is a path in V from a point in C to a point in X. The first point on such a path that belongs to X must be adjacent to C.

A connected set of points X in a picture $\mathscr{P} = (V, \beta, \omega, B)$ is said to *surround* a (not necessarily connected) set of points Y in $\mathscr{P}$ if every point in Y is contained in a finite component of $\bar{X}$ (i.e., a component of $\bar{X}$ consisting of just finitely many points). Note that since V has no accumulation points a subset of V is finite if and only if it is bounded. In a picture on a connected DPS if X surrounds Y, then Y does not surround X—this is Proposition 3.5.1 below.

A white component of a picture $\mathscr{P}$ which is both adjacent to and surrounded by a black component C of $\mathscr{P}$ is called a *hole* of (or in) C if $\mathscr{P}$ is a 2-d picture, and a *cavity* of (or in) C if $\mathscr{P}$ is a 3-d picture. By a hole in $\mathscr{P}$ we mean a hole in any black component of $\mathscr{P}$.

A white component of $\mathscr{P}$ that surrounds the set of *all* black points is called a *background component* of $\mathscr{P}$. The background component may be the only white component of $\mathscr{P}$. On the other hand, $\mathscr{P}$ may have no background component. This is so when $\mathscr{P} = (\mathbb{Z}^2, 8, 4, B)$ and B is the set of all lattice points whose x and y coordinates are both positive. But in a picture on a connected DPS the background component, if it exists, is unique—this is Proposition 3.5.2.

We now prove Propositions 3.5.1 and 3.5.2. The proofs are fairly easy, and would also be good exercises for the reader.

Proposition 3.5.1. *In a picture on a connected DPS if a connected set of points X surrounds a connected set of points Y, then Y does not surround X.*

Proof. Let X and Y be connected sets of points in a picture on the connected DPS (V, β, ω) such that X surrounds Y. Then the component of $\bar{X}$ that contains Y is finite. Therefore, since V is infinite, either some other component of $\bar{X}$ is infinite, or there are infinitely many other components of $\bar{X}$, or X itself is infinite. As V is connected each component of $\bar{X}$ is adjacent to X. So in each of the three cases X is contained in an infinite component of $\bar{Y}$. Therefore Y does not surround X. $\square$

Proposition 3.5.2. *A picture on a connected DPS has no more than one background component.*

Proof. Let D be a background component of a picture $\mathscr{P}$ on the connected DPS (V, β, ω), and let F be any other white component of $\mathscr{P}$. Each component of $\bar{D}$

must contain a black point adjacent to D (at the border of the component). So each component of $\bar{D}$ is finite, since D must surround the black points in that component. As V is infinite it follows that either D is infinite, or $\bar{D}$ has infinitely many components in which case D is adjacent to infinitely many black points. Hence the black points adjacent to D belong to an infinite component of $\bar{F}$, and so F is not a background component. $\square$

Suppose $\mathscr{P}$ is a picture on a connected DPS such that each point in $\mathscr{P}$ is adjacent to only finitely many other points. Then the complement of any finite set of points in $\mathscr{P}$ has only finitely many components, one of which must be infinite. But all components of the complement of the background component of $\mathscr{P}$ must be finite. Hence the background component of $\mathscr{P}$, if it exists, is infinite.

3.6. Regular digital picture spaces

To avoid awkward, or "pathological", digital picture spaces that are incompatible with our definition of digital fundamental groups, we shall have to impose two restrictions on the sets β and ω of a DPS (V, β, ω). We call the DPS's that satisfy these conditions *regular*, and in the rest of Section 3 we will confine our attention to regular DPS's.

Definition 3.6.1. A DPS (V, β, ω) is said to be *regular* if it satisfies both of the following conditions:

(1) no β-adjacency or ω-adjacency passes through any point in V other than its endpoints,

(2) no β-adjacency meets an ω-adjacency with which it does not share an endpoint.

The second condition in this definition essentially says that no β-adjacency ever "crosses" an ω-adjacency. Notice that if a DPS $\mathscr{S}$ is regular, then so is its complement $\bar{\mathscr{S}}$.

For a DPS $\mathscr{S} = (V, \beta, \omega)$ satisfying condition (1), condition (2) is equivalent to the following condition:

(2') If the points a, b, c, d in V are the corners of a convex quadrilateral, where a is diagonally opposite to c, then in any picture on $\mathscr{S}$ in which a and c are black points and b and d are white points, the sets $\{a, c\}$ and $\{b, d\}$ are not both connected.

The motivation for condition (2') is that Euclidean space has an analogous property: if a closed convex quadrilateral in Euclidean space with corners a, b, c, d, where a is diagonally opposite to c, is partitioned into two subsets in such a way that a and c belong to one subset and b and d to the other, then the two subsets are not both arcwise connected.

The DPS's $(\mathbb{Z}^2, 8, 8)$, $(\mathbb{Z}^3, 18, 18)$, $(\mathbb{Z}^3, 18, 26)$ and $(\mathbb{Z}^3, 26, 26)$ are not regular. If V is the set of grid points of the face-centered cubic grid, then $(V, 18, 18)$ is not a regular DPS.

In the rest of Section 3 all digital picture spaces will be regular. For brevity, this assumption will rarely be made explicit. Thus "a picture $\mathscr{P}$" will actually mean "a picture $\mathscr{P}$ on a regular DPS".

3.7. $\mathscr{P}$-walks and $\mathscr{P}$-loops; the digital fundamental group

A $\mathscr{P}$-walk is a curve $\gamma : [0, 1] \to E^n$, where $n = 2$ or 3 according as $\mathscr{P}$ is 2-d or 3-d, such that $\gamma(0)$ and $\gamma(1)$ are black points of $\mathscr{P}$, and there exists a positive integer k such that for all nonnegative integers $i < k$:
   (1) $\gamma(i/k)$ is a black point, and
   (2) $\gamma(i/k)$ is equal or adjacent to $\gamma((i+1)/k)$, and
   (3) γ is linear on the closed interval $[i/k, (i+1)/k]$.
A $\mathscr{P}$-walk γ is said to be a $\mathscr{P}$-walk *from* $\gamma(0)$ *to* $\gamma(1)$.

A $\mathscr{P}$-walk that is a constant map will be called *trivial*; all other $\mathscr{P}$-walks will be called *nontrivial*. If γ is a nontrivial $\mathscr{P}$-walk, then it follows from the first condition in Definition 3.6.1 that there is just one positive integer k such that the conditions (1), (2) and (3) in the definition of a $\mathscr{P}$-walk are satisfied for all nonnegative integers $i < k$. This value of k will be called the *length* of γ. For a trivial $\mathscr{P}$-walk all positive integers k satisfy conditions (1), (2) and (3), so this definition cannot be used. We define the length of a trivial $\mathscr{P}$-walk to be 1.

If γ_1 is a $\mathscr{P}$-walk of length m from p to q and γ_2 is a $\mathscr{P}$-walk of length n from q to r, then the *product of* γ_1 *and* γ_2, written $\gamma_1 \cdot \gamma_2$, is the $\mathscr{P}$-walk from p to r obtained by catenating the curves γ_1 and γ_2 in the following way:

$$\gamma_1 \cdot \gamma_2(x) = \begin{cases} \gamma_1((m+n)x/m), & \text{if } 0 \le x \le m/(m+n), \\ \gamma_2((m+n)x/n - m/n), & \text{if } m/(m+n) \le x \le 1. \end{cases}$$

Note that this operation is associative. The length of $\gamma_1 \cdot \gamma_2$ is the sum of the lengths of γ_1 and γ_2, provided at least one of γ_1 and γ_2 is nontrivial.

A $\mathscr{P}$-walk from a point p to itself is called a $\mathscr{P}$-loop, and is said to be *based at* p; we also call p the *base point* of the $\mathscr{P}$-loop. A trivial $\mathscr{P}$-walk is a $\mathscr{P}$-loop, and is called a *trivial $\mathscr{P}$-loop*; all other $\mathscr{P}$-loops are called *nontrivial*. The trivial $\mathscr{P}$-loop based at p is denoted by e_p.

Now let $\mathscr{P}$ be a picture on an n-dimensional DPS, where $n = 2$ or 3. Two $\mathscr{P}$-loops with the same base point are called *equivalent* if they are fixed base point homotopic in $E^n - W$, where W is the union of all white points of $\mathscr{P}$ if $n = 2$, and the union of all white adjacencies of $\mathscr{P}$ if $n = 3$. This is of course an equivalence relation.

We write $[\lambda]_{\mathscr{P}}$ for the equivalence class consisting of all $\mathscr{P}$-loops which have the same base point as λ and which are equivalent to λ. If the $\mathscr{P}$-loops λ and λ' have the same base point, then define $[\lambda]_{\mathscr{P}} \cdot [\lambda']_{\mathscr{P}}$ to be the equivalence class $[\lambda \cdot \lambda']_{\mathscr{P}}$. This is a well-defined associative binary operation on equivalence classes.

Definition 3.7.1. Let $\mathcal{P}$ be a picture on a regular DPS. The *digital fundamental group of $\mathcal{P}$ with base point p*, denoted by $\pi(\mathcal{P}, p)$, is the group of all equivalence classes $[\lambda]_{\mathcal{P}}$ where λ is a $\mathcal{P}$-loop based at p, under the $\cdot$ operation.[5]

It is readily confirmed that if p_1 and p_2 are points in the same black component of a picture $\mathcal{P}$ on a regular DPS, then $\pi(\mathcal{P}, p_1)$ and $\pi(\mathcal{P}, p_2)$ are isomorphic groups.

Digital fundamental groups are invariant under isomorphism of pictures. In fact, if f is any isomorphism of a picture $\mathcal{P}_1$ to a picture $\mathcal{P}_2$, then, for each black point p in $\mathcal{P}_1$, f induces a group isomorphism of $\pi(\mathcal{P}_1, p)$ to $\pi(\mathcal{P}_2, f(p))$.

Although the definition of $\mathcal{P}$-loop equivalence, and hence the definition of the digital fundamental group, involves global continuous deformation, we shall see in Section 4.4 that for pictures on a large class of well-behaved DPS's—the strongly normal digital picture spaces—there is a group with a purely discrete definition that is naturally isomorphic to the digital fundamental group.

4. Strongly normal digital picture spaces

4.1. General discussion

A strongly normal digital picture space is a DPS $\mathcal{S} = (\mathbb{Z}^n, \beta, \omega)$ (where $n = 2$ or 3) in which there is a certain duality between the β-adjacencies and the ω-adjacencies. As a result of this duality the digital topology of $\mathcal{S}$ is in many ways analogous to the topology of the Euclidean plane or Euclidean 3-space.

We will show in Section 7 that strongly normal DPS's have many good properties. A strongly normal DPS provides a possible basis for topology-related image processing operations such as thinning and border following. Digital picture spaces that are not strongly normal, such as $(\mathbb{Z}^2, 4, 4)$, may be quite unsuitable for this purpose.

4.2. Definition of a strongly normal DPS

Definition 4.2.1. $\mathcal{S} = (V, \beta, \omega)$ is *strongly normal* if it is regular and also satisfies all of the following conditions:

(1) $V = \mathbb{Z}^2$ (the 2-d case) or $V = \mathbb{Z}^3$ (the 3-d case).

(2) In the 2-d case every 4-adjacency and in the 3-d case every 6-adjacency is both a β-adjacency and an ω-adjacency.

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[5] In [19], $\pi(\mathcal{P}, p)$ is defined as the subgroup of the fundamental group $\pi_1(E^n - \mathcal{P}_W, p)$ consisting of those homotopy classes that contain a loop in $\mathcal{P}_B$, where $\mathcal{P}_W$ is the union of the white points and white adjacencies of $\mathcal{P}$ and $\mathcal{P}_B$ is the union of the black points and black adjacencies of $\mathcal{P}$. This is equivalent to the present definition up to a natural group isomorphism. For, firstly, the Simplicial Approximation Theorem implies that every loop in $\mathcal{P}_B$ based at p is fixed base point homotopic in $\mathcal{P}_B$ to a $\mathcal{P}$-loop. Secondly, two $\mathcal{P}$-loops are fixed base point homotopic in $E^n - \mathcal{P}_W$ if and only if they are equivalent in our sense—by the definition of equivalence when $n = 3$, and as a consequence of Lemma 6.2.2 (since $\mathcal{P}_W \subseteq C'(\mathcal{P})$) when $n = 2$.

(3) All β-adjacencies and ω-adjacencies are 8-adjacencies in the 2-d case and 26-adjacencies in the 3-d case.

(4) In any given unit lattice square either both diagonals are β-adjacencies or both diagonals are ω-adjacencies or one of the diagonals is both a β-adjacency and an ω-adjacency.

(5) Every picture $\mathcal{P}$ on $\mathcal{S}$ has the property that whenever a black component of $\mathcal{P}$ is either β-adjacent or ω-adjacent to a white component of $\mathcal{P}$, the black component is in the 2-d case 4-adjacent and in the 3-d case 6-adjacent to the white component.

It is easy to see that the familiar DPS's $(\mathbb{Z}^2, 8, 4)$ and $(\mathbb{Z}^3, 26, 6)$ are strongly normal. Other examples will be given in Section 4.3. Notice that if a DPS $\mathcal{S}$ is strongly normal, then so is its complement $\bar{\mathcal{S}}$.

Conditions (1) and (2) imply that a strongly normal DPS is connected. Regarding condition (4), note that if both diagonals are β-adjacencies (ω-adjacencies), then neither is an ω-adjacency (a β-adjacency) because $\mathcal{S}$ is regular. Conditions (1), (2) and (5) imply that a black component and a white component of a picture on a strongly normal DPS (V, β, ω) are β-adjacent if and only if they are ω-adjacent. We leave it to the reader to verify that for a 2-d DPS conditions (1), (2) and (3) imply condition (5).

Given that $\mathcal{S}$ satisfies condition (1), it is easily seen that condition (4) is equivalent to each of the following conditions:

(4') In any picture on $\mathcal{S}$, if two diagonally opposite corners a, c of a unit lattice square are black points and the other two corners b, d are white points, then one of the sets $\{a, c\}$ and $\{b, d\}$ is connected.

(4*) If either diagonal of a unit lattice square is not a β-adjacency, the other is an ω-adjacency.

When $V = \mathbb{Z}^2$ and $\mathcal{S}$ satisfies conditions (2) and (3) we want condition (4') (and hence (4) and (4*)) to hold because if it does not, then we can construct a "connectivity paradox". For if (4') fails, then we may suppose w.l.o.g. that $\{a, c\} = \{(0, 0), (1, 1)\}$. Then, in the picture on $\mathcal{S}$ with black point set $B = \{(0, n) \mid n \in \mathbb{Z}, n \leq 0\} \cup \{(1, n) \mid n \in \mathbb{Z}, n > 0\}$, B has two components and neither component separates $\mathbb{Z}^2$ (i.e., if we remove either black component by changing its points into white points, then the white point set becomes connected). Now[6] if a closed set in the Euclidean plane E^2 has just two components and neither component separates E^2, then the set itself does not separate E^2. However, B does separate $\mathbb{Z}^2$ (in the sense that $\mathbb{Z}^2 - B$ is not connected). To avoid this "connectivity paradox", we must require condition (4') to hold when $V = \mathbb{Z}^2$.

We want a 3-d strongly normal DPS to meet each coordinate plane in a strongly normal 2-dimensional DPS. So we also require condition (4') to hold when $V = \mathbb{Z}^3$.

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[6] Here we are appealing to one of the Phragmen–Brouwer properties of Euclidean space [26, Chapter II], namely that in E^n if neither of two disjoint closed sets separates two points, then the union of the sets does not separate those points.

