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Optimal inventory policy for capacitated systems with two supply sources

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ABSTRACT

We study a periodic review, dual-sourcing inventory problem where suppliers are differentiated based on their variable and fixed costs as well as their order size constraints. Based on the preservation of quasi-convexity and the proposed preservation of strong *CK*-convexity, we partially characterize the structure of the optimal policy.

1. Introduction

When replenishing inventories, there is often more than one supplier from which to choose. In this paper, we study an integrated procurement and inventory control problem with two suppliers. Procuring from the first supplier involves a high unit purchase cost but negligible fixed cost. Procuring from the second supplier involves a low unit purchase cost but a high fixed cost, as well as a constraint on the maximum order size. The problem arises in a variety of settings. For example, firms in industries such as chemical manufacturing may have to decide whether to produce some of the input ingredients in-house or to procure them from outside vendors. Producing them in-house can be cheaper but may require a significant setup cost and is subject to capacity constraints. Firms that are involved in the procurement of commodities may have to choose between purchasing from a long term supplier that offers preferential pricing but is geographically distant or purchasing from a local spot market with lower transportation cost but at higher unit price.

The problem we consider is related to the growing literature dealing with inventory control with dual (and in some cases multiple) suppliers. Most of this literature considers systems where suppliers are differentiated instead based on their variable costs and their lead times; see [5,7,10]. There is also literature that considers sourcing from multiple suppliers when some of the suppliers are unreliable, either in terms of supply lead times or quantities delivered [1,8,9]. Another stream of the multi-sourcing literature considers the difference of suppliers based on the fixed cost and procurement cost [2,11].

In this paper, we consider a periodic review, dual-sourcing inventory system where ordering from the supplier with lower variable cost is subject to both fixed cost and order size constraint. Based on the preservations of quasi-convexity and strong *CK*-convexity, we partially characterize the structure of optimal inventory policy (in fact, even for single-sourcing systems with both fixed cost and order size constraint, the existing literature just partially characterizes the optimal inventory policy; see [3,6]).

The rest of the paper is organized as follows. In Section 2, we formulate the problem as a stochastic dynamic program. In Section 3, we introduce the notions of quasi-convexity and strong *CK*-convexity, and show their preservations. In Section 4, we partially characterize the structure of optimal inventory policy. In Section 5, we give some concluding remarks.

2. Problem formulation

We consider an inventory control problem of a single item over a finite planning horizon consisting of N discrete time periods. Demand in each period can be described by a non-negative random variable D. Demand realizations in different periods are independently and identically distributed with a probability density function φ that is strongly unimodal (i.e., $\log(\varphi(\xi))$ is concave). This class of distributions is quite general and covers many of the commonly used distributions [11]. In each period, an order can be placed with one of two suppliers, supplier H and supplier L, or with both suppliers. The purchase costs per unit from supplier H and supplier L are c^H and c^L ($c^L < c^H$), respectively. There is a fixed cost K ($K > 0$) to purchasing from supplier L, but there is no fixed cost to purchasing from supplier H. There is a limit on the order size that can be placed with supplier L, which we denote by C, where $C < \infty$, but no limit on order sizes with supplier H. Orders placed in one period from any of the two suppliers can be used to satisfy demand from that period.

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At the beginning of each period, the firm observes the starting inventory level x, and places an order of size q^L with supplier L and q^H with supplier H. Orders placed are then delivered, bringing the inventory level to $y = x + q$, where $q = q^L + q^H$ denotes the total order quantity. Finally demand realizes, leftover inventory is carried to the next period with unit holding cost h, and excess demand is backlogged with unit backlogging cost b. We assume that unit backlogging cost satisfies that $b > c^H - \alpha c^L$ ($\alpha \in (0, 1)$ is the discount factor), which means that the cost of backlogging a unit for one period exceeds the saving from delaying the purchase of the unit by one period. The objective of the system manager is to minimize the expected discounted cost over the planning horizon.

It is easy to show that the problem degenerates to a classic single supplier problem (supplier H) with no order size constraint if $C \le C_0 = K/(c^H - c^L)$. To avoid triviality, we assume throughout this paper that $C > C_0$. We define the ordering cost function, $OC(q)$, as follows:

$$OC(q) = \begin{cases} c^H q & q \in [0, C_0] \\ K + c^L q & q \in [C_0, C] \\ K + (c^L - c^H)C + c^H q & q \in [C, \infty). \end{cases} \quad (1)$$

We number periods in reverse order so that period 1 is the last period in the planning horizon, while period N is the first period. Let $f_n(x)$ be the optimal (expected discounted) cost from period n to the end of the planning horizon. Then, $f_n(x)$ satisfies the following dynamic programming recursion:

$$f_n(x) = \min_{y \ge x}\{OC(y - x) + g_n(y)\}, \quad (2)$$

$$g_n(y) = L(y) + \alpha E f_{n-1}(y - D), \qquad f_0(x) = p(-x)^+, \quad (3)$$

where $L(y) = hE(y - D)^+ + bE(D - y)^+ (u^+ = \max(u, 0))$ is the holding-backlogging cost, and p is the unit penalty cost of the unfulfilled demand at the end of planning horizon. We assume that this unit penalty cost is larger than the unit purchase cost from supplier L, i.e., $p \ge c^L$. Denote by $y_n^*(x)$ the optimal order-up-to level with beginning inventory level x in period n.

3. Preliminary results

In this section, we introduce the notions of quasi-convexity and strong CK-convexity, and show their preservations, respectively.

Definition 1. A function $f(x)$ is quasi-convex if $f(\lambda x_1 + (1 - \lambda)x_2) \le \max\{f(x_1), f(x_2)\}$ for all $\lambda \in [0, 1]$.

An implication of this definition is that the quasi-convex function $f(x)$ is either always increasing in x, always decreasing in x, or first decreasing and then increasing. Throughout this paper, we use "increasing" and "decreasing" in the non-strict sense to mean "nondecreasing" and "nonincreasing", respectively. From [4], we have the following preservation of quasi-convexity:

Lemma 1 ([4]). *Suppose that $f(x)$ is quasi-convex. If the distribution of random variable D is strongly unimodal, then $Ef(x-D)$ is also quasi-convex.*

Lemma 1 indicates that quasi-convexity is preserved for integral convolutions, provided that the distribution of random variable is strongly unimodal. This lemma will be used in characterizing the optimal supplier selection and the optimal ordering from supplier L.

