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European Journal of Operational Research xxx (2005) xxx–xxx

EUROPEAN JOURNAL OF OPERATIONAL RESEARCH

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Linear programming with fuzzy parameters: An interactive method resolution ☆

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Abstract

This paper proposes a method for solving linear programming problems where all the coefficients are, in general, fuzzy numbers. We use a fuzzy ranking method to rank the fuzzy objective values and to deal with the inequality relation on constraints. It allows us to work with the concept of feasibility degree. The bigger the feasibility degree is, the worst the objective value will be. We offer the decision-maker (DM) the optimal solution for several different degrees of feasibility. With this information the DM is able to establish a fuzzy goal. We build a fuzzy subset in the decision space whose membership function represents the balance between feasibility degree of constraints and satisfaction degree of the goal. A reasonable solution is the one that has the biggest membership degree to this fuzzy subset. Finally, to illustrate our method, we solve a numerical example.
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Keywords: Fuzzy sets; Fuzzy linear programming; Ranking fuzzy numbers; Decision making

1. Introduction

Linear programming (LP) is the optimisation technique most frequently applied in real-world problems and therefore it is very important to introduce new tools in the approach that allow the model to fit into the real world as much as possible.

☆ This work was supported by the Spanish Ministry of Education and Science (project MEC-04-MTM-07478) and by the University of the Basque Country. This support is gratefully acknowledged.
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doi:10.1016/j.ejor.2005.10.002

Any linear programming model representing real-world situations involves a lot of parameters whose values are assigned by experts, and in the conventional approach, they are required to fix an exact value to the aforementioned parameters. However, both experts and the DM frequently do not precisely know the value of those parameters. If exact values are suggested these are only statistical inference from past data and their stability is doubtful, so the parameters of the problem are usually defined by the DM in a uncertain way or by means of language statement parameters. Therefore, it is useful to consider the knowledge of experts about the parameters as fuzzy data.

This paper considers LP problems whose parameters are fuzzy numbers but whose decision variables are crisp. The aim of this paper is to introduce a resolution method for this type of problems that permits the interactive participation of DM in all steps of decision process, expressing his/her opinions in linguistic terms.

Two key questions may be found in these kinds of problems: how to handle the relationship between the fuzzy left and the fuzzy right hand side of the constraints, and how to find the optimal value for the fuzzy objective function. The answer is related to the problem of ranking fuzzy numbers.

A variety of methods for comparing or ranking fuzzy numbers have been reported in the literature (Wang and Kerre, 1996) and ranking methods do not always agree with each other. Different properties have been applied to justify ranking methods, such as: distinguishability (Bortolan and Degani, 1985), rationality (Nakamura, 1986), fuzzy or linguistic presentation (Delgado et al., 1988; Tong and Bonissone, 1980) and robustness (Yuan, 1991). In this paper we use a method (Jiménez, 1996) that verifies all the above properties and that, besides, is computationally efficient to solve an LP problem, because it preserves its linearity.

Looking at the property of representing the preference relationship in linguistic or fuzzy terms, ranking methods can be classified into two approaches. One of them associates, by means of different functions, each fuzzy number to a single point of the real line and then a total crisp order relationship between fuzzy numbers is established. The other approach ranks fuzzy numbers by means of a fuzzy relationship. It allows DM to present his/her preferences in a gradual way, which in an LP problem allows it to be handled with different degrees of satisfaction of constraints and, with regard to objective value, it allows us to look for a non-dominated satisfying solution. In Section 3 we show how we use our method to rank fuzzy numbers in order to define the feasibility degree of the decision vector and to define the acceptable optimal solution concept.

Obviously if the DM establishes a high degree of satisfaction of constraints for a solution, the feasible solution set becomes smaller and, consequently, the objective optimal value is worse. So, the DM has to find a balanced solution between two objectives in conflict: to improve the objective function value and to improve the degree of satisfaction of constraints. In Section 4 we show how we can operate in an interactive way in order to evaluate the two aforementioned conflicting factors. Finally in Section 5 we solve a numerical example.