Condition (5) may seem unsatisfactory for two reasons. First, it may not be clear why one might expect condition (5) to hold in a well-behaved DPS. Second, it looks as though one might have to do some work to determine whether a given DPS satisfies condition (5) or not. Our next result (Proposition 4.2.2 below), eliminates these apparent drawbacks of condition (5) by giving two alternate formulations of that condition. It asserts that if $\mathscr{S}$ satisfies conditions (1)-(4), then condition (5) is equivalent to each of the following conditions:

(5') In any picture on $\mathscr{S}$, a one-point black component $\{p\}$ and a one-point white component $\{q\}$ cannot be β-adjacent or ω-adjacent to each other.

(5*) In the case $V = \mathbb{Z}^3$, if p and q are diametrically opposite corners of a unit lattice cube in which p is not β-adjacent to any 6-neighbor of q, and q is not ω-adjacent to any 6-neighbor of p, then p and q are neither β- nor ω-adjacent.

Here is an informal, intuitive, argument in support of condition (5') (and hence of (5) and (5*)). A one-point white component $\{q\}$ corresponds to a very small cavity Q in some object in E^3. A one-point black component $\{p\}$ corresponds to a very small object P in E^3. Since (assuming condition (2)) $\{p\}$ does not surround $\{q\}$ and $\{q\}$ does not surround $\{p\}$, the object P should neither surround nor be surrounded by the cavity Q. Thus removing the object P should not affect the cavity Q, so changing p to a white point should not enlarge the white component $\{q\}$. Hence $\{p\}$ should not be ω-adjacent to $\{q\}$. Similarly, filling in the cavity Q should not affect the object P, so changing q to a black point should not enlarge the black component $\{p\}$. Hence $\{p\}$ should not be β-adjacent to $\{q\}$.

Proposition 4.2.2. *Suppose $\mathscr{S} = (\mathbb{Z}^3, \beta, \omega)$ satisfies conditions (2)-(4) in the definition of a strongly normal DPS. Then conditions (5), (5') and (5*) are equivalent.*

Proof. To see that (5') implies (5*), suppose (5') holds. Let p and q be diametrically opposite corners of the unit lattice cube K. Let a, b, c be the three 6-neighbors of q in K. If q is not ω-adjacent to any 6-neighbor of p, and p is not β-adjacent to any 6-neighbor of q, then when B is the set consisting of p, a, b, c, and the nineteen 26-neighbors of q outside K the sets $\{p\}$ and $\{q\}$ are respectively a black and white component of $(\mathbb{Z}^3, \beta, \omega, B)$, and so (5') implies p is neither β- nor ω-adjacent to q. It is equally straightforward to verify that (5*) implies (5) and that (5) implies (5'). $\square$

4.3. Examples of strongly normal DPS's

The DPS's $(\mathbb{Z}^2, 8, 4)$, $(\mathbb{Z}^2, 4, 8)$, $(\mathbb{Z}^3, 6, 26)$, $(\mathbb{Z}^3, 26, 6)$, $(\mathbb{Z}^3, 6, 18)$ and $(\mathbb{Z}^3, 18, 6)$ are all strongly normal. Both the 2-d and the 3-d "Khalimsky digital picture spaces", in which $\beta = \omega =$ the Khalimsky adjacencies, are strongly normal. (However, the DPS's $(\mathbb{Z}^2, 4, 4)$, $(\mathbb{Z}^2, 8, 8)$, $(\mathbb{Z}^3, 6, 6)$, $(\mathbb{Z}^3, 18, 18)$ and $(\mathbb{Z}^3, 26, 26)$ are *not* strongly normal.) It is not difficult to show that each of the following five DPS's is isomorphic to a strongly normal DPS (for further details, see [11, Section 4.3]):

(1) $(V, 6, 6)$ where $V =$ grid points of the 2-d isometric hexagonal grid,

(2) $(V, 12, 12)$ where $V =$ grid points of the 3-d face-centered cubic grid,

(3) $(V, 12, 18)$ where $V =$ grid points of the 3-d face-centered cubic grid,

(4) $(V, 18, 12)$ where $V =$ grid points of the 3-d face-centered cubic grid,

(5) $(V, 14, 14)$ where $V =$ grid points of the 3-d body-centered cubic grid.

In a DPS that is isomorphic to a strongly normal DPS every point has at least four β-neighbors and at least four ω-neighbors (each point has at least six of each in the 3-d case). Thus neither of the DPS's $(V, 12, 3)$ and $(V, 3, 12)$ where V is the set of grid points of the 2-d triangular grid (described in Section 2) is isomorphic to a strongly normal DPS.

4.4. Black digital walks and loops; the discrete digital fundamental group

Given two finite sequences c_1, c_2, where the final point of c_1 is the same as the initial point of c_2, the *product* of c_1 and c_2, written $c_1 \cdot c_2$, is the sequence obtained by removing the initial element of c_2 and appending the resulting sequence onto the end of c_1. Thus $\langle p, p, q, r, a \rangle \cdot \langle a, x, y, a \rangle = \langle p, p, q, r, a, x, y, a \rangle$.

The *reduced form* of a finite sequence c is the subsequence of c that is obtained when we remove from c all but one point from every set of consecutive equal points. If all members of a sequence are equal to p, then the reduced form of the sequence is $\langle p \rangle$. Otherwise, if $c = \langle p_1, p_2, \ldots, p_m \rangle$, then the reduced form of c is the longest sequence of the form $\langle p_1, p_{i_1}, \ldots, p_{i_n} \rangle$, where $n \geq 1$, i_1 is the smallest value of i such that $p_i \neq p_1$, and each of the other i_k is the smallest value of i greater than i_{k-1} such that $p_i \neq p_{i_{k-1}}$. Thus the reduced form of $\langle p, p, p, q, r, r, q, q, q, q, p, p \rangle$ is $\langle p, q, r, q, p \rangle$ if p, q and r are distinct.

For any digital picture $\mathcal{P}$ and black points p, p' of $\mathcal{P}$, a *black digital walk of $\mathcal{P}$ from p to p'* is a sequence $\langle p_1, p_2, \ldots, p_n \rangle$ of black points of $\mathcal{P}$ where $n \geq 1$, $p_1 = p$, $p_n = p'$ and each point p_i is equal or adjacent to p_{i-1} $(1 < i \leq n)$.[7] A black digital walk is said to be *trivial* if all its points are equal, *nontrivial* otherwise. A black digital walk of $\mathcal{P}$ from p to p is called a *black digital loop of $\mathcal{P}$ based at p*, and we call p its *base point*.

Now suppose $\mathcal{P}$ is a picture on a strongly normal DPS. If K is any unit lattice square or unit lattice cube, then we say that one black digital walk is *K-equivalent* to another with the same initial and final points if the two are equal or if the first is $\langle x = p_1, p_2, \ldots, p_m = y \rangle$, the second is $\langle x = q_1, q_2, \ldots, q_n = y \rangle$ and the following three conditions are satisfied:

(1) $m = n$, and

(2) for $1 \leq i \leq n$, $p_i \in K$ if $q_i \in K$, and

(3) for $1 < i < n$, $p_i = q_i$ if $q_{i-1} \notin K$ or $q_i \notin K$ or $q_{i+1} \notin K$.

K-equivalence is a symmetric relation, since (3) implies that $p_i \notin K$ if $q_i \notin K$ and so (2) and (3) together imply $p_i \in K$ if and only if $q_i \in K$.

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[7] In general a black digital walk is not a path because consecutive points may be equal.

Say that two black digital loops of $\mathscr{P}$ with the same base point are *immediately equivalent* if they have the same reduced form, or if they are K-equivalent for some unit lattice square K, or if they are K-equivalent for some unit lattice cube K in which no two diametrically opposed corners are white points that are adjacent to one another. Obviously the third case applies only if $\mathscr{P}$ is 3-dimensional.

Define *equivalence* of black digital loops of $\mathscr{P}$ to be the transitive closure of immediate equivalence. This is of course an equivalence relation. Given a black digital loop c, write $[c]_{\mathscr{P}}$ for the equivalence class consisting of all black digital loops of $\mathscr{P}$ that have the same base point as c and which are equivalent to c. If c_1 and c_2 are black digital loops with the same base point, then we define $[c_1]_{\mathscr{P}} \cdot [c_2]_{\mathscr{P}} = [c_1 \cdot c_2]_{\mathscr{P}}$. This is a well-defined associative binary operation on equivalence classes.

Definition 4.4.1. Let $\mathscr{P}$ be a picture on a strongly normal DPS and let p be a black point of $\mathscr{P}$. The *discrete digital fundamental group* of $\mathscr{P}$ with base point p, written $\pi^d(\mathscr{P}, p)$, is the group of all equivalence classes $[c]_{\mathscr{P}}$ in which c is a black digital loop of $\mathscr{P}$ with base point p, under the $\cdot$ operation.

This is in essence the discrete definition of the digital fundamental group given in [11, Section 4.4]. (However, we have introduced a new name and notation for the group defined in this definition, to avoid any possible confusion with the digital fundamental group $\pi(\mathscr{P}, p)$.)

The *black digital loop* of a $\mathscr{P}$-loop λ of length k is the black digital loop $\langle \lambda(i/k) | 0 \leqslant i \leqslant k \rangle$. This is a base point preserving 1-1 mapping of $\mathscr{P}$-loops to black digital loops. In Section 7.9 we show that this mapping induces an isomorphism of the digital fundamental group $\pi(\mathscr{P}, p)$ to the discrete digital fundamental group $\pi^d(\mathscr{P}, p)$.

5. Continuous analogs of digital pictures

Every digital picture space we consider in Section 5 will be strongly normal, and every digital picture we consider will be a picture on a strongly normal DPS. For brevity these hypotheses will not always be stated explicitly.

5.1. Continuous analog properties

We now relate the digital topology of strongly normal digital picture spaces to the topology of Euclidean space. Specifically, we shall associate each picture $\mathscr{P} = (\mathbb{Z}^n, \beta, \omega, B)$, where $n = 2$ or 3, with a polyhedron $C(\mathscr{P}) \supseteq B$ (a "continuous analog" of $\mathscr{P}$). The polyhedron $C(\mathscr{P})$ will have all of the following properties, which we shall refer to as *continuous analog properties*:

(1) All black points and all black adjacencies of $\mathscr{P}$ are contained in $C(\mathscr{P})$.

(2) All white points and all white adjacencies of $\mathscr{P}$ are contained in $E^n - C(\mathscr{P})$.

(3) Each component of $C(\mathscr{P})$ meets $\mathbb{Z}^n$ in a black component of $\mathscr{P}$.

(4) Each component of $E^n - C(\mathscr{P})$ meets $\mathbb{Z}^n$ in a white component of $\mathscr{P}$.

(5) The boundary of a component X of $C(\mathscr{P})$ meets the boundary of a component Y of $E^n - C(\mathscr{P})$ if and only if there is a black point in X that is adjacent to a white point in Y.

(6) For each black point p *in* $\mathscr{P}$, the inclusion of the black points and black adjacencies of $\mathscr{P}$ in $C(\mathscr{P})$ induces an isomorphism of the digital fundamental group $\pi(\mathscr{P}, p)$ to the (classical) fundamental group $\pi_1(C(\mathscr{P}), p)$.

(7) For each white point q in $\mathscr{P}$, the inclusion of the white points and white adjacencies of $\mathscr{P}$ in $E^n - C(\mathscr{P})$ induces an isomorphism of the digital fundamental group $\pi(\bar{\mathscr{P}}, q)$ to the fundamental group $\pi_1(E^n - C(\mathscr{P}), q)$.

5.2. Ordinary and special unit lattice cubes. $\mathscr{P}$-simplexes

Our construction of a polyhedron $C(\mathscr{P})$ having properties (1)–(7) is based on a triangulation of the Euclidean plane or Euclidean 3-space that is consistent with the adjacencies of $\mathscr{P}$. This triangulation involves dividing each unit lattice square into two $(1, 1, \sqrt{2})$ triangles along an appropriate diagonal, and appropriately subdividing each unit lattice cube, as we now explain.

Every unit lattice square must satisfy exactly one of the following three conditions:

(1) The four corners are all black points or are all white points of $\mathscr{P}$.

(2) The corners are not all white points or all black points, but one of the two diagonals is a black adjacency or a white adjacency of $\mathscr{P}$.

(3) The corners are not all white points or all black points, and neither diagonal is a white adjacency or a black adjacency.

When condition (1) is satisfied it turns out that either diagonal may in principle be used to subdivide the unit lattice square. For definiteness we choose the diagonal each of whose endpoints has coordinates that sum to an even number. When condition (2) holds the appropriate diagonal for subdividing the square is the one which is a black or a white adjacency. (There is only one such diagonal, since $\mathscr{P}$ is regular.) When condition (3) holds the appropriate diagonal is a diagonal that joins a black point to a white point (at least one diagonal has this property, by strong normality); if both diagonals join a white point to a black point, then in principle either diagonal may be used, and we again choose the diagonal each of whose endpoints has coordinates that sum to an even number.

Regardless of whether $\mathscr{P}$ is 2-d or 3-d, let $T_2(\mathscr{P})$ be the set of closed $(1, 1, \sqrt{2})$ triangles obtained by subdividing all unit lattice squares in accordance with these rules.

For $\mathscr{P} = (\mathbb{Z}^3, \beta, \omega, B)$, we now define a collection $T_3(\mathscr{P})$ of 3-simplexes. The following are some important properties of $T_3(\mathscr{P})$:

(1) Every simplex in $T_3(\mathscr{P})$ is contained in a unit lattice cube.

(2) For each simplex σ in $T_3(\mathscr{P})$, every 2-d face of σ that is contained in a unit lattice square is in $T_2(\mathscr{P})$.

(3) Every simplex in $T_2(\mathscr{P})$ is the common face of two simplexes in $T_3(\mathscr{P})$.

Say that a unit lattice cube K is *special* (with respect to $\mathscr{P}$) if there are three black adjacencies and three white adjacencies of $\mathscr{P}$ in K both of which form a $(\sqrt{2}, \sqrt{2}, \sqrt{2})$ equilateral triangle. (Here regularity and strong normality condition (5*) imply that the two triangles lie in parallel planes perpendicular to a diameter[8] of K that is neither a β- nor an ω-adjacency.) Observe that a unit lattice cube is special with respect to $\mathscr{P}$ if and only if it is special with respect to $\bar{\mathscr{P}}$. Call a unit lattice cube *ordinary* if it is not special.

In a special unit lattice cube K let e_1, e_2, e_3, e_4, e_5 and e_6 be the edges of the two equilateral triangles formed by black and by white adjacencies (in any order). We want e_7 to be one of the three diameters of K that join a corner of one triangle to a corner of the other triangle; for definiteness we choose e_7 as follows[9]: if the diameter of K parallel to the vector $(1, 1, 1)$ is not perpendicular to the triangles, then let e_7 be this diameter; otherwise let e_7 be the diameter of K parallel to the vector $(1, -1, 1)$. We subdivide K in the obvious manner into six 3-simplexes whose edges are the edges of K and the e_i.

Every ordinary unit lattice cube is subdivided into twelve congruent 3-simplexes, each of which has a vertex at the centroid of the unit lattice cube and a face in $T_2(\mathscr{P})$. We define $T_3(\mathscr{P})$ to be the set of 3-simplexes produced by subdividing all unit lattice cubes, ordinary and special, in the ways just described.

For $n = 2$ or 3, if $\mathscr{P}$ is any picture on a strongly normal n-dimensional DPS, then a nonempty face of a simplex in $T_n(\mathscr{P})$ is called a *$\mathscr{P}$-simplex*.[10] An r-dimensional $\mathscr{P}$-simplex is also called an *r-simplex of $\mathscr{P}$*. When $n = 2$ the 0-simplexes of $\mathscr{P}$ are just the points of $\mathscr{P}$. When $n = 3$ the 0-simplexes of $\mathscr{P}$ are the points of $\mathscr{P}$ and the centroids of ordinary lattice cubes.

