



Analytical solution of forced-convective boundary-layer flow over a flat plate

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In this letter, the problem of forced convection heat transfer over a horizontal flat plate is investigated by employing the Adomian Decomposition Method (ADM). The series solution of the nonlinear differential equations governing on the problem is developed. Comparison between results obtained and those of numerical solution shows excellent agreement, illustrating the effectiveness of the method. The solution obtained by ADM gives an explicit expression of temperature distribution and velocity distribution over a flat plate.

Keywords: convection heat transfer, nonlinear equations, Adomian decomposition method, numerical method (NM)

1. Introduction

Most scientific problems such as heat transfer are inherently of nonlinearity. We know that except a limited number of these problems, most of them do not have analytical solutions. Therefore, these nonlinear equations should be solved by using other methods. Some of them are solved by using numerical techniques and some of them are solved by using perturbation method. Since there are some limitations with the common perturbation method, and also because the basis of the common perturbation method is upon the existence of a small parameter, developing the method for different applications is very difficult. Most boundary-layer models can be reduced to systems of nonlinear ordinary differential equations which are usually solved by numerical methods. It is however interesting to find solutions to boundary layer problems using analytical approach. Analytical methods have significant advantages over numerical methods in providing analytic, verifiable, rapidly convergent approximation. The Adomian decomposition method based on series approximation is the newly developed method for strongly nonlinear problems. The Homotopy Perturbation Method uses functions to obtain series solutions to boundary-layer equations [1–6] while the

series in ADM [7] are derived from functions consisting of terms corresponding to the initial conditions. The analytic ADM has been proven successful in solving a wide class of nonlinear differential equations [7–13]. Hashim [8] applied ADM to the classical Blasius' equation. Wazwaz [14] used ADM to solve the boundary layer equation of viscous flow due to a moving sheet. Awang Kechil and Hashim [15] extended the applicability of ADM to obtain approximate analytical solution of an unsteady boundary layer problem over an impulsively stretching sheet. The first application of ADM to a 2-by-2 system of nonlinear ordinary differential equations of free-convective boundary layer equation was presented by Awang Kechil and Hashim [16]. Hayat et al. [17] studied the MHD flow over a nonlinearly stretching sheet by employing the Modified Adomian Decomposition Method.

In this paper, we revisit the steady two-dimensional laminar forced convection in a flow of viscous fluid against a flat plate with uniform wall temperature. Fluid is assumed to have constant properties. In this letter, we are interested in applying ADM to obtain an approximate analytical solution of this problem and the results obtained will be validated by those of numerical simulation.

2. Governing equations

Consider steady flow, with constant free stream velocity u_∞ without turbulence over a semi-infinite flat plate aligned with the flow. All fluid properties are considered to be constant. The continuity, Navier–Stokes, and energy equations of this flow are as follows [16]:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad (1)$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{dP}{dx} + \nu \frac{\partial^2 u}{\partial x^2}, \quad (2)$$

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2}, \quad (3)$$

with following boundary conditions:

$$T = 0 \text{ at } y = 0, \quad (4a)$$

$$T \rightarrow T_\infty \text{ when } y \rightarrow \infty, \quad (4b)$$

$$T = T_\infty \text{ at } x = 0, \quad (4c)$$

$$u = 0, \quad v = 0 \text{ at } y = 0, \quad (4d)$$

$$u = u_\infty \text{ at } x = 0, \quad (4e)$$

$$u \rightarrow u_\infty \text{ when } y \rightarrow \infty. \quad (4f)$$

The solution to the momentum equation is decoupled from the energy solution. However, the solution of the energy equation is still linked to the momentum solution. The following dimensionless variables are introduced in the transformation:

$$\eta = \frac{y}{\sqrt{x}} Re_x^{0.5}, \quad (5)$$

$$\theta(\eta) = \frac{T - T_\infty}{T_w - T_\infty}. \quad (6)$$

The Reynolds number is defined as:

$$Re = \frac{u_\infty x}{\nu}. \quad (7)$$

Using Equations (1) through (6), the governing equations can be reduced to two equations where f is a function of the similarity variable (η) [18]:

$$f''' + \frac{1}{2} f f'' = 0, \quad (8a)$$

$$\theta'' + \frac{Pr}{2} f \theta' = 0, \quad (8b)$$

where f is related to the u velocity by [18]:

$$f' = \frac{u}{u_\infty}. \quad (9)$$

The reference velocity is the free stream velocity of forced convection. The boundary conditions are obtained from the similarity variables. For the forced convection case [18]:

$$f(0) = 0, \quad f'(0) = 1, \quad \theta'(0) = 1, \quad f'(\infty) = 1, \quad \theta(\infty) = 0, \quad (10)$$