2. Notation and basic definitions

A *fuzzy set* $\widetilde{A}$ of a universe Ω is characterized by its membership function $\mu_{\widetilde{A}} : \Omega \to [0, 1]$. Where $r = \mu_{\widetilde{A}}(x)$; $x \in \Omega$, is the membership degree of x to $\widetilde{A}$ (Zadeh, 1965).

When $\widetilde{A}$ is an uncertain value parameter, the membership degree $\mu_{\widetilde{A}}(x)$ can be viewed as the plausibility degree of $\widetilde{A}$ taking value x. Zadeh (1978) defines a possibility distribution associated with $\widetilde{A}$ as numerically equal to $\mu_{\widetilde{A}}$.

A *fuzzy number* is a fuzzy set $\tilde{a}$ on the real line R whose membership function $\mu_{\tilde{a}}$ is upper semi-continuous (we will suppose that it is continuous) and such that

$$r = \mu_{\tilde{a}}(x) = \begin{cases} 0 & \forall x \in (-\infty, a_1], \\ f_a(x) & \text{increasing on } [a_1, a_2], \\ 1 & \forall x \in [a_2, a_3], \\ g_a(x) & \text{decreasing on } [a_3, a_4], \\ 0 & \forall x \in [a_4, \infty). \end{cases} \tag{1}$$

A *r-cut* of a fuzzy number $\tilde{a}$ is defined by $a_r = \{x \in \Omega | \mu_{\tilde{a}}(x) \geqslant r\}$. Since $\mu_{\tilde{a}}$ is upper semi-continuous the *r-cuts* are closed and bounded intervals and we represent them by $a_r = [f_a^{-1}(r), g_a^{-1}(r)]$.

A fuzzy number is *trapezoidal* if f_a and g_a are linear functions. We will denote it by $\tilde{a} = (a_1, a_2, a_3, a_4)$. If $a_2 = a_3$, we obtain a *triangular* fuzzy number.

Following Heilpern (1992) we define the *expected interval* of a fuzzy number $\tilde{a}$, noted $EI(\tilde{a})$, by

$$EI(\tilde{a}) = [E_1^a, E_2^a] = \left[\int_0^1 f_a^{-1}(r)\,\mathrm{d}r, \int_0^1 g_a^{-1}(r)\,\mathrm{d}r\right]. \tag{2}$$

The *expected value* of a fuzzy number $\tilde{a}$, noted $EV(\tilde{a})$, is the half point of its expected interval (Heilpern, 1992)

$$EV(\tilde{a}) = \frac{E_1^a + E_2^a}{2}. \tag{3}$$

From (2) if a fuzzy number $\tilde{a}$ is trapezoidal or triangular, its expected interval and its expected value are easily calculated as follows:

$$EI(\tilde{a}) = \left[\frac{1}{2}(a_1 + a_2), \frac{1}{2}(a_3 + a_4)\right]; \quad EV(\tilde{a}) = \frac{1}{4}(a_1 + a_2 + a_3 + a_4). \tag{4}$$

Fuzzy arithmetic: given tow fuzzy numbers $\tilde{a}, \tilde{b}$, any arithmetic operation $\tilde{a} * \tilde{b}$, can be aggregated to a fuzzy number by Zadeh's minimum extension principle (Zadeh, 1978):

$$\mu_{\tilde{a}*\tilde{b}}(z) = \sup_{z=x*y} \min\{\mu_{\tilde{a}}(x), \mu_{\tilde{b}}(y)\}.$$

When the extended minimum principle is used to aggregate fuzzy numbers Dubois and Prade (1978) show the following relationship for the r-cuts:

$$[f_{\lambda a+\gamma b}^{-1}(r), g_{\lambda a+\gamma b}^{-1}(r)] = [\lambda f_a^{-1}(r) + \gamma f_b^{-1}(r), \lambda g_a^{-1}(r) + \gamma g_b^{-1}(r)], \tag{5}$$

where $\tilde{a}, \tilde{b}$ are fuzzy numbers and λ, γ are non-negative real numbers.