We next introduce strong CK-convexity, which is first proposed in [3].

Definition 2 ([3]). A function $f : \mathbb{R} \to \mathbb{R}$ is called strongly CK-convex ($C > 0,\ K \ge 0$) if

$$K + f(y + z) \ge f(y) + (z/b)[f(y - a) - f(y - a - b)] \quad (4)$$

for all y, $a \ge 0$, $b > 0$ and $z \in [0, C]$.

From Definition 2, it is easy to verify that: (i) any linear or convex function is strongly CK-convex for all C and K; (ii) if f is strongly CK-convex, then αf is strongly CK'-convex with $K' = \alpha K$; and (iii) a strongly CK-convex function is also strongly $\tilde{C}\tilde{K}$-convex with $\tilde{C} \le C$ and $\tilde{K} \ge K$.

The following technical lemma describes the preservation of strong CK-convexity when the left part of a strongly CK-convex function undergoes a certain modification.

Lemma 2. *Suppose that strong CK-convex function $g(y)$ satisfies that $g(y)+cy$ is increasing when $y \ge A$, where c is a constant. The modified function of $g(y)$,*

$$f(y) = \begin{cases} r(y) - r(A) + g(A) & y \le A \\ g(y) & y \ge A, \end{cases}$$

is strongly CK-convex if the function $r(y)$ is strong CK-convex, and $r(y) + cy$ is decreasing when $y \le A$.

Proof. We just focus on the case with $z \in (0, C]$, because Eq. (4) clearly holds when $z = 0$. Define $\tilde{r}(y) = r(y) - r(A) + g(A)$. It is easy to verify that $\tilde{r}(y)$ is also strong CK-convex, $\tilde{r}(A) = g(A)$ and $\tilde{r}(y) + cy$ is decreasing when $y \le A$. The strong CK-convexities of $g(y)$ and $\tilde{r}(y)$ implies that they satisfy Eq. (4) for $A \le y - a - b < y - a \le y \le y + z$ and $y - a - b < y - a \le y \le y + z \le A$, respectively. So, we only need to verify whether Eq. (4) holds or not when $y - a - b < A$ and $y + z > A$. We discuss in the following three cases depending on the relationship among A, $y - a$ and y.

Case 1. $y - a - b < y - a \le y < A < y + z$.

Since $f(y) = \tilde{r}(y)$ when $y \le A$, and $\tilde{r}(y)$ satisfies Eq. (4) for $y - a - b < y - a \le y \le y + z \le A$,

$$K + f(A) - f(y) \ge [(A - y)/b][f(y - a) - f(y - a - b)]. \quad (5)$$

Here A can be viewed as $y + z'$ with $0 < z' = A - y < z \le C$. It can be verified that

$$\begin{aligned} f(y + z) - f(A) &= g(y + z) - g(A) \ge -c(y + z - A) \\ &\ge [(y + z - A)/b][\tilde{r}(y - a) - \tilde{r}(y - a - b)] \\ &= [(y + z - A)/b][f(y - a) - f(y - a - b)]. \quad (6) \end{aligned}$$

Note that the first and second inequalities hold because $g(y) + cy$ is increasing when $y \ge A$, and $\tilde{r}(y) + cy$ is decreasing when $y \le A$, respectively. Eqs. (5) and (6) indicate that Eq. (4) holds in Case 1.

Case 2. $y - a - b < y - a \le A \le y < y + z$.

From the monotonicities of $\tilde{r}(y) + cy$ when $y \le A$ and $g(y) + cy$ when $y \ge A$, we have

$$\begin{aligned} & f(y - a) - f(y - a - b) \\ & \quad = g(y - a) - g(y - a - b) \le -cb, \quad \text{and} \\ & f(y + z) - f(y) = g(y + z) - g(y) \ge -cz. \end{aligned}$$

So, $K + f(y + z) - f(y) \ge K - cz > -cz = -(z/b)(cb) \ge (z/b)[f(y - a) - f(y - a - b)] (K > 0)$.

Case 3. $y - a - b < A < y - a \le y < y + z$.

Since $f(y) = g(y)$ when $y \ge A$, and $g(y)$ satisfies Eq. (4) for $A \le y - a - b < y - a \le y \le y + z$,

$$K + f(y + z) \ge f(y) + [z/(y - a - A)][f(y - a) - f(A)]. \quad (7)$$

Here A can be viewed as $y - a - b'$ with $b' = y - a - A > 0$. Since $f(y) + cy = g(y) + cy$ is increasing when $y \ge A$, and $f(y) + cy = \tilde{r}(y) + cy$ is decreasing when $y \le A$, we have

$$\begin{aligned} & [f(y - a) - f(y - a - b)]/b \\ & \quad \le [f(y - a) - f(A)]/(y - a - A). \quad (8) \end{aligned}$$

Eqs. (7) and (8) indicates that Eq. (4) holds in Case 3. In summary, Lemma 2 holds. □

Lemma 2 indicates that the strong CK-convexity is preserved if the undergone modification is strongly CK-convex and satisfies a certain monotonic property. This property will be used to characterize the optimal ordering from supplier L, which is new and has potential applicability to relevant problems where strong CK-convexity holds, e.g., the problems that integrate random purchase cost(s) into our dual-sourcing problem.

In fact, by replacing the strong CK-convexities of $g(y)$ and $r(y)$ with slack conditions, Lemma 2 still holds. That is, if (i) $g(y)$ just satisfies Eq. (4) for all $A \le y-a-b < y-a \le y \le y+z$, and (ii) $r(y)$ just satisfies Eq. (4) for all $y-a-b < y-a \le y \le y+z \le A$, then Lemma 2 can be also proved by following the same process in the proof of Lemma 2.

4. Structure of the optimal policy

Based on Lemma 1, we have the following lemma.

