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Induced continuous Choquet integral operators and their application to group decision making ☆

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ABSTRACT

With respect to multi attribute group decision making, in this study two induced continuous Choquet integral operators named as the induced continuous Choquet weighted averaging (ICCWA) operator and the induced continuous Choquet geometric mean (ICCGM) operator are defined, which reflect the interactive characteristics between elements. Meantime, some associated desirable properties are studied to provide assurance in applications. In order to globally reflect the interactions between elements, we further define the probabilistic generalized semivalue ICCWA (PGS ICCWA) operator and the probabilistic generalized semivalue ICCGM (PGS ICCGM) operator. If the information about the weights of experts and attributes is incompletely known, the models for the optimal fuzzy measures on experts set and on attribute set based on consistency principle and TOPSIS method are respectively established. Moreover, an approach to uncertain multi attribute group decision making with incomplete weight information and interactive conditions is developed. Finally, a numerical example is provided to illustrate the practicality and feasibility of the developed procedure.

1. Introduction

As one of the most important aggregation operators, the ordered weighted averaging (OWA) operator proposed by Yager (1988) has been widely used in many different areas (Calvo, Mayor, & Mesiar, 2002; Liu, 2006; Merigó & Casanovas, 2009; Merigo & Gil Lafuente, 2009; Merigó, 2010; Wei, 2010a, 2010b; Wei & Zhao, 2012; Xu & Da, 2003; Xu, 2005; Yager & Kacprzyk, 1997; Yager, 2004a, 2004b, 1988; Zhang & Chu, 2009). Since it was first introduced in 1988, many generalized forms have been developed, such as the ordered weighted operator (Chiclana, Herrera, & Herrera Viedma, 2001; Xu & Da, 2002, 2003), the continuous ordered weighted operator (Yager, 2004a; Yager & Xu, 2006; Chen, Liu, & Wang, 2008), the generalized OWA operator (Yager, 2004b), the continuous generalized ordered weighted operator (Zhou & Chen, 2011), the induced ordered weighted operator (Yager & Filev, 1999; Yager, 2003; Xu & Da, 2003; Chen, Liu, & Sheng, 2004; Chiclana et al., 2007), the induced generalized ordered weighted operator (Merigo & Gil Lafuente, 2009; Su, Xia, Chen, & Wang, 2012), the induced continuous ordered weighted operator (Wu, Li, Li, & Duan, 2009; Chen & Zhou, 2011) and the induced generalized continuous OWA operator (Chen & Zhou, 2011).

All above mentioned aggregation operators only consider situations where all the elements in a set are independent, i.e., they only consider the addition of the importance of individual elements. However, in many practical situations, the elements are usually correlative, for example, Grabisch (1995, 1996) gave the following classical example: "We are to evaluate a set of students in relation to three subjects: {mathematics, physics, literature}, we want to give more importance to science related subjects than to literature, but on the other hand we want to give some advantage to students that are good both in literature and in any of the science related subjects". When there exist inter dependent or correlative characteristics between attributes or between experts, it is unreasonable to aggregate the alternative values by using additive measures. Fuzzy measures (Sugeno, 1974), as an effective tool to measure the interactions between elements, have been widely used in many different fields, such as game theory and decision making. Corresponding to fuzzy measures, fuzzy integrals are important operators to aggregate fuzzy information. One of the most important fuzzy integrals is the Choquet integral (Choquet, 1953), which has been deeply studied by many scholars. Yager (2003) introduced the Choquet integral operator on fuzzy sets. Tan and Chen (2010), Tan (2011) and Xu (2010) studied some Choquet integral operators on intuitionistic fuzzy sets (IFSs) and on interval valued intuitionistic fuzzy sets (IVIFSs), respectively. Further, Yager

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(2004b) defined the generalized Choquet OWA operator. Zhou and Chen (2011) introduced the combined continuous generalized Choquet integral aggregation (C-GCIA) operator. Meanwhile, the application of the Choquet integral is also studied by many researchers (Yager, 2003; Labreuche & Grabisch, 2003; Grabisch & Labreuche, 2008; Tan & Chen, 2010, 2011; Tan, 2011; Xu, 2010).

Although many operators based on fuzzy measures have been defined, most of them cannot reflect the global interactions between elements in a set. Further, the research on aggregation operators with fuzzy measures mainly focuses on the decision making problems with known information about the fuzzy measures on the attribute set and on the expert set. When the weight information is incompletely known, then we need to find some new ways to deal with these issues in which the decision data in question are correlative. To deal with these issues, this study defines two induced continuous Choquet integral operators called the ICCWA and ICCGM operators, which can be seen as an extension of the ICOWA operator (Chen & Zhou, 2011) and the ICOWG operator (Wu et al., 2009), respectively. In order to overall reflect interactions between elements in a set, the probabilistic generalized semivalue ICCWA (PGS-ICCWA) operator and the probabilistic generalized semivalue ICCGM (PGS-ICCGM) operator are presented. As a series of development, the models for the optimal fuzzy measures on the attribute set and on the expert set are established, respectively. Consequently, a procedure to uncertain multi-attribute group decision making is developed to provide a comprehensive and applicable framework.

This paper is organized as follows: In Section 2, some basic concepts and definitions are reviewed, which will be used in the following. In Section 3, the ICCWA and ICCGM operators are defined. Meanwhile some desirable properties are studied. In Section 4, the PGS-ICCWA and PGS-ICCGM operators are defined, which do not only globally cover the significance of elements or their ordered positions, but also overall reflect the correlations between them or their ordered positions. Further, an important case of the PGS-ICCWA and PGS-ICCGM operators is studied. In Section 5, based on the Shapley function, consistency principle, and TOPSIS method, the models for the optimal fuzzy measures on the attribute set and on the expert set are established, respectively. Then, an approach to uncertain multi-attribute group decision making with incomplete weight information and interactive conditions is developed. In Section 6, an example is provided to illustrate the developed procedure. The conclusions are made in the last section.

2. Basic concepts

2.1. Some aggregation operators

Yager (1988) introduced the ordered weighted averaging (OWA) operator for aggregating a finite collection of arguments, whose fundamental aspect is the reordering step. An OWA operator (Yager, 1988) of dimension n is a mapping f: $R^n \to R$ which has an associated weight vector $w = (w_1, w_2, \ldots, w_n)^T$ such that $w_j \in [0,1]$ and $\sum_{j=1}^n w_j = 1$, where

$$f(a_1, a_2, \ldots, a_n) = \sum_{j=1}^{n} w_j b_j,$$

with b_j being the jth largest of a_i ($i = 1, 2, \ldots, n$), R^n and R are the sets of dimension n real numbers and real numbers, respectively.

In a similar way to the OWA operator, Xu and Yager (2006) defined the ordered weighted geometric (OWG) operator, described as follows:

An OWG operator (Xu & Yager, 2006) of dimension n is a mapping f: $R^{n+} \to R^+$ which has associated with it an exponential weight vector $w = (w_1, w_2, \ldots, w_n)^T$, with $w_j \in [0,1]$, $\sum_{j=1}^n w_j = 1$, such that

$$g(a_1, a_2, \ldots, a_n) = \prod_{j=1}^{n} b_j^{w_j},$$

where b_j is the jth largest of the a_i ($i = 1, 2, \ldots, n$), R^{n+} and R^+ are the sets of dimension n positive real numbers and positive real numbers, respectively.

Later, Yager (2004b) presented the continuous ordered weighted averaging (COWA) operator, which was defined as follows:

Definition 1 Yager (2004b). A COWA operator of dimension n is a mapping F: $\Omega^+ \to R^+$ which has associated with it a basic unit-interval monotonic (BUM) function Q: $[0,1] \to [0,1]$, and it is monotonic with $Q(0) = 0$ and $Q(1) = 1$, such that

$$F_Q([a,b]) = \int_0^1 \frac{dQ(y)}{dy}(b - y(b - a))dy, \tag{1}$$

where Ω^+ is the set of positive interval numbers, namely, $\Omega^+ = \{[a,b] | a, b \in R^+, a \leqslant b\}$.