From expressions (2), (3) and (5) it is easily deduced that

$$EI(\lambda\tilde{a} + \gamma\tilde{b}) = \lambda EI(\tilde{a}) + \gamma EI(\tilde{b}), \tag{6}$$

$$EV(\lambda\tilde{a} + \gamma\tilde{b}) = \lambda EV(\tilde{a}) + \gamma EV(\tilde{b}). \tag{7}$$

3. Presentation of the problem

Let us consider the following linear programming problem with fuzzy parameters:

$$\begin{aligned} &\text{minimize} \quad z = \tilde{c}^t x \\ &\text{subject to} \quad x \in \aleph(\widetilde{A}, \tilde{b}) = \{x \in R^n | \tilde{a}_i x \geqslant \tilde{b}_i, \quad i = 1, \ldots, m, \ x \geqslant 0\}, \end{aligned} \tag{8}$$

where $\tilde{c} = (\tilde{c}_1, \tilde{c}_2, \dots, \tilde{c}_n), \tilde{A} = [\tilde{a}_{ij}]_{m\times n}, \tilde{b} = (\tilde{b}_1, \tilde{b}_2, \dots, \tilde{b}_m)^t$ represent, respectively, fuzzy parameters involved in the objective function and constraints. The possibility distribution of fuzzy parameters is assumed to be characterized by fuzzy numbers. $x = (x_1, x_2, \dots, x_n)$ is the crisp decision vector.

The uncertain and/or imprecise nature of the parameters of the problem leads us to compare fuzzy numbers that involves two main problems: feasibility and optimality, therefore it is necessary to answer two questions:

(1) How to define the feasibility of a decision vector x, when the constraints involve fuzzy numbers.
(2) How to define the optimality for an objective function with fuzzy coefficients.

Several focuses have been developed to solve this problem and to answer these questions (see i.e. Sakawa, 1993; Lai and Hwang, 1994; Rommelfanger and Slowinski, 1998). A variety of methods for comparing or ranking fuzzy numbers has been reported in the literature (Wang and Kerre, 1996). Different properties have been applied to justify ranking methods, such as: distinguishability, rationality, fuzzy or linguistic representation and robustness. In this paper we use a fuzzy relationship to compare fuzzy numbers [Jiménez, 1996] that verifies all the above suitable properties and that, besides, is computationally efficient to solve linear problems because it preserves its linearity.

Definition 1 Jiménez, 1996. For any pair of fuzzy numbers $\tilde{a}$ and $\tilde{b}$, the degree in which $\tilde{a}$ is bigger than $\tilde{b}$ is the following:

$$\mu_M(\tilde{a}, \tilde{b}) = \begin{cases} 0 & \text{if } E_2^a - E_1^b < 0, \\ \dfrac{E_2^a - E_1^b}{E_2^a - E_1^b - (E_1^a - E_2^b)} & \text{if } 0 \in [E_1^a - E_2^b, E_2^a - E_1^b], \\ 1 & \text{if } E_1^a - E_2^b > 0, \end{cases} \tag{9}$$

where $[E_1^a, E_2^b]$ and $[E_1^b, E_2^b]$ are the expected intervals of $\tilde{a}$ and $\tilde{b}$. When $\mu_M(\tilde{a}, \tilde{b}) = 0.5$ we will say that $\tilde{a}$ and $\tilde{b}$ are indifferent.

When $\mu_M(\tilde{a}, \tilde{b}) \geqslant \alpha$ we will say that $\tilde{a}$ is bigger than, or equal, to $\tilde{b}$ at least in a degree α and we will represent it by $\tilde{a} \geqslant_\alpha \tilde{b}$

Definition 2. Given a decision vector $x \in R^n$, we will say that it is feasible in degree α (or α-feasible) if

$$\min_{i=1,\dots,m} \{\mu_M(\tilde{a}_i x, \tilde{b}_i)\} = \alpha, \tag{10}$$

where $\tilde{a}_i = (\tilde{a}_{i1}, \tilde{a}_{i2}, \dots, \tilde{a}_{in})$.