Lemma 3. *The function $g_n^H(y) = g_n(y) + c^H y$ is quasi-convex for all $n \ge 1$.*

Proof. We first show the monotonocities of $f_n(x) + c^H x$ and $f_n(x) + c^L x$, and then prove the quasi-convexity of $g_n^H(y)$ based on Lemma 1. It is clear that $f_0(x) + c^H x$ is increasing when $x \ge 0$. It can be verified that $OC(y-x) + c^H x$ is increasing in x ($x \le y$) for all y. Hence, for all $n \ge 1$ and $x^2 > x^1$,

$$\begin{aligned} f_n(x^2) + c^H x^2 &= \min_{y \ge x^2}\{OC(y-x^2) + c^H x^2 + g_n(y)\} \\ &\ge \min_{y \ge x^2}\{OC(y-x^1) + c^H x^1 + g_n(y)\} \\ &\ge \min_{y \ge x^1}\{OC(y-x^1) + c^H x^1 + g_n(y)\} \\ &= f_n(x^1) + c^H x^1. \end{aligned}$$

That is, $f_n(x) + c^H x$ is increasing for all $n \ge 1$. So $f_n(x) + c^H x$ is increasing when $x \ge 0$ for all $n \ge 0$.

We show the monotonicity of $f_n(x) + c^L x$ when $x < 0$ by induction. It is clear that $f_0(x) + c^L x = p(-x)^+ + c^L x$ is decreasing ($p \ge c^L$) when $x < 0$. Suppose that $f_{n-1}(x) + c^L x$ is decreasing when $x < 0$. We first consider $y_n^*(x)$ for $n \ge 1$. It is clear that $y_n^*(x) \ge x \ge 0$ if $x \ge 0$. For any $x < 0$,

$$\begin{aligned} f_n(x) &= \min_{y \ge x}\{OC(y-x) + g_n(y)\} \\ &= \min_{y \ge x}\{OC(y-x) + L(y) + \alpha E f_{n-1}(y-D)\} \\ &= \min_{y \ge x'}\{OC(y-x) - c^H y + (c^H - \alpha c^L)y + L(y) \\ &\quad + \alpha[E f_{n-1}(y-D) + c^L y]\}. \end{aligned}$$

It is easy to verify that (i) $OC(y-x) - c^H y$ is decreasing, (ii) $(c^H - \alpha c^L)y + L(y) = (c^H - \alpha c^L)y - by + bED$ is strictly decreasing when $y < 0$ ($b > c^H - \alpha c^L$), and (iii) $E f_{n-1}(y-D) + c^L y$ is decreasing when $y < 0$ from the monotonicity of $f_{n-1}(x) + c^L x$. So $OC(y-x) + g_n(y)$ is strictly decreasing when $y < 0$. This implies that $y_n^*(x) \ge 0$ when $x < 0$. So $y_n^*(x) \ge 0$ for all x. When $x < x' < 0 \le y_n^*(x)$, we have

$$\begin{aligned} f_n(x) + c^L x &= OC[y_n^*(x) - x] + c^L x + g_n[y_n^*(x)] \\ &\ge OC[y_n^*(x) - x'] + c^L x' + g_n[y_n^*(x)] \\ &\ge \min_{y \ge x'}\{OC(y-x') + c^L x + g_n(y)\} = f_n(x') + c^L x'. \end{aligned}$$

Note that the first inequality holds because $OC(y-x) + c^L x$ is decreasing in x. So, $f_n(x) + c^L x$ is decreasing when $x < 0$ for all $n \ge 0$.

Rewrite functions $g_n^H(y)$ as $g_n^H(y) = c^H E(D) + E R_n^H(y-D)$, where E represents expectation, and

$$\begin{aligned} R_n^H(u) &= hu^+ + bu^- + c^H u + \alpha f_n(u) \\ &= \begin{cases} -(b - c^H - \alpha c^L)u + \alpha(f_{n-1}(u) + c^L) & u < 0 \\ (h + c^H - \alpha c^H)u + \alpha[f_{n-1}(u) + c^H u] & u \ge 0. \end{cases} \end{aligned}$$

Since $f_n(x) + c^H x$ is increasing when $x \ge 0$ for all $n \ge 0$, and $f_n(x) + c^L x$ is decreasing when $x < 0$ for all $n \ge 0$, we have that $R_n^H(u)$ is quasi-convex (first decreasing and then increasing) for all $n \ge 1$. Lemma 1 implies that $g_n^H(y)$ is also quasi-convex for all $n \ge 1$. □

From the proof of Lemma 3, we see that $b > c^H - \alpha c^L$ implies that $y_n^*(x) \ge 0$ for all $n \ge 1$. That is, any backlogged demand in one period can be fulfilled by the total order quantity in the next period.

Define $S_n^H \triangleq \arg\min_{y \in \mathbb{R}^+} g_n^H(y)$, $s_n^1 \triangleq S_n^H - C$ and $s_n^2 \triangleq S_n^H - C_0$. It is easy to verify that $s_n^1 < s_n^2 < S_n^H$ because $C > C_0 > 0$. Based on the quasi-convexity of $g_n^H(y)$, the following theorem characterizes the optimal ordering from supplier H and the optimal supplier selection.

Theorem 1. *In period n, the optimal ordering from supplier H follows a base-stock policy with the base-stock level S_n^H. The optimal supplier selection satisfies:*

(i) *If $x < s_n^1$, then it is optimal to order from both suppliers L and H.*
(ii) *If $s_n^1 \le x < s_n^2$, then it is optimal to order exclusively from supplier L.*
(iii) *If $s_n^2 \le x < S_n^H$, then it is optimal to order exclusively either from supplier L or from supplier H.*
(iv) *If $x \ge S_n^H$, then it is optimal to order exclusively from supplier L.*

Proof. Define $F_n^1(x) = \min_{y \in [x, x+C_0]} g_n^H(y) - c^H x$, $F_n^2(x) = \min_{y \in [x+C_0, x+C]} g_n^L(y) - c^L x + K$,

$$F_n^3(x) = \min_{y \in [x+C, \infty)} g_n^H(y) - c^H x + (c^L - c^H)C + K,$$

where $g_n^L(y) = g_n(y) + c^L y$. Note that $F_n^1(x) \le \min_{y \in [x, x+C_0]}\{g_n^L(y) + K\delta(y-x)\} - c^L x$ ($\delta(u) = 1$ if $u > 0$, otherwise $\delta(u) = 0$) because it is optimal to order from supplier H when the total order quantity is smaller than C_0. From Eqs. (1) and (2), we have $f_n(x) = \min[F_n^1(x), F_n^2(x), F_n^3(x)]$. Here $F_n^1(x)$ and $F_n^2(x)$ are the optimal costs of ordering exclusively from supplier H and supplier L, respectively, and $F_n^3(x)$ is the optimal cost of ordering simultaneously from both suppliers.