Further, Xu and Yager (2006) proposed the continuous ordered weighted geometric (COWG) operator, which was defined as follows:

Definition 2 Xu and Yager (2006). A COWG operator of dimension n is a mapping G: $\Omega^+ \to R^+$ associated with it a BUM function Q, such that

$$G_Q([a,b]) = b\left(\frac{a}{b}\right)^{\int_0^1 \frac{dQ(y)}{dy} y\, dy}, \tag{2}$$

where Q and Ω^+ as given in Definition 1.

Remark 1. If $\lambda = \int_0^1 Q(y)dy$, then Eqs. (1) and (2) can be expressed by $F_Q([a,b]) = (1 - \lambda)a + \lambda b$ and $G_Q([a,b]) = a^{1-\lambda} b^{\lambda}$, respectively.

Based on the COWA operator, Chen and Zhou (2011) developed the induced continuous OWA (ICOWA) operator ICOWA: $\Omega^{n+} \to R^+$, which is defined to aggregate the set of second arguments of two-tuples $\langle u_1, [a_1, b_1] \rangle, \langle u_2, [a_2, b_2] \rangle, \ldots, \langle u_n, [a_n, b_n] \rangle$, denoted by

$$\begin{aligned} &\text{ICOWA}(\langle u_1, [a_1, b_1] \rangle, \langle u_2, [a_2, b_2] \rangle, \ldots, \langle u_n, [a_n, b_n] \rangle) \\ &= \text{ICOWA}(\langle u_1, F_Q[a_1, b_1] \rangle, \langle u_2, F_Q[a_2, b_2] \rangle, \ldots, \langle u_n, F_Q[a_n, b_n] \rangle) \\ &= \sum_{j=1}^{n} (w_j F_Q([a_{(j)}, b_{(j)}])), \end{aligned} \tag{3}$$

where Ω^{n+} is the set of dimension n positive interval numbers, σ is a permutation on $\{1, 2, \ldots, n\}$ such that $u_{\sigma(j)} \geqslant u_{\sigma(j+1)}$, $u_{\sigma(j)}$ is the jth largest value of $u_i (i = 1, 2, \ldots, n)$, $w = (w_1, w_2, \ldots, w_n)^T$ is the associated weight vector, with $w_j \in [0,1]$, $\sum_{j=1}^n w_j = 1$, and $F_Q([a_{\sigma(j)}, b_{\sigma(j)}])$ given as Eq. (1).

According to the COWG operator, Wu et al. (2009) developed the induced continuous OWG (ICOWG) operator ICOWG: $\Omega^{n+} \to R^+$, which is defined to aggregate the set of second arguments of two-tuples $\langle u_1, [a_1, b_1] \rangle, \langle u_2, [a_2, b_2] \rangle, \ldots, \langle u_n, [a_n, b_n] \rangle$, denoted by

$$\begin{aligned} &\text{ICOWG}(\langle u_1, [a_1, b_1] \rangle, \langle u_2, [a_2, b_2] \rangle, \ldots, \langle u_n, [a_n, b_n] \rangle) \\ &= \text{ICOWG}(\langle u_1, G_Q[a_1, b_1] \rangle, \langle u_2, G_Q[a_2, b_2] \rangle, \ldots, \langle u_n, G_Q[a_n, b_n] \rangle) \\ &= \prod_{j=1}^{n} G_Q([a_{\sigma(j)}, b_{\sigma(j)}])^{w_j}, \end{aligned} \tag{4}$$

where Ω^{n+} is the set of dimension n positive interval numbers, σ is a permutation on $\{1, 2, \ldots, n\}$ such that $u_{\sigma(j)} \geqslant u_{\sigma(j+1)}$, $u_{\sigma(j)}$ is the jth largest value of $u_i (i = 1,2,\ldots,n)$, $w = (w_1, w_2, \ldots, w_n)^T$ is the associated weight vector, with $w_j \in [0,1]$, $\sum_{j=1}^{n} w_j = 1$, and $G_Q([a_{\sigma(j)}, b_{\sigma(j)}])$ given as Eq. (2).

2.2. Fuzzy measure and the Choquet integral

In many practical situations, the elements in a set are usually correlative. Thus, it is unsuitable to use the additive measure to measure their importance. In 1974, Sugeno (1974) introduced the concept of fuzzy measures, which is a powerful tool to measure the interactions phenomena between elements (Grabisch & Roubens, 1999; Kojadinovic, 2003, 2005) and to deal with decision making problems (Grabisch, 1995, 1996; Labreuche & Grabisch, 2003; Grabisch & Labreuche, 2008; Xu, 2010; Tan & Chen, 2010, 2011).

Definition 3 Sugeno (1974). A fuzzy measure μ on finite set $N = \{1,2,\ldots,n\}$ is a set function μ: $P(N) \to [0,1]$ satisfying

(1) $\mu(\emptyset) = 0$, $\mu(N) = 1$,
(2) $A \subseteq B$ implies $\mu(A) \leqslant \mu(B)$,
where $P(N)$ is the power set of N.

In the multi attribute group decision making, $\mu(A)$ can be viewed as the importance of the attribute (or expert) set A. Thus, in addition to the usual weights on the attribute (or expert) set taken separately, weights on any combination of the attributes (or experts) are also defined.

Corresponding to fuzzy measures, fuzzy integrals are important aggregation operators for uncertain information, which are studied by many researchers (Sugeno, 1974; Grabisch, 1997; Miranda, Grabisch, & Gil, 2002; Dubois & Prade, 1988). One of the most important fuzzy integrals is the Choquet integral (Choquet, 1953). As a generalization of the OWA operator, the Choquet integral on discrete sets is defined as follows (Grabisch, 1997):

Definition 4 Grabisch (1997). Let f be a positive real valued function on $X = \{x_1, x_2, \ldots, x_n\}$, and μ be a fuzzy measure on X. The discrete Choquet integral of f w.r.t. μ is defined by

$$C_\mu(f(x_{(1)}), f(x_{(2)}), \ldots, f(x_{(n)})) = \sum_{i=1}^{n} f(x_{(i)})(\mu(A_{(i)}) - \mu(A_{(i+1)})),$$

where ( ) indicates a permutation on $N = \{1,2,\ldots,n\}$ such that $f(x_{(1)}) \leqslant f(x_{(2)}) \leqslant \ldots \leqslant f(x_{(n)})$, and $A_{(i)} = \{x_{(i)}, \ldots, x_{(n)}\}$, with $A_{(n+1)} = \emptyset$.

Based on the definition of the Choquet integral, many Choquet integral operators are defined, such as the Choquet integral operator on fuzzy sets (Yager, 2003), the Choquet integral operators on IFSs and IVIFSs (Tan & Chen, 2010; Tan, 2011; Xu, 2010). Further, Yager (2004b) defined the following generalized Choquet integral OWA operator

$$\mathrm{GCOWA}(a_1, a_2, \ldots, a_n) = \left(\sum_{j=1}^{n} \left((\mu(A_{(j)}) - \mu(A_{(j+1)}))b_{(j)}^{\gamma}\right)\right)^{1/\gamma},$$

where $\gamma \in \mathrm{R}\backslash\{0\}$, ( ) indicates a permutation on $N = \{1,2,\ldots,n\}$, with $b_{(j)}$ being the jth least value of a_i $(i = 1,2,\ldots,n)$, and $A_{(i)} = \{b_{(i)}, \ldots, b_{(n)}\}$ with $A_{(n+1)} = \emptyset$.