3. Adomian decomposition method

We follow the standard procedure of ADM [7] by introducing two linear differential operators $L_1 = d^3/d\eta^3$ and $L_2 = d^2/d\eta^2$ with inverse operators $L_1^{-1}(\cdot) = \int_0^\eta \int_0^\eta \int_0^\eta (\cdot) dt dt dt$ and $L_2^{-1}(\cdot) = \int_0^\eta \int_0^\eta (\cdot) dt dt$. Thus, Equation (8) in operator form,

$$L_1(f) = -\frac{1}{2}ff'' \quad (11)$$

$$L_2(\theta) = -\frac{Pr}{2}f\theta' \quad (12)$$

Applying the inverse operators on Equations (11) and (12) and let $f''(0) = \alpha_1$, $\theta'(0) = \alpha_2$, we obtain,

$$f = \frac{1}{2}\alpha_1\eta^2 + \frac{1}{2}L_1^{-1}(P(f)), \quad (13)$$

$$\theta = 1 + \alpha_2\eta + \frac{Pr}{2}L_2^{-1}(Q(f, \theta)), \quad (14)$$

where the nonlinear terms in Equations (13) and (14),

$$P(f) = ff'', \quad (15)$$

$$Q(f, \theta) = f\theta', \quad (16)$$

and their respective decompositions,

$$P(f) = \sum_{i=0}^{\infty} A_i, \quad (17)$$

$$Q(f, \theta) = \sum_{i=0}^{\infty} E_i, \quad (18)$$

A_i, E_i are so-called Adomian polynomials [7], given by

$$A_i = \frac{1}{i!} \left[\frac{d^i}{d\lambda^i} P \left(\sum_{j=0}^{\infty} \lambda^j f_j \right) \right]_{\lambda=0}, \quad i \geq 0. \quad (19)$$

This yields

$$A_0 = f_0 f_0'', \quad (20)$$

$$E_0 = f_0 \theta_0', \quad (21)$$

and for $i \geq 1$

$$A_i = \sum_{m=0}^i f_m f_{i-m}'', \quad (22)$$

$$E_i = \sum_{m=0}^i f_m \theta_{i-m}', \quad (23)$$

In ADM [14], f and θ are defined as infinite series,

$$f = \sum_{i=0}^{\infty} f_i(\eta), \quad (24)$$

$$\theta = \sum_{i=0}^{\infty} \theta_i(\eta). \quad (25)$$

Substituting Equations (17) and (18) and Equations (24) and (25) into Equations (13) and (14), we obtain

$$\sum_{i=0}^{\infty} f_i(\eta) = \frac{1}{2} \alpha_1 \eta^2 - \frac{1}{2} L_1^{-1} \left(\sum_{i=0}^{\infty} A_i \right), \quad (26)$$

$$\sum_{i=0}^{\infty} \theta_i(\eta) = 1 + \alpha_2 \eta - \frac{Pr}{2} L_2^{-1} \left(\sum_{i=0}^{\infty} E_i \right), \quad (27)$$

and the individual terms for f and θ are obtained from the recursive relations

$$f_0 = \frac{1}{2} \alpha_1 \eta^2, \quad (28)$$

$$\theta_0 = 1 + \alpha_2 \eta, \quad (29)$$

$$f_{i+1} = -\frac{1}{2}L_1^{-1}(A_i), \quad i \geq 0, \quad (30)$$

$$\theta_{i+1} = -\frac{Pr}{2}L_2^{-1}(E_i), \quad i \geq 0, \quad (31)$$

For practical numerical computation, we will compute the j -term approximation of $f(\eta)$, $\theta(\eta)$ which are $\phi_j(\eta) = \sum_{i=0}^{j-1} f_i$, $\psi_j(\eta) = \sum_{i=0}^{j-1} \theta_i$, respectively, as the j -term approximations converge to the true series as j approaches infinity.

4. Results and discussion

The Adomian polynomials (20–23) and the recursive relations (28–31) are then coded in the Maple environment computer package with the controlling significant digits set to 11. We obtain 10-term approximation to both f and θ given by $\phi_{10}(\eta) = \sum_{i=0}^9 f_i$, and $\psi_{10}(\eta) = \sum_{i=0}^9 \theta_i$, respectively, but for lack of space, only the first 3 terms produced from (28–31) are given below:

$$f_0(\eta) = \frac{1}{2}\alpha_1\eta^2, \quad (32)$$

$$f_1(\eta) = -\frac{1}{240}\alpha_1^2\eta^5, \quad (33)$$

$$f_2(\eta) = \frac{11}{161280}\alpha_1^3\eta^8, \quad (34)$$

$$\theta_0(\eta) = \alpha_2\eta + 1, \quad (35)$$

$$\theta_1(\eta) = -\frac{Pr}{48}\alpha_1\alpha_2\eta^4, \quad (36)$$

$$\theta_2(\eta) = \frac{11}{20160}Pr\alpha_1^2\alpha_2\eta^7, \quad (37)$$

The undetermined values of α_1 and α_2 are calculated from the boundary conditions at infinity in (10). The difficulty at infinity is overcome by employing the diagonal

