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Finite Deformations and Motions of Radially Inextensible Hollow Spheres

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Abstract Radial inflation–compaction and radial oscillation solutions are presented for hollow spheres of isotropic elastic material that are radially inextensible. The solutions for radial inflation–compaction and radial oscillation are obtained also for everted radially inextensible hollow spheres of isotropic elastic material. The static and dynamic results for everted and uneverted radially inextensible hollow spheres are then compared. Harmonic and compressible Varga materials are used to demonstrate the solutions.

Keywords Finite elasticity · Inextensible · Deformations · Motions

Mathematics Subject Classifications (2000) 74B20

1 Introduction

Solid materials reinforced by fibers are frequently modeled as homogeneous solids that are inextensible in the fiber direction. The fibers are identified with a unit tangent vector field $\mathbf{N}(\mathbf{X})$ in the initial state that deforms with the body and so is identified with a unit tangent vector field $\mathbf{n}(\mathbf{x})$ after deformation. The material is inextensible in the fiber direction, so that

$$\mathbf{N} \cdot \mathbf{CN} = 1, \tag{1}$$

where $\mathbf{F}=\mathrm{Grad}\mathbf{x}$ is the deformation gradient tensor and $\mathbf{C}=\mathbf{F}^{\mathbf{T}}\mathbf{F}$ is the right Cauchy–Green deformation tensor. The constraint reaction is an arbitrary uniaxial stress $q\mathbf{n} \otimes \mathbf{n}$ in the fiber direction and the arbitrary scalar field $q(\mathbf{x})$ is a Lagrange multiplier.

While the mechanical response of such inextensible elastic solids has been treated in several papers, there does not seem to be a plentitude of solutions involving finite deformation. Beskos [1], using a semi-inverse method, identified some controllable or universal deformations, i.e., deformations that can be supported without body force in *every*

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isotropic solid with appropriate inextensibility constraint. He gave expressions for the stress components but he did not solve specific boundary value problems.

The present paper treats what is, perhaps, the simplest family of inhomogeneous finite deformations, namely, spherically radial deformations. The material is assumed to be homogeneous, inextensible in the radial direction, and either isotropic or transversely isotropic about the radial direction. Five different problems of finite deformation or motion are considered:

(1) radial inflation or compaction
(2) radial oscillation
(3) eversion
(4) radial inflation or compaction of an everted sphere
(5) radial oscillation of an everted sphere.

The material response is described by a single function of one variable – the axi-symmetric stress response function – and the conditions of spherical symmetry and radial inextensibility determine the deformations or motions *a priori*, independently of the form of this function, so that they are controllable for isotropic or transversely isotropic elastic solids with radial inextensibility.

The eversion process may be envisioned as cutting a hollow sphere in half, subjecting each part to a snap-through bending, and gluing the newly created surfaces together to form the everted hollow sphere. This process preserves the spherical symmetry but it introduces a pre-stress. The effect of this pre-stress on the mechanical response may be assessed by comparing the solutions (4) and (5) above with the solutions (1) and (2).

The problems treated here for radially inextensible materials also provide controllable deformations and motions for incompressible materials (Green and Shield [2], Ericksen [3], Knowles [4], Zhong-Heng and Solecki [5]). However, the static and dynamic solutions for everted hollow spheres do not appear to have been treated.

In addition to the work of Beskos [1], Ogden [6] has given a brief treatment of the inflation problem (Section 4) for a radially inextensible hollow sphere. Other models of fiber reinforcement consider the fibers as introducing a local transverse isotropy but not inextensibility (Spencer [7]).