3. Two new induced continuous Choquet integral operators

3.1. The ICCWA and ICCGM operators

According to the ICOWA and ICOWG operators (Chen & Zhou, 2011; Wu et al., 2009), we define the ICCWA and ICCGM operators as follows:

Definition 5. An ICCWA operator of dimension n is a mapping ICCWA: $\Omega^{n+} \to \mathrm{R}^+$ defined on the set of second arguments of two tuples $\langle u_1, [a_1, b_1]\rangle, \langle u_2, [a_2, b_2]\rangle, \ldots, \langle u_n, [a_n, b_n]\rangle$, denoted by

$$\begin{aligned} &\mathrm{ICCWA}_\mu(\langle u_1, [a_1, b_1]\rangle, \langle u_2, [a_2, b_2]\rangle, \ldots, \langle u_n, [a_n, b_n]\rangle) \\ &= \mathrm{ICCWA}_\mu(\langle u_1, F_Q[a_1, b_1]\rangle, \langle u_2, F_Q[a_2, b_2]\rangle, \ldots, \langle u_n, F_Q[a_n, b_n]\rangle) \\ &= \sum_{j=1}^{n}((\mu(A_{\sigma(j)}) - \mu(A_{\sigma(j+1)}))F_Q([a_{\sigma(j)}, b_{\sigma(j)}])), \end{aligned} \tag{5}$$

where Ω^{n+} is the set of dimension n positive interval numbers, μ is a fuzzy measure on $\{[a_i, b_i]\}_{i=1,2,\ldots,n}$, σ is a permutation on $N = \{1,2,\ldots,n\}$ such that $u_{\sigma(j)} \leqslant u_{\sigma(j+1)}$, $u_{\sigma(j)}$ is the jth least value of u_i $(i = 1,2,\ldots,n)$, $F_Q([a_{\sigma(j)}, b_{\sigma(j)}])$ given as Eq. (1), and $A_{\sigma(i)} = \{[a_{\sigma(j)}, b_{\sigma(j)}], \ldots, [a_{\sigma(n)}, b_{\sigma(n)}]\}$, with $A_{\sigma(n+1)} = \emptyset$.

Definition 6. An ICCGM operator of dimension n is a mapping ICCGM: $\Omega^{n+} \to \mathrm{R}^+$ defined on the set of second arguments of two tuples $\langle u_1, [a_1, b_1]\rangle, \langle u_2, [a_2, b_2]\rangle, \ldots, \langle u_n, [a_n, b_n]\rangle$, denoted by

$$\begin{aligned} &\mathrm{ICCGM}_\mu(\langle u_1, [a_1, b_1]\rangle, \langle u_2, [a_2, b_2]\rangle, \ldots, \langle u_n, [a_n, b_n]\rangle) \\ &= \mathrm{ICCGM}_\mu(\langle u_1, G_Q[a_1, b_1]\rangle, \langle u_2, G_Q[a_2, b_2]\rangle, \ldots, \langle u_n, G_Q[a_n, b_n]\rangle) \\ &= \prod_{j=1}^{n}\left(G_Q([a_{\sigma(j)}, b_{\sigma(j)}])^{\mu(A_{\sigma(j)}) - \mu(A_{\sigma(j+1)})}\right), \end{aligned} \tag{6}$$

where Ω^{n+} is the set of dimension n positive interval numbers, μ is a fuzzy measure on $\{[a_i, b_i]\}_{i=1,2,\ldots,n}$, σ is a permutation on $N = \{1,2,\ldots,n\}$ such that $u_{\sigma(j)} \leqslant u_{\sigma(j+1)}$, $u_{\sigma(j)}$ is the jth least value of u_i $(i = 1,2,\ldots,n)$, $G_Q([a_{\sigma(j)}, b_{\sigma(j)}])$ given as Eq. (2), and $A_{\sigma(i)} = \{[a_{\sigma(j)}, b_{\sigma(j)}], \ldots, [a_{\sigma(n)}, b_{\sigma(n)}]\}$, with $A_{\sigma(n+1)} = \emptyset$.

When the fuzzy measure μ is additive, namely, $\mu(S) = \sum_{[a_i, b_i] \in S} \mu([a_i, b_i])$ for any $S \subseteq \{[a_i, b_i]\}_{i=1,2,\ldots,n}$, then the ICCWA and ICCGM operators degenerate to be the ICOWA and ICOWG operators, respectively.

3.2. Some properties

Proposition 1 (*Monotonicity*). *Let $[a_i, b_i]$ and $[a_i', b_i']$ $(i = 1,2,\ldots,n)$ be two collections of positive interval numbers, and μ be a fuzzy measure on $\{[a_i, b_i]\}_{i=1,2,\ldots,n}$ and $\{[a_i', b_i']\}_{i=1,2,\ldots,n}$, with $\mu(S) = \mu(T)$, S and T having the same subscript for $S \subseteq \{[a_i, b_i]\}_{i=1,2,\ldots,n}$ and $T \subseteq \{[a_i', b_i']\}_{i=1,2,\ldots,n}$. If $a_i' \leqslant a_i$ and $b_i' \leqslant b_i$ for all i = 1, 2, …, n, then*

$$\begin{aligned} &\mathrm{ICCWA}_\mu(\langle u_1, [a_1, b_1]\rangle, \langle u_2, [a_2, b_2]\rangle, \ldots, \langle u_n, [a_n, b_n]\rangle) \\ &\geq \mathrm{ICCWA}_\mu(\langle u_1, [a_1', b_1']\rangle, \langle u_2, [a_2', b_2']\rangle, \ldots, \langle u_n, [a_n', b_n']\rangle) \end{aligned} \tag{7}$$

and

$$\begin{aligned} &\mathrm{ICCGM}_\mu(\langle u_1, [a_1, b_1]\rangle, \langle u_2, [a_2, b_2]\rangle, \ldots, \langle u_n, [a_n, b_n]\rangle) \\ &\geq \mathrm{ICCGM}_\mu(\langle u_1, [a_1', b_1']\rangle, \langle u_2, [a_2', b_2']\rangle, \ldots, \langle u_n, [a_n', b_n']\rangle). \end{aligned} \tag{8}$$

Proof. For Eq. (7): By $a_i' \leqslant a_i, b_i' \leqslant b_i$ and $F_Q([a,b]) = (1-\lambda)a + \lambda b$ where $\lambda = \int_0^1 Q(y)dy$, we have

$$F_Q([a_i, b_i]) \geqslant F_Q([a_i', b_i'])$$

for all i = 1, 2,…, n.

Namely, $F_Q([a_{\sigma(j)}, b_{\sigma(j)}]) \geqslant F_Q\left([a_{\sigma(j)}', b_{\sigma(j)}']\right)$ for all j = 1, 2, …, n.

From $\mu(A_{\sigma(j)}) - \mu(A_{\sigma(j+1)}) \geqslant 0$ for all j = 1, 2, …, n, we get

$$\begin{aligned} &\sum_{j=1}^{n}((\mu(A_{\sigma(j)}) - \mu(A_{\sigma(j+1)}))F_Q([a_{\sigma(j)}, b_{\sigma(j)}])) \\ &\geqslant \sum_{j=1}^{n}\left((\mu(A_{\sigma(j)}) - \mu(A_{\sigma(j+1)}))F_Q\left([a_{\sigma(j)}', b_{\sigma(j)}']\right)\right). \end{aligned}$$

For Eq. (8): By $a_i' \leqslant a_i, b_i' \leqslant b_i$ and $G_Q([a,b]) = a^{1-\lambda}b^{\lambda}$ where $\lambda = \int_0^1 Q(y)dy$, we have

$$G_Q([a_i,b_i]) \geqslant G_Q\left(\left[a_i',b_i'\right]\right)$$

for all $i = 1, 2, \ldots, n$. Namely, $G_Q([a_{\sigma(j)}, b_{\sigma(j)}]) \geqslant G_Q\left(\left[a'_{\sigma(j)}, b'_{\sigma(j)}\right]\right)$ for all $j = 1, 2, \ldots, n$. From $\mu(A_{\sigma(j)}) - \mu(A_{\sigma(j+1)}) \geqslant 0$ for all $j = 1, 2, \ldots, n$, we have

$$prod_{j=1}^{n}\left(G_Q([a_{\sigma(j)}, b_{\sigma(j)}])^{\mu(A_{\sigma(j)})-\mu(A_{\sigma(j+1)})}\right) \geqslant \prod_{j=1}^{n}\left(G_Q\left(\left[a'_{\sigma(j)}, b'_{\sigma(j)}\right]\right)^{\mu(A_{\sigma(j)})-\mu(A_{\sigma(j+1)})}\right).$$

□

Proposition 2 (*Idempotency*). *Let $[a_i, b_i]$ $(i = 1,2,\ldots,n)$ be a collection of positive interval numbers, and μ be a fuzzy measure on $\{[a_i, b_i]\}_{i=1,2,\ldots,n}$. If $[a_i, b_i] = [a, b]$ for all $i = 1, 2, \ldots, n$, then*

$$\mathrm{ICCWA}_\mu(\langle u_1,[a_1,b_1]\rangle, \langle u_2,[a_2,b_2]\rangle, \ldots, \langle u_n,[a_n,b_n]\rangle) = (1-\lambda)a + \lambda b \quad (9)$$

and

$$\mathrm{ICCGM}_\mu(\langle u_1,[a_1,b_1]\rangle, \langle u_2,[a_2,b_2]\rangle, \ldots, \langle u_n,[a_n,b_n]\rangle) = a^{1-\lambda}b^{\lambda}, \quad (10)$$

where $\lambda = \int_0^1 Q(y)dy$.