Recall that the quasi-convexity of $g_n^H(y)$ implies that $g_n^H(y)$ is first decreasing when $y \le S_n^H$, and then increasing when $y \ge S_n^H$. Since $c^H > c^L$, $g_n^L(y)$ is also decreasing when $y \le S_n^H$. If $x < s_n^1$, then the monotonicities of $g_n^H(y)$ and $g_n^L(y)$ imply that

$$\begin{aligned} F_n^1(x) &= g_n^H(x+C_0) - c^H x = g_n(x+C_0) + c^H C_0, \\ F_n^2(x) &= g_n^L(x+C) - c^L x + K \le g_n^L(x+C_0) - c^L x + K \\ &= g_n(x+C_0) + c^H C_0, \\ F_n^3(x) &= g_n^H(S_n^H) - c^H x + (c^L - c^H)C + K \\ &\le g_n^H(x+C) - c^H x + (c^L - c^H)C + K \\ &= g_n^L(x+C) - c^L x + K. \end{aligned}$$

That is, $F_n^3(x) \le F_n^2(x) \le F_n^1(x)$. Here $g_n^L(y)$ is decreasing when $x + C_0 \le y \le x + C$ because $x + C \le S_n^H$ when $x < s_n^1 = S_n^H - C$. So, when $x < s_n^1$, $f_n(x) = F_n^3(x) = g_n^H(S_n^H) - c^H x + (c^L - c^H)C + K$ and $y_n^*(x) = S_n^H$. That is, it is optimal to order simultaneously from both suppliers (C units from supplier L and the rest ($S_n^H - C - x$) units from supplier H).

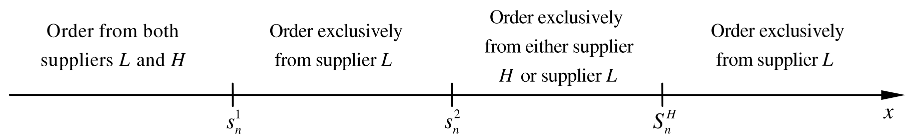

Fig. 1. The optimal supplier selection for all starting inventory levels.

If $x \geq s_n^1$, then $F_n^3(x) = g_n^H(x+C) - c^H x + (c^L - c^H)C + K = g_n^L(x+C) - c^L x + K \geq F_n^2(x)$. Here the first and second equalities hold because $g_n^H(y)$ is increasing when $y \geq S_n^H$, and $g_n^H(y) = g_n^L(y) + (c^H - c^L)y$, respectively. So, we only need to compare $F_n^1(x)$ and $F_n^2(x)$ if $x \geq s_n^1$.

If $s_n^1 \leq x < s_n^2$, then

$$F_n^1(x) = g_n^H(x+C_0) - c^H x = g_n(x+C_0) + c^H C_0, \quad \text{and}$$
$$F_n^2(x) = \min_{y\in[x+C_0,x+C]} g_n^L(y) - c^L x + K$$
$$\leq g_n^L(x+C_0) - c^L x + K = g_n(x+C_0) + c^H C_0.$$

Hence, $f_n(x) = F_n^2(x)$ when $s_n^1 \leq x < s_n^2$. That is, it is optimal to order exclusively from supplier L, and $y_n^*(x) = \min_{y\in[x+C_0,x+C]} g_n^L(y)$. Since $F_n^2(x) \leq F_n^1(x) \leq \min_{y\in[x,x+C_0]}\{g_n^L(y) + K\delta(y-x)\} - c^L x$, the optimal order-up-to level can be rewritten as $y_n^*(x) = \arg\min_{y\in[x,x+C]}\{g_n^L(y) + K\delta(y-x)\}$.

If $s_n^2 \leq x < S_n^H$, then it is optimal to order exclusively either from supplier L or from supplier H because $f_n(x) = \min[F_n^1(x), F_n^2(x)]$, where $F_n^1(x) = g_n^H(S_n^H) - c^H x$ and $F_n^2(x) = \min_{[x+C_0,x+C]} g_n^L(y) - c^L x + K$. Here $y_n^*(x) = S_n^H$ if $F_n^1(x) \leq F_n^2(x)$. Similarly, we also have $y_n^*(x) = \arg\min_{y\in[x,x+C]}\{g_n^L(y) + K\delta(y-x)\}$ if $F_n^1(x) > F_n^2(x)$. Note that it is optimal to order exclusively from supplier L when $x = s_n^2$, because

$$F_n^1(s_n^2) = \min_{y\in[s_n^2,s_n^2+C_0]} g_n^H(y) - c^H x = g_n(s_n^2+C_0) + c^H C_0$$
$$= g_n^L(s_n^2+C_0) - c^L s_n^2 + K$$
$$\geq \min_{y\in[s_n^2+C_0,s_n^2+C]} g_n^L(y) - c^L s_n^2 + K = F_n^2(s_n^2). \quad (9)$$

If $x \geq S_n^H$, then $F_n^1(x) = g_n(x)$, and $F_n^2(x) = \min_{[x+C_0,x+C]} g_n^L(y) - c^L x + K$. Since "ordering nothing" can be viewed as order nothing from supplier L, it is optimal to order exclusively from supplier L if $x \geq S_n^H$. It can be verified that $y_n^*(x) = \arg\min_{y\in[x,x+C]}\{g_n^L(y) + K\delta(y-x)\}$ because it is not optimal to order no more than C_0 units from supplier L. □

Fig. 1 graphically illustrates the optimal supplier selection for each possible starting inventory level. As we can see, sourcing simultaneously from both suppliers occurs only if the starting inventory level is smaller than s_n^1; otherwise, it is optimal to order exclusively from only one of the suppliers.

To determine which of two suppliers is optimal to select if $s_n^2 \leq x < S_n^H$ (case (iii) in Theorem 1), we need to further compare the costs of ordering from suppliers L and H. So we define the following function

$$G_n(x) \triangleq \min_{y\in[x+C_0,x+C]} g_n^L(y) - c^L x + K - g_n^H(S_n^H) + c^H x,$$

where $\min_{y\in[x+C_0,x+C]} g_n^L(y) - c^L x + K$ is the optimal cost of ordering exclusively from supplier L, and $g_n^H(S_n^H) - c^H x$ is the optimal cost of ordering exclusively from supplier H when $s_n^2 \leq x < S_n^H$ as shown in the proof of Theorem 1. Proposition 1 summarizes the optimal supplier selection if $s_n^2 \leq x < S_n^H$.