2 The Axi-symmetric Stress Response Function

The strain energy W for any isotropic elastic solid may be represented as a function of the principal invariants (I_1, I_2, I_3) of the deformation tensor, or of the principal invariants (i_1, i_2, i_3) of the stretch tensor $\mathbf{V}=(\mathbf{F}\mathbf{F}^{\mathbf{T}})^{1/2}$, or as a fully symmetric function of the principal stretches (λ_1, λ_2, λ_3), thus

$$W = \widehat{W}(I_1, I_2, I_3) = \widehat{w}(i_1, i_2, i_3) = \widetilde{w}(\lambda_1, \lambda_2, \lambda_3) \tag{2}$$

with

$$I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \quad I_2 = \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2 + \lambda_1^2\lambda_2^2, \quad I_3 = \lambda_1^2\lambda_2^2\lambda_3^2 \tag{3}$$

and

$$i_1 = \lambda_1 + \lambda_2 + \lambda_3, \quad i_2 = \lambda_2\lambda_3 + \lambda_3\lambda_1 + \lambda_1\lambda_2, \quad i_3 = \lambda_1\lambda_2\lambda_3. \tag{4}$$

The representation $(2)_3$ may be obtained from $(2)_1$ or $(2)_2$ by substituting for the principal invariants from (3) or (4).

The principal vectors of the Cauchy stress tensor **T** are those of **V** and the principal stresses (t_1, t_2, t_3) are found from (2)$_3$ as

$$t_i = \frac{1}{\lambda_j \lambda_k} \frac{\partial}{\partial \lambda_i} \widetilde{w}(\lambda_1, \lambda_2, \lambda_3) \quad i \neq j \neq k \neq i. \tag{5}$$

If the material is inextensible in a particular direction and is subjected to equal stretches λ normal to that direction, then the principal stretches are (λ, λ, 1) and the principal stresses may be written as

$$t_1 = t_2 = T(\lambda), \quad t_3 = q, \quad with \;\; T(\lambda) = \frac{1}{\lambda_2 \lambda_3} \frac{\partial}{\partial \lambda_i} \widetilde{w}(\lambda_1, \lambda_2, \lambda_3) \Big|_{\lambda_1 = \lambda_2 = \lambda, \lambda_3 = 1}. \tag{6}$$

If the material is transversely isotropic about the fiber direction, then the strain energy has the form

$$W = \overline{W}\,(I_1, I_2, I_3, I_4, I_5), \quad with \quad I_4 = N \cdot CN, \quad I_5 = N \cdot C^2 N. \tag{7}$$

The axi-symmetric stress response function $T(\lambda)$ is again given by (6)$_2$, with

$$\widetilde{w}(\lambda_1, \lambda_2, \lambda_3) = \overline{W}\,\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2,\; \lambda_2^2 \lambda_3^2 + \lambda_3^2 \lambda_1^2 + \lambda_1^2 \lambda_2^2,\; \lambda_1^2 \lambda_2^2 \lambda_3^2, 1, 1\right). \tag{8}$$

3 Spherically Symmetric Deformations

Consider a deformation

$$r = \widehat{r}(R), \quad \theta = \Theta, \quad \varphi = \Phi; \quad \frac{dr}{dR} > 0 \tag{9}$$

of a hollow sphere, with inner and outer radii R_1 and R_2. Here (R, Θ, Φ) and (r, θ, ϕ) are the spherical polar coordinates of a typical particle before and after deformation. This deformation is irrotational and it describes radial inflation or compaction. The coordinate directions are principal directions of stretch and the principal stretches are (dr/dR, r/R, r/R). Thus, if the hollow sphere is inextensible in the radial direction, the deformation must have the form

$$r = R + \beta, \quad \theta = \Theta, \quad \varphi = \Phi. \tag{10}$$

The coordinate directions are also principal directions of stress and the principal stresses are

$$T_{rr} = q, \quad T_{\theta\theta} = T_{\varphi\varphi} = T(\lambda), \quad \lambda = r/R = 1 + \beta/R. \tag{11}$$

The equation of equilibrium with no body force,

$$div\mathbf{T} = 0, \tag{12}$$

reduces to the radial equation

$$\frac{dT_{rr}}{dr} + \frac{2}{r}(T_{rr} - T_{\theta\theta}) = 0, \tag{13}$$

which is satisfied because q is arbitrary. Integrating this equation determines the remaining principal stress T_{rr} to within an additive constant

$$T_{rr} = \frac{2}{r^2}\int T(\lambda) r dr, \quad \lambda = r/(r-\beta). \tag{14}$$