Proof. For (9): We have

$$\begin{aligned}
&\mathrm{ICCWA}_\mu(\langle u_1,[a_1,b_1]\rangle, \langle u_2,[a_2,b_2]\rangle, \ldots, \langle u_n,[a_n,b_n]\rangle) \\
&= \sum_{j=1}^{n}((\mu(A_{\sigma(j)}) - \mu(A_{\sigma(j+1)}))F_Q([a_{\sigma(j)}, b_{\sigma(j)}])) \\
&= \sum_{j=1}^{n}((\mu(A_{\sigma(j)}) - \mu(A_{\sigma(j+1)}))F_Q([a,b])) \\
&= F_Q([a,b])\sum_{j=1}^{n}(\mu(A_{\sigma(j)}) - \mu(A_{\sigma(j+1)})) = F_Q([a,b]) = (1-\lambda)a + \lambda b.
\end{aligned}$$

For (10): We get

$$\begin{aligned}
&\mathrm{ICCGM}_\mu(\langle u_1,[a_1,b_1]\rangle, \langle u_2,[a_2,b_2]\rangle, \ldots, \langle u_n,[a_n,b_n]\rangle) \\
&= \prod_{j=1}^{n}\left(G_Q([a_{\sigma(j)}, b_{\sigma(j)}])^{\mu(A_{\sigma(j)})-\mu(A_{\sigma(j+1)})}\right) = \prod_{j=1}^{n}\left(G_Q([a,b])^{\mu(A_{\sigma(j)})-\mu(A_{\sigma(j+1)})}\right) \\
&= G_Q([a,b])^{\sum_{j=1}^{n}(\mu(A_{\sigma(j)})-\mu(A_{\sigma(j+1)}))} = G_Q([a,b]) = a^{1-\lambda}b^{\lambda}. \quad \square
\end{aligned}$$

Proposition 3 (*Boundary*). *Let $[a_i, b_i]$ $(i = 1, 2, \ldots, n)$ be a collection of positive interval numbers, and μ be a fuzzy measure on $\{[a_i, b_i]\}_{i=1,2,\ldots,n}$, then*

$$\min_j a_j \leqslant \mathrm{ICCWA}_\mu(\langle u_1,[a_1,b_1]\rangle, \langle u_2,[a_2,b_2]\rangle, \ldots, \langle u_n,[a_n,b_n]\rangle) \leqslant \max_j b_j \quad (11)$$

and

$$\min_j a_j \leqslant \mathrm{ICCGM}_\mu(\langle u_1,[a_1,b_1]\rangle, \langle u_2,[a_2,b_2]\rangle, \ldots, \langle u_n,[a_n,b_n]\rangle) \leqslant \max_j b_j. \quad (12)$$

Proof. For all $i = 1, 2, \ldots, n$, since $F_Q([a_i, b_i]) = (1-\lambda)a_i + \lambda b_i$, we get

$$a_i \leqslant F_Q([a_i, b_i]) \leqslant b_i.$$

Thus, $\min_j a_{\sigma(j)} \leqslant F_Q([a_{\sigma(j)}, b_{\sigma(j)}]) \leqslant \max_j b_{\sigma(j)}$ for all $j = 1,2, \ldots, n$. Namely,

$$\min_j a_{\sigma(j)} \leqslant F_Q([a_{\sigma(j)}, b_{\sigma(j)}]) \leqslant \max_j b_{\sigma(j)}$$

for all $j = 1, 2, \ldots, n$. By $\sum_{j=1}^{n}(\mu(A_{\sigma(j)}) - \mu(A_{\sigma(j+1)})) = 1$, we get Eq. (11). Similarly, one can easily get Eq. (12). □

Proposition 4 (*Linearity 1*). *Let $\left[a_i^k, b_i^k\right]$ $(i = 1,2,\ldots,n; k = 1,2,\ldots,m)$ be a collection of positive interval numbers, and μ be a fuzzy measure on $\left\{\left[a_i^k, b_i^k\right]\right\}_{i=1,2,\ldots,n}$, with $\mu(S) = \mu(T)$, S and T having the same subscript for $S \subseteq \left\{\left[a_i^k, b_i^k\right]\right\}_{i=1,2,\ldots,n}$ and $T \subseteq \left\{\left[a_i^l, b_i^l\right]\right\}_{i=1,2,\ldots,n}$, $k, l = 1,2,\ldots,m, k \neq l$. Then,*

$$\begin{aligned}
&\mathrm{ICCWA}_\mu\left(\left\langle u_1, \sum_{k=1}^{m}\alpha_k\left[a_1^k, b_1^k\right] + [c,d]\right\rangle, \left\langle u_2, \sum_{k=1}^{m}\alpha_k\left[a_2^k, b_2^k\right] + [c,d]\right\rangle, \ldots, \left\langle u_n, \sum_{k=1}^{m}\alpha_k\left[a_n^k, b_n^k\right] + [c,d]\right\rangle\right) = (1-\lambda)c + \lambda d \\
&+ \sum_{k=1}^{m}\alpha_k \mathrm{ICCWA}_\mu\left(\left\langle u_1, \left[a_1^k, b_1^k\right]\right\rangle, \left\langle u_2, \left[a_2^k, b_2^k\right]\right\rangle, \ldots, \left\langle u_n, \left[a_n^k, b_n^k\right]\right\rangle\right) \quad (13)
\end{aligned}$$

and

$$\begin{aligned}
&\mathrm{ICCGM}_\mu\left(\left\langle u_1, \prod_{k=1}^{m}\alpha_k\left[a_1^k, b_1^k\right]\cdot[c,d]\right\rangle, \left\langle u_2, \prod_{k=1}^{m}\alpha_k\left[a_2^k, b_2^k\right]\cdot[c,d]\right\rangle, \ldots, \left\langle u_n, \prod_{k=1}^{m}\alpha_k\left[a_n^k, b_n^k\right]\cdot[c,d]\right\rangle\right) = c^{1-\lambda}d^{\lambda}\prod_{k=1}^{m}\alpha_k \mathrm{ICCGM}_\mu\left(\left\langle u_1, \left[a_1^k, b_1^k\right]\right\rangle,\right. \\
&\left.\left\langle u_2, \left[a_2^k, b_2^k\right]\right\rangle, \ldots, \left\langle u_n, \left[a_n^k, b_n^k\right]\right\rangle\right), \quad (14)
\end{aligned}$$

where $\lambda = \int_0^1 Q(y)dy$, $\alpha_k \in \mathrm{R}_+$ and $[c, d]$ is a positive interval number.