Proposition 1. *If $s_n^2 \leq x < S_n^H$, then the optimal supplier selection can be specified as follows.*

(i) *If $G_n(S_n^H) \leq 0$, then it is optimal to order exclusively from supplier L if $s_n^2 \leq x < S_n^H$.*
(ii) *If $G_n(S_n^H) > 0$, then there exists a $s_n^3 \in [s_n^2, S_n^H)$ such that $G_n(s_n^3) = 0$. It is optimal to order exclusively from supplier L if $s_n^2 \leq x < s_n^3$, and to order exclusively from supplier H if $s_n^3 \leq x < S_n^H$.*

Proof. We first prove the following result: If $f(x) + kx$ ($k \geq 0$) is increasing when $x \geq A$, then $\min_{y\in[x+a,x+b]} f(y) + kx$ is increasing when $x \geq A - a$, where $b > a \geq 0$.

Let $\tilde{f}(x) = \min_{y\in[x+a,x+b]} f(y)$. For any $x^2 \geq x^1 \geq A - a$, let $\tilde{f}(x^i) = f(x^i + \Delta x^i)$, where $\Delta x^i \in [a, b],\ i = 1, 2$. If $\tilde{f}(x^1) \leq \tilde{f}(x^2)$, then $\tilde{f}(x^2) + kx^2 - [\tilde{f}(x^1) + kx^1] = \tilde{f}(x^2) - \tilde{f}(x^1) + k(x^2 - x^1) \geq k(x^2 - x^1) \geq 0$.

We next consider the case with $\tilde{f}(x_1) > \tilde{f}(x_2)$. Note that $x^2 + \Delta x^2 \notin [x^1 + a, x^1 + b]$, otherwise $\tilde{f}(x^2) = f(x^2 + \Delta x^2) \geq \min_{y\in[x_1+a,x_1+b]} f(y) = \tilde{f}(x^1)$. So, $x^2 + \Delta x^2 > x^1 + b$ because $x^2 + \Delta x^2 \geq x^2 + a > x^1 + a$.

From the monotonicity of $f(x) + kx$ and $x^2 + \Delta x^2 > x^1 + b \geq A - a + b > A$, we have

$$0 \leq f(x^2 + \Delta x^2) - f(x^1 + b) + k(x^2 + \Delta x^2 - x^1 - b)$$
$$= \tilde{f}(x^2) - f(x^1 + b) + k(x^2 + \Delta x^2 - x^1 - b)$$
$$\leq \tilde{f}(x^2) - \tilde{f}(x^1) + k(x^2 + \Delta x^2 - x^1 - b)$$
$$\leq \tilde{f}(x^2) - \tilde{f}(x^1) + k(x^2 - x^1).$$

The first inequality holds because $f(x) + kx$ is increasing, and the second inequality can be verified from the definition of $\tilde{f}(x^1)$.

Recall that $g_n^L(x) + (c^H - c^L)x = g_n^H(x)$ is increasing when $x \geq S_n^H$. Hence, $\min_{y\in[x+C_0,x+C]} g_n^L(y) + (c^H - c^L)x$ is also increasing when $s_n^2 \leq x < S_n^H(s_n^2 + C_0 = S_n^H)$. That is, $G_n(x)$ is increasing when $s_n^2 \leq x < S_n^H$. From Eq. (9), we know that $G_n(s_n^2) \leq 0$. So, the optimal supplier selection satisfies:

(i) $G_n(S_n^H) \leq 0$. It is optimal to order exclusively from supplier L when $s_n^2 \leq x < S_n^H$.
(ii) $G_n(S_n^H) > 0$. Then when $s_n^2 \leq x < s_n^3$, $G_n(x) \leq 0$, and it is optimal to order exclusively from supplier L; when $s_n^3 \leq x < S_n^H$, $G_n(x) \geq 0$, and it is optimal to order exclusively from supplier H. □

So far, we have fully characterized the optimal supplier selection, and the optimal ordering from supplier H (ordering up to S_n^H), see Theorem 1 and Proposition 1. In the following analysis, we partially characterize the optimal ordering from supplier L, and then the structure of optimal policy.

Based on the proposed preservation of strong CK-convexity in Lemma 2, we can prove the following key result that paves the way for characterizing the optimal ordering from supplier L.

Lemma 4. *The function $g_n^L(y) = g_n(y) + c^L y$ is strongly CK-convex for all $n \geq 1$.*

Proof. We prove it by induction. It is clear that $g_1^L(y)$ is strongly CK-convex. Suppose that $g_n^L(y)$ is strongly CK-convex. Define $\bar{g}_n^L(x) \triangleq \min_{y\in[x,x+C]}\{g_n^L(y) + K\delta(y-x)\} - c^L x$. By applying Proposition 3.1 and Lemma 4.1 in [3], we have that $\bar{g}_n^L(x)$ is strongly CK-convex.

When $G_n(S_n^H) \le 0$, from Proposition 1 and Theorem 1, we have

$$f_n(x) = \begin{cases} g_n^H(S_n^H) - c^H x + (c^L - c^H)C + K & x < s_n^1 \\ \bar{g}_n^L(x) & x \ge s_n^1, \end{cases}$$

where the function $g_n^H(S_n^H) - c^H x + (c^L - c^H)C + K$ is linear with slope $-c^H$. Since (i) $\bar{g}_n^L(x)$ is strongly CK-convex, and (ii) $\bar{g}_n^L(x)+c^H x$ is increasing when $x \ge s_n^1$ from the monotonicity of $f_n(x) + c^H x$, Lemma 2 implies that $f_n(x)$ is strongly CK-convex.