It follows that the function $F(\beta;\, R_1,\, R_2)$, defined as

$$F(\beta; R_1, R_2) = 2\beta^2 \int_{1+\beta/R_2}^{1+\beta/R_1} \frac{T(\lambda)\lambda d\lambda}{(\lambda-1)^3}, \tag{15}$$

describes the radial response of a radially inextensible hollow sphere, with inner and outer radii R_1 and R_2, in radial inflation or compaction. The solution corresponding to the boundary conditions

$$T_{rr} = -P_1 \; at \; r = r_1 = R_1 + \beta, \quad T_{rr} = -P_2 \; at \; r = r_2 = R_2 + \beta \tag{16}$$

is

$$r_1^2 P_1 - r_2^2 P_2 = F(\beta; R_1, R_2). \tag{17}$$

3.1 Eversion

The deformation gradient tensor corresponding to the deformation

$$r = \widetilde{r}(R), \quad \theta = \pi - \Theta, \quad \varphi = \Phi; \quad \frac{dr}{dR} < 0 \tag{18}$$

is

$$\boldsymbol{F} = \frac{dr}{dR}\boldsymbol{e}_r \otimes \boldsymbol{e}_R - \frac{r}{R}\boldsymbol{e}_\theta \otimes \boldsymbol{e}_\Theta + \frac{r}{R}\boldsymbol{e}_\varphi \otimes \boldsymbol{e}_\Phi. \tag{19}$$

This admits the polar decomposition **F=VR**, with rotation tensor

$$\boldsymbol{R} = -\boldsymbol{e}_r \otimes \boldsymbol{e}_r - \boldsymbol{e}_\theta \otimes \boldsymbol{e}_\Theta + \boldsymbol{e}_\varphi \otimes \boldsymbol{e}_\Phi \tag{20}$$

and stretch tensor

$$\boldsymbol{V} = -\frac{dr}{dR}\boldsymbol{e}_r \otimes \boldsymbol{e}_r + \frac{r}{R}\boldsymbol{e}_\theta \otimes \boldsymbol{e}_\theta + \frac{r}{R}\boldsymbol{e}_\varphi \otimes \boldsymbol{e}_\varphi, \tag{21}$$

so that the coordinate directions are again principal directions of stretch and the principal stretches are ($-dr/dR$, r/R, r/R). Thus, the deformation has the form

$$r = \beta - R, \quad \theta = \pi - \Theta, \quad \varphi = \Phi; \quad (\beta > R_2) \tag{22}$$

and it describes radial inflation or compaction of an everted hollow sphere.

The coordinate directions are also principal directions of stress and the principal stresses are

$$T_{rr} = q, \quad T_{\theta\theta} = T_{\varphi\varphi} = T(\lambda), \quad \lambda = r/R = \beta/R - 1. \tag{23}$$

Integrating the radial equation of equilibrium (13) now gives

$$T_{rr} = \frac{2}{r^2}\int T(\lambda) r dr, \quad \lambda = r/(\beta - r), \tag{24}$$

and the function $G(\beta; R_1, R_2)$, defined as

$$G(\beta; R_1, R_2) = 2\beta^2 \int_{\beta/R_2-1}^{\beta/R_1-1} \frac{T(\lambda)\lambda d\lambda}{(\lambda+1)^3}, \tag{25}$$

describes the radial response of an everted radially inextensible hollow sphere, with inner and outer radii R_1 and R_2, in radial inflation or compaction. The solution corresponding to the boundary conditions

$$T_{rr} = -P_1 \text{ at } r = r_1 = \beta - R_1, \quad T_{rr} = -P_2 \text{ at } r = r_2 = \beta - R_2 \tag{26}$$

is

$$r_1^2 P_1 - r_2^2 P_2 = G(\beta; R_1, R_2). \tag{27}$$

It should be kept in mind that P_1 denotes the pressure on the *outer* boundary of the everted sphere.