Proof. For (13): By Eq. (5), we have

$$\begin{aligned}
&\mathrm{ICCWA}_\mu\left(\left\langle u_1, \sum_{k=1}^{m}\alpha_k\left[a_1^k, b_1^k\right] + [c,d]\right\rangle, \left\langle u_2, \sum_{k=1}^{m}\alpha_k\left[a_2^k, b_2^k\right] + [c,d]\right\rangle, \ldots, \left\langle u_n, \sum_{k=1}^{m}\alpha_k\left[a_n^k, b_n^k\right] + [c,d]\right\rangle\right) \\
&= \sum_{j=1}^{n}\left((\mu(A_{\sigma(j)}) - \mu(A_{\sigma(j+1)}))F_Q\left(\sum_{k=1}^{m}\alpha_k\left[a_{\sigma(j)}^k, b_{\sigma(j)}^k\right] + [c,d]\right)\right) \\
&= \sum_{j=1}^{n}\left((\mu(A_{\sigma(j)}) - \mu(A_{\sigma(j+1)}))F_Q\left(\left[c + \sum_{k=1}^{m}\alpha_k a_{\sigma(j)}^k, d + \sum_{k=1}^{m}\alpha_k b_{\sigma(j)}^k\right]\right)\right) \\
&= \sum_{j=1}^{n}\left((\mu(A_{\sigma(j)}) - \mu(A_{\sigma(j+1)}))\left((1-\lambda)\left(c + \sum_{k=1}^{m}\alpha_k a_{\sigma(j)}^k\right) + \lambda\left(d + \sum_{k=1}^{m}\alpha_k b_{\sigma(j)}^k\right)\right)\right) \\
&= \sum_{j=1}^{n}\left((\mu(A_{\sigma(j)}) - \mu(A_{\sigma(j+1)}))\left(((1-\lambda)c + \lambda d) + \sum_{k=1}^{m}\alpha_k\left((1-\lambda)a_{\sigma(j)}^k + \lambda b_{\sigma(j)}^k\right)\right)\right) \\
&= \left(1-\lambda)c + \lambda d + \sum_{j=1}^{n}\sum_{k=1}^{m}\alpha_k\left((\mu(A_{\sigma(j)}) - \mu(A_{\sigma(j+1)}))\left((1-\lambda)a_{\sigma(j)}^k + \lambda b_{\sigma(j)}^k\right)\right)\right. \\
&= \left(1-\lambda)c + \lambda d + \sum_{j=1}^{n}\sum_{k=1}^{m}\alpha_k\left((\mu(A_{\sigma(j)}) - \mu(A_{\sigma(j+1)}))F_Q\left(\left[a_{\sigma(j)}^k, b_{\sigma(j)}^k\right]\right)\right)\right. \\
&= \left(1-\lambda)c + \lambda d + \sum_{k=1}^{m}\alpha_k \mathrm{ICCWA}_\mu\left(\langle u_1, \left[a_1^k, b_1^k\right]\rangle, \langle u_2, \left[a_2^k, b_2^k\right]\rangle, \ldots, \langle u_n, \left[a_n^k, b_n^k\right]\rangle\right)\right..
\end{aligned}$$

For (14): By Eq. (6), we get

$$
\begin{aligned}
&\mathrm{ICCGM}_{\mu}\left(\left\langle u_1,\prod_{k=1}^{m}\alpha_k\left[a_1^k,b_1^k\right]\cdot[c,d]\right\rangle,\left\langle u_2,\prod_{k=1}^{m}\alpha_k\left[a_2^k,b_2^k\right]\cdot[c,d]\right\rangle,\ldots,\right.\\
&\left.\left\langle u_n,\prod_{k=1}^{m}\alpha_k\left[a_n^k,b_n^k\right]\cdot[c,d]\right\rangle\right)\\
&=\prod_{j=1}^{n}\left(G_Q\left(\prod_{k=1}^{m}\alpha_k[a_{\sigma(j)},b_{\sigma(j)}]\cdot[c,d]\right)^{\mu(A_{\sigma(j)})-\mu(A_{\sigma(j+1)})}\right)\\
&=\prod_{j=1}^{n}\left(G_Q\left([c\prod_{k=1}^{m}\alpha_k a_{\sigma(j)}^k,d\prod_{k=1}^{m}\alpha_k b_{\sigma(j)}^k]\right)^{\mu(A_{\sigma(j)})-\mu(A_{\sigma(j+1)})}\right)\\
&=\prod_{j=1}^{n}\left(\left(\left(c\prod_{k=1}^{m}\alpha_k a_{\sigma(j)}^k\right)^{1-\lambda}\left(d\prod_{k=1}^{m}\alpha_k b_{\sigma(j)}^k\right)^{\lambda}\right)^{\mu(A_{\sigma(j)})-\mu(A_{\sigma(j+1)})}\right)\\
&=\prod_{j=1}^{n}\left(\left(c^{1-\lambda}d^{\lambda}\right)^{\mu(A_{\sigma(j)})-\mu(A_{\sigma(j+1)})}\left(\prod_{k=1}^{m}\alpha_k^{1-\lambda}\left(a_{\sigma(j)}^k\right)^{1-\lambda}\prod_{k=1}^{m}\alpha_k^{\lambda}b_{\sigma(j)}^{k\,\lambda}\right)^{\mu(A_{\sigma(j)})-\mu(A_{\sigma(j+1)})}\right)\\
&=\prod_{j=1}^{n}\left(\left(c^{1-\lambda}d^{\lambda}\right)^{\mu(A_{\sigma(j)})-\mu(A_{\sigma(j+1)})}\left(\prod_{k=1}^{m}\alpha_k\left(a_{\sigma(j)}^k\right)^{1-\lambda}\left(b_{\sigma(j)}^k\right)^{\lambda}\right)^{\mu(A_{\sigma(j)})-\mu(A_{\sigma(j+1)})}\right)\\
&=\left(c^{1-\lambda}d^{\lambda}\right)^{\sum_{j=1}^{n}(\mu(A_{\sigma(j)})-\mu(A_{\sigma(j+1)}))}\prod_{k=1}^{m}\alpha_k^{\sum_{j=1}^{n}(\mu(A_{\sigma(j)})-\mu(A_{\sigma(j+1)}))}\prod_{j=1}^{n}\\
&\quad\times\left(\left(a_{\sigma(j)}^k\right)^{1-\lambda}\left(b_{\sigma(j)}^k\right)^{\lambda}\right)^{\mu(A_{\sigma(j)})-\mu(A_{\sigma(j+1)})}\\
&=c^{1-\lambda}d^{\lambda}\prod_{k=1}^{m}\alpha_k\prod_{j=1}^{n}\left(G_Q\left(\left[a_{\sigma(j)}^k,b_{\sigma(j)}^k\right]\right)^{\mu(A_{\sigma(j)})-\mu(A_{\sigma(j+1)})}\right)\\
&=c^{1-\lambda}d^{\lambda}\prod_{k=1}^{m}\alpha_k\mathrm{ICCGM}_{\mu}\left(\left\langle u_1,\left[a_1^k,b_1^k\right]\right\rangle,\left\langle u_2,\left[a_2^k,b_2^k\right]\right\rangle,\ldots,\left\langle u_n,\left[a_n^k,b_n^k\right]\right\rangle\right).\quad\square
\end{aligned}
$$

Proposition 5 (*Linearity 2*). *Let $[a_i,b_i]$ ($i = 1,2,\ldots,n$) be a collection of positive interval numbers, and μ_l ($l = 1,2,\ldots,q$) be a collection of fuzzy measures on $\{[a_i,b_i]\}_{i=1,2,\ldots,n}$. Then,*

$$\mathrm{ICCWA}_{\sum_{l=1}^{q}\beta_l\mu_l+\varepsilon_l}(\langle u_1,[a_1,b_1]\rangle,\langle u_2,[a_2,b_2]\rangle,\ldots,\langle u_n,[a_n,b_n]\rangle)=\sum_{l=1}^{q}\beta_l\mathrm{ICCWA}_{\mu_l}(\langle u_1,[a_1,b_1]\rangle,\langle u_2,[a_2,b_2]\rangle,\ldots,\langle u_n,[a_n,b_n]\rangle)\quad(15)$$

and

$$\mathrm{ICCGM}_{\sum_{l=1}^{q}\beta_l\mu_l+\varepsilon_l}(\langle u_1,[a_1,b_1]\rangle,\langle u_2,[a_2,b_2]\rangle,\ldots,\langle u_n,[a_n,b_n]\rangle)=\prod_{l=1}^{q}\beta_l\mathrm{ICCGM}_{\mu_l}(\langle u_1,[a_1,b_1]\rangle,\langle u_2,[a_2,b_2]\rangle,\ldots,\langle u_n,[a_n,b_n]\rangle),\quad(16)$$

where $\beta_l \geqslant 0$ with $\sum_{l=1}^{q}\beta_l = 1$, and $\varepsilon_l \in R$.