When $G_n(S_n^H) > 0$, we define

$$\tilde{g}_n^L(x) \triangleq \begin{cases} g_n^H(S_n^H) - c^H x & s_n^3 \le x < S_n^H \\ \bar{g}_n^L(x) & \text{otherwise.} \end{cases} \tag{10}$$

Here $\tilde{g}_n^L(x)$ is continuous. Proposition 1 and Theorem 1 imply that $\bar{g}_n^L(x) \ge \tilde{g}_n^L(x)$ for all x, and

$$f_n(x) = \begin{cases} g_n^H(S_n^H) - c^H x + (c^L - c^H)C + K & x < s_n^1 \\ \tilde{g}_n^L(x) & \text{otherwise.} \end{cases} \tag{11}$$

For all $s_n^1 \le y - a - b < y - a \le y < y + z$, we first verify whether

$$K + \tilde{g}_n^L(y+z) \ge \tilde{g}_n^L(y) + (z/b)[\tilde{g}_n^L(y-a) - \tilde{g}_n^L(y-a-b)], \tag{12}$$

holds or not (Eq. (12) clearly holds when $z = 0$). First, Eq. (12) clearly holds when $y+z < s_n^3$ because $\bar{g}_n^L(x)$ is strongly CK-convex. Second, if $y - a - b \ge s_n^3$, then Eq. (12) also holds according to the same process in proving Lemma 2 because (i) $\tilde{g}_n^L(x) + c^H x$ is a constant when $s_n^3 \le x < S_n^H$, and (ii) $\tilde{g}_n^L(x)+c^H x$ is increasing when $x \ge s_n^H (\tilde{g}_n^L(x) + c^H x = f_n(x) + c^H x)$. Third, if $y - a - b < s_n^3$ and $y + z \ge S_n^H$, then Eq. (12) also holds because $\tilde{g}_n^L(y) \le \bar{g}_n^L(y)$ and $\tilde{g}_n^L(y-a) \le \bar{g}_n^L(y-a) (\bar{g}_n^L(x) \ge \tilde{g}_n^L(x)$ for all x). So, we only need to verify the case with $s_n^1 \le y - a - b < s_n^3$ and $s_n^3 \le y + z < S_n^H$.

Case i. $s_n^1 \le y - a - b < y - a \le y < s_n^3 \le y + z < S_n^H$.

It is clear that $S_n^H - y < S_n^H - s_n^1 = C$. So, Eq. (10) and the strong CK-convexity of $\bar{g}_n^L(y)$ indicates

$$K + \tilde{g}_n^L(S_n^H) - \tilde{g}_n^L(y) \\ \ge [(S_n^H - y)/b][\tilde{g}_n^L(y-a) - \tilde{g}_n^L(y-a-b)]. \tag{13}$$

From the monotonicity of $\tilde{g}_n^L(y)$, we have $[\tilde{g}_n^L(S_n^H) - \tilde{g}_n^L(y)]/(S_n^H - y) \le [\tilde{g}_n^L(y+z) - \tilde{g}_n^L(y)]/z$. So,

$$[K + \tilde{g}_n^L(y+z) - \tilde{g}_n^L(y)]/z \\ > K/(S_n^H - y) + [\tilde{g}_n^L(S_n^H) - \tilde{g}_n^L(y)]/(S_n^H - y). \tag{14}$$

Note that $(S_n^H - y > z)$.

Consequently, Eqs. (13) and (14) indicates that Eq. (12) holds if $y \ge s_n^1$.

Case ii. $s_n^1 \le y - a - b < y - a < s_n^3 \le y < y + z < S_n^H$.

From Eq. (10) and the strong CK-convexity of $\bar{g}_n^L(y)$, we have

$$K + \tilde{g}_n^L(S_n^H) - \tilde{g}_n^L(s_n^3) \\ \ge [(S_n^H - s_n^3)/b][\tilde{g}_n^L(y-a) - \tilde{g}_n^L(y-a-b)].$$

Please note that $(C > S_n^H - s_n^3 > z)$.

Recall that $\tilde{g}_n^L(x) + c^H x$ is a constant when $s_n^3 \le x < S_n^H$. Since

$$[K + \tilde{g}_n^L(y+z) - \tilde{g}_n^L(y)]/z = K/z - c^H \ge K/(S_n^H - s_n^3) \\ + [\tilde{g}_n^L(S_n^H) - \tilde{g}_n^L(s_n^3)]/S_n^H - s_n^3.$$

Eq. (12) holds when $y - a - b < y - a < s_n^3 \le y < y + z < S_n^H$.

Case iii. $s_n^1 \le y - a - b < s_n^3 \le y - a \le y < y + z < S_n^H$.

Case ii implies that $K + \tilde{g}_n^L(y+z) - \tilde{g}_n^L(y) \ge [z/(s_n^3 - y + a + b)][\tilde{g}_n^L(s_n^3) - \tilde{g}_n^L(y-a-b)]$. Since (i) $\tilde{g}_n^L(x) + c^H x$ is increasing when $x \ge s_n^1$, and (ii) $\tilde{g}_n^L(y-a) - \tilde{g}_n^L(s_n^3) = -c^H(y-a-s_n^3)$,

$$[\tilde{g}_n^L(s_n^3) - \tilde{g}_n^L(y-a-b)]/(s_n^3 - y + a + b) \\ \ge [\tilde{g}_n^L(y-a) - \tilde{g}_n^L(y-a-b)]/b.$$

Consequently, Eq. (12) holds if $y - a - b \ge s_n^1$. Now we have that $\tilde{g}_n^L(x)$ satisfies Eq. (12) for all cases. Since $g_n^H(S_n^H) - c^H x + (c^L - c^H)C + K$ is linear with slope $-c^H$, and $\tilde{g}_n^L(x) + c^H x$ is increasing when $x \ge s_n^1$, Eq. (11) implies that $f_n(x)$ is also strongly CK-convex by applying Lemma 2.

Based on the strong CK-convexity of $f_n(x)$, it is easy to verify that $g_{n+1}^L(y)$ is strongly CK-convex by applying Proposition 3.1 and Lemma 4.1 in [3]. So, $g_n^L(y)$ is strongly CK-convex for all $n \ge 1$. □

We next show the structure of the optimal ordering from supplier L based on the strong CK-convexity of $g_n^L(y)$. Define $S_n^L \triangleq \min\{\arg\min_y g_n^L(y)\}$. It is easy to verify that $g_n^L(y)$ reaches its minimum at a finite point, i.e., $S_n^L < \infty$ because $\lim_{y\to\infty}[L(y) + c^L y] = \infty$. We define two new thresholds, $\bar{s}_n^L$ and $\tilde{s}_n^L$:

$$\bar{s}_n^L \triangleq \max\{x \le S_n^L : \min_{y\in[x,x+C]} g_n^L(y) + K \le g_n^L(x)\},$$

$$\tilde{s}_n^L \triangleq \inf\{\min_{y\in[x,x+C]} g_n^L(y) + K \ge g_n^L(x)\}.$$

From [3] we know that $\tilde{s}_n^L \le \bar{s}_n^L \le S_n^L$. Parameter $\bar{s}_n^L$ and $\tilde{s}_n^L$ can be interpreted as reorder points for the inventory problem with a single supplier (supplier L). If $x \ge \bar{s}_n^L$, then it is optimal to order nothing from supplier L, and if $x \le \tilde{s}_n^L$, it is optimal to order some units from supplier L.