4 Inflation or compaction

Equation (17) gives the solution for inflation of a hollow sphere under internal pressure P as

$$P = \frac{1}{(R_1 + \beta)^2} F(\beta; R_1, R_2) \tag{28}$$

and that for compaction under external pressure P as

$$P = -\frac{1}{(R_2 + \beta)^2} F(\beta; R_1, R_2). \tag{29}$$

5 Radial Oscillation

Consider a finite radial motion described by (10), with $\beta = \beta(t)$ now being a function of time. The principal stresses are again given by (11). The equation of motion is

$$div\ \boldsymbol{T} = \rho \ddot{\boldsymbol{x}} = \frac{1}{i_3}\rho_0 \ddot{\boldsymbol{x}}, \tag{30}$$

where the superposed dots denote material time derivatives and ρ_o and ρ denote the mass densities before and after deformation. This reduces to the radial equation

$$\frac{\partial T_{rr}}{\partial r} + \frac{2}{r}(T_{rr} - T_{\theta\theta}) = \rho_0 \left(\frac{R}{r}\right)^2 \ddot{\beta}. \tag{31}$$

Integrating this equation gives the principal stress $\mathbf{T}_{rr}$ to within an additive function of time as

$$r^2 T_{rr} = 2\int T_{\theta\theta} r dr + \frac{1}{3}\rho_o R^3 \ddot{\beta}. \tag{32}$$

The boundary conditions (26), with $P_1=P_1(t)$ and $P_2=P_2(t)$, give

$$r_1^2 P_1 - r_2^2 P_2 = F(\beta; R_1, R_2) + \frac{1}{4\pi} M\ddot{\beta}, \quad M = \frac{4}{3}\pi\rho_0\left(R_2^3 - R_1^3\right), \tag{33}$$

where M is the mass of the hollow sphere.

Introducing the function $\Psi(\beta;\, R_1,\, R_2)$, defined as

$$\Psi(\beta; R_1, R_2) = \int F(\beta; R_1, R_2) d\beta, \tag{34}$$

leads to the energy balance equation

$$4\pi\dot{\beta}\left(r_1^2 P_1 - r_2^2 P_2\right) = \frac{d}{dt}\left\{4\pi\Psi(\beta; R_1, R_2) + \frac{1}{2}M\dot{\beta}^2\right\}. \tag{35}$$

The case of free oscillations ($P_1=P_2=0$) is of particular interest. Equation (35) leads to the energy integral

$$4\pi\Psi(\beta; R_1, R_2) + \frac{1}{2}M\dot{\beta}^2 = C, \tag{36}$$

with C constant. Equation (36) may be integrated to obtain an inverse description $t=t(\beta)$ of the motion

$$t = \sqrt{\frac{M}{2}}\int \frac{d\beta}{\sqrt{C - 4\pi\Psi(\beta; R_1, R_2)}}. \tag{37}$$

It is expected that, for admissible values of C, the denominator of the integrand in (37) will have just two zeros $\beta_{\min}$ ($> -R_1$) and $\beta_{\max}$, which represent the extreme excursions of the inner and outer boundaries. The period τ of the free oscillation is found from (37) as

$$\tau = \sqrt{2M}\int_{\beta_{\min}}^{\beta_{\max}} \frac{d\beta}{\sqrt{C - 4\pi\Psi(\beta; R_1, R_2)}}. \tag{38}$$

6 Eversion

The eversion solution of special interest is that which leaves the boundaries traction free after deformation. Rewriting (22) as

$$r = B - R, \quad \theta = \pi - \Theta, \quad \varphi = \Phi \tag{39}$$

and using (27) leads to the condition

$$G(B; R_1, R_2) = 0, \tag{40}$$

which serves to determine B. The value $B = R_1 + R_2$ is of special interest because it implies that $r_1 = R_2$ and $r_2 = R_1$, so that the dimensions of the hollow sphere are unchanged by eversion. The class of materials for which this is true for all initial radii R_1 and R_2 is that for which

$$G(R_1 + R_2; R_1, R_2) = 0, \tag{41}$$

is an identity, i.e., for which

$$\int_{1/\gamma}^{\gamma} \frac{T(\lambda)\lambda d\lambda}{(\lambda+1)^3} = 0 \qquad \textit{for all } \gamma. \tag{42}$$