Proof. For (15): By Eq. (5), we get

$$
\begin{aligned}
&\mathrm{ICCWA}_{\sum_{l=1}^{q}\beta_l\mu_l+\varepsilon_l}(\langle u_1,[a_1,b_1]\rangle,\langle u_2,[a_2,b_2]\rangle,\ldots,\langle u_n,[a_n,b_n]\rangle)\\
&=\sum_{j=1}^{n}\left(\left(\sum_{l=1}^{q}\beta_l\mu_l+\varepsilon_l\right)(A_{\sigma(j)})-\left(\sum_{l=1}^{q}\beta_l\mu_l+\varepsilon_l\right)(A_{\sigma(j+1)})\right)F_Q([a_{\sigma(j)},b_{\sigma(j)}])\\
&=\sum_{j=1}^{n}\left(\sum_{l=1}^{q}\beta_l(\mu_l(A_{\sigma(j)})-\mu_l(A_{\sigma(j+1)}))\right)F_Q([a_{\sigma(j)},b_{\sigma(j)}])\\
&=\sum_{l=1}^{q}\beta_l\sum_{j=1}^{n}(\mu_l(A_{\sigma(j)})-\mu_l(A_{\sigma(j+1)}))F_Q([a_{\sigma(j)},b_{\sigma(j)}])\\
&=\sum_{l=1}^{q}\beta_l\mathrm{ICCWA}_{\mu_l}(\langle u_1,[a_1,b_1]\rangle,\langle u_2,[a_2,b_2]\rangle,\ldots,\langle u_n,[a_n,b_n]\rangle).
\end{aligned}
$$

For (16): By Eq. (6), we obtain

$$
\begin{aligned}
&\mathrm{ICCGM}_{\sum_{l=1}^{q}\beta_l\mu_l+\varepsilon_l}(\langle u_1,[a_1,b_1]\rangle,\langle u_2,[a_2,b_2]\rangle,\ldots,\langle u_n,[a_n,b_n]\rangle)\\
&=\prod_{j=1}^{n}(G_Q([a_{\sigma(j)},b_{\sigma(j)}]))^{(\sum_{l=1}^{q}\beta_l\mu_l+\varepsilon_l)(A_{\sigma(j)})-(\sum_{l=1}^{q}\beta_l\mu_l+\varepsilon_l)(A_{\sigma(j+1)})}\\
&=\prod_{j=1}^{n}(G_Q([a_{\sigma(j)},b_{\sigma(j)}]))^{\sum_{l=1}^{q}\beta_l(\mu_l(A_{\sigma(j)})-\mu_l(A_{\sigma(j+1)}))}\\
&=\prod_{l=1}^{q}\beta_l\prod_{j=1}^{n}(G_Q([a_{\sigma(j)},b_{\sigma(j)}])^{\sum_{l=1}^{q}\beta_l(\mu_l(A_{\sigma(j)})-\mu_l(A_{\sigma(j+1)}))})\\
&=\prod_{l=1}^{q}\beta_l\mathrm{ICCGM}_{\mu_l}(\langle u_1,[a_1,b_1]\rangle,\langle u_2,[a_2,b_2]\rangle,\ldots,\langle u_n,[a_n,b_n]\rangle).\quad\square
\end{aligned}
$$

Corollary 1. *Let $\left[a_i^k,b_i^k\right]$ ($i = 1,2,\ldots,n; k = 1,2,\ldots,m$) be a collection of positive interval numbers, and μ_l ($l = 1,2,\ldots,q$) be a collection of fuzzy measures on $\left\{\left[a_i^k,b_i^k\right]\right\}_{i=1,2,\ldots,n}$, with $\mu(S) = \mu(T)$, S and T having the same subscript for $S \subseteq \{[a_i^k,b_i^k]\}_{i=1,2,\ldots,n}$ and $T \subseteq \{[a_i^r,b_i^r]\}_{i=1,2,\ldots,n}$, $k,r = 1,2,\ldots,m, k \neq r$. Then,*

$$
\begin{aligned}
&\mathrm{ICCWA}_{\sum_{l=1}^{q}\beta_l\mu_l+\varepsilon_l}\left(\left\langle u_1,\sum_{k=1}^{m}\alpha_k\left[a_1^k,b_1^k\right]+[c,d]\right\rangle,\right.\\
&\left.\left\langle u_2,\sum_{k=1}^{m}\alpha_k\left[a_2^k,b_2^k\right]+[c,d]\right\rangle,\ldots,\left\langle u_n,\sum_{k=1}^{m}\alpha_k\left[a_n^k,b_n^k\right]+[c,d]\right\rangle\right)\\
&=(1-\lambda)c+\lambda d+\sum_{l=1}^{q}\sum_{k=1}^{m}\alpha_k\beta_l\mathrm{ICCWA}_{\mu_l}\left(\left\langle u_1,\left[a_1^k,b_1^k\right]\right\rangle,\right.\\
&\left.\left\langle u_2,\left[a_2^k,b_2^k\right]\right\rangle,\ldots,\left\langle u_n,\left[a_n^k,b_n^k\right]\right\rangle\right)
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathrm{ICCGM}_{\sum_{l=1}^{q}\beta_l\mu_l+\varepsilon_l}\left(\left\langle u_1,\prod_{k=1}^{m}\alpha_k\left[a_1^k,b_1^k\right]\cdot[c,d]\right\rangle,\right.\\
&\left.\left\langle u_2,\prod_{k=1}^{m}\alpha_k\left[a_2^k,b_2^k\right]\cdot[c,d]\right\rangle,\ldots,\left\langle u_n,\prod_{k=1}^{m}\alpha_k\left[a_n^k,b_n^k\right]\cdot[c,d]\right\rangle\right)\\
&=c^{1-\lambda}d^{\lambda}\prod_{l=1}^{q}\prod_{k=1}^{m}\alpha_k\beta_l\mathrm{ICCGM}_{\mu_l}\left(\left\langle u_1,\left[a_1^k,b_1^k\right]\right\rangle,\left\langle u_2,\left[a_2^k,b_2^k\right]\right\rangle,\ldots,\right.\\
&\left.\left\langle u_n,\left[a_n^k,b_n^k\right]\right\rangle\right),
\end{aligned}
$$

where the notations as given in Propositions 4 and 5.

Definition 7. Let μ be a fuzzy measure on $N = \{1,2,\ldots,n\}$. An element $i \in N$ is said to be inessential if $\mu(S \cup i) = \mu(S)$ for any $S \subseteq N\backslash i$, and $i \in N$ is said to be independent if $\mu(S \cup i) = \mu(S) + \mu(i)$ for any $S \subseteq N\backslash i$.

From the definition of the inessential element, we know if an element i is inessential, then its contribution to any other combination $S \subseteq N\backslash i$ is equal 0. Further, if an element i is independent, then its contribution to any other combination $S \subseteq N\backslash i$ is equal to the importance of its own.

Proposition 6. *Let $[a_i,b_i]$ ($i = 1,2,\ldots,n$) be a collection of positive interval numbers, and μ be a fuzzy measure on $\{[a_i,b_i]\}_{i=1,2,\ldots,n}$. If $[a_p,b_p] \in \{[a_i,b_i]\}_{i=1,2,\ldots,n}$ is an independent element, then*

$$\begin{aligned}&\mathrm{ICCWA}_\mu(\langle u_1,[a_1,b_1]\rangle,\langle u_2,[a_2,b_2]\rangle,\ldots,\langle u_n,[a_n,b_n]\rangle)\\&\quad=\mathrm{ICCWA}_\mu(\ldots,\langle u_{p-1},[a_{p-1},b_{p-1}]\rangle,\langle u_{p+1},[a_{p+1},b_{p+1}]\rangle,\ldots)\\&\quad+\mu([a_p,b_p])F_Q([a_p,b_p])\end{aligned}\tag{17}$$

and

$$\begin{aligned}&\mathrm{ICCGM}_\mu(\langle u_1,[a_1,b_1]\rangle,\langle u_2,[a_2,b_2]\rangle,\ldots,\langle u_n,[a_n,b_n]\rangle)\\&\quad=G_Q([a_p,b_p])^{\mu([a_p,b_p])}\mathrm{ICCGM}_\mu(\ldots,\langle u_{p-1},[a_{p-1},b_{p-1}]\rangle,\\&\quad\langle u_{p+1},[a_{p+1},b_{p+1}]\rangle,\ldots).\end{aligned}\tag{18}$$