Lemma 5.2.1. *The following are true for any picture $\mathscr{P} = (\mathbb{Z}^n, \beta, \omega, B)$ on a strongly normal DPS:*

(1) *The set of all $\mathscr{P}$-simplexes is a triangulation of E^n. In particular, every point x in E^n lies in the relative interior[11] of exactly one $\mathscr{P}$-simplex, and that $\mathscr{P}$-simplex is a face of every $\mathscr{P}$-simplex that contains x.*

(2) *Let e be a straight line segment joining two black or two white points of $\mathscr{P}$ which is not a diagonal of a unit lattice square whose corner points are all black or all white. Then e is a 1-simplex of $\mathscr{P}$ if and only if e is contained in a unit lattice square and e's endpoints are adjacent to each other.*

(3) *Every $\mathscr{P}$-simplex is a $\bar{\mathscr{P}}$-simplex, and vice versa.*

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[8] By a *diameter* of a cube we mean a straight line segment joining two diametrically opposite corners of the cube.

[9] In [11], e_7 was allowed to be any one of the three diameters. But in this paper we specify which diameter is to be e_7 to ensure that $T_3(\mathscr{P}) = T_3(\bar{\mathscr{P}})$.

[10] In other words, the $\mathscr{P}$-simplexes are the members of $T_n(\mathscr{P})$, the 2-d faces of members of $T_n(\mathscr{P})$ if $n = 3$, the edges of members of $T_n(\mathscr{P})$, and the vertices of members of $T_n(\mathscr{P})$.

[11] The *relative interior* of a simplex σ is the set of points in σ which do not belong to any proper face of σ. Note that the relative interior of a 0-simplex is the 0-simplex itself. Every point in a simplex σ lies in the relative interior of exactly one face of σ.

This lemma follows from the definitions of $T_2(\mathcal{P})$ and $T_3(\mathcal{P})$. We leave the details to the reader.

We emphasize that part (2) of the lemma only applies when the endpoints of e are both black points or both white points.

5.3. The augmented black and white point sets; black, white and semi-black $\mathcal{P}$-simplexes

The *augmented black point set* of a 3-d picture $\mathcal{P}$, denoted by $B'(\mathcal{P})$, is the union of the black point set of $\mathcal{P}$ with the set of all centroids of ordinary unit lattice cubes K that satisfy at least one of the following two conditions:

(1) One of the four diameters of K is a black adjacency of $\mathcal{P}$.

(2) K contains a black simple closed curve of $\mathcal{P}$ which is not contained in any one of the six faces of K, and no diameter of K is a white adjacency of $\mathcal{P}$.

To avoid having to distinguish the 2-d and the 3-d cases all the time, when $\mathcal{P}$ is a 2-d picture we define its augmented black point set $B'(\mathcal{P})$ to be its black point set.

The augmented black point sets of $\mathcal{P}$ and $\bar{\mathcal{P}}$ are disjoint. For when $\mathcal{P}$ is a 3-d picture it is easily verified that if a unit lattice cube K in which no diameter is a black or a white adjacency of $\mathcal{P}$ contains a black simple closed curve of $\mathcal{P}$ and a black simple closed curve of $\bar{\mathcal{P}}$ (i.e., a white simple closed curve of $\mathcal{P}$), neither of which is contained in a face of K, then K is special.

Every point in the augmented black point set of $\mathcal{P}$ is a 0-simplex of $\mathcal{P}$. The *augmented white point set* of a picture $\mathcal{P}$, denoted by $\bar{B}'(\mathcal{P})$, is the set of all 0-simplexes of $\mathcal{P}$ that do not belong to the augmented black point set $B'(\mathcal{P})$. When $n = 2$, since the set of 0-simplexes of $\mathcal{P}$ is just $\mathbb{Z}^2$ and since $B'(\mathcal{P})$ is just the black point set of $\mathcal{P}$, the augmented white point set $\bar{B}'(\mathcal{P})$ is just the white point set of $\mathcal{P}$. When $n = 3$, $\bar{B}'(\mathcal{P})$ is the union of the white point set with the set of centroids of *ordinary* unit lattice cubes which do not belong to the augmented black point set. Note that in the 3-d case the centroid of a special unit lattice cube is not a 0-simplex of $\mathcal{P}$ and so is not in the augmented black point set or the augmented white point set.

Since the set of $\mathcal{P}$-simplexes is the same as the set of $\bar{\mathcal{P}}$-simplexes, and the augmented black point sets of $\mathcal{P}$ and $\bar{\mathcal{P}}$ are disjoint, the augmented white point set of $\mathcal{P}$ includes the augmented black point set of $\bar{\mathcal{P}}$ (and the inclusion may be strict).

The augmented black and augmented white point sets provide important information about the connectedness of the set of black and the set of white points in each ordinary unit lattice cube:

Proposition 5.3.1. *Let $\mathcal{P}$ be a 3-d picture on a strongly normal DPS, and let K be a unit lattice cube with centroid c. If c is in the augmented white point set of $\mathcal{P}$, then the set of white points of $\mathcal{P}$ in K is connected. If c is in the augmented black point set of $\mathcal{P}$, then the set of black points of $\mathcal{P}$ in K is connected.*

Proof. To prove the first assertion, suppose the set of white points in K has more than one component. Then no diameter of K can be a white adjacency. Moreover,

either one of the components contains just one point or there are precisely two components, each of which consists of two 18-adjacent points. In both cases one sees (after some case checking using the definition of a strongly normal DPS) that there must be a black simple closed curve of $\mathcal{P}$ in K that is not contained in any one face of K. So either K is special or c is in the augmented black point set, whence c is not in the augmented white point set. This proves the first assertion.

Since the augmented white point set of $\bar{\mathcal{P}}$ includes the augmented black point set of $\mathcal{P}$, we can deduce the second assertion by applying the first assertion to the picture $\bar{\mathcal{P}}$. □

A *black* $\mathcal{P}$-simplex is a $\mathcal{P}$-simplex all of whose vertices lie in the augmented black point set of $\mathcal{P}$. A *white* $\mathcal{P}$-simplex is a $\mathcal{P}$-simplex all of whose vertices lie in the augmented white point set of $\mathcal{P}$. A *semi-black* $\mathcal{P}$-simplex is a $\mathcal{P}$-simplex which is neither black nor white.

5.4. $C(\mathcal{P})$ and $C'(\mathcal{P})$

$C(\mathcal{P})$ will denote the union of all black $\mathcal{P}$-simplexes, and $C'(\mathcal{P})$ will denote the union of all white $\mathcal{P}$-simplexes.

Since no black $\mathcal{P}$-simplex ever meets a white $\mathcal{P}$-simplex, $C'(\mathcal{P})$ and $C(\mathcal{P})$ are disjoint. Also, $C'(\mathcal{P}) \supseteq C(\bar{\mathcal{P}})$, since the augmented white point set of $\mathcal{P}$ contains the augmented black point set of $\bar{\mathcal{P}}$. When $\mathcal{P}$ is a 2-d picture, $C'(\mathcal{P}) = C(\bar{\mathcal{P}})$ since the augmented black and augmented white point sets are just the black and the white point sets respectively.

If $\mathcal{P} = (\mathbb{Z}^n, \beta, \omega, B)$ is any picture on a strongly normal DPS and $\mathcal{P} \cap K$ denotes the picture $(\mathbb{Z}^n, \beta, \omega, B \cap K)$ where K is a unit lattice square or cube, then $C(\mathcal{P} \cap K) = C(\mathcal{P}) \cap K$. For the black $(\mathcal{P} \cap K)$-simplexes are then all contained in K, and the black $(\mathcal{P} \cap K)$-simplexes in K are just the black $\mathcal{P}$-simplexes in K.

We end this section with two useful results about $C(\mathcal{P})$ and $C'(\mathcal{P})$.

Lemma 5.4.1. *Let $\mathcal{P}$ be a 2-d or 3-d picture on a strongly normal DPS, and let each of A and D be a union of $\mathcal{P}$-simplexes. Then $A - (C(\mathcal{P}) \cup C'(\mathcal{P}) \cup D)$ is the union of the relative interiors of all semi-black $\mathcal{P}$-simplexes that are contained in A but are not contained in D.*

Proof. The set A is the union of the relative interiors of the $\mathcal{P}$-simplexes contained in A, and D is the union of the relative interiors of the $\mathcal{P}$-simplexes contained in D. So, since the relative interiors of distinct $\mathcal{P}$-simplexes are disjoint, $A - D$ is the union of the relative interiors of the $\mathcal{P}$-simplexes that are contained in A but are not contained in D. Similarly, since $C(\mathcal{P}) \cup C'(\mathcal{P})$ is the union of the relative interiors of all $\mathcal{P}$-simplexes that are not semi-black, $A - (C(\mathcal{P}) \cup C'(\mathcal{P}) \cup D) = (A - D) - (C(\mathcal{P}) \cup C'(\mathcal{P}))$ is the union of the relative interiors of all semi-black $\mathcal{P}$-simplexes that are contained in A but not contained in D. □

Proposition 5.4.2. *Let $\mathcal{P} = (\mathbb{Z}^n, \beta, \omega, B)$ be a 2-d or 3-d picture on a strongly normal DPS. Then there are strong deformation retractions $f : E^n - C'(\mathcal{P}) \rightarrow C(\mathcal{P})$ and $g : E^n - C(\mathcal{P}) \rightarrow C'(\mathcal{P})$. Moreover, if K is any unit lattice square or cube, then the restrictions of f to $K - C'(\mathcal{P})$ and g to $K - C(\mathcal{P})$ are strong deformation retractions onto $C(\mathcal{P}) \cap K$ and $C'(\mathcal{P}) \cap K$ respectively.*

Proof. For each point y in E^n let $\sigma(y)$ be the $\mathcal{P}$-simplex whose relative interior contains y.

We define the retraction mapping f as follows, using the fact that if $x \in E^n - C'(\mathcal{P})$, then at least one vertex of $\sigma(x)$ is in $B'(\mathcal{P})$. For all x in $E^n - C'(\mathcal{P})$ let $f(x)$ be the point in $\sigma(x)$ such that:

(1) each barycentric coordinate of $f(x)$ associated with a vertex of $\sigma(x)$ that is in $\overline{B'}(\mathcal{P})$ is 0, and

(2) each barycentric coordinate of $f(x)$ associated with a vertex of $\sigma(x)$ that is in $B'(\mathcal{P})$ is the quotient of the corresponding barycentric coordinate of x divided by the sum of the barycentric coordinates of x that are associated with vertices of $\sigma(x)$ that are in $B'(\mathcal{P})$.

It is readily confirmed that f is continuous, even at points x in $E^n - C'(\mathcal{P})$ that belong to two or more $\mathcal{P}$-simplexes. Plainly f maps $E^n - C'(\mathcal{P})$ to $C(\mathcal{P})$, maps each point in $C(\mathcal{P})$ to itself, and is homotopic relative to $C(\mathcal{P})$ to the identity map on $E^n - C'(\mathcal{P})$ (by a linear homotopy). Also, $f(K - C'(\mathcal{P})) \subseteq K$ for any unit lattice square or cube K.

We define the retraction mapping g analogously, but with the roles of $B'(\mathcal{P})$ and $\overline{B'}(\mathcal{P})$ interchanged. $\square$

6. The main theorem

As in Section 5, every digital picture space we consider in Section 6 will be strongly normal, and every digital picture we consider will be a picture on a strongly normal DPS. Again, these hypotheses will not always be explicitly stated.

6.1. Statement of the result

The following theorem is the principal result of our paper:

Theorem 6.1.1. *If $\mathcal{P}$ is a 2-d or 3-d picture on a strongly normal DPS, then $C(\mathcal{P})$ has all seven of the continuous analog properties listed in Section 5.1.*

Properties (1) and (2) are easy consequences of the definition of $C(\mathcal{P})$. We proceed to establish the other five properties.

6.2. Well-defined homomorphisms

In this section we show that the inclusion of the black points and adjacencies of $\mathscr{P}$ in $C(\mathscr{P})$ and the inclusion of the white points and adjacencies of $\bar{\mathscr{P}}$ in $E^n - C(\mathscr{P})$ induce *well-defined* group homomorphisms $i_1: \pi(\mathscr{P}, p) \to \pi_1(C(\mathscr{P}), p)$ and $i_2: \pi(\bar{\mathscr{P}}, q) \to \pi_1(E^n - C(\mathscr{P}), q)$.

Recall that a closed curve $\gamma: [0, 1] \to X$ is *null-homotopic in X* if γ is freely homotopic in X to a trivial loop (i.e., to a constant map), or, equivalently, if γ is fixed base point homotopic in X to a trivial loop.

Proposition 6.2.1. *Let S be an open set in a topological space Z, and let $\gamma: [0, 1] \to S$ be a closed curve in S that is not null-homotopic in S. Let T be an open set in Z such that each component of $S \cap T$ is simply connected. Then γ is not null-homotopic in $S \cup T$.*

This proposition is a special case of Brown's fundamental groupoid version of the van Kampen Theorem [2]. However, we will give a detailed sketch of an elementary proof.

Proof (Sketch). Suppose γ is null-homotopic in $S \cup T$. Then there exists a fixed base point homotopy $h: [0, 1] \times [0, 1] \to S \cup T$ such that
  (1) $h(x, 0) = \gamma(x)$ for all x in $[0, 1]$, and
  (2) $h(x, 1) = \gamma(0) = \gamma(1)$ for all x in $[0, 1]$, and
  (3) $h(0, t) = h(1, t) = \gamma(0) = \gamma(1)$ for all t in $[0, 1]$.
The proposition is proved by showing that γ is null-homotopic in S, contrary to hypothesis.

Since h is continuous and S, T are open, $\{h^{-1}(S), h^{-1}(T)\}$ is an open cover of the compact set $[0, 1] \times [0, 1]$. So by Lebesgue's lemma there is a positive integer k such that every closed square in $[0, 1] \times [0, 1]$ with side length $1/k$ is contained either in the open set $h^{-1}(S)$ or in the open set $h^{-1}(T)$. Subdivide $[0, 1] \times [0, 1]$ into k^2 closed squares of side length $1/k$ in the obvious manner. Call a small square a *white square* if it is contained in $h^{-1}(S)$, and a *black square* otherwise. Then every black square is contained in $h^{-1}(T)$. Call an edge of a small square a *white edge* if it is not the common edge of two adjacent black squares. Call a corner of a small square a *white corner* if it is not the common corner of four black squares. Thus every white edge and white corner is contained in $h^{-1}(S)$.

Define an *S-edge loop* to be a closed curve $\lambda: [0, 1] \to h^{-1}(S)$ satisfying $\lambda(0) = \lambda(1) = (0, 0)$ for which there exist numbers $0 = x_0 < x_1 < \cdots < x_n = 1$ such that λ is linear on each interval $[x_i, x_{i+1}]$ and maps every such interval onto a white edge or onto a white corner. (Thus, for each i, $h(x_i)$ is a white corner.) An *S*-edge loop λ will be called *degenerate* if $\lambda([0, 1]) = (0, 0)$, *nondegenerate* otherwise. An *S*-edge loop λ will be said to *enclose* a small square X if the interior of X is contained in a bounded component of $E^2 - \lambda([0, 1])$.

For S-edge loops $\lambda_1, \lambda_2 : [0, 1] \to h^{-1}(S)$, say that λ_1 is *immediately equivalent* to λ_2 if there are a, b in $[0, 1]$ with $a \leq b$, such that $\lambda_1 = \lambda_2$ on $[0, a] \cup [b, 1]$, both $\lambda_1(a) = \lambda_2(a)$ and $\lambda_1(b) = \lambda_2(b)$ are white corners, and at least one of the following conditions holds:

(1) there is a white square whose boundary contains both $\lambda_1([a, b])$ and $\lambda_2([a, b])$, or

(2) there is a collection of black squares such that the boundary of the union of those squares contains both $\lambda_1([a, b])$ and $\lambda_2([a, b])$.