That is to say

$$\tilde{a}_i x \geqslant_\alpha \tilde{b}_i, \quad i = 1, \dots, m, \tag{11}$$

keeping in mind (9) it is equivalent to:

$$\frac{E_2^{a_i x} - E_1^{b_i}}{E_2^{a_i x} - E_1^{a_i x} + E_2^{b_i} - E_1^{b_i}} \geqslant \alpha, \quad i = 1, \dots, m \tag{12}$$

or (bearing in mind (6)):

$$[(1-\alpha)E_2^{a_i} + \alpha E_1^{a_i}]x \geqslant \alpha E_2^{b_i} + (1-\alpha)E_1^{b_i}. \tag{13}$$

The set of the decision vectors that are α-feasible will be denoted by $\aleph(\alpha)$. It is evident that:

$$\alpha_1 < \alpha_2 \Rightarrow \aleph(\alpha_1) \supset \aleph(\alpha_2). \tag{14}$$

By means of Definition 2 we have answered the first question, that is, how to define the feasibility of a vector decision. From that it is deduced that $1-\alpha$ provides a measure of the risk of unfeasibility of a decision vector.

Regarding the second question, that is to say: how to define the optimality for an objective function with fuzzy coefficients, let us consider the following problem:

$$\begin{array}{ll} \min & z = \tilde{c}^t x \\ \text{subject to} & x \in \aleph(A,b) = \{x \in R^n | a_i x \geqslant b_i, \quad i = 1, \ldots m,\ x \geqslant 0\}, \end{array} \tag{15}$$

where the only fuzzy coefficients are in the objective function. That is, $\tilde{c}$ is a fuzzy vector but A and b are crisp. According to the ranking method that we have introduced in Definition 1, we can define a solution of the model (15) as follows:

Definition 3. A vector $x^0 \in \aleph(A,b)$ is an *acceptable optimal solution* of the model (15) if it is verified that:

$$\mu_M(\tilde{c}^t x, \tilde{c}^t x^0) \geqslant \frac{1}{2} \quad \forall x \in \aleph(A,b),$$

thus:

$$\tilde{c}^t x \geqslant_{1/2} \tilde{c}^t x^0 \quad \forall x \in \aleph(A,b),$$

therefore x^0 is a better choice (with the objective of minimizing) at least in degree 1/2 as opposed to the other feasible vectors.

If we apply Definition 1, the previous expression can be written as:

$$\frac{E_2^{c^t x} - E_1^{c^t x^0}}{E_2^{c^t x} - E_1^{c^t x} + E_2^{c^t x^0} - E_1^{c^t x^0}} \geqslant \frac{1}{2}$$

or

$$\frac{E_2^{c^t x} + E_1^{c^t x}}{2} \geqslant \frac{E_2^{c^t x^0} + E_1^{c^t x^0}}{2}.$$

Keeping in mind (3) and (7), this expression allows us to set the following proposition:

Proposition 1. *A vector $x^0 \in R^n$ is an acceptable optimal solution of the model (15) if it is a optimal solution to the following crisp linear program:*

$$\begin{array}{ll} \text{minimize} & EV(\tilde{c})x \\ \text{subject to} & x \in \aleph(A,b), \end{array}$$

where $EV(\tilde{c}) = (EV(\tilde{c}_1), EV(\tilde{c}_2), \ldots, EV(\tilde{c}_n))$ represents the expected value of the fuzzy vector $\tilde{c}$.

According to this result we may establish the following definition related to the initial fuzzy model (8).

Definition 4. A vector $x^0(\alpha) \in R^n$ is a α-*acceptable optimal solution* of the model (8) if it is an optimal solution to the following problem:

$$\begin{array}{ll} \text{minimize} & EV(\tilde{c})x \\ \text{subject to} & x \in \aleph_\alpha(\tilde{A},\tilde{b}) = \{x \in R^n | \tilde{a}_i x \geqslant_\alpha \tilde{b}_i, \quad i = 1, \ldots m,\ x \geqslant 0\}. \end{array} \tag{16}$$

Model (16) is a crisp α-parametric linear program (see expressions (11) and (13)), very difficult to solve. In order to handle it we propose an interactive procedure in the next section.