Padé approximants [19] that approximate $f'(\eta)$ and $\theta(\eta)$ using $\phi_{10}(\eta)$ and $\psi'_{10}(\eta)$, respectively. The numerical results of α_1 and α_2 from $\lim_{\eta \rightarrow \infty} \phi'_{10} = 1$ and $\lim_{\eta \rightarrow \infty} \psi'_{10} = 0$ for selected m in the range from 8 to 11 are presented in Table 1 for $Pr = 1$. Since Equation (8) can not be easily solved by the analytical method; Equation (8) is, therefore, solved by the numerical method using the software MAPLE whose results are given in Tables 2 and 3, and also the consequent results of the numerical and Adomian Decomposition are compared in Figures 1, 2 and 3. As you can see in $Pr = 1$, the ADM has a high accuracy.

Table 1. Numerical values of $f''(0), \theta'(0)$, for $Pr = 1$

	[8.8]	[9.9]	[10.10]	[11.11]	Exact
α_1	0.2683198	0.44927	0.3293037	0.3293037	0.3293037
α_2	-0.2797056	-0.426513	-0.3487661	-0.3487661	-0.3487661

Table 2. The results of ADM, HPM and NM for $f(\eta), f'(\eta)$ if $Pr = 1$

η	$f(\eta)$		$f'(\eta)$	
	ADM	NM	ADM	NM
0	0	0	0	0
0.2	0.0065859	0.0066412	0.0658571	0.0664077
0.4	0.0263396	0.0266762	0.1316636	0.1327641
0.6	0.0592395	0.0597215	0.1972899	0.1989372
0.8	0.1052295	0.1061082	0.2625216	0.2647094
1	0.1642024	0.1655717	0.3270638	0.32978
1.2	0.2359847	0.2379487	0.3905487	0.3937761
1.4	0.3203229	0.3229819	0.4525474	0.4562617
1.6	0.4168731	0.4203207	0.5125869	0.5167567
1.8	0.525194	0.529518	0.5701715	0.5747581
2	0.6447451	0.6500243	0.6248074	0.6297657
2.2	0.7748894	0.7811933	0.6760307	0.6813103
2.4	0.9149026	0.9222901	0.7234349	0.7289819
2.6	1.0639869	1.0725059	0.7666961	0.772455
2.8	1.2212899	1.2309773	0.8055934	0.8115096
3	1.3859262	1.3968082	0.8400224	0.8460444
3.2	1.5570018	1.5690949	0.8699999	0.8760814
3.4	1.7336381	1.7469501	0.89565596	0.9017612
3.6	1.9149935	1.9295251	0.9172393	0.9233296
3.8	2.1002831	2.1160298	0.9350592	0.9411181
4	2.2887911	2.3057464	0.9494855	0.9555182
4.2	2.4798731	2.4980396	0.9608389	0.966957
4.4	2.6729204	2.6923609	0.969051	0.9758708
4.6	2.8671706	2.888248	0.9722134	0.9826835
4.8	3.0608776	3.0853206	0.9815741	0.9877895
5	3.2479336	3.2832736	0.9896845	0.9915419