We claim that if λ_1 and λ_2 are immediately equivalent S-edge loops, then the closed curve $h\lambda_1$ is fixed base point homotopic in S to the closed curve $h\lambda_2$. Every white square is contained in $h^{-1}(S)$, so our claim is valid if condition (1) holds because λ_1 and λ_2 are then fixed base point homotopic in $h^{-1}(S)$. If condition (2) holds, then $\lambda_1([a, b])$ and $\lambda_2([a, b])$ are contained in $h^{-1}(S) \cap h^{-1}(T)$; so the restrictions to $[a, b]$ of $h\lambda_1$ and $h\lambda_2$ are curves in $S \cap T$ with the same endpoints, and these curves must be fixed endpoint homotopic because each component of $S \cap T$ is simply connected. Thus our claim is valid in this case also.

Suppose λ is a nondegenerate S-edge loop. Let *is equivalent to* be the transitive closure of "is immediately equivalent to". If λ encloses one or more small squares, then it is not hard to show that λ is equivalent to an S-edge loop λ' which encloses strictly fewer small squares: we leave the details to the reader. If λ encloses no small squares, then it is easy to see that λ is equivalent to an S-edge loop λ' whose image contains strictly fewer edges of small squares.

Hence if λ is any S-edge loop, then λ is equivalent to a degenerate S-edge loop, which implies that $h\lambda$ is null-homotopic in S.

But the closed curve γ is certainly fixed base point homotopic in S to $h\lambda$ where λ is an S-edge loop which winds around the boundary of $[0, 1] \times [0, 1]$ just once in an anti-clockwise direction. Hence γ is null-homotopic in S contrary to our hypothesis. $\square$

This result will now be used to prove two key lemmas.

Lemma 6.2.2. *Let $\mathscr{P} = (\mathbb{Z}^2, \beta, \omega, B)$ be a picture on a strongly normal 2-d DPS, and let λ be a $\mathscr{P}$-loop with base point p that is not null-homotopic in $E^2 - C'(\mathscr{P})$. Then λ is not equivalent to a trivial $\mathscr{P}$-loop.*

Proof. Let $S = E^2 - C'(\mathscr{P})$. Then λ lies in S and is not null-homotopic in S. We must show that λ is not null-homotopic in $E^2 - \bar{B}$. In view of Proposition 6.2.1, we can do this by finding an open set T such that $S \cup T \supseteq E^2 - \bar{B}$ and each component of $S \cap T$ is simply connected.

Let T be the open set $E^2 - (C(\mathscr{P}) \cup X)$, where X is the union of all semi-black 1-simplexes of $\mathscr{P}$. Since a semi-black 1-simplex meets $C'(\mathscr{P})$ only at its endpoint in $\bar{B}$, $S \cup T = E^2 - (C'(\mathscr{P}) \cap (C(\mathscr{P}) \cup X)) = E^2 - (C'(\mathscr{P}) \cap X) \supseteq E^2 - \bar{B}$.

Also, $S \cap T = E^2 - (C(\mathcal{P}) \cup C'(\mathcal{P}) \cup X)$ is the union of the interiors of all semi-black 2-simplexes of $\mathcal{P}$ by Lemma 5.4.1. So each component of $S \cap T$ is the interior of a 2-simplex and is therefore simply connected. $\square$

Next, we prove a 3-d version of this lemma. The basic idea of the proof is similar; however, the details are more involved.

Lemma 6.2.3. *Let* $\mathcal{P} = (\mathbb{Z}^3, \beta, \omega, B)$ *be a picture on a strongly normal 3-d DPS, and let* λ *be a* $\mathcal{P}$-*loop with base point p that is not null-homotopic in* $E^3 - C'(\mathcal{P})$. *Then* λ *is not equivalent to a trivial* $\mathcal{P}$-*loop.*

Proof. Let $S = E^3 - C'(\mathcal{P})$, and let $\bar{B}_1$ denote the union of all white points and white adjacencies of $\mathcal{P}$. Then λ lies in S and is not null-homotopic in S. We will show that λ is not null-homotopic in $E^3 - \bar{B}_1$. (This will clearly imply that λ is also not null-homotopic in the complement of the union of the white adjacencies alone, and hence not equivalent to a trivial loop.) In view of Proposition 6.2.1, it is enough to find an open set T such that $S \cup T \supseteq E^3 - \bar{B}_1$ and each component of $S \cap T$ is simply connected.

It follows from part (2) of Lemma 5.2.1 that if two vertices of a semi-black 2-simplex of $\mathcal{P}$ are white points of $\mathcal{P}$, then the edge joining those vertices is always a white adjacency. For, firstly, that edge is not the diameter of a unit lattice cube, since two diametrically opposite corners of an ordinary unit lattice cube cannot be the vertices of a 1-simplex of $\mathcal{P}$, and two diametrically opposite corners of a special unit lattice cube that are both white points cannot be the vertices of a 1-simplex of $\mathcal{P}$. Secondly, that edge cannot be a diagonal of a unit lattice square all of whose corners are white points: any unit lattice cube K with such a unit lattice square as a face is ordinary and the centroid of K is in the augmented white point set, so a diagonal of a face of K all four of whose corners are white points is not the edge of any semi-black 2-simplex of $\mathcal{P}$ in K.

Call a semi-black 2-simplex of $\mathcal{P}$ *unexceptional* if its intersection with $C'(\mathcal{P})$ is a white point of $\mathcal{P}$, or a white adjacency of $\mathcal{P}$, or half of a white adjacency of $\mathcal{P}$ that joins two diametrically opposite corners of a unit lattice cube; otherwise call it *exceptional*. It follows from the preceding paragraph that a semi-black 2-simplex of $\mathcal{P}$ is exceptional if and only if one of its vertices is the centroid of a unit lattice cube K in the augmented white point set, and each of its other two vertices either is a black point, or is diametrically opposite a black point in K, or is diametrically opposite a white point in K that is not adjacent to it. Let Y be the union of all unexceptional semi-black 2-simplexes of $\mathcal{P}$.

Let Z be the union of all semi-black 2-simplexes σ of $\mathcal{P}$ such that one vertex of σ is the centroid of a unit lattice cube in the augmented white point set, the other two vertices are black points, and two diametrically opposite corners of the unit lattice cube containing σ are white points adjacent to one another. (Thus Z is a union of exceptional 2-simplexes, and every point in $Z \cap C'(\mathcal{P})$ is the midpoint of a white adjacency of $\mathcal{P}$.)

Let T be the open set $E^3 - (C(\mathcal{P}) \cup Y \cup Z)$. Since every point in $C'(\mathcal{P}) \cap (Y \cup Z)$ is a white point or lies on a white adjacency of $\mathcal{P}$, $S \cup T = E^3 - (C'(\mathcal{P}) \cap (C(\mathcal{P}) \cup Y \cup Z)) = E^3 - (C'(\mathcal{P}) \cap (Y \cup Z)) \supseteq E^3 - \bar{B}_1$ as required.

It remains to show that each component of $S \cap T = E^3 - (C'(\mathcal{P}) \cup C(\mathcal{P}) \cup Y \cup Z)$ is simply connected. Suppose A is a unit lattice cube or a face of a unit lattice cube. Then by Lemma 5.4.1, $S \cap T \cap A = A - (C'(\mathcal{P}) \cup C(\mathcal{P}) \cup Y \cup Z)$ is the union of the relative interiors of all semi-black $\mathcal{P}$-simplexes that are contained in A but are not contained in $Y \cup Z$. Moreover, if A is a face of a unit lattice cube, or if A is a unit lattice cube whose centroid is *not* in the augmented white point set, then all semi-black 2- and 1-simplexes of $\mathcal{P}$ contained in A are in fact contained in Y. (Here the semi-black 1-simplexes are contained in Y because each of them is an edge of some unexceptional semi-black 2-simplex of $\mathcal{P}$.) Hence:

(1) If A is a face of a unit lattice cube, then $S \cap T \cap A = \emptyset$.

(2) If A is a unit lattice cube whose centroid is *not* in the augmented white point set, then $S \cap T \cap A$ is the union of the interiors of all semi-black 3-simplexes of $\mathcal{P}$ contained in A.

Now (1) shows that each component of $S \cap T$ is contained in the interior of some unit lattice cube, while (2) shows that every component contained in a unit lattice cube whose centroid is not in the augmented white point set is the interior of a 3-simplex and hence is simply connected.

Let U be the interior of a unit lattice cube K whose centroid c is contained in the augmented white point set. To complete the proof, it suffices to show that each component of $S \cap T \cap U$ is simply connected.

Since K's centroid c is in the augmented white point set, $C(\mathcal{P}) \cap U = \emptyset$. Let $W = (Y \cup Z) \cap U$ and let $D = C'(\mathcal{P}) \cap U$. Then $S \cap T \cap U = U - (C(\mathcal{P}) \cup C'(\mathcal{P}) \cup Y \cup Z) = U - (C'(\mathcal{P}) \cup Y \cup Z) = U - (D \cup W)$.

Let F be the boundary of K. Now $D \cup W$ is a union of sets $\sigma_1 - F, \sigma_2 - F, \ldots, \sigma_k - F$ where each σ_i is a 1-, 2- or 3-simplex with one vertex at the centroid c of K and all other vertices in F. Thus $U - (D \cup W)$ is the union of all straight line segments (without their endpoints) that join c to $F - \mathrm{cl}(D \cup W)$. If M is the set of midpoints of those straight line segments, then M is a strong deformation retract of $U - (D \cup W)$, and M is homeomorphic to the set of endpoints $F - \mathrm{cl}(D \cup W)$. Hence $U - (D \cup W)$ and $F - \mathrm{cl}(D \cup W)$ are homotopy equivalent.

Thus it is enough to show that each component of $F - \mathrm{cl}(D \cup W)$ is simply connected. But F is a polyhedral 2-sphere, and it is well known that in a polyhedral 2-sphere each component of the complement of a connected polyhedron is simply connected. Hence we need only show that $\mathrm{cl}(D \cup W) \cap F$ is connected. Since $\mathrm{cl}(D \cup W) \cap F$ is a union of 0-, 1- and 2-simplexes whose vertices are corners of K, to prove that this set is connected it is enough to show that all corners of K that are in the set belong to the same component of the set.

Suppose no two diametrically opposite corners of K are white points that are adjacent to one another. Then $Z \cap U = Y \cap U = \emptyset$, so $W = \emptyset$. Thus $\mathrm{cl}(D \cup W) \cap F = \mathrm{cl}(D) \cap F = C'(\mathcal{P}) \cap F$, and a corner of K is in this set if and only if it is a white

point. By Proposition 5.3.1 the set of white points in K is connected. So by continuous analog property (1), applied to $C(\bar{\mathscr{P}})$ (which is a subset of $C'(\mathscr{P})$), all white points in $C'(\mathscr{P}) \cap F$ belong to the same component of $C'(\mathscr{P}) \cap F$, as required.

Now suppose two diametrically opposite corners p and q of K are white points that are adjacent to one another. We will show that all corners of K are in $\mathrm{cl}(D \cup W) \cap F$ and belong to the same component of that set.

The set $\mathrm{cl}(W) \cap F$ contains each line segment joining one of p and q to a black 6-neighbor in K (because every such line segment is an edge of a semi-black 2-simplex of $\mathscr{P}$ the rest of which is contained in $Y \cap U$). The set $\mathrm{cl}(D) \cap F = C'(\mathscr{P}) \cap F$ certainly contains each straight line segment joining two 6-adjacent white points in K. It follows that p and its three 6-neighbors in K belong to one component of $\mathrm{cl}(D \cup W) \cap F$, and q and its three 6-neighbors in K belong to one component of $\mathrm{cl}(D \cup W) \cap F$.

The set $\mathrm{cl}(W) \cap F$ contains each line segment joining two 6-adjacent black points in K (because every such line segment is an edge of a semi-black 2-simplex of $\mathscr{P}$ the rest of which is contained in $Z \cap U$). So if among the corners of K other than p and q there are two 6-adjacent points x, y that are both black or both white, then we are done. For x and y then belong to the same component of $\mathrm{cl}(D \cup W) \cap F$, and moreover one must be a 6-neighbor of p, the other a 6-neighbor of q.

Now suppose such points x and y do not exist. Then we may assume w.l.o.g. that the three 6-neighbors of q in K are black and that the three 6-neighbors of p in K are white. Since p is ω-adjacent to q, it follows from strong normality condition (5*) that either p is β-adjacent to a 6-neighbor of q in K, a say, or q is ω-adjacent to a 6-neighbor of p in K, b say. In the first case the line segment joining a to p is a 1-simplex of $\mathscr{P}$, and so it is an edge of a 2-simplex of $\mathscr{P}$ the rest of which is in $Y \cap U$; thus that line segment lies in $\mathrm{cl}(W) \cap F$. In the second case the line segment joining q to b is a white adjacency and is contained in $C'(\mathscr{P}) \cap F = \mathrm{cl}(D) \cap F$. In either case it follows that p and its 6-neighbors belong to the same component of $\mathrm{cl}(D \cup W) \cap F$ as q and its 6-neighbors. $\quad\square$

Corollary 6.2.4. *Let $\mathscr{P} = (\mathbb{Z}^n, \beta, \omega, B)$ be a picture on a strongly normal DPS, and let λ be a $\mathscr{P}$-loop with base point p that is not null-homotopic in $C(\mathscr{P})$. Then λ is not equivalent to a trivial $\mathscr{P}$-loop.*

Proof. By Proposition 5.4.2, λ is not null-homotopic in $E^n - C'(\mathscr{P})$, so the result follows from Lemmas 6.2.2 and 6.2.3. $\quad\square$

We have now shown that both in two and in three dimensions, the inclusion of the black points and black adjacencies of $\mathscr{P}$ in $C(\mathscr{P})$ and the inclusion of the white points and white adjacencies of $\mathscr{P}$ in $E^n - C(\mathscr{P})$ induce well-defined group homomorphisms $i_1 : \pi(\mathscr{P}, p) \to \pi_1(C(\mathscr{P}), p)$ and $i_2 : \pi(\bar{\mathscr{P}}, q) \to \pi_1(E^n - C(\mathscr{P}), q)$. Here i_1 is well defined by Corollary 6.2.4; and i_2 is well defined by Lemmas 6.2.2 and 6.2.3, which we apply to $\bar{\mathscr{P}}$ instead of $\mathscr{P}$, noting that $C'(\bar{\mathscr{P}}) \supseteq C(\mathscr{P})$. It is plain that i_1 and i_2 are 1-1. But we still have to prove that they are onto.

6.3. T-adjacency, T-walks, T-loops and $\bar{T}$-walks

Let $\mathcal{P}$ be any picture on a strongly normal DPS. Two points p and q that are both in the augmented black point set of $\mathcal{P}$ or are both in the augmented white point set of $\mathcal{P}$ will be said to be *T-adjacent* (with respect to $\mathcal{P}$) if the straight line segment from p to q is a 1-simplex of $\mathcal{P}$.[12] Note that if p and q are T-adjacent black points or T-adjacent white points, then p and q are contained in some unit lattice square.

Let $\mathcal{P}$ be any n-dimensional ($n = 2$ or 3) picture on a strongly normal DPS. A *T-walk of $\mathcal{P}$* is a curve $\gamma : [0, 1] \to E^n$ such that $\gamma(0)$ and $\gamma(1)$ are black points of $\mathcal{P}$, and there exists a strictly increasing sequence $\langle x_i \mid 0 \le i \le k \rangle$ in $[0, 1]$ with $x_0 = 0$ and $x_k = 1$ such that for all nonnegative integers $i < k$:

(1) $\gamma(x_i)$ is in the augmented black point set of $\mathcal{P}$, and

(2) $\gamma(x_i)$ is equal or T-adjacent to $\gamma(x_{i+1})$, and

(3) γ is linear on the closed interval $[x_i, x_{i+1}]$.