4. Interactive resolution method

From (14) the obtaining of a better value to the optimal objective function implies a lesser degree of feasibility of the constraints. Then the DM runs into two conflicting objectives: to improve the objective function value and to improve the degree of satisfaction of constraints. In Jiménez et al. (2000) we proposed to solve this problem through compromise programming, now we show how it can be solved in an interactive way as well.

The best way to reflect DM preferences is to express them through natural language, establishing a semantic correspondence for the different degrees of feasibility (Zadeh, 1975). The number of elements on the semantic scale depends on the number of linguistic labels the DM is able to distinguish. Following Kaufmann and Gil Aluja (1992), we are inclined to establish 11 scales, which allow for sufficient distinction between levels without being excessive. Then the term set is:

0 Unacceptable solution
0.1 Practically unacceptable solution
0.2 Almost unacceptable solution
0.3 Very unacceptable solution
0.4 Quite unacceptable solutions
0.5 Neither acceptable nor unacceptable solution
0.6 Quite acceptable solution
0.7 Very acceptable solution
0.8 Almost acceptable solution
0.9 Practically acceptable
1 Completely acceptable solution

Obviously, depending of wishes of DM, other scales can be used.

If α_0 is the minimum constraint feasibility degree that the DM is willing to admit, the feasibility interval of α is reduced to $\alpha_0 \leqslant \alpha \leqslant 1$ and, according to the semantic scale, we will work with discrete values of α:

$$M = \left\{\alpha_k = \alpha_0 + 0.1k | k = 0, 1, \ldots, \frac{1-\alpha_0}{0.1}\right\} \subset [0, 1].$$

In the *first step* of our method, we solve the corresponding ordinary linear program (16) for each α_k. We obtain the space $O = \{x^0(\alpha_k), \alpha_k \in M\}$ of the α_k-acceptable optimal solution of the original problem (8), and the corresponding possibility distribution of the objective value: $\tilde{z}^0(\alpha_k) = \tilde{c}x^0(\alpha_k)$ (see Fig. 1).

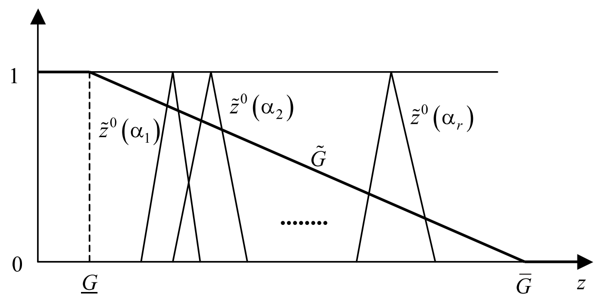

Fig. 1. Possibility distribution of the objective values and the fuzzy goal provided by the DM.

In order to get a decision vector that complies with the expectations of the DM, we should evaluate two conflicting factors: the feasibility degree and the reaching of an acceptable value for the objective function. After seeing the information given by the different $\tilde{z}^0(\alpha_k)$, the DM is asked to specify a goal $\underline{G}$ and its tolerance threshold $\overline{G}$. So, if $z \leqslant \underline{G}$ he will find it totally satisfactory, but if $z \geqslant \overline{G}$, his degree of satisfaction will be null. Then the goal is expressed by means of a fuzzy set $\tilde{G}$ whose membership function is as follows (see Fig. 1):

$$\mu_{\tilde{G}}(z) = \begin{cases} 1 & \text{if } z \leqslant \underline{G}, \\ \lambda \in [0,1] & \text{decreasing on } \underline{G} \leqslant z \leqslant \overline{G}, \\ 0 & \text{if } z \geqslant \overline{G}. \end{cases}$$

The DM wants to obtain a maximum satisfaction degree. But a lower level of fulfilment of constraints will be achieved, in order to get a better objective value. Given these circumstances the DM might want a lower degree of satisfaction of his/her objectives in exchange for a better constraints fulfilment degree.