Proof. For (17): We have

$$\begin{aligned}&\mathrm{ICCWA}_\mu(\langle u_1,[a_1,b_1]\rangle,\langle u_2,[a_2,b_2]\rangle,\ldots,\langle u_n,[a_n,b_n]\rangle)\\&=\sum_{j=1}^{n}((\mu(A_{\sigma(j)})-\mu(A_{\sigma(j+1)}))F_Q([a_{\sigma(j)},b_{\sigma(j)}]))\\&=\sum_{j=1,j\neq p}^{n}((\mu(A_{\sigma(j)})-\mu(A_{\sigma(j+1)}))F_Q([a_{\sigma(j)},b_{\sigma(j)}]))\\&\quad+\mu([a_p,b_p])F_Q([a_p,b_p])\\&=\mathrm{ICCWA}_\mu(\ldots,\langle u_{p-1},[a_{p-1},b_{p-1}]\rangle,\langle u_{p+1},[a_{p+1},b_{p+1}]\rangle,\ldots)\\&\quad+\mu([a_p,b_p])F_Q([a_p,b_p]).\end{aligned}$$

Similarly, one can easily get Eq. (18). □

Corollary 2. *Let $[a_i,b_i]$ $(i=1,2,\ldots,n)$ be a collection of positive interval numbers, and μ be a fuzzy measure on $\{[a_i,b_i]\}_{i=1,2,\ldots,n}$. If $[a_p,b_p]\in\{[a_i,b_i]\}_{i=1,2,\ldots,n}$ is an inessential element, then*

$$\begin{aligned}&\mathrm{ICCWA}_\mu(\langle u_1,[a_1,b_1]\rangle,\langle u_2,[a_2,b_2]\rangle,\ldots,\langle u_n,[a_n,b_n]\rangle)\\&\quad=\mathrm{ICCWA}_\mu(\ldots,\langle u_{p-1},[a_{p-1},b_{p-1}]\rangle,\langle u_{p+1},[a_{p+1},b_{p+1}]\rangle,\ldots)\end{aligned}$$

and

$$\begin{aligned}&\mathrm{ICCGM}_\mu(\langle u_1,[a_1,b_1]\rangle,\langle u_2,[a_2,b_2]\rangle,\ldots,\langle u_n,[a_n,b_n]\rangle)\\&\quad=\mathrm{ICCGM}_\mu(\ldots,\langle u_{p-1},[a_{p-1},b_{p-1}]\rangle,\langle u_{p+1},[a_{p+1},b_{p+1}]\rangle,\ldots).\end{aligned}$$

4. The PGS-ICCWA and PGS-ICCGM operators

Although the ICCWA and ICCGM operators can reflect the interactions between elements, they give no more than a fuzzy measure on a set. Moreover, they only reflect interactions between two "adjacent" coalitions $A_{\sigma(i)}$ and $A_{\sigma(i+1)}$ $(i=1,2,\ldots,n)$, which seems to be unreasonable.

4.1. The probabilistic generalized semivalue

In order to measure the power or the strength of each coalition in a game rather than the power of each of these players, Marichal (2000) introduced the probabilistic generalized semivalue on any finite set $N=\{1,2,\ldots,n\}$ as follows:

$$\varphi_p(\mu,S)=\sum_{T\subseteq N\setminus S}p_t^s(n)(\mu(T\cup S)-\mu(T)),\tag{19}$$

where $\sum_{t=0}^{n-s}C_{n-s}^t p_t^s(n)=1$ for all $S,T\subseteq N$ with $S\cap T\neq 0$, s, t and n denote the cardinalities of S, T and N, respectively.

For any $S\subseteq N$, Eq. (19) is an expectation value of the overall marginal contributions between the coalition S and any coalition $T\subseteq N\setminus S$.

Theorem 1. *Let μ is a fuzzy measure on any finite set $N=\{1,2,\ldots,n\}$, then φ_p given as Eq. (19) is also a fuzzy measure.*

Proof. By Eq. (19), we easily get $\varphi_p(\mu,\emptyset)=0$ and $\varphi_p(\mu,N)=\mu(N)=1$. In the following, we show $\varphi_p(\mu,A)\leqslant\varphi_p(\mu,B)$ for all A, $B\subseteq N$, with $A\subseteq B$.

Case (1) When $a=b-1$, with a and b being the cardinalities of A and B, respectively. Without loss of generality, suppose $A\cup i=B$.

From Eq. (19), we have

$$\begin{aligned}\varphi_p(\mu,A)&=\sum_{T\subseteq N\setminus A}p_t^a(n)(\mu(T\cup A)-\mu(T))\\&=\sum_{T\subseteq N\setminus A\cup i}p_t^a(n)(\mu(T\cup A)-\mu(T))+\sum_{T\subseteq N\setminus A\cup i}p_{t+1}^a(n)(\mu(T\cup A\cup i)-\mu(T\cup i))\\&=\sum\nolimits_{T\subseteq N\setminus A\cup i}(p_t^a(n)(\mu(T\cup A)-\mu(T))+p_{t+1}^a(n)(\mu(T\cup A\cup i)-\mu(T\cup i)))\end{aligned}$$

and

$$\begin{aligned}\varphi_p(\mu,B)&=\sum_{T\subseteq N\setminus B}p_t^b(n)(\mu(T\cup B)-\mu(T))\\&=\sum_{T\subseteq N\setminus A\cup i}p_t^{a+1}(n)(\mu(T\cup A\cup i)-\mu(T)).\end{aligned}$$

Since $\sum_{t=0}^{n-a-1}C_{n-a-1}^t(p_t^a(n)+p_{t+1}^a(n))=1$ and $\sum_{t=0}^{n-b}C_{n-b}^t p_t^b(n)=\sum_{t=0}^{n-a-1}C_{n-a-1}^t p_t^{a+1}(n)=1$, we get

$$p_t^a(n)+p_{t+1}^a(n)=p_t^{a+1}(n)$$

for any $T\subseteq N\setminus A\cup i$.

Since $\mu(T\cup A\cup i)\geqslant\mu(T\cup A)$ and $\mu(T)\leqslant\mu(T\cup i)$, we obtain

$$\begin{aligned}&p_t^a(n)(\mu(T\cup A)-\mu(T))+p_{t+1}^a(n)(\mu(T\cup A\cup i)-\mu(T\cup i))\\&\leqslant p_t^a(n)(\mu(T\cup A\cup i)-\mu(T))+p_{t+1}^a(n)(\mu(T\cup A\cup i)-\mu(T))\\&=(p_t^a(n)+p_{t+1}^a(n))(\mu(T\cup A\cup i)-\mu(T))\\&=p_t^{a+1}(n)(\mu(T\cup A\cup i)-\mu(T))=p_t^b(n)(\mu(T\cup b)-\mu(T))\end{aligned}$$

or any $T\subseteq N\setminus A\cup i$.

Thus, $\varphi_p(\mu,A)\leqslant\varphi_p(\mu,B)$ for all $A,B\subseteq N$ with $a=b-1$.

Case (2) For any A, $B\subseteq N$, without loss of generality, suppose $a=b-q(q\leqslant n-a)$ and $A\cup\{i_1,i_2,\ldots,i_q\}=B$. Let $A_1=A\cup\{i_1\},A_2=A_1\cup\{i_2\},\ldots,A_q=A_{q-1}\cup\{i_q\}$.

From case (1), we get

$$\varphi_p(\mu,A)\leqslant\varphi_p(\mu,A_1)\leqslant\ldots\leqslant\varphi_p(\mu,A_q)=\varphi_p(\mu,B).$$

From induction, we obtain $\varphi_p(\mu,A)\leqslant\varphi_p(\mu,B)$ for all A, $B\subseteq N$, $A\subseteq B$. From Definition 3, we get the conclusion. □

From Theorem 1, we know $\{\varphi_p(\mu,A_{(i)})-\varphi_p(\mu,A_{(i+1)})\}_{i\in N}$ is a weight vector on $N=\{1,2,\ldots,n\}$, where $A_{(i)}=\{i,\ldots,n\}$ with $A_{(n+1)}=\emptyset$.