A *$\bar{T}$-walk of $\mathcal{P}$* is defined in the same way except that $\gamma(0)$ and $\gamma(1)$ must be white points of $\mathcal{P}$ and the augmented *white* point set replaces the augmented black point set in condition (1). Thus every T-walk of $\bar{\mathcal{P}}$ is a $\bar{T}$-walk of $\mathcal{P}$, but the converse is false. A T-walk or $\bar{T}$-walk γ is said to be a T-walk or $\bar{T}$-walk *from $\gamma(0)$ to $\gamma(1)$*.

A *T-walk of $\mathcal{P}$* from a point p to itself is called a *T-loop of $\mathcal{P}$* and is said to be *based at p*; we also call p the *base point* of the T-loop. A T-loop is said to be *trivial* if it is a constant map onto its base point.

Proposition 6.3.1. *Let $\mathcal{P} = (\mathbb{Z}^n, \beta, \omega, B)$ be a picture on a strongly normal DPS. If γ is a T-walk of $\mathcal{P}$, then γ is fixed endpoint homotopic in $C(\mathcal{P})$ to a $\mathcal{P}$-walk. Similarly, if γ is a $\bar{T}$-walk of $\mathcal{P}$, then γ is fixed endpoint homotopic in $E^n - C(\mathcal{P})$ to a $\bar{\mathcal{P}}$-walk.*

Proof. We will say that a T-walk $\gamma : [0, 1] \to E^n$ is *elementary* if it satisfies one of the following conditions, and that it is *of type k* (where $k = 1, 2$ or 3) if it satisfies condition k:

(1) $\gamma([0, 1]) = \{p\}$ for some black point p of $\mathcal{P}$.

(2) $\gamma(0)$ and $\gamma(1)$ are T-adjacent black points of $\mathcal{P}$ and γ is linear.

(3) $\gamma(0)$ and $\gamma(1)$ are black points of $\mathcal{P}$ in a unit lattice cube K with centroid c, and there are $x, y \in (0, 1)$, $x \le y$, such that $\gamma([x, y]) = \{c\}$ and γ is linear on $[0, x]$ and on $[y, 1]$.

Define an *elementary $\bar{T}$-walk of type k* in the same way, but with "white" instead of "black". Then every T-walk ($\bar{T}$-walk) can be obtained by catenating a finite collection of elementary T-walks (elementary $\bar{T}$-walks). So it is enough to prove the proposition for elementary T-walks and $\bar{T}$-walks. Now we claim that:

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[12] T-adjacency is so called because the 1-simplexes of $\mathcal{P}$ are just the edges of members of $T_2(\mathcal{P})$ or $T_3(\mathcal{P})$.

(1) If the black (white) points p and q of $\mathcal{P}$ are T-adjacent but not adjacent, then p and q must be diagonally opposite corners of a unit lattice square S with four black (white) corners. In this case $S \subseteq C(\mathcal{P})$ $(S \subseteq E^n - C(\mathcal{P}))$.

(2) If the centroid of a unit lattice cube K is in the augmented black (augmented white) point set of $\mathcal{P}$, then the set $C(\mathcal{P}) \cap K$ (the set $K - C(\mathcal{P})$) is simply connected.

Claim (1) follows from part (2) of Lemma 5.2.1 and the definition of $C(\mathcal{P})$. As for claim (2), note that if the centroid c of a unit lattice cube K is in the augmented black point set, then for all $x \in C(\mathcal{P}) \cap K$ the closed straight line segment joining c to x is contained in $C(\mathcal{P}) \cap K$: so $C(\mathcal{P}) \cap K$ is a star body and is therefore simply connected. Similarly, if the centroid of K is in the augmented white point set, then $C'(\mathcal{P}) \cap K$ is a star body and is therefore simply connected, whence $K - C(\mathcal{P})$ is also simply connected by Proposition 5.4.2. So claim (2) is also correct.

The proposition is trivially valid for elementary T-walks and $\bar{T}$-walks of type 1. For elementary T-walks and $\bar{T}$-walks of type 2, it is an immediate consequence of claim (1).

Now suppose γ is an elementary T-walk (an elementary $\bar{T}$-walk) of type 3. Then $\gamma([0,1]) \subseteq C(\mathcal{P}) \cap K$ $(\gamma([0,1]) \subseteq K - C(\mathcal{P}))$ for some unit lattice cube K whose centroid is in the augmented black (white) point set. By Proposition 5.3.1 there is a $\mathcal{P}$-walk (a $\bar{\mathcal{P}}$-walk) Γ from $\gamma(0)$ to $\gamma(1)$ with $\Gamma([0,1]) \subseteq K$. By continuous analog property (1) (property (2)), $\Gamma([0,1]) \subseteq C(\mathcal{P}) \cap K$ $(\Gamma([0,1]) \subseteq K - C(\mathcal{P}))$, so it follows from claim (2) that γ is fixed endpoint homotopic to Γ. $\square$

6.4. Proof of the main theorem

We already know that $C(\mathcal{P})$ has properties (1) and (2). Also it is readily confirmed that each component of $C(\mathcal{P})$ or of $E^n - C(\mathcal{P})$ meets $\mathbb{Z}^n$.

Let $\gamma : [0,1] \to E^n - C(\mathcal{P})$ be a curve whose endpoints $\gamma(0)$ and $\gamma(1)$ are white points of $\mathcal{P}$. Here $n = 2$ or 3 according as $\mathcal{P}$ is 2-d or 3-d. We will show that γ is fixed endpoint homotopic in $E^n - C(\mathcal{P})$ to a $\bar{\mathcal{P}}$-walk. This will establish property (4). It will also show that the well-defined homomorphism $i_2 : \pi(\bar{\mathcal{P}}, q) \to \pi_1(E^n - C(\mathcal{P}), q)$ induced by the inclusion of the white points and adjacencies of $\mathcal{P}$ in $E^n - C(\mathcal{P})$ is onto. So property (7) will also be established—since i_2 is plainly 1-1. Our argument will be a variant of a special case of the usual proof of the Simplicial Approximation Theorem.

The *star* with respect to $\mathcal{P}$ of a point p in E^n, written $\mathrm{st}_{\mathcal{P}}(p)$, is the union of the relative interiors of all $\mathcal{P}$-simplexes that contain p. Thus $\mathrm{st}_{\mathcal{P}}(p)$ is an open neighborhood of p, $\mathrm{st}_{\mathcal{P}}(p) - \{p\}$ does not contain any 0-simplex of $\mathcal{P}$, and $\mathrm{st}_{\mathcal{P}}(x) \cap \mathrm{st}_{\mathcal{P}}(y)$ is nonempty if and only if there is a $\mathcal{P}$-simplex that contains both x and y.

Let x be any point in $E^n - C(\mathcal{P})$. Then $x \in \mathrm{st}_{\mathcal{P}}(p)$ for some point p in the augmented white point set $\overline{B'}(\mathcal{P})$. (For x lies in the relative interior of a $\mathcal{P}$-simplex σ, and $x \in \mathrm{st}_{\mathcal{P}}(v)$ for each vertex v of σ. Since $x \notin C(\mathcal{P})$, σ is not a black $\mathcal{P}$-simplex, so at least one vertex of σ is in $\overline{B'}(\mathcal{P})$.)

Thus $\{\gamma^{-1}(\mathrm{st}_{\mathscr{P}}(y)) \mid y \in \overline{B'}(\mathscr{P})\}$ is an open cover of the compact set $[0, 1]$. So by Lebesgue's lemma there is an integer $k \geq 2$ such that, for all positive integers $i < k$, $[(i-1)/k, (i+1)/k] \subseteq \gamma^{-1}(\mathrm{st}_{\mathscr{P}}(w_i))$ for some point $w_i \in \overline{B'}(\mathscr{P})$. Observe that, for all positive integers $i < k-1$, $\mathrm{st}_{\mathscr{P}}(w_i) \cap \mathrm{st}_{\mathscr{P}}(w_{i+1})$ is nonempty (it contains $\gamma([i/k, (i+1)/k])$): so either $w_i = w_{i+1}$ or w_i is T-adjacent to w_{i+1}.

Note that $\gamma(0) \in \mathrm{st}_{\mathscr{P}}(w_1)$, which implies $w_1 = \gamma(0)$ (since $\gamma(0)$ is a 0-simplex of $\mathscr{P}$). Similarly $w_{k-1} = \gamma(1)$. Now define $w_0 = \gamma(0)$ and $w_k = \gamma(1)$. Then by "joining the w_i" we can produce a $\bar{T}$-walk γ_1 from $\gamma(0)$ to $\gamma(1)$ such that, for all nonnegative integers $i < k$, $\gamma_1(i/k) = w_i$ and γ_1 is linear on $[i/k, (i+1)/k]$.

For all x in $[0, 1]$ there is a white or semi-black $\mathscr{P}$-simplex that contains both $\gamma(x)$ and $\gamma_1(x)$. (Indeed, suppose $x \in [i/k, (i+1)/k]$ where $i < k$ is a nonnegative integer. Then $\gamma(x) \in \mathrm{st}_{\mathscr{P}}(w_i) \cap \mathrm{st}_{\mathscr{P}}(w_{i+1})$, which implies that the $\mathscr{P}$-simplex whose relative interior contains $\gamma(x)$ contains both w_i and w_{i+1}. Since that $\mathscr{P}$-simplex contains both $w_i = \gamma_1(i/k)$ and $w_{i+1} = \gamma_1((i+1)/k)$, it contains $\gamma_1([i/k, (i+1)/k])$ and therefore contains $\gamma_1(x)$.) Consequently γ is fixed endpoint homotopic to γ_1 in $E^n - C(\mathscr{P})$ by a linear homotopy. By Proposition 6.3.1, γ_1 is fixed endpoint homotopic in $E^n - C(\mathscr{P})$ to a $\bar{\mathscr{P}}$-walk. This implies that γ is fixed endpoint homotopic in $E^n - C(\mathscr{P})$ to a $\bar{\mathscr{P}}$-walk. So properties (4) and (7) are established.

A very similar compactness argument, or an appeal to the Simplicial Approximation Theorem, shows that if $\gamma : [0, 1] \to C(\mathscr{P})$ is any curve whose endpoints $\gamma(0)$ and $\gamma(1)$ are black points of $\mathscr{P}$, then γ is fixed endpoint homotopic in $C(\mathscr{P})$ to a T-walk of $\mathscr{P}$. So by Proposition 6.3.1, γ is fixed endpoint homotopic in $C(\mathscr{P})$ to a $\mathscr{P}$-walk. This establishes property (3). It also shows that the well-defined homomorphism $i_1 : \pi(\mathscr{P}, p) \to \pi_1(C(\mathscr{P}), p)$ induced by the inclusion of the black points and adjacencies of $\mathscr{P}$ in $C(\mathscr{P})$ is onto. Since i_1 is plainly 1-1, property (6) is established.

It remains only to establish property (5). For this purpose let X be a component of $C(\mathscr{P})$ where $\mathscr{P} = (\mathbb{Z}^n, \beta, \omega, B)$, and let Y be a component of $E^n - C(\mathscr{P})$. By properties (3) and (4), the sets $X \cap B = X \cap \mathbb{Z}^n$ and $Y \cap \bar{B} = Y \cap \mathbb{Z}^n$ are respectively a black and a white component.

Suppose a point p in $X \cap \mathbb{Z}^n$ is adjacent to a point q in $Y \cap \mathbb{Z}^n$. If there is a unit lattice square containing both p and q—there must be such a square in the 2-d case—then p is a common point of the boundaries of X and Y, for the straight line segment joining p to q meets $C(\mathscr{P})$ only at p (if p and q are diagonally opposite corners of the square, then since $\mathscr{P}$ is regular the other diagonal is not a black adjacency and is not contained in $C(\mathscr{P})$, whence $C(\mathscr{P})$ does not meet the interior of the square). Now consider the case where no unit lattice square contains both p and q. Then p and q are diametrically opposite corners of a unit lattice cube K. If K is special, then, since p is adjacent to q, p is a corner of the equilateral triangle in K whose edges are black adjacencies, and q is a corner of the equilateral triangle in K whose edges are white adjacencies: so the straight line segment joining p to q meets $C(\mathscr{P})$ only at p, and p is a common point of the boundaries of X and Y. Now suppose K is ordinary and c is its centroid. If c is in the augmented white

point set, then the straight line segment joining p to q meets $C(\mathcal{P})$ only at p, so p is a common point of the boundaries of X and Y. If c is in the augmented black point set, then the straight line segment joining p to c is contained in $C(\mathcal{P})$ and the straight line segment joining c to q meets $C(\mathcal{P})$ only at c: so c is a common point of the boundaries of X and Y.

Conversely, suppose the boundaries of X and Y meet. Then in the 2-d case there is a unit lattice square that meets both X and Y. We claim such a unit lattice square exists in the 3-d case also. For suppose otherwise. Since the boundaries of X and Y meet there is a unit lattice cube K such that the boundaries of $X \cap K$ and $Y \cap K$ meet. Each of X and Y must contain a corner of K. So (since no face of K meets both X and Y) $X \cap K \cap \mathbb{Z}^3 = \{x\}$ and $Y \cap K \cap \mathbb{Z}^3 = \{y\}$, where x and y are diametrically opposite corners of K. Since the 6-neighbors of x in K are not in the black component $X \cap \mathbb{Z}^3$, those 6-neighbors are all white points. None of those 6-neighbors is in the white component $Y \cap \mathbb{Z}^3$, so none of them is adjacent to y. Similarly the 6-neighbors of y in K are all black points, and none of them is adjacent to x. Now strong normality condition (4′) implies K is special and it follows from the definition of $C(\mathcal{P})$ that the boundaries of $X \cap K$ and $Y \cap K$ do not meet—a contradiction.

Thus there is a unit lattice square which meets both X and Y, and which therefore meets both the black component $X \cap \mathbb{Z}^n$ and the white component $Y \cap \mathbb{Z}^n$. Since each component of black points in a unit lattice square is 4- or 6-adjacent to each component of white points in that square, it follows that $X \cap \mathbb{Z}^n$ is adjacent to $Y \cap \mathbb{Z}^n$. □

7. Topological properties of strongly normal digital picture spaces

7.1. Introductory remarks

In Section 7 we show that a number of important topological results about Euclidean space have analogs for any DPS that is isomorphic to a strongly normal DPS. Many of these results are generalizations of known results about the DPS's $(\mathbb{Z}^2, 8, 4)$, $(\mathbb{Z}^2, 4, 8)$, $(\mathbb{Z}^3, 26, 6)$ and $(\mathbb{Z}^3, 6, 26)$ which are given, for example, in [24]. We will also prove that for pictures on a strongly normal DPS the discrete digital fundamental group is naturally isomorphic to the digital fundamental group.

7.2. A digital Jordan curve theorem

Our first result is a digital analog of the celebrated Jordan Curve Theorem.

Proposition 7.2.1. *Let* $\mathcal{P} = (\mathbb{Z}^2, \beta, \omega, B)$ *be a picture on a strongly normal DPS, where B is a black simple closed curve of $\mathcal{P}$ that is not contained in any unit lattice square. Then $\mathcal{P}$ has just two white components, and each point in B is adjacent to both of them.*

Proof. Let p be any point in B and define $\mathscr{P}' = (\mathbb{Z}^2, \beta, \omega, B-\{p\})$. Since B is a black simple closed curve of $\mathscr{P}$, B is β-connected and each point in B is β-adjacent to just two others. Hence no four points in B are the four corners of a unit lattice square. (For otherwise none of the points in question is β-adjacent to any other point in B and so, since B is β-connected, B contains no other point and is therefore contained in a unit lattice square, contrary to the hypothesis.) Similarly, since three pairwise β-adjacent points are pairwise 8-adjacent and are therefore contained in a unit lattice square, no three points in B are pairwise β-adjacent.