Fig. 1 shows the different $\tilde{z}^0(\alpha_k)$ and the goal $\tilde{G}$ (for simplicity we assume that $\mu_{\tilde{G}}$ is a linear function, but any other shape can be used without increasing the resolution difficulty).

The aim is to obtain a crisp solution x^* that fulfils the DM's wishes.

In the *second step* of our method we have to compute the degree of satisfaction of the fuzzy goal $\tilde{G}$ by each α-*acceptable optimal solution*, that is to say the membership degree of each fuzzy number $\tilde{z}^0(\alpha_k)$ to the fuzzy set $\tilde{G}$.

There are several methods to do this (see i.e. Dubois et al., 2000). We suggest using an index proposed by Yager (1979)

$$K_{\tilde{G}}(z^0(\alpha)) = \frac{\int_{-\infty}^{+\infty} \mu_{\tilde{z}^0(\alpha)}(z) \cdot \mu_{\tilde{G}}(z)\,\mathrm{d}z}{\int_{-\infty}^{+\infty} \mu_{\tilde{z}^0(\alpha)}(z)\,\mathrm{d}z}, \tag{17}$$

where the denominator is the area under $\mu_{\tilde{z}^0(\alpha)}$ and, in the numerator, the possibility of occurrence $\mu_{\tilde{z}^0(\alpha)}(z)$ of each crisp value z is weighted by its satisfaction degree $\mu_{\tilde{G}}(z)$ of the goal $\tilde{G}$ (see Fig. 2). This is an extension of the widely accepted center of gravity defuzzification method, using the goal function $\mu_{\tilde{G}}$ as a weighting value. Yager index has the following suitable properties: (a) the two sides of the fuzzy numbers are exploited, (b) it is compensatory, that is to say, a low membership degree, $\mu_{\tilde{z}^0(\alpha)}(z)$, can be compensated by a high satisfaction degree $\mu_{\tilde{G}}(z)$, (c) the fuzzy goal $\tilde{G}$ is understood as a fuzzy constraint and not as an ideal fuzzy set or a utility function and (d) it is very easy to figure it out, even though $\mu_{\tilde{z}^0}$ and $\mu_{\tilde{G}}$ were nonlinear functions.

In the *third step* of our method we look for a balanced solution between the feasibility degree and the degree of satisfaction. Thus, we consider the space of the α_k-acceptable optimal solutions O and over it we define two fuzzy sets: $\tilde{F}$ and $\tilde{S}$ with the following membership functions: $\mu_{\tilde{F}}(x^0(\alpha_k)) = \alpha_k$ and $\mu_{\tilde{S}}(x^0(\alpha_k)) = K_{\tilde{G}}(\tilde{z}^0(\alpha_k))$, respectively.

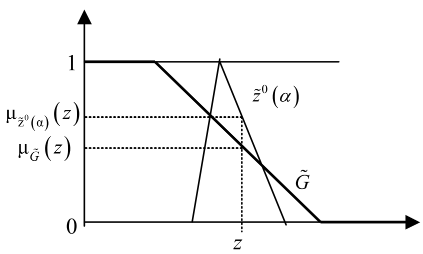

Fig. 2. Occurrence possiblity of a crisp objective value z and its goal satisfaction degree.

Then we define the fuzzy decision $\widetilde{D} = \widetilde{F} \cap \widetilde{S}$ (Bellman and Zadeh, 1970), i.e.:

$$\mu_{\tilde{D}}(x^0(\alpha_k)) = \alpha_k * K_{\tilde{G}}(\tilde{z}^0(\alpha_k)), \tag{18}$$

where $*$ represents a t-norm which can be the minimum, the algebraic product, etc.