7 Inflation or Compaction of an Everted Sphere

The deformation

$$r = B + \beta - R, \quad \theta = \pi - \Theta, \quad \varphi = \Phi, \tag{43}$$

where B is the root of (40), describes radial inflation ($\beta > 0$) or compaction ($\beta < 0$) of an everted hollow sphere. The solution for internal pressure P_2 and external pressure P_1 is

$$r_1^2 P_1 - r_2^2 P_2 = G(B + \beta; R_1, R_2). \tag{44}$$

This solution may be compared with the corresponding solution (17) for a natural (uneverted) sphere.

In the case of inflation under internal pressure P, for example, the two solutions for natural (uneverted) and everted spheres are, respectively,

$$P = \frac{1}{(R_1 + \beta)^2} F(\beta; R_1, R_2) \tag{45}$$

and

$$P = \frac{1}{(B + \beta - R_2)^2} G(B + \beta; R_1, R_2). \tag{46}$$

8 Radial Oscillation of an Everted Sphere

Equation (43), with $\beta = \beta(t)$, describes a radial motion of an everted hollow sphere. An analysis paralleling that in Section 5 leads to the energy balance equation

$$4\pi\dot{\beta}\left(r_2^2 P_2 - r_1^2 P_1\right) = \frac{d}{dt}\left\{4\pi\,\Gamma(B + \beta; R_1, R_2) + \frac{1}{2} M \dot{\beta}^2\right\}, \tag{47}$$

with

$$\Gamma(B + \beta; R_1, R_2) = \int G(B + \beta; R_1, R_2) d\beta. \tag{48}$$

The energy integral

$$4\pi T(B+\beta;R_1,R_2)+\frac{1}{2}M\dot{\beta}^2=C, \tag{49}$$

pertains to free oscillations, which are described in inverse form by

$$t=\sqrt{\frac{M}{2}}\int\frac{d\beta}{\sqrt{C-4\pi T(B+\beta;R_1,R_2)}}, \tag{50}$$

where roots $\beta_{\min}$ and $\beta_{\max}$, are the admissible roots of the denominator of the integrand in (50). The period of free oscillation is found from (50) as

$$\tau=\sqrt{2M}\int_{B+\beta_{\min}}^{B+\beta_{\max}}\frac{d\beta}{\sqrt{C-4\pi T(B+\beta;R_1,R_2)}}. \tag{51}$$

9 Harmonic and Compressible Varga Materials

9.1 Harmonic Material

The strain energy function for a harmonic material has the form (cf., Carroll and Rooney [8])

$$W=2\mu\{f(i_1-3)-\zeta(i_2-3)-(1-\zeta)(i_3-1)\}. \tag{52}$$

Here f( ) is a nonlinear function with

$$f(0)=0,\quad f'(0)=1+\zeta,\quad f''(0)=\frac{1-\nu}{1-2\nu}, \tag{53}$$

μ and ν are the shear modulus and Poisson's ratio for infinitesimal deformation, and ζ is a non-dimensional parameter. Equations (4), $(6)_2$ and (52) give the axi-symmetric stress response function as

$$T(\lambda)=\frac{2\mu}{\lambda}\{f'(2\lambda-2)-\lambda-\zeta\}. \tag{54}$$

If the function $f(\ )$ is a polynomial of degree n, so that

$$f(i_1-3)=\sum_{k=1}^{n}a_k(i_1-3)^k,\quad a_1=1+\varsigma,\quad a_2=\frac{1-\nu}{1-2\nu}, \tag{55}$$

then

$$T(\lambda)=\frac{2\mu}{\lambda}\left[\frac{\lambda-1}{1-2\nu}+\sum_{k=3}^{n}ka_k\,\{2(\lambda-1)\}^{k-1}\right]. \tag{56}$$