For both $\mathscr{P}$ and $\mathscr{P}'$, it follows from part (2) of Lemma 5.2.1 that two black points are T-adjacent if and only if they are adjacent (or, equivalently, β-adjacent). So in $\mathscr{P}$ and $\mathscr{P}'$ there do not exist three black points that are pairwise T-adjacent. Hence $C(\mathscr{P})$ and $C(\mathscr{P}')$ contain no 2-simplexes of $\mathscr{P}$ or $\mathscr{P}'$, only 1-simplexes. It follows that $C(\mathscr{P})$ is the union of all straight line segments joining T-adjacent (or, equivalently, adjacent) black points of $\mathscr{P}$; and similarly $C(\mathscr{P}')$ is the union of all black adjacencies of $\mathscr{P}'$. Recalling that in $\mathscr{P}$ each black point is adjacent to just two others and the black point set is connected, it is readily confirmed that $C(\mathscr{P})$ is a Jordan curve; and $C(\mathscr{P}') \subset C(\mathscr{P}) - \{p\}$. So by the (classical) Jordan curve theorem $E^2 - C(\mathscr{P})$ has just two components and $E^2 - C(\mathscr{P}')$ is connected. Hence, by continuous analog property (4), $\mathscr{P}$ has just two white components, and $\mathscr{P}'$ has just one white component. Since $\mathscr{P}'$ has just one white component, $(\mathbb{Z}^2 - B) \cup \{p\}$ is ω-connected, whence p is adjacent in $\mathscr{P}$ to both white components of $\mathscr{P}$. $\quad\square$

7.3. The adjacency graph. Weak normality

For any picture $\mathscr{P}$ let $\mathrm{adj}(\mathscr{P})$ denote the adjacency graph of the black and white components of $\mathscr{P}$. Each vertex of $\mathrm{adj}(\mathscr{P})$ represents a different black or white component of $\mathscr{P}$, and two vertices are adjacent whenever they represent adjacent black and white components.

If Z is a black or white component of $\mathscr{P}$, then let $\mathrm{adj}(\mathscr{P}) - Z$ denote the (usually disconnected) graph obtained by removing the vertex that corresponds to Z from $\mathrm{adj}(\mathscr{P})$. For any black or white component Z of $\mathscr{P}$, the union of the black and white components of $\mathscr{P}$ that are represented by the vertices in any single component of the graph $\mathrm{adj}(\mathscr{P}) - Z$ is a component of $\bar{Z}$. This is a natural 1-1 correspondence between the components of $\bar{Z}$ and the components of the graph $\mathrm{adj}(\mathscr{P}) - Z$.

Note that if $\mathscr{P}$ is a picture on a strongly normal DPS, then $\mathrm{adj}(\mathscr{P})$ is isomorphic to $\mathrm{adj}(\bar{\mathscr{P}})$. For by condition (5) in the definition of strong normality, adjacent black and white components of $\mathscr{P}$ are adjacent white and black components of $\bar{\mathscr{P}}$.

A fundamental property of strongly normal DPS's is that the adjacency graph of any picture on such a DPS is a tree. In fact we prove a more general version of this result, which asserts that the adjacency graph is a tree under much weaker hypotheses. We will use the more general result to prove the proposition in the next section about the connectedness of borders.

Call a DPS (V, β, ω) *weakly normal* if there is a strongly normal DPS (V, β_0, ω_0) such that $\beta_0 \subseteq \beta \subseteq \beta_0 \cup \omega_0$ and $\omega_0 \subseteq \omega \subseteq \beta_0 \cup \omega_0$. Every strongly normal DPS is weakly normal. Two other examples of weakly normal DPS's are $(\mathbb{Z}^2, 8, 8)$ and $(\mathbb{Z}^3, 26, 18)$. The DPS's $(\mathbb{Z}^2, 4, 4)$ and $(\mathbb{Z}^3, 6, 6)$ are not weakly normal.

Observe that a weakly normal DPS satisfies conditions (1)-(5) in the definition of a strongly normal DPS (and hence also conditions (4'), (4*), (5') and (5*)). However, a weakly normal DPS need not be regular.

Proposition 7.3.1. *Let $\mathcal{P} = (V, \beta, \omega, B)$ where $\mathcal{S} = (V, \beta, \omega)$ is isomorphic to a weakly normal DPS. Then* $\mathrm{adj}(\mathcal{P})$ *is a tree.*

Proof. We first assume that $\mathcal{S}$ is strongly normal. Suppose, for the purpose of getting a contradiction, that $\mathrm{adj}(\mathcal{P})$ contains a cycle. Pick a black component C of $\mathcal{P}$ in that cycle, and let Y and Z be the white components of $\mathcal{P}$ that are adjacent to C in the cycle. Let C_0 be the component of $C(\mathcal{P})$ that contains C. Since Y and Z are different white components contained in the same component of $V - C$ and in the same component of $V - (B - C)$, by continuous analog property (4), Y and Z are in different components of $E^n - C(\mathcal{P})$, but by properties (1)-(5) they are in the same component of $E^n - C_0$ and in the same component of $E^n - (C(\mathcal{P}) - C_0)$. This is impossible[13] since C_0 and $C(\mathcal{P}) - C_0$ are disjoint closed sets whose union is $C(\mathcal{P})$, and the contradiction proves the result in the case when $\mathcal{S}$ is strongly normal.

To prove the general result, assume without loss of generality that $\mathcal{S}$ is weakly normal. Let β_0 and ω_0 be sets of adjacencies such that $\beta_0 \subseteq \beta \subseteq \beta_0 \cup \omega_0, \omega_0 \subseteq \omega \subseteq \beta_0 \cup \omega_0$ and (V, β_0, ω_0) is strongly normal. Then $\mathrm{adj}((V, \beta_0, \omega_0, B))$ is a tree. We assume $\beta - \beta_0$ and $\omega - \omega_0$ are finite: it is not hard to show that the truth of the result in this special case implies the truth of the result in general.

Let $\mathcal{P}_1 = (V, \beta_1, \omega_1, B)$ where $\beta_0 \subseteq \beta_1 \subseteq \beta_0 \cup \omega_0$, and $\omega_0 \subseteq \omega_1 \subseteq \beta_0 \cup \omega_0$. Let e be any member of $\beta_0 \cup \omega_0$, let $\mathcal{P}_1' = (V, \beta_1, \omega_1 \cup \{e\}, B)$ and let $\mathcal{P}_1'' = (V, \beta_1 \cup \{e\}, \omega_1, B)$. Suppose $\mathrm{adj}(\mathcal{P}_1)$ is a tree. We will show this implies $\mathrm{adj}(\mathcal{P}_1')$ and $\mathrm{adj}(\mathcal{P}_1'')$ are trees too. This will prove the proposition (since β_0 can be converted to β, and ω_0 to ω, in a finite number of steps where each step adds a single adjacency in $\beta_0 \cup \omega_0$).

Suppose e joins two white points of $\mathcal{P}_1$. Then $\mathrm{adj}(\mathcal{P}_1'') = \mathrm{adj}(\mathcal{P}_1)$ is a tree. If the endpoints of e are in the same white component of $\mathcal{P}_1$, then $\mathrm{adj}(\mathcal{P}_1') = \mathrm{adj}(\mathcal{P}_1)$ is a tree. So we may suppose the endpoints of e are in different white components X and Y of $\mathcal{P}_1$. Then X is 26-adjacent (3-d case) or 8-adjacent (2-d case) to Y, so each of X and Y is 6- or 4-adjacent, and hence ω_1-adjacent, to some black component Z of $\mathcal{P}_1$. (For there is a 6- or 4-path from X to Y containing at most two points other than its endpoints in X and Y; that path passes through a black component of $\mathcal{P}_1$ that is 6- or 4-adjacent to each of the white components X and Y.) Thus $\mathrm{adj}(\mathcal{P}_1')$ is obtained from the tree $\mathrm{adj}(\mathcal{P}_1)$ by identifying two neighbors of the vertex that represents Z. This shows that $\mathrm{adj}(\mathcal{P}_1')$ is a tree. We have now shown that if e

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[13] We are again appealing to the Phragmen-Brouwer property of E^n stated in an earlier footnote.

joins two white points of $\mathscr{P}_1$, then $\mathrm{adj}(\mathscr{P}_1')$ and $\mathrm{adj}(\mathscr{P}_1'')$ are trees. By a symmetrical argument the same is true if e joins two black points of $\mathscr{P}_1$.

Now suppose e joins a point in a black component X of $\mathscr{P}_1$ to a point in a white component Y of $\mathscr{P}_1$. Then $\mathrm{adj}(\mathscr{P}_1'') = \mathrm{adj}(\mathscr{P}_1)$ is a tree. Also, X and Y are β_0-adjacent or ω_0-adjacent (because $e \in \beta_0 \cup \omega_0$), so they are β_1-adjacent or ω_1-adjacent, and hence 6- or 4-adjacent (by normality condition (5) and the fact that (V, β_1, ω_1) is weakly normal). Thus X is adjacent to Y in $\mathscr{P}_1$ and therefore $\mathrm{adj}(\mathscr{P}_1') = \mathrm{adj}(\mathscr{P}_1)$ is a tree. $\square$

7.4. Connectedness of borders

In this section we use Proposition 7.3.1 to prove a fundamental result about borders:

Proposition 7.4.1. *Let $\mathscr{P} = (V, \beta, \omega, B)$, where (V, β, ω) is isomorphic to a weakly normal DPS. Then the border of a black component X of $\mathscr{P}$ with respect to an adjacent white component Y of $\mathscr{P}$ is connected.*

Proof. Let F be the border of X with respect to Y. Let β_0 be the set of all β-adjacencies joining pairs of points in X. Let $\mathscr{P}_1$ be the picture $(V, \beta, \omega \cup \beta_0, F \cup (B - X))$. Then the white points of $\mathscr{P}_1$ are $\bar{B} \cup (X - F)$. Notice that Y is a white component of $\mathscr{P}_1$, since Y is a white component of $\mathscr{P}$ and by definition of F, Y is not ω-adjacent to $X - F$.

Now suppose F is not β-connected. Then since X is β-connected there must be a β-path in X that joins two distinct β-components F_1 and F_2 of F, and which meets F only at its endpoints. Let E be the white component of $\mathscr{P}_1$ that this path passes through. Then E, F_1, Y and F_2 form a cycle in $\mathrm{adj}(\mathscr{P}_1)$, contrary to Proposition 7.3.1. This contradiction proves that F is β-connected. $\square$

It is now clear why we had to prove Proposition 7.3.1 for *weakly* normal DPS's. The above proof depended on the fact that if (V, β, ω) is weakly normal and $\beta_0 \subseteq \beta$, then $(V, \beta, \omega \cup \beta_0)$ is also weakly normal. However, it is not true that if (V, β, ω) is strongly normal and $\beta_0 \subseteq \beta$, then $(V, \beta, \omega \cup \beta_0)$ is strongly normal.

7.5. Topological independence of distinct components I

In this section and the next we give two results which show that distinct black components of a picture on a strongly normal DPS "do not interfere with each other's digital topology". Our result in this section shows that the border of a black component is not affected by other black components. This result is true even on weakly normal DPS's.

Proposition 7.5.1. *Let* $\mathcal{P} = (V, \beta, \omega, B)$, *where* (V, β, ω) *is isomorphic to a weakly normal DPS. Let C be a black component of* $\mathcal{P}$, *let* $\mathcal{P}'$ *be the picture* (V, β, ω, C) *and let Y be any white component of* $\mathcal{P}'$. *Then Y contains just one white component of* $\mathcal{P}$ *that is adjacent to C, and the border of C in* $\mathcal{P}$ *with respect to that white component of* $\mathcal{P}$ *is the same as the border of C with respect to Y in* $\mathcal{P}'$.

Proof. Assume w.l.o.g. that (V, β, ω) is weakly normal. Note that Y is a union of white components of $\mathcal{P}$ with a (possibly empty) set of black points of $\mathcal{P}$. Also, Y contains a point x that is 4- or 6-adjacent to C (in any 4- or 6-path in V that joins a point in C to a point in Y, take the first point that is in Y). Since C is a black component of $\mathcal{P}$, x is a white point of $\mathcal{P}$. So Y contains a white component of $\mathcal{P}$ that is adjacent to C (namely the white component of $\mathcal{P}$ containing x). We claim that Y contains no more than one white component of $\mathcal{P}$ that is adjacent to C.

Let β_1 be the union of β with the set of all ω-adjacencies both of whose endpoints are in B. Then (V, β_1, ω) is isomorphic to a weakly normal DPS. Let $\mathcal{P}_1$ be the picture (V, β_1, ω, B). Then each white component of $\mathcal{P}'$ is a connected subset of $\bar{C}$ in $\mathcal{P}_1$. By Proposition 7.3.1, $\mathrm{adj}(\mathcal{P}_1)$ is a tree. So, by the natural 1–1 correspondence between the components of $\bar{C}$ in $\mathcal{P}_1$ and the components of the graph $\mathrm{adj}(\mathcal{P}_1) - C$, no connected subset of $\bar{C}$ in $\mathcal{P}_1$ meets more than one white component of $\mathcal{P}_1$ that is adjacent to C. Thus no white component of $\mathcal{P}'$ contains more than one white component of $\mathcal{P}_1$ that is adjacent to C. Since the white components of $\mathcal{P}_1$ are precisely the same as the white components of $\mathcal{P}$, our claim is justified.

Now let Y_0 be the unique white component of $\mathcal{P}$ that is contained in Y and adjacent to C. Let p be any point in C. To complete the proof, we need to show that if p is ω-adjacent to Y, then p is ω-adjacent to Y_0. But C is not ω-adjacent to any white component of $\mathcal{P}$ that meets Y other than Y_0. Thus it is enough to show that if p is ω-adjacent to Y, then p is ω-adjacent to a white point of $\mathcal{P}$ in Y. To prove this, suppose p is ω-adjacent to some black point q of $\mathcal{P}$ in Y. Then there must be a unit lattice square or cube that contains both p and q.

Suppose there is a unit lattice square K that contains both p and q. Then since p and q are in different black components of $\mathcal{P}$, they are diagonally opposite corners of K. Moreover, the two other corners of K must be white points of $\mathcal{P}$, and as they are 4- or 6-adjacent to q (which is in Y) they are both in Y. Since p is 4- or 6-adjacent to those two corners, p is indeed ω-adjacent to a white point of $\mathcal{P}$ in Y.

Now suppose p and q are diametrically opposite corners of a unit lattice cube K. If a 6-neighbor r of q in K is also a black point of $\mathcal{P}$, then the common 6-neighbors of r and p in K must be white points of $\mathcal{P}$ in Y (for $q \in Y$ which implies $q \notin C$ which implies $r \notin C$ and $r \in Y$)—so p is ω-adjacent to a white point of $\mathcal{P}$ in Y, as required. Hence we may assume that all three 6-neighbors of q in K are white points of $\mathcal{P}$ and are therefore in Y. If any 6-neighbor of p in K is a white point of $\mathcal{P}$, then, since that point is 6-adjacent to a 6-neighbor of q in K, it is in Y, which again shows that p is ω-adjacent to a white point of $\mathcal{P}$ in Y. So we may assume all three 6-neighbors of p in K are black points of $\mathcal{P}$ and are therefore in C. Since q

is a black point of $\mathcal{P}$ and $q \notin C$, q is not β-adjacent to any 6-neighbor of p in K. But p is ω-adjacent to q, so, by normality condition (5*), p is ω-adjacent to one of the 6-neighbors of q in K, all of which are white points of $\mathcal{P}$ in Y. $\square$

Corollary 7.5.2. *Let $\mathcal{P}$, $\mathcal{P}'$ and C be defined as in the above proposition. Then if $\mathcal{P}$ is 2-d, each hole of C in $\mathcal{P}'$ contains just one hole of C in $\mathcal{P}$. If $\mathcal{P}$ is 3-d, then each cavity of C in $\mathcal{P}'$ contains just one cavity of C in $\mathcal{P}$.*

Proof. For if C surrounds a white component Y of $\mathcal{P}'$, then *a fortiori* C surrounds the unique white component of $\mathcal{P}$ that is contained in Y and adjacent to C. $\square$

7.6. Topological independence of distinct components II

Our result in this section shows that in a picture on a strongly normal DPS the digital fundamental groups associated with each black component are unaffected by other black components.