As we want to have a crisp decision, we propose as a solution, to the fuzzy linear program (8), those, $x^* \in O$, with the highest membership degree in the fuzzy set decision:

$$\mu_{\tilde{D}}(x^*) = \max_{\alpha_k \in M}\{\alpha_k * K_{\tilde{G}}(\tilde{z}^0(\alpha_k))\}. \tag{19}$$

5. Numerical example

To illustrate our method, we will solve the following linear programming problem with fuzzy parameters, which is the same as that proposed in Jiménez et al. (2000):

$$\begin{aligned}
&\min\ (19,20,21)x_1 + (29,30,31)x_2\\
&\text{s.t.}\\
&(4.5,5,5.5)x_1 + (2.5,3,4)x_2 \geqslant (194,200,206),\\
&(3,4,5)x_1 + (6.5,7,7.5)x_2 \geqslant (230,240,250),\\
&x_1 \geqslant 0, x_2 \geqslant 0.
\end{aligned} \tag{20}$$

For simplicity we have supposed that all imprecise parameters are triangular fuzzy numbers, but any other fuzzy number could be used.

Bearing in mind the expression (13) and the Definition 4, we will calculate the α-*acceptable optimal solutions* of the problem (20) with the following ordinary α-parametric linear program:

$$\begin{aligned}
&\min\ 20x_1 + 30x_2\\
&\text{subject to,}\\
&((1-\alpha)5.25 + \alpha 4.75)x_1 + ((1-\alpha)3.5 + \alpha 2.75)x_2 \geqslant \alpha 203 + (1-\alpha)197,\\
&((1-\alpha)4.5 + \alpha 3.5)x_1 + ((1-\alpha)7.25 + \alpha 6.75)x_2 \geqslant \alpha 245 + (1-\alpha)235,\\
&x_1 \geqslant 0,\ x_2 \geqslant 0.
\end{aligned}$$

Let us suppose that the feasibility degrees α, that the DM is willing to consider, are the following ones, (we suppose that the DM will not be willing to admit high risks in the violation of the constraints):

$$M = \{0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1\} \tag{21}$$

The results are as in Table 1.

Table 1
α-acceptable optimal solutions

Feasibility degree, α	Decision vector, $x^0(\alpha)$	Possibility distribution of objective value, $\tilde{z}^0(\alpha) = \tilde{c}_1x_1 + \tilde{c}_2x_2$
0.4	$x_1 = 28.51,\ x_2 = 17.32$	$\tilde{z}^0(0.4) = (1043.97, 1089.80, 1135.63)$
0.5	$x_1 = 28.89,\ x_2 = 17.78$	$\tilde{z}^0(0.5) = (1064.53, 1111.20, 1157.87)$
0.6	$x_1 = 29.28,\ x_2 = 18.24$	$\tilde{z}^0(0.6) = (1085.28, 1132.80, 1159.32)$
0.7	$x_1 = 29.70,\ x_2 = 18.72$	$\tilde{z}^0(0.7) = (1107.18, 1155.60, 1204.02)$
0.8	$x_1 = 30.13,\ x_2 = 19.20$	$\tilde{z}^0(0.8) = (1129.27, 1178.60, 1227.93)$
0.9	$x_1 = 30.58,\ x_2 = 19.70$	$\tilde{z}^0(0.9) = (1152.32, 1202.60, 1252.88)$
1	$x_1 = 31.04,\ x_2 = 20.20$	$\tilde{z}^0(1) = (1175.56, 1226.80, 1278.04)$

As a result of this, the DM is asked to establish an aspiration level $\tilde{G}$. We will suppose that the DM is fully satisfied with an objective value lower than 1044 and that he will not be able to assume a cost of more than 1278, and if, for simplicity, we suppose that the membership function is linear, the goal will be expressed by the following fuzzy subset:

$$\mu_{\tilde{G}}(z) = \begin{cases} 1 & \text{if } z \leqslant 1044, \\ \dfrac{1278 - z}{1278 - 1044} & \text{if } 1044 \leqslant z \leqslant 1278, \\ 0 & \text{if } z \geqslant 1278. \end{cases}$$