It follows from (56) and (15) that the function $F(\beta;\, R_1,\, R_2)$, for harmonic materials with polynomial response functions described by (52) and (55), is

$$F(\beta_1; R_1, R_2) = 4\mu\beta^2\left[\frac{R_2 - R_1}{(1-2\nu)\beta} + 12a_3 \ln\frac{R_2}{R_1} + \sum_{k=4}^{n}\frac{k}{k-3}2^{k-1}a_k\left\{\left(\frac{\beta}{R_1}\right)^{k-3} - \left(\frac{\beta}{R_2}\right)^{k-3}\right\}\right]. \tag{57}$$

Observe that in the simplest case when a_k=0 for k>2, i.e., for an harmonic material with strain energy

$$W = 2\mu\left\{(i_1 - 3) + \frac{1-\nu}{2(1-2\nu)}(i_1 - 3)^2 - \zeta(i_2 - 3) - (1-\zeta)(i_3 - 1)\right\}, \tag{58}$$

the pressure attains its maximum value

$$P_{\max} = \frac{\mu}{1-2\nu}\left(\frac{R_2}{R_1} - 1\right) \tag{59}$$

when β=R_1, i.e., when the inner radius attains twice its original value, and then tends to zero as β continues to increase. For this strain energy, it follows from (29) and (57) that the hollow sphere compacts fully (β=−R_1) at finite external pressure

$$P_{cr} = \frac{4\mu}{1-2\nu}\left(\frac{R_2}{R_1} - 1\right)^{-1}. \tag{60}$$

For an harmonic material with strain energy (58), a straightforward calculation from (57), (34), (37) and (38) shows that the oscillation is sinusoidal, with amplitude-independent period

$$\tau = \sqrt{\frac{(1-2\nu)M}{4\pi\mu(R_2 - R_1)}}. \tag{61}$$

The function $G(\ )$ is found for harmonic materials with polynomial response functions by using the substitution $\lambda - 1 = (\lambda + 1) - 2$ to rewrite (56) as

$$\lambda T(\lambda) = \sum_{k=0}^{n-1} b_k(\lambda + 1)^k. \tag{62}$$

Substituting from this equation in (25) gives

$$G(\beta; R_1, R_2) = b_o\left(R_2^2 - R_1^2\right) + 2b_1(R_2 - R_1) + 2b_2\ln(R_2/R_1) + 2\sum_{k=1}^{n-2}\frac{1}{k}b_{k+2}\beta^{k+2}\left\{\left(\frac{1}{R_1}\right)^k - \left(\frac{1}{R_2}\right)^k\right\}. \tag{63}$$

For the special strain energy (58),

$$G(\beta; R_1, R_2) = \frac{2\mu}{1-2\nu}(R_2 - R_1)(\beta - R_1 - R_2). \tag{64}$$

This has the property (41), so that, in this case, the dimensions of the hollow sphere are unchanged by eversion. For the same strain energy, both uneverted and everted solutions reduce to

$$P = \frac{4\mu\beta(R_2 - R_1)}{(1-2\nu)(R_1+\beta)^2}, \tag{65}$$

so that, in this case, and the response is unaffected by eversion. Results for the radial oscillation of the everted sphere are the same as uneverted results, as expected.

9.2 Compressible Varga Material

The strain energy for the compressible Varga material has the form

$$W = 2\mu\{\zeta(i_1-3) + (1-\zeta)(i_2-3) - h(i_3-1)\}, \tag{66}$$

where $h(\ )$ is a nonlinear function with

$$h(0) = 0, \quad h'(0) = \zeta - 2, \quad h''(0) = \frac{1-\nu}{2(1-2\nu)}. \tag{67}$$

ν is Poisson's ratio for infinitesimal deformation, μ is the shear modulus and ζ is a non-dimensional parameter. From (4), $(6)_2$ and (66) the stress response function can be found as

$$T(\lambda) = \frac{2\mu}{\lambda}\left\{1 + (1-\zeta)\lambda + \lambda h'\left(\lambda^2-1\right)\right\}. \tag{68}$$