Define $\bar{y}_n^*(x) \triangleq \arg\min_{y\in[x,x+C_L]}\{g_n^L(y)+K\delta(y-x)\}$. From Lemma 4.1 in [3], the strong CK-convexity of $g_n^L(y)$ indicates that

$$\bar{y}_n^*(x) = x \quad \text{when } x \ge \bar{s}_n^L. \tag{15}$$

$$\bar{y}_n^*(x) = x + C \quad \text{when } x < \hat{s}_n^L = \min\{\tilde{s}_n^L, \bar{s}_n^L - C\}. \tag{16}$$

Eqs. (15) and (16) characterize the structure of the ordering from supplier L. When $x \in [\hat{s}_n^L, \bar{s}_n^L)$, Lemma 4.1 in [3] implies that $\bar{y}_n^*(x) = x$, $\bar{y}_n^*(x) = x + C$ or $\bar{y}_n^*(x) \ge \bar{s}_n^L$.

Based on Eqs. (15) and (16), and the results in Theorem 1 and Proposition 1, we are now ready to characterize the structure of the optimal policy for any possible starting inventory level.

Theorem 2. *The optimal policy in each period has the following structure:*

(a) *If $\bar{s}_n^L < S_n^H$, then it is optimal to order C units from supplier L and $S_n^H - C - x$ units from supplier H when $x < s_n^1$, order up to $\bar{y}_n^*(x)$ from supplier L when $s_n^1 \le x < s_n^3$, order up to S_n^H from supplier H when $s_n^3 \le x < S_n^H$, and order nothing when $x \ge S_n^H$.*

(b) *If $\bar{s}_n^L \ge S_n^H$, there are two subcases:*

(1) *If $G_n(S_n^H) \le 0$, then it is optimal to order C units from supplier L and $S_n^H - C - x$ units from supplier H when $x < s_n^1$, order C units from supplier L when $s_n^1 \le x < \max(s_n^1, \hat{s}_n^L)$, order up to $\bar{y}_n^*(x)$ from supplier L when $\max(s_n^1, \hat{s}_n^L) \le x < \bar{s}_n^L$, and order nothing when $x \ge \bar{s}_n^L$.*

(2) *If $G_n(S_n^H) > 0$, then it is optimal to order C units from supplier L and $S_n^H - C - x$ units from supplier H when $x < s_n^1$, order C units from supplier L when $s_n^1 \le x < \min(s_n^3, \hat{s}_n^L)$, order up to $\bar{y}_n^*(x)$ from supplier L when $\max(s_n^1, \min(s_n^3, \hat{s}_n^L)) \le x < s_n^3$*

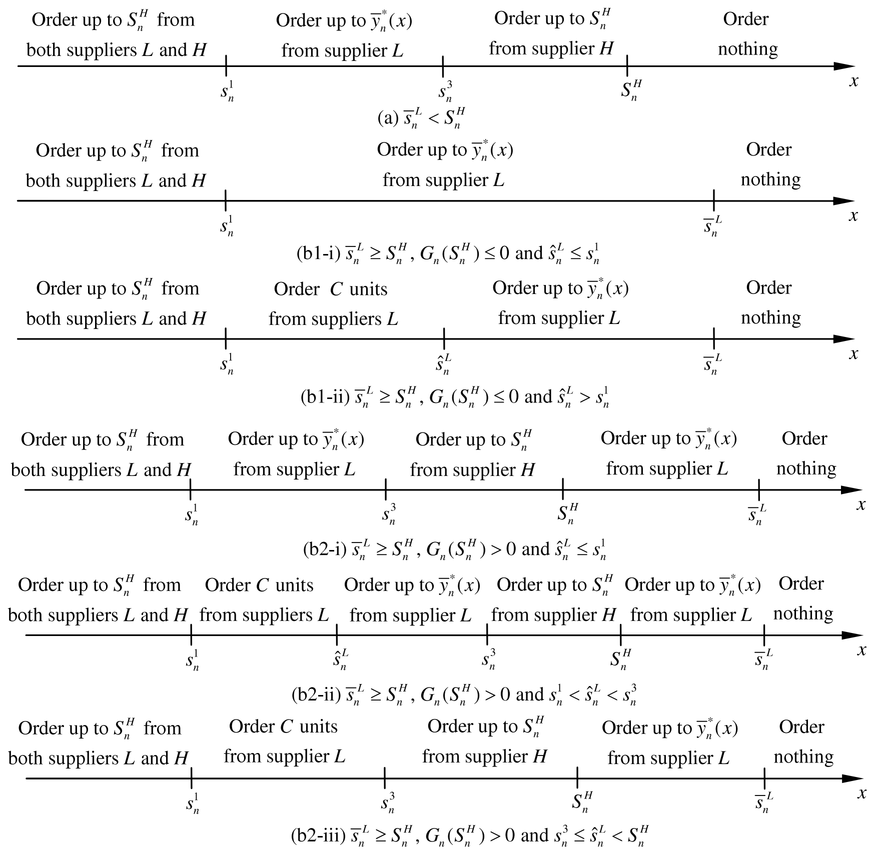

Fig. 2. The structure of the optimal policy.

and $S_n^H \le x < \bar{s}_n^L$, order up to S_n^H from supplier H when $s_n^3 \le x < S_n^H$, and order nothing when $x \ge \bar{s}_n^L$.