Proposition 7.6.1. *Let $\mathcal{P} = (V, \beta, \omega, B)$, where (V, β, ω) is isomorphic to a strongly normal DPS. Let D be a black component of $\mathcal{P}$ and let $\mathcal{P}'$ be the picture (V, β, ω, D). Then $C(\mathcal{P}')$ is the component of $C(\mathcal{P})$ that contains D, and $\pi(\mathcal{P}, p) \equiv \pi(\mathcal{P}', p)$ for all points p in D.*

Proof. The second assertion follows from the first assertion and Theorem 6.1.1. We prove the first assertion.

Claim 1. *In the 3-d case, if a unit lattice cube K contains a point in D, then the centroid of K is in the augmented black point set of $\mathcal{P}$ if and only if it is in the augmented black point set of $\mathcal{P}'$; if K contains more than one point in D, then K is special with respect to $\mathcal{P}$ if and only if K is special with respect to $\mathcal{P}'$.*

We begin by justifying the second assertion of Claim 1. First observe that if K is special with respect to $\mathcal{P}'$, then K is special with respect to $\mathcal{P}$. For in this case no corner of the equilateral triangle in K whose edges are white adjacencies of $\mathcal{P}'$ can be in B, since each of those corners is in $\bar{D}$ but is 6-adjacent to D, and D is a black component of $\mathcal{P}$. Now suppose K is special with respect to $\mathcal{P}$. Then, since D is a black component of $\mathcal{P}$, either D contains all three corners of the equilateral triangle in K whose edges are black adjacencies of $\mathcal{P}$ (in which case K is special with respect to $\mathcal{P}'$), or D contains no corner of K except perhaps the corner which is not 6-adjacent to any corner of that triangle. So if K contains more than one point in D, then K is special with respect to $\mathcal{P}'$. Thus we have justified the second assertion.

To justify the first assertion of Claim 1, suppose first that the centroid of K is in the augmented black point set of $\mathcal{P}$. Then by Proposition 5.3.1 the set $B \cap K$ is connected so, since D is a black component of $\mathcal{P}$, either $D \cap K = \emptyset$ or $D \cap K = B \cap K$. Therefore if K contains a point in D, then the centroid of K is also in the augmented black point set of $\mathcal{P}'$.

Next, suppose the centroid of K is not the augmented black point set of $\mathcal{P}$. If K is special with respect to $\mathcal{P}$, then by the second assertion either K is special with respect to $\mathcal{P}'$ or K contains at most one point in D; in neither case is the centroid of K in the augmented black point set of $\mathcal{P}'$. Now suppose K is not special with respect to $\mathcal{P}$. Then, since K's centroid is not in the augmented black point set of $\mathcal{P}$, no diameter of K is a black adjacency of $\mathcal{P}$, and either one of the diameters of K is a white adjacency of $\mathcal{P}$ or there is no black simple closed curve of $\mathcal{P}$ in K that is not contained in one face of K. *A fortiori* the same is true with $\mathcal{P}'$ in place of $\mathcal{P}$, so the centroid of K is not in the augmented black point set of $\mathcal{P}'$. Thus we have justified Claim 1.

Claim 2. *Let p and q be points in the augmented black point set of $\mathcal{P}'$. Then p and q are T-adjacent with respect to $\mathcal{P}'$ if and only if p and q are T-adjacent with respect to $\mathcal{P}$.*

To justify Claim 2, we may assume that p and q are T-adjacent with respect to $\mathcal{P}'$ or with respect to $\mathcal{P}$. Then by part (2) of Lemma 5.2.1 at least one of the following is true:

(1) (3-d case only) one of p and q is the centroid of a unit lattice cube K, or

(2) p and q are diagonally opposite corners of a unit lattice square all four of whose corners are in B, or

(3) p and q are β-adjacent in some unit lattice square.

If (1) is true, or if (1) and (2) are both false (so (3) is true), then p and q are T-adjacent with respect to $\mathcal{P}'$ and with respect to $\mathcal{P}$, so Claim 2 holds. (If (1) is true, then the centroid of K is in the augmented black point set of $\mathcal{P}'$, so by Claim 1 it is also in the augmented black point set of $\mathcal{P}$.) If (2) is true, then, since D is a black component of $\mathcal{P}$, (2) remains true when B is replaced by D: so for p to be T-adjacent to q in one of $\mathcal{P}$ and $\mathcal{P}'$ the sum of the coordinates of p must be even, which makes p T-adjacent to q in both of $\mathcal{P}$ and $\mathcal{P}'$. Thus Claim 2 holds in this case also. Hence Claim 2 is true in all cases.

Define a $C(\mathcal{P})$-*simplex* to be a black $\mathcal{P}$-simplex that has at least one vertex in V, and a $C(\mathcal{P}')$-*simplex* to be a black $\mathcal{P}'$-simplex that has at least one vertex in V. Then $C(\mathcal{P})$ is the union of all $C(\mathcal{P})$-simplexes and $C(\mathcal{P}')$ is the union of all $C(\mathcal{P}')$-simplexes.

It is readily confirmed that a simplex σ is a $C(\mathcal{P})$-simplex if and only if the following three conditions hold, and is a $C(\mathcal{P}')$-simplex if and only if the three conditions hold when $\mathcal{P}$ is replaced by $\mathcal{P}'$:

(1) Every vertex of σ is in the augmented black point set of $\mathcal{P}$, and at least one vertex is in V.

(2) Every two vertices of σ are T-adjacent with respect to $\mathcal{P}$.

(3) In the 3-d case *either* σ is contained in a unit lattice square, *or* one vertex of σ is the centroid of a unit lattice cube and all other vertices lie on one face of that cube, *or* σ is a 2- or 3-simplex in a unit lattice cube that is special with respect to $\mathcal{P}$.

It follows from this characterization, and Claims 1 and 2, that a simplex σ with a vertex in D and no vertex in $B - D$ is a $C(\mathcal{P}')$-simplex if and only if it is a $C(\mathcal{P})$-simplex. So since every $C(\mathcal{P}')$-simplex has a vertex in V and hence in D but has no vertex in $B - D$, and since no $C(\mathcal{P})$-simplex has vertices in both of D and $B - D$ (e.g., by continuous analog property (3) and the fact that D is a black component of $\mathcal{P}$), the $C(\mathcal{P}')$-simplexes are precisely the $C(\mathcal{P})$-simplexes that have a vertex in D.

Let X be the component of $C(\mathcal{P})$ that contains D. Then, by continuous analog property (3), $X \cap V = D$. Since every $C(\mathcal{P})$-simplex is contained in $C(\mathcal{P})$, every $C(\mathcal{P})$-simplex which meets X is contained in X. So, since every point in X is in $C(\mathcal{P})$ and therefore lies in a $C(\mathcal{P})$-simplex, X is the union of all $C(\mathcal{P})$-simplexes that meet X.

Since a $C(\mathcal{P})$-simplex must have a vertex in V and hence in B, if a $C(\mathcal{P})$-simplex has no vertex in D, then it has a vertex in $B - D$ and cannot be contained in X (because $X \cap V = D$). Thus a $C(\mathcal{P})$-simplex meets X, or, equivalently, is contained in X, if and only if it has a vertex in D. So X is the union of all $C(\mathcal{P})$-simplexes that have a vertex in D. Hence X is the union of all $C(\mathcal{P}')$-simplexes. Therefore $X = C(\mathcal{P}')$. $\square$

7.7. The Euler characteristic. Tunnels

As usual, we write $\chi(S)$ for the Euler characteristic of a finite polyhedron S. Recall that if S is any finite polyhedron, then $\chi(S) = \sum_n (-1)^n h_n(S)$ where h_n denotes the nth Betti number and, for any triangulation K of S, $\chi(S) = \sum_n (-1)^n c_n$ where c_n is the number of n-simplexes in K.

Now suppose S is a finite polyhedron in 3-space. Then $h_0(S)$ is the number of components of S, and by Alexander duality $h_2(S)$ is the number of cavities in S (i.e., the number of bounded components of $E^3 - S$). Note that $h_1(S) = 0$ if and only if S is simply connected [17]. Also, if the polyhedron S' is obtained from S by attaching n "solid handles" to S or by removing the interior of an n-holed solid polyhedral torus from the interior of S, then $h_1(S') = h_1(S) + n$. For these reasons we call the topological invariant $h_1(S)$ the *number of tunnels* in S. Then $\chi(S) =$ no. of components of $S +$ no. of cavities in $S -$ no. of tunnels in S.

We now define an analogous quantity for digital pictures that is invariant under DPS isomorphisms:

Definition 7.7.1. Let $\mathcal{P}$ be a picture with finitely many black points. Suppose further that $\mathcal{P}$ is isomorphic to a picture $\mathcal{P}^*$ on a strongly normal DPS. Then the *Euler characteristic* of $\mathcal{P}$, denoted by $\chi(\mathcal{P})$, is $\chi(C(\mathcal{P}^*))$.

For any given $\mathcal{P}$ there may well be more than one picture $\mathcal{P}^*$ that satisfies the condition in this definition. However, $\chi(\mathcal{P})$ is well defined because the value of $\chi(C(\mathcal{P}^*))$ is the same for any valid choice of $\mathcal{P}^*$—as we now show.

In the 2-d case $\chi(\mathscr{P}) = \chi(C(\mathscr{P}^*)) = $ (no. of components of $C(\mathscr{P}^*)$) − (no. of holes in $C(\mathscr{P}^*)$). So by Theorem 6.1.1, $\chi(\mathscr{P}) = $ (no. of black components of $\mathscr{P}^*$) − (no. of holes in $\mathscr{P}^*$) = (no. of black components of $\mathscr{P}$) − (no. of holes in $\mathscr{P}$).

For any 3-d picture $\mathscr{P}$ we define $h_0(\mathscr{P})$, $h_1(\mathscr{P})$ and $h_2(\mathscr{P})$ as follows. Let $B_1, B_2, \ldots, B_n$ be the black components of $\mathscr{P}$. For $1 \le i \le n$ let p_i be a point in B_i. Then we define $h_0(\mathscr{P}) = n$, $h_2(\mathscr{P}) = $ (no. of cavities in $\mathscr{P}$), and $h_1(\mathscr{P}) = $ (sum of the ranks of the abelianizations of the groups $\pi(\mathscr{P}, p_i)$). We now have:

Proposition 7.7.2. *Let $\mathscr{P}$ be a 3-d picture with finitely many black points on a DPS that is isomorphic to a strongly normal DPS. Then $\chi(\mathscr{P}) = h_0(\mathscr{P}) - h_1(\mathscr{P}) + h_2(\mathscr{P})$.*

Proof. Suppose $\mathscr{P}$ is isomorphic to a picture $\mathscr{P}^*$ on the strongly normal DPS $(\mathbb{Z}^3, \beta, \omega)$. Let $\mathscr{P}_i^* = (\mathbb{Z}^3, \beta, \omega, B_i^*)$, where $B_1^*, B_2^*, \ldots, B_k^*$ are the black components of $\mathscr{P}^*$. For $1 \le i \le k$ pick $p_i^* \in B_i^*$. Then $\pi(\mathscr{P}^*, p_i^*) \cong \pi(\mathscr{P}_i^*, p_i^*) \cong \pi_1(C(\mathscr{P}_i^*), p_i^*)$, by Proposition 7.6.1 and Theorem 6.1.1. Since the rank of the abelianization of $\pi_1(C(\mathscr{P}_i^*), p_i^*)$ is just $h_1(C(\mathscr{P}_i^*))$, $h_1(\mathscr{P}) = h_1(\mathscr{P}^*) = \sum_i h_1(C(\mathscr{P}_i^*)) = h_1(C(\mathscr{P}^*))$, where the last equality follows from the fact that the $C(\mathscr{P}_i^*)$ are the components of $C(\mathscr{P}^*)$ (by Proposition 7.6.1). Also, $h_0(\mathscr{P}) = h_0(\mathscr{P}^*) = h_0(C(\mathscr{P}^*))$, and $h_2(\mathscr{P}) = h_2(\mathscr{P}^*) = h_2(C(\mathscr{P}^*))$ by the Alexander Duality Theorem. Thus $\chi(\mathscr{P}) = \chi(C(\mathscr{P}^*)) = \sum_n (-1)^n h_n(C(\mathscr{P}^*)) = h_0(\mathscr{P}) - h_1(\mathscr{P}) + h_2(\mathscr{P})$. $\square$

This proposition suggests an alternative and more general definition of $\chi(\mathscr{P})$ for 3-d pictures $\mathscr{P}$, namely $\chi(\mathscr{P}) = h_0(\mathscr{P}) - h_1(\mathscr{P}) + h_2(\mathscr{P})$.

If $\mathscr{P}$ is a 3-d picture with finitely many black points that is isomorphic to a picture $\mathscr{P}^*$ on a strongly normal DPS, then as we observed in the above proof $h_1(\mathscr{P}) = h_1(C(\mathscr{P}^*))$. Thus $h_1(\mathscr{P}) = $ (no. of tunnels in $C(\mathscr{P}^*)$). So we call $h_1(\mathscr{P})$ the *number of tunnels* in $\mathscr{P}$. Thus $\chi(\mathscr{P}) = $ (no. of black components of $\mathscr{P}$) + (no. of cavities in $\mathscr{P}$) − (no. of tunnels in $\mathscr{P}$).

7.8. Computing Euler characteristics

For any unit lattice square or unit lattice cube K let K_0 be the set of corners of K, let K_1 be the union of the edges of K, and if K is a unit lattice cube, let K_2 be the union of the six faces of K.

Let $\mathscr{P}$ be a picture on a strongly normal DPS and let $C = C(\mathscr{P})$. If $\mathscr{P}$ is 2-dimensional, then for all unit lattice squares K define

$$\chi(\mathscr{P}; K) = \chi(C \cap K) - \chi(C \cap K_1)/2 - \chi(C \cap K_0)/4. \tag{1}$$

If $\mathscr{P}$ is 3-dimensional, then for all unit lattice cubes K define

$$\chi(\mathscr{P}; K) = \chi(C \cap K) - \chi(C \cap K_2)/2 - \chi(C \cap K_1)/4$$

$$- \chi(C \cap K_0)/8. \tag{2}$$

It is not difficult to show using the Inclusion–Exclusion Principle that if $\mathcal{P}$ has only finitely many black points, then $\chi(\mathcal{P})$ is just the sum of the $\chi(\mathcal{P}; K)$ over all unit lattice squares K or all unit lattice cubes K, according as $\mathcal{P}$ is 2-d or 3-d. This is essentially a finite sum, for if $\mathcal{P}$ has only finitely many black points, then there are only finitely many K for which $\chi(\mathcal{P}; K)$ is nonzero. Note that the value of $\chi(\mathcal{P}; K)$ is always completely determined by the configuration of black and white points and black and white adjacencies of $\mathcal{P}$ in K, and does not depend on the rest of $\mathcal{P}$.

For other possible approaches to the problem of computing $\chi(\mathcal{P})$ see [18].