If the function $h(\ )$ is a polynomial of degree n, so that

$$h(i_3-1) = \sum_{j=1}^{n} a_j(i_3-1)^j, \quad a_1 = \varsigma - 2, \quad a_2 = \frac{1-\nu}{2(1-2\nu)}, \tag{69}$$

then

$$T(\lambda) = \frac{2\mu}{\lambda}\left[1 - \lambda + \frac{1-\nu}{1-2\nu}\left(\lambda^3-\lambda\right) + \lambda\sum_{j=3}^{n} ja_j\left(\lambda^2-1\right)\}^{j-1}\right]. \tag{70}$$

For the simplest case when a_j=0 for j>2, i.e., for a compressible Varga material with strain energy

$$W = 2\mu\left\{\zeta(i_1-3) + (1-\zeta)(i_2-3) - (\zeta-2)(i_3-1) + \frac{1-\nu}{2(1-2\nu)}(i_3-1)^2\right\}, \tag{71}$$

from (70) and (15) the function $F(\beta;\, R_1,\, R_2)$ is found as

$$F(\beta; R_1, R_2) = 4\mu\beta^2\left[-\frac{1-2\kappa}{\beta}(R_2-R_1) + \kappa\beta\frac{R_2-R_1}{R_2R_1} + 3\kappa\ln\frac{R_2}{R_1}\right], \quad \kappa = \frac{1-\nu}{1-2\nu}. \tag{72}$$

Figure 1 shows the relation between normalized β and normalized pressure for different t=R_2/R_1 values. There is no maximum pressure point for the compressible Varga material, as $\beta\to\infty$, $P\to\infty$.

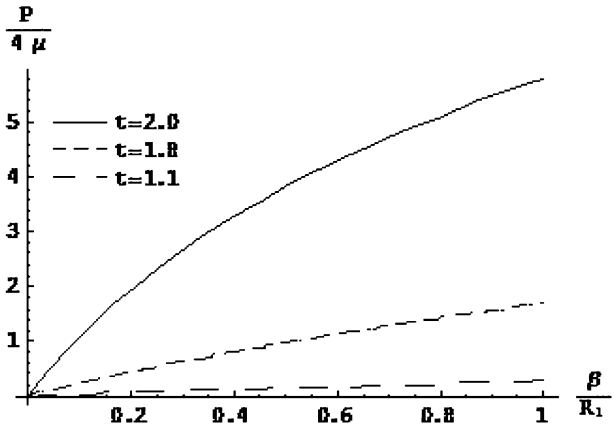

Fig. 1 Relation between normalized β and normalized pressure for different $t=R_2/R_1$ and $\kappa=2$

The function $G(\ )$ for the simplest compressible Varga material is found by using (25) and (70) as

$$G(\beta;R_1,R_2)=4\mu\beta^2\left[-\frac{1-2\kappa}{\beta}(R_2-R_1)+\kappa\beta\frac{R_2-R_1}{R_2R_1}-3\kappa\ln\frac{R_2}{R_1}+\frac{R_2^2-R_1^2}{\beta^2}\right],\quad \kappa=\frac{1-\nu}{1-2\nu}. \tag{73}$$

This function does not have the property (41), so the dimensions of the sphere are changed by eversion. For the special strain energy (71) the pressures required to inflate the everted