Proof. The optimal policy when $x < s_n^1$ has been shown in the proof of Theorem 1. In what follows, we focus on the case with $x \ge s_n^1$. When $\bar{s}_n^L < S_n^H$, the definition of $\bar{s}_n^L$ implies $\min_{y\in(S_n^H,S_n^H+C_L]} g_n^L(y) + K > g_n^L(S_n^H)$. It is clear that $\min_{y\in[S_n^H+C_0,S_n^H+C]} g_n^L(y) \ge \min_{y\in(S_n^H,S_n^H+C]} g_n^L(y)$. So,

$$\min_{y\in[S_n^H+C_0,S_n^H+C]} g_n^L(y) + (c^H - c^L)S_n^H + K - g_n^H(S_n^H)$$
$$= G_n(S_n^H) > 0.$$

From Theorem 1 and Proposition 1, we have that: (a) $y_n^*(x) = \bar{y}_n^*(x)$ (order from supplier L) when $s_n^1 \le x < s_n^3$ (note that $\hat{s}_n^L \le \bar{s}_n^L - C < S_n^H - C = s_n^1$); (b) $y_n^*(x) = S_n^H$ (order from supplier H) when $x \in [s_n^3, S_n^H)$. In addition, since $\bar{s}_n^L < S_n^H$, Theorem 1 and Eq. (15) implies that $y_n^*(x) = x$ when $x \ge S_n^H$.

We next consider the case when $\bar{s}_n^L \ge S_n^H$ in two subcases: $G_n(S_n^H) \le 0$ and $G_n(S_n^H) > 0$.

Subcase 1. $G_n(S_n^H) \le 0$.

In this subcase, Eq. (15) implies that $y_n^*(x) = x$ when $x \ge \bar{s}_n^L$, and Proposition 1 and Theorem 1 indicates that $y_n^*(x) = \bar{y}_n^*(x)$ (order from supplier L) when $x \ge s_n^1$. Hence, if $\hat{s}_n^L \le s_n^1$, then $y_n^*(x) = \bar{y}_n^*(x)$ when $s_n^1 \le x < \bar{s}_n^L$, otherwise ($\hat{s}_n^L > s_n^1$), $y_n^*(x) = x + C$ when $s_n^1 \le x < \hat{s}_n^L$, $y_n^*(x) = \bar{y}_n^*(x)$ when $\hat{s}_n^L \le x < \bar{s}_n^L$.

Subcase 2. $G_n(S_n^H) > 0$.

Theorem 1 and Proposition 1 indicates that $y_n^*(x) = S_n^H$ (order from supplier H) when $s_n^3 \le x < S_n^H$, and $y_n^*(x) = \bar{y}_n^*(x)$ (order from supplier L) when $s_n^1 \le x < s_n^3$ and $x \ge S_n^H$. Note that $y_n^*(x) = \bar{y}_n^*(x) = x$ when $x \ge \bar{s}_n^L$. So we focus on the optimal policy when $s_n^1 \le x < s_n^3$ and $S_n^H \le x < \bar{s}_n^L$.

We first prove that $\hat{s}_n^L < S_n^H$ by contradiction. If $\hat{s}_n^L \ge S_n^H$, then Eq. (16) implies that

$$\min_{y\in[S_n^H,S_n^H+C]} \{g_n^L(y) + K\delta(S_n^H - x)\} = g_n^L(S_n^H + C) + K \le g_n^L(S_n^H).$$

So, $\min_{y\in[S_n^H+C_0,S_n^H+C]} g_n^L(y) = \min_{y\in[S_n^H,S_n^H+C]}\{g_n^L(y) + K\delta(S_n^H - x)\} - K = g_n^L(S_n^H + C)$, and

$$G_n(S_n^H) = \min_{y\in[S_n^H+C_0,S_n^H+C]} g_n^L(y) + (c^H - c^L)S_n^H$$
$$+ K - g_n^H(S_n^H) = g_n^L(S_n^H + C) + K - g_n^L(S_n^H) \le 0.$$

This leads to a contradiction. So, $\hat{s}_n^L < S_n^H$.

From Eqs. (15) and (16), the optimal policy when $s_n^1 \le x < s_n^3$ and $S_n^H \le x < \bar{s}_n^L$ has the following structure: (i) If $\hat{s}_n^L \le s_n^1$, then $y_n^*(x) = \bar{y}_n^*(x)$ when $s_n^1 \le x < s_n^3$ and $S_n^H \le x < \bar{s}_n^L$; (ii) If $s_n^1 < \hat{s}_n^L < s_n^3$, then $y_n^*(x) = \bar{y}_n^*(x) = x + C$ when $s_n^1 \le x < \hat{s}_n^L$, and $y_n^*(x) = \bar{y}_n^*(x)$ when $\hat{s}_n^L \le x < s_n^3$ and $S_n^H \le x < \bar{s}_n^L$; (iii) If $s_n^3 \le \hat{s}_n^L < S_n^H$, then $y_n^*(x) = \bar{y}_n^*(x) = x + C$ when $s_n^1 \le x < s_n^3$, and $y_n^*(x) = \bar{y}_n^*(x)$ when $S_n^H \le x < \bar{s}_n^L$. □

Note that the interval $u \le x < v$ is empty if $u \ge v$. The possible structures of the optimal policy in different cases are graphically illustrated in Fig. 2. Fig. 2(b2) describes a specific setting where switching back and forth to ordering from supplier L occurs as the beginning inventory level x increases. That is, as x increases, it is optimal to first order exclusively from supplier L, then order exclusively from supplier H, and finally order exclusively from supplier L. To the best of our knowledge, this feature has not been documented previously in the literature for other types of inventory systems, and is driven by the complex tradeoffs that arise in our problem from the presence of fixed costs, order size constraints, and differences in the purchase costs.

5. Conclusion

In this paper, we study a periodic review inventory system with two heterogeneous suppliers differentiated by their fixed and variable costs as well as their order size constraints. Our research can be extended in several directions. Based on the properties of strong CK-convexity after Definition 2, our results can be directly extended to the systems with non-stationary fixed costs and order size constraints of supplier L as long as the fixed costs K_n and order size constraints C_n satisfy $K_n \geq \alpha K_{n-1}$ and $C_n \leq C_{n-1}$ ($n = 1, \ldots, N$). It can be verified that our results hold for the systems with non-stationary strongly unimodal demand distribution. Our research can be also extended to the infinite horizon systems with finite demand because (i) the optimal cost function for the finite horizon systems with finite demand is uniformly convergent, and (ii) both quasi-convexity and strong CK-convexity are preserved under limit operations.

As to future research directions, the optimal policy for the systems with multiple capacitated suppliers and/or different lead times is interesting. We expect that the analysis of these systems becomes intractable with additional complexity.

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