Let us calculate the compatibility index of each solution with DM's aspirations (see (17))

$$\begin{aligned} &K_{\tilde{G}}(z^0(0.4)) = 0.80; \quad K_{\tilde{G}}(z^0(0.5)) = 0.71; \quad K_{\tilde{G}}(z^0(0.6)) = 0.65, \\ &K_{\tilde{G}}(z^0(0.8)) = 0.58; \quad K_{\tilde{G}}(z^0(0.8)) = 0.42; \quad K_{\tilde{G}}(z^0(0.9)) = 0.32, \\ &K_{\tilde{G}}(z^0(1)) = 0.22, \end{aligned} \tag{22}$$

so, according to (18), if we use the t-norm algebraic product, the membership degree of each α-acceptable optimal solution to $\tilde{D}$ (the fuzzy set that represents the balance between feasibility degree of constraints and satisfaction degree of the goal) is the following:

$$\begin{aligned} &\mu_{\tilde{D}}(x^0(0.4)) = 0.4 \cdot 0.80 = 0.32; \quad \mu_{\tilde{D}}(x^0(0.5)) = 0.5 \cdot 0.71 = 0.36; \\ &\mu_{\tilde{D}}(x^0(0.6)) = 0.6 \cdot 0.65 = 0.39; \quad \mu_{\tilde{D}}(x^0(0.7)) = 0.7 \cdot 0.58 = 0.41; \\ &\mu_{\tilde{D}}(x^0(0.8)) = 0.8 \cdot 0.42 = 0.34; \quad \mu_{\tilde{D}}(x^0(0.9)) = 0.9 \cdot 0.32 = 0.29; \\ &\mu_{\tilde{D}}(x^0(1)) = 1 \cdot 0.22 = 0.22 \end{aligned} \tag{23}$$

and, in agreement with (19), the solution of the fuzzy problem (20) will be the one which has the greatest membership degree. In (23) we observe that the 0.7-feasible optimal solution, $x_1 = 29.70$, $x_2 = 18.72$ (see Table 1), has the greatest membership degree: 0.41. (This solution is practically the same as what we have obtained, by compromise programming, to L_∞ metric in Jiménez et al. (2000).) If the DM is not satisfied with this solution, he can change the goal and its tolerance threshold, or refine the values of the different degree of feasibility that he considered in (21).

In general, we think that the method introduced in this paper provides the following advantages in respect to others:

- We use a fuzzy relation to compare fuzzy numbers, while most of the methods in the literature use comparison relations that simply report that a fuzzy number is bigger than others, which does not give any information about the risk of violation of constraints.
- Although we use a fuzzy relation to compare fuzzy numbers, our method allows us to work easily with nonlinear fuzzy numbers, while other methods widely cited in the literature, i.e. Tanaka and Asai (1984), only work with symmetrical triangular fuzzy numbers.
- In some methods, the DM is requested to initially give his aspiration level without providing any information about it, which, as Cadenas and Verdegay (1995, 1997) say, is unrealistic. However, through the information provided by optimal solutions that are feasible in different degrees, we provide the DM with the determination of the goal and its tolerance threshold.
- Lastly, in the sense recommended by Rommelfanger (1996), our method allows the DM to consider, in an interactive way, two important factors when making a decision: the degree of achievement of his aspiration level and the risk of violation of the constraints.

In short our approach possesses the following features: it allows the DM to have information about the risk of violation of constraints in all the steps of the solution process; the model allows us to work with nonlinear membership functions; it allows the DM to determine his/her aspiration levels and finally, it provides a balanced solution between the degree of achievement of the aspiration levels and the risk of violation of the constraints.

6. Conclusions

In this paper we have proposed a resolution method, for a linear programming problem with fuzzy parameters, which allows us to take a decision interactively with the DM. Through the idea of feasible optimal solution in degree α, the DM has enough information to fix an aspiration level. The DM can also choose the degrees of feasibility that he/she is willing to admit depending on the context. It is important to highlight that the acceptable optimal solutions in degree α are not fuzzy quantities, which makes it easier to take a decision in a simple way by solving a crisp parametric linear program. The DM also has additional information about the risk of violation of the constraints, and about the compatibility of the cost of the solution with his wishes for the values of the objective function. The DM can intervene in all the steps of the decision process which makes our approach very useful to be applied in a lot of real-world problems where the information is uncertain or incomplete, like environmental management, project investment, marketing, etc.

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