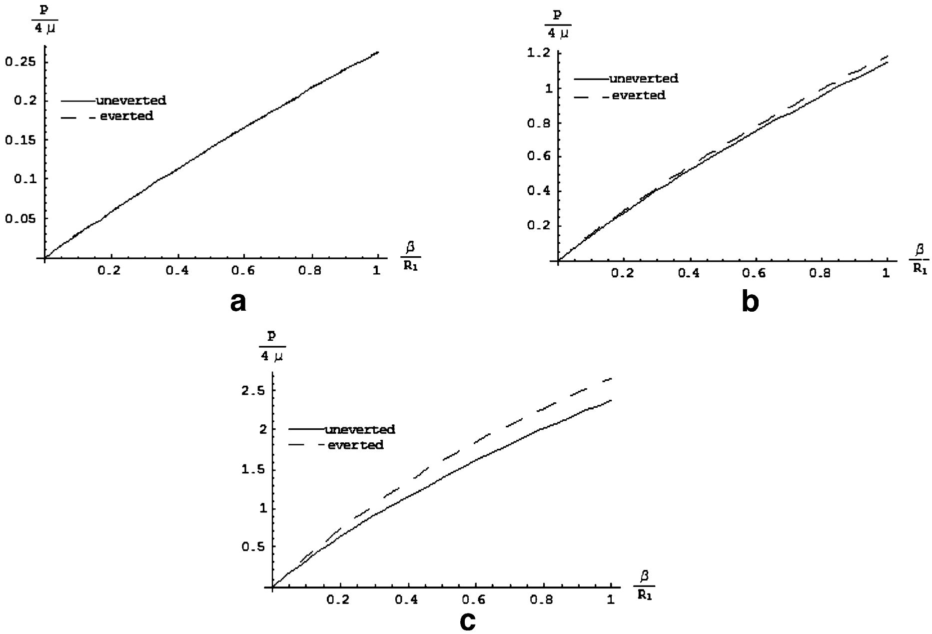

Fig. 2 Relation between normalized β and normalized pressure for everted and uneverted spheres for **a** $R_2/R_1=1.1$, $\kappa=2$, **b** $R_2/R_1=1.5$, $\kappa=2$ and **c** $R_2/R_1=2.0$, $\kappa=2.3$

and uneverted spheres are shown in Fig 2(a–c) for admissible values of R_2/R_1 and κ. It is observed that as the thickness of the sphere increases it is harder to inflate the everted sphere.

10 Limiting Cases

The results obtained in Sections 4 and 5 pertain to spherical cavities in infinite media, in the limit as $R_2 \to \infty$ and to thin-walled spherical shells, in the limit as $R_2 \to R_1$.

For a spherical cavity, the function $F(\beta;\ R_1,\ R_2)$, defined in (15), reduces to the function

$$\widehat{F}(\beta; R_1) = 2\beta^2 \int_1^{1+\beta/R_1} \frac{T(\lambda)\lambda d\lambda}{(\lambda-1)^3}. \tag{74}$$

This is singular unless $T''(1)=0$, which is an unrealistic condition, since it would imply vanishing of the shear modulus of the non-reinforced material in infinitesimal deformation (cf., the first two terms on the right hand side in (57)). Thus, a radially inextensible material of infinite extent containing a spherical cavity behaves under internal pressure as a rigid body.

For a thin-walled shell with mean radius R and thickness h, the function $F(\beta;\ R_1,\ R_2)$ reduces to the function

$$\widetilde{F}(\beta; R, h) = 2Rh\lambda T(\lambda), \quad \lambda = 1 + \beta/R \tag{75}$$

and the solution for internal pressure P is

$$P = \frac{2hT(1+\beta/R)}{R(1+\beta/R)}. \tag{76}$$

The solutions for time-dependent pressurization and for free oscillation are given by (39)–(43), with $\Psi(\beta;\ R_1,\ R_2)$ replaced by

$$\widetilde{\Psi}(\beta; R, h) = \int \widetilde{F}(\beta; R, h) d\beta. \tag{77}$$

11 Conclusion

Radial deformations and motions, with or without eversion, are controllable for radially inextensible hollow spheres that are isotropic or locally transversely isotropic. The relevant material response property is the axi-symmetric stress response function $T(\lambda)$ and integration of the equations of equilibrium or motion introduces additional functions $F(\beta;\ R_1,\ R_2)$ and $G(\beta;\ R_1,\ R_2)$, the latter describing eversion and post-eversion behavior. Comparison of the static and dynamic solutions in Sections 4 and 5 with their post-eversion counterparts in Sections 7 and 8 allows an assessment of the effects of pre-stress on mechanical response for any particular strain energy, i.e., for any particular $T(\lambda)$. The stiffness of the response in radial deformation naturally increases as the wall thickness increases and it becomes effectively rigid for a spherical cavity in an infinite medium.

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