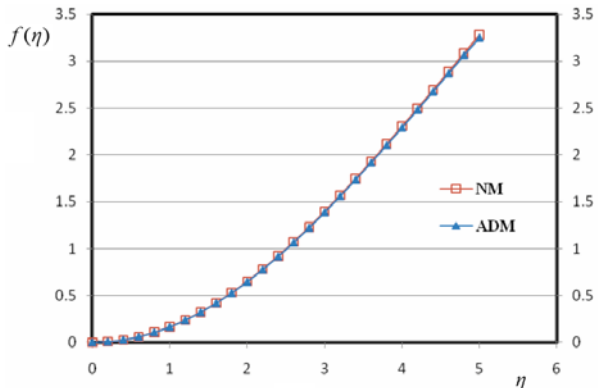


Fig. 1. The comparison of the answers resulted by *ADM* and *NM* for $f(\eta)$

Table 3. The results of *ADM*, *HPM* and *NM* for $\theta(\eta)$ if $Pr = 1$

η	ADM	NM	η	ADM	NM
0	1	1	2.6	0.2333038	0.2275449
0.2	0.9341428	0.9335922	2.8	0.1944065	0.1884903
0.4	0.8683363	0.8672358	3	0.1599775	0.1439554
0.6	0.80271	0.8010627	3.2	0.13012	0.1239183
0.8	0.7374783	0.7352908	3.4	0.1043403	0.0882386
1	0.6729361	0.6702199	3.6	0.0827606	0.0666702
1.2	0.6094512	0.6062238	3.8	0.0649412	0.0588819
1.4	0.5474525	0.5437381	4	0.0505144	0.0314817
1.6	0.487413	0.4832432	4.2	0.03916103	0.0330429
1.8	0.4298284	0.4252418	4.4	0.0309489	0.0241292
2	0.3751925	0.3702342	4.6	0.0277865	0.0173165
2.2	0.3239692	0.3186896	4.8	0.0186443	0.0122105
2.4	0.276565	0.271018	5	0.01264024	0.0084581

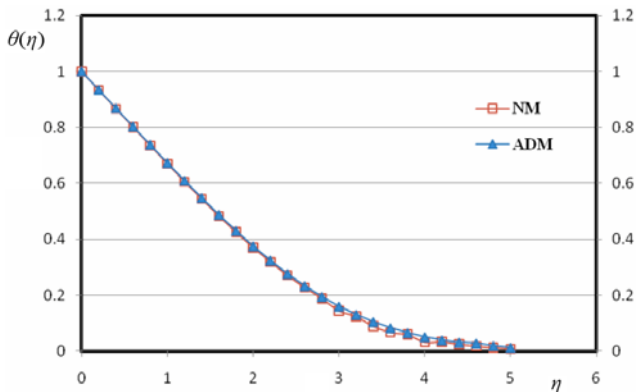


Fig. 2. The comparison of the answers resulted by *ADM* and *NM* for $\theta(\eta)$

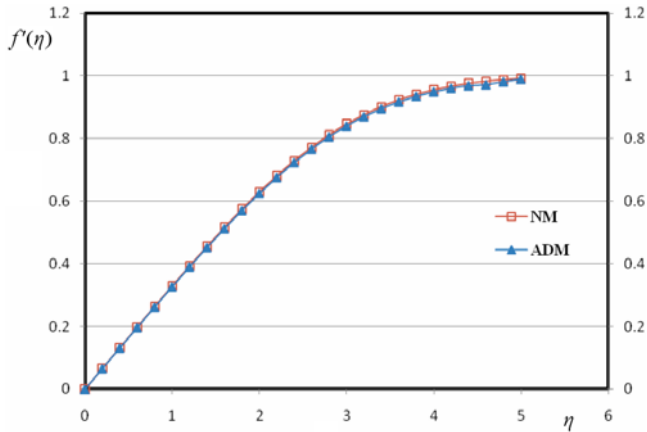


Fig. 3. The comparison of the answers resulted by *ADM* and *NM* for $f'(\eta)$

5. Conclusions

In this letter, Adomian Decomposition Method has been successfully applied to natural convection heat transfer problem with specified boundary conditions. The obtained solutions are compared with ones from numerical method and Homotopy Perturbation Method. The excellent agreement of the ADM solutions and the exact solutions shows the reliability and the efficiency of the method. This new method accelerated the convergence to the solutions. The ADM combined with the Padé approximant provide efficient alternative tools in solving nonlinear models.

Nomenclature

- g – gravitational force
- v – velocity component in the y direction
- ADM – Adomian Decomposition Method
- x – dimensional vertical coordinate
- HPM – Homotopy Perturbation Method
- y – dimensional horizontal coordinate
- NM – numerical method
- P – pressure
- Pr – Prandtl number
- ρ – density
- T – temperature
- T_W – temperature imposed on the plate
- ν – kinematic viscosity
- T_∞ – local ambient temperature
- α – thermal diffusivity
- u – velocity component in the x direction
- θ – dimensionless temperature

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Analityczne rozwiązanie wymuszonego konwekcyjnie przepływu w warstwie przyściennej płaskiej płyty

W artykule przedstawiono zastosowanie metody dekompozycji Adomiana do wymuszonego, konwekcyjnie przepływu ciepła w poziomej, płaskiej płycie. Rozwiązania nieliniowych równań różniczkowych opisujących zagadnienie poszukiwana w postaci szeregów Adomiana. Z porównania otrzymanych wyników z wynikami innych metod numerycznych wynika doskonała ich zgodność, która potwierdza skuteczność zastosowanej metody. Otrzymane rozwiązanie pozwoliło jednoznacznie wyznaczyć rozkład i prędkości mian temperatury w analizowanej płycie.