7.9. Equivalence of the discrete and continuous definitions of the digital fundamental group

Let $\mathcal{P}$ be a picture on a strongly normal DPS. The $\mathcal{P}$-loop of a black digital loop c of $\mathcal{P}$ is the $\mathcal{P}$-loop obtained by joining up the points of c in the obvious way. Formally, the $\mathcal{P}$-loop of the black digital loop $\langle c_i \mid 0 \leq i \leq m \rangle$ is the $\mathcal{P}$-loop λ given by $\lambda((i+h)/m) = (1-h)c_i + hc_{i+1}$ for h in $[0, 1]$ and nonnegative integers $i < m$. Every nontrivial black digital loop is the black digital loop of its $\mathcal{P}$-loop. Every $\mathcal{P}$-loop is the $\mathcal{P}$-loop of its black digital loop.

We can now prove:

Proposition 7.9.1. *Let $\mathcal{P}$ be a picture on a strongly normal DPS. Then two $\mathcal{P}$-loops are equivalent if and only if their black digital loops are equivalent.*

Proof. We shall assume $\mathcal{P}$ is a 3-d picture. A simplified version of this argument can be used to prove the result in the case when $\mathcal{P}$ is a 2-d picture.

Two $\mathcal{P}$-loops are certainly equivalent if their black digital loops are immediately equivalent, so (since equivalence of $\mathcal{P}$-loops is transitive) two $\mathcal{P}$-loops are equivalent if their black digital loops are equivalent. Thus the "if" part of the proposition is true. Now we have to prove the "only if" part.

Say that a black digital loop c_1 is *contiguous* to a black digital loop c_2 if there are immediately equivalent black digital loops d_1 and d_2 which have the same reduced forms as c_1 and c_2 respectively. Plainly, the immediate equivalence and contiguity relations have the same transitive closure.

Claim 1. *Let a and a' be black digital walks in K from x to y, where K either is a unit lattice square or is a unit lattice cube in which no diameter is a white adjacency of $\mathcal{P}$. Let g and g' be black digital walks from p to x and from y to p respectively. Then the black digital loops $g \cdot a \cdot g'$ and $g \cdot a' \cdot g'$ are contiguous.*

Suppose w.l.o.g. that a' contains at least as many points as a, and the difference in the number of points is exactly k. Let e_y^k denote the trivial black digital loop with exactly $k+1$ points all of which are equal to y. Then the black digital loops $g \cdot (a \cdot e_y^k) \cdot g'$ and $g \cdot a' \cdot g'$ are K-equivalent, and hence immediately equivalent. Moreover, $g \cdot (a \cdot e_y^k) \cdot g'$ has the same reduced form as $g \cdot a \cdot g'$. Thus Claim 1 is valid.

Claim 2. *Let* λ *be a* $\mathscr{P}$-*loop and let* c *be the black digital loop of* λ. *Then* λ *is equivalent to a* $\mathscr{P}$-*loop* λ' *such that* λ' *is a T-loop and such that* c *is related to the black digital loop of* λ' *by the transitive closure of contiguity.*

Let $c = \langle c_i \mid 1 \leq i \leq n \rangle$. Suppose two consecutive points of c, c_j and c_{j+1}, are diagonally opposite corners of a unit lattice square K whose corners are all black points. Let p be one of the two common 6-neighbors of c_j and c_{j+1}. Then since $a = \langle c_j, c_{j+1} \rangle$ and $a' = \langle c_j, p, c_{j+1} \rangle$ are black digital walks in the unit lattice square K, it follows from Claim 1 that c is contiguous to the black digital loop c' obtained from c by replacing its subsequence a with a' (i.e., by inserting the point p between c_j and c_{j+1}). Now the number of pairs of consecutive points of c' that are diagonally opposite corners of a unit lattice square whose corners are all black points is one less than the number of such pairs of consecutive points of c.

The argument in the previous paragraph shows that the transitive closure of contiguity relates c to a black digital loop $d = \langle d_i \mid 1 \leq i \leq m \rangle$ in which no two consecutive points are diagonally opposite corners of a unit lattice square whose corners are all black points. Since d is a black digital loop, each point d_i is equal or adjacent to d_{i+1}. So, by part (2) of Lemma 5.2.1, d_i and d_{i+1} either are equal or are T-adjacent or are diametrically opposite corners of a unit lattice cube. Note that in the third case the cube cannot be special (since no diameter of a special unit lattice cube is a black adjacency); so it is ordinary, its centroid is in the augmented black point set, and its centroid is T-adjacent to d_i and to d_{i+1}. Thus in all three cases the $\mathscr{P}$-walk of length 1 from d_i to d_{i+1} is a T-walk: so the $\mathscr{P}$-loop of d (which is just the $\cdot$ product of those $\mathscr{P}$-walks in order of increasing i) is also a T-loop.

Let λ' be the $\mathscr{P}$-loop of d. Then d is the black digital loop of λ' unless both are trivial, but even in the latter case d has the same reduced form as the black digital loop of λ'. Moreover, λ is equivalent to λ' since c and d are equivalent. This justifies Claim 2.

An *augmented black T-sequence* is a sequence of points in the augmented black point set such that the first and last points are black points and every pair of consecutive points are either equal or T-adjacent. An augmented black T-sequence is *closed* if its initial and final points are equal. The *black point sequence* of an augmented black T-sequence c is the subsequence of c that contains just those points of c that are black points.

Define a *valid quadruple* to be an ordered quadruple (p, q, i, t) in which t is an augmented black T-sequence, p and q are black points such that q is the immediate successor of p in the black point sequence of t, and i is a positive integer such that the ith point of t is p and the first black point in t after that point is q. We now define a function Θ which maps each valid quadruple (p, q, i, t) to a black digital walk from p to q that is contained in every unit lattice square and cube that contains p, q and the $(i+1)$th point of t.

Let (p, q, i, t) be any valid quadruple. If p and q are equal or adjacent black points, then define $\Theta(p, q, i, t)$ to be $\langle p, q \rangle$. Otherwise, if p and q are T-adjacent, then (since they are not adjacent) part (2) of Lemma 5.2.1 implies they are diagonally opposite corners of a unit lattice square K whose corner points are all black. In this case we define $\Theta(p, q, i, t)$ to be $\langle p, r, q \rangle$ where r is the common 6-neighbor of p and q that is given by some rule (e.g., r is the common 6-neighbor for which $x + 2y$ is greater, where x and y denote the x- and y-coordinates). Notice that if p and q are equal, adjacent or T-adjacent black points, then $\Theta(p, q, i, t)$ is completely determined by p and q (if it is defined at all) and does not depend on i or t. Moreover, in these cases $\Theta(p, q, i, t)$ is certainly contained in every unit lattice square and cube that contains both p and q.

Now suppose p and q are not equal, adjacent or T-adjacent. Then q is not the immediate successor of p in t. So, since the ith point of t is p and the first black point in t after that point is q, q is the jth point of t where $j \geq i + 2$ and for all integers k such that $i < k < j$ the kth point of t is not a black point. It follows that for all integers k such that $i < k < j$ the kth point of t is the centroid of a unit lattice cube K that contains p and q. Since the centroid of K is in the augmented black point set, Proposition 5.3.1 implies that there is a black digital walk in K from p to q. Let $\Theta(p, q, i, t)$ be such a black digital walk in K (if there are two or more such black digital walks, then we do not care which one is given by $\Theta(p, q, i, t)$). Note that in this case K is the unique unit lattice cube that contains the $(i+1)$th point of t.

Let n be the number of terms of t that are black points. For positive integers $j \leq n$, let l_j denote the jth smallest integer l such that the lth point of t is a black point, and let p_j denote the l_jth point of t. (So the sequence of p_j's is the black point sequence of t.) We define $\theta(t)$ to be the black digital walk given by the $\cdot$ product, over all integers j in the range $1 \leq j < n$, of the black digital walks $\Theta(p_j, p_{j+1}, l_j, t)$ taken in order of increasing j.

For any $\mathscr{P}$-loop λ that is also a T-loop, the *augmented black T-sequence of λ* is the augmented black T-sequence obtained from the black digital loop of λ as follows. Let $\langle p_i \mid 1 \leq i \leq n \rangle$ be the black digital loop of λ. (Note that since λ is a T-loop and a $\mathscr{P}$-loop, p_i is equal or T-adjacent to p_{i+1} unless p_i and p_{i+1} are diametrically opposite corners of a unit lattice cube.) Whenever p_i and p_{i+1} are diametrically opposite corners of a unit lattice cube, replace the two consecutive points p_i, p_{i+1} with p_i, x, p_{i+1} where x is the centroid of that cube.

Thus there is a strictly increasing finite sequence of points $\langle y_i \mid 1 \leq i \leq m \rangle$ in $[0, 1]$ with $y_1 = 0, y_m = 1$, such that $\langle \lambda(y_i) \mid 1 \leq i \leq m \rangle$ is the augmented black T-sequence of λ.

Claim 3. *Suppose the $\mathscr{P}$-loop λ is a T-loop and the augmented black T-sequence of λ is t. Let c be the black digital loop of λ. Then $c = \theta(t)$.*

Here c is the black point sequence of t. Thus in the above definition of $\theta(t)$ the sequence of p_j's is just c; and, for $1 \leq j < n$, p_j and p_{j+1} are equal or adjacent black

points so $\Theta(p_j, p_{j+1}, l_j, t)$ is just $\langle p_j, p_{j+1} \rangle$. Now $\theta(t)$ is defined to be the product of these in order of increasing j, and this is just c. Hence Claim 3 is true.

Say that two augmented black T-sequences are *T-contiguous* if one of the sequences can be obtained from the other by replacing one of its points t_i with t_i, t_i, or if one of the sequences can be obtained from the other by replacing two consecutive points t_i, t_{i+1} with t_i, x, t_{i+1} where x, t_i, and t_{i+1} are (not necessarily distinct) vertices of a black 2-simplex of $\mathcal{P}$.

Claim 4. *Let t and t' be T-contiguous closed augmented black T-sequences. Then $\theta(t)$ and $\theta(t')$ are contiguous black digital loops.*

Let $t = \langle t_i \mid 1 \le i \le m \rangle$. If t' can be obtained from t by replacing one of its points t_i with t_i, t_i, then $\theta(t)$ has the same reduced form as $\theta(t')$ (and the two are equal if t_i is the centroid of a unit lattice cube).

Now suppose t' can be obtained from t by replacing two consecutive points t_i, t_{i+1} with t_i, x, t_{i+1} where x, t_i, and t_{i+1} are (not necessarily distinct) vertices of a black 2-simplex of $\mathcal{P}$. Then any two of x, t_i and t_{i+1} are equal or T-adjacent, and so at least one of the following is true:

(1) $\{x, t_i, t_{i+1}\}$ lies in a unit lattice square K.

(2) $\{x, t_i, t_{i+1}\}$ lies in a special unit lattice cube K.

(3) $\{x, t_i, t_{i+1}\}$ lies in an ordinary unit lattice cube K in which no diameter is a white adjacency.

(4) $\{x, t_i, t_{i+1}\}$ lies in an ordinary unit lattice cube K in which some diameter is a white adjacency.

Consider case (4). Here the centroid of K is in the augmented white point set, so none of x, t_i and t_{i+1} is the centroid. Hence x, t_i and t_{i+1} are corners of K, and since they are vertices of a black 2-simplex of $\mathcal{P}$ it follows from the way ordinary unit lattice cubes are subdivided that they all lie on one face of K. Thus case (1) also applies. So it suffices to consider cases (1), (2) and (3). Note that in case (2) the fact that K is special implies that no diameter of K is a white adjacency.

Suppose case (1), (2) or (3) holds. Let j be the greatest integer $\le i$ such that t_j is a black point, and let k be the least integer $\ge i+1$ such that t_k is a black point. Then t_j and t_k are consecutive points in the black point sequence of t. Now either t_{i+1} is a black point in which case $k = i+1$ and $t_k = t_{i+1}$ is a corner of K, or else t_{i+1} is the centroid of K. If the latter is true (which is only possible in case (3)) and R is the least integer $> i+1$ such that t_R is not equal to the centroid of K, then t_R is a corner of K so t_R is a black point and $R = k$ (whence $t_k = t_R$ is a corner of K). Therefore t_k is a corner of K and, for all integers r such that $i+1 \le r < k$, t_r is the centroid of K. A similar argument applies to t_j. Thus t_j and t_k are corners of K and, for all integers r such that $j < r < k$, t_r is the centroid of K.

If x is not a black point, then x is the centroid of K: in this case t_j and t_k are consecutive points in the black point sequence of t' as well as that of t, so $\theta(t')$ is obtained from $\theta(t)$ by replacing a subsequence of consecutive points $a = \Theta(t_j, t_k, j, t)$

with the sequence $a' = \Theta(t_j, t_k, j, t')$. If x is a black point, then t_j, x and t_k are consecutive points in the black point sequence of t', and the inserted point x is the $(i+1)$th point of t': so $\theta(t')$ is obtained from $\theta(t)$ by replacing a subsequence of consecutive points $b = \Theta(t_j, t_k, j, t)$ with the sequence $b' = \Theta(t_j, x, j, t') \cdot \Theta(x, t_k, i+1, t')$.

Now x, t_j, t_k and t_{j+1} are in K; and since t_r is the centroid of K for all integers r such that $j < r < k$, the $(j+1)$th and $(i+2)$th points of t' are also in K. So (since $\Theta(p, q, l, \tau)$ is always a black digital walk that is contained in every unit lattice square and cube that contains p, q and the $(l+1)$th point of τ) the black digital walks a, a', b and b' defined in the previous paragraph are black digital walks in K. Hence $\theta(t)$ is contiguous to $\theta(t')$ by Claim 1, and Claim 4 is justified.

By Claim 2 it suffices to prove the proposition for $\mathscr{P}$-loops which are also T-loops. Let λ and λ' be two equivalent $\mathscr{P}$-loops that are also T-loops. By Theorem 6.1.1, λ is fixed base point homotopic to λ' in $C(\mathscr{P})$, and hence[14] as a fairly straightforward consequence of the Simplicial Approximation Theorem the augmented black T-sequences of λ and λ' are related by the transitive closure of T-contiguity. So by Claims 3 and 4 the black digital loops of λ and λ' are related by the transitive closure of contiguity and hence of immediate equivalence. $\square$

Let $\mathscr{P}$ be a picture on a strongly normal DPS. It follows from Proposition 7.9.1 that the operation of taking the black digital loop of a $\mathscr{P}$-loop induces a well-defined injection of the digital fundamental group $\pi(\mathscr{P}, p)$ to the discrete digital fundamental group $\pi^d(\mathscr{P}, p)$. This injection is in fact a bijection, since every trivial black digital loop $\langle p, p \rangle$ and every nontrivial black digital loop is the black digital loop of its $\mathscr{P}$-loop, and all trivial black digital loops with the same base point are equivalent. Now if λ_1 and λ_2 are $\mathscr{P}$-loops based at p, and their black digital loops are respectively c_1 and c_2, then the black digital loop of $\lambda_1 \cdot \lambda_2$ is equivalent to $c_1 \cdot c_2$ (in fact either they are both trivial or they are equal). So the induced bijection is an isomorphism of $\pi(\mathscr{P}, p)$ onto $\pi^d(\mathscr{P}, p)$.

8. Concluding remarks

We have shown that a large class of binary digital picture spaces, namely the strongly normal digital picture spaces, have good topological properties that provide a suitable foundation for image processing operations such as thinning, border following and contour filling. One such good property is that on a strongly normal digital picture space it is possible to define a digital fundamental group that behaves very much like the fundamental group of polyhedral topology. Most combinations of grids and adjacency relations that have been considered in the literature are isomorphic to a strongly normal digital picture space.

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[14] For further details see, for example, the proof of Theorem 3.3.9 in [23].

Acknowledgement

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