When we replace the fuzzy measure with the probabilistic generalized semivalue to the ICCWA and ICCGM operators, we get the probabilistic generalized semivalue ICCWA (PGS-ICCWA) operator and the probabilistic generalized semivalue ICCGM (PGS-ICCGM) operator as follows:

Definition 8. A PGS-ICCWA operator of dimension n is a mapping PGS-ICCA:$\Omega^{n+}\to R^+$ defined on the set of second arguments of two tuples $\langle u_1,[a_1,b_1]\rangle,\langle u_2,[a_2,b_2]\rangle,\ldots,\langle u_n,[a_n,b_n]\rangle$, denoted by

$$\begin{aligned}&\text{PGS-ICCWA}_{\varphi_p}(\langle u_1,[a_1,b_1]\rangle,\langle u_2,[a_2,b_2]\rangle,\ldots,\langle u_n,[a_n,b_n]\rangle)\\&\quad=\text{PGS-ICCWA}_{\varphi_p}(\langle u_1,F_Q[a_1,b_1]\rangle,\langle u_2,F_Q[a_2,b_2]\rangle,\ldots,\langle u_n,F_Q[a_n,b_n]\rangle)\\&\quad=\sum_{j=1}^{n}\Big((\varphi_p(\mu,A_{\sigma(j)})-\varphi_p(\mu,A_{\sigma(j+1)}))F_Q([a_{\sigma(j)},b_{\sigma(j)}])\Big),\end{aligned}\tag{20}$$

where Ω^{n+} is the set of dimension n positive interval numbers, φ_p is the probabilistic generalized semivalue w.r.t. the fuzzy measure μ on $\{[a_i,b_i]\}_{i=1,2,\ldots,n}$, σ is a permutation on $N=\{1,2,\ldots,n\}$ such that $u_{\sigma(j)}\leqslant u_{\sigma(j+1)}$, $u_{\sigma(j)}$ is the jth least value of u_i $(i=1,2,\ldots,n)$, $F_Q([a_{\sigma(j)},$

$b_{\sigma(j)}]$) given as Eq. (1), and $A_{\sigma(i)}\ = \{[a_{\sigma(j)}, b_{\sigma(j)}], \ldots, [a_{\sigma(n)}, b_{\sigma(n)}]\}$, with $A_{\sigma(n+1)} = \emptyset$.

Definition 9. A PGS ICCGM operator of dimension n is a mapping PGS ICCGM: $\Omega^{n+} \to R^{+}$ defined on the set of second arguments of two tuples $\langle u_1, [a_1, b_1]\rangle, \langle u_2, [a_2, b_2]\rangle, \ldots, \langle u_n, [a_n, b_n]\rangle$, denoted by

$$\text{PGS-ICCGM}_{\varphi_p}(\langle u_1,[a_1,b_1]\rangle,\langle u_2,[a_2,b_2]\rangle,\ldots,\langle u_n,[a_n,b_n]\rangle)$$
$$\text{PGS-ICCGM}_{\varphi_p}(\langle u_1,G_Q[a_1,b_1]\rangle,\langle u_2,G_Q[a_2,b_2]\rangle,\ldots,\langle u_n,G_Q[a_n,b_n]\rangle)$$
$$\sum_{j\ 1}^{n} G_Q([a_{\sigma(j)},b_{\sigma(j)}])^{\varphi_p(\mu,A_{\sigma(j)})\quad \varphi_p(\mu,A_{\sigma(j+1)})}, \tag{21}$$

where Ω^{n+} is the set of dimension n positive interval numbers, φ_p is the probabilistic generalized semivalue w.r.t. the fuzzy measure μ on $\{[a_i, b_i]\}_{i=1,2,\ldots,n}$, σ is a permutation on $N = \{1, 2, \ldots, n\}$ such that $u_{\sigma(j)} \leqslant u_{\sigma(j+1)}, u_{\sigma(j)}$ is the jth least value of u_i ($i = 1, 2, \ldots, n$), $G_Q([a_{\sigma(j)}, b_{\sigma(j)}])$ given as Eq. (2), and $A_{\sigma(i)} = \{[a_{\sigma(j)}, b_{\sigma(j)}], \ldots, [a_{\sigma(n)}, b_{\sigma(n)}]\}$, with $A_{\sigma(n+1)} = \emptyset$.

From Theorem 1, we know φ_p is a fuzzy measure, which means that the PGS ICCWA and PGS ICCGM operators satisfy the proper ties studied in Section 3.2. When each $[a_i, b_i]$ ($i = 1, 2, \ldots, n$) degen erates to be a real number, namely, $a_i = b_i$, we get the following two aggregation operators.

The probabilistic generalized semivalue induced Choquet weighted averaging (PGS ICWA) operator

$$\text{PGS-ICWA}_{\varphi_p}(\langle u_1,a_1\rangle,\langle u_2,a_2\rangle,\ldots,\langle u_n,a_n\rangle)$$
$$\sum_{j\ 1}^{n}((\varphi_p(\mu,A_{\sigma(j)})\quad \varphi_p(\mu,A_{\sigma(j+1)}))a_{\sigma(j)}).$$

The probabilistic generalized semivalue induced Choquet geometric mean (PGS ICGM) operator

$$\text{PGS-ICGM}_{\varphi_p}(\langle u_1,a_1\rangle,\langle u_2,a_2\rangle,\ldots,\langle u_n,a_n\rangle)\quad \sum_{j\ 1}^{n} a_{\sigma(j)}^{\varphi_p(\mu,A_{\sigma(j)})\quad \varphi_p(\mu,A_{\sigma(j+1)})}.$$

4.2. An important case

In this section, we give an important case of the PGS ICCWA and PGS ICCGM operators, where the probabilistic generalized semi value is the so called generalized Shapley index, denoted by (Mari chal, 2000):

$$\varphi^{Sh}(\mu,S)\quad \sum_{T\subseteq N\setminus S}\frac{(n\quad s\quad t)!t!}{(n\quad s+1)!}(\mu(S\cup T)\quad \mu(T))\quad \forall S\subseteq N. \tag{22}$$

From Eq. (22), we know it is an expectation value of the overall mar ginal contributions between the coalition S and every coalition $T \subseteq N \setminus S$.

Based on the generalized Shapley index, we introduce the fol lowing two aggregation operators.

The generalized Shapley index ICCWA (GSI ICCWA) operator

$$\text{GSI-ICCWA}_{\varphi^{Sh}}(\langle u_1,[a_1,b_1]\rangle,\langle u_2,[a_2,b_2]\rangle,\ldots,\langle u_n,[a_n,b_n]\rangle)$$
$$\sum_{j\ 1}^{n}((\varphi^{Sh}(\mu,A_{\sigma(j)})\quad \varphi^{Sh}(\mu,A_{\sigma(j+1)}))F_Q([a_{\sigma(j)},b_{\sigma(j)}])). \tag{23}$$

The generalized Shapley index ICCGM (GSI ICCGM) operator

$$\text{GSI-ICCGM}_{\varphi^{Sh}}(\langle u_1,[a_1,b_1]\rangle,\langle u_2,[a_2,b_2]\rangle,\ldots,\langle u_n,[a_n,b_n]\rangle)$$
$$\sum_{j\ 1}^{n} G_Q([a_{\sigma(j)},b_{\sigma(j)}])^{\varphi^{Sh}(\mu,A_{\sigma(j)})\quad \varphi^{Sh}(\mu,A_{\sigma(j+1)})}. \tag{24}$$

Remark 2. From Theorem 1, we know the generalized Shapley index is a fuzzy measure, which means that the GSI ICCWA and GSI ICCGM operators satisfy the properties discussed in Section 3.2.

5. An approach to uncertain multi-attribute group decision making

With economic development, the decision making problems are becoming more complicated, uncertain and fuzzy than ever (Chiclana, Herrera, & Herrera Viedma, 1998; Herrera & Martínez, 2001). In many situations, because of time pressure, lack of knowl edge, and people's limited expertise related with problem domain, it is apparent that an increasing amount of information provided for decision making will be given in interval arguments. Based on the induced continuous Choquet integral operators, we develop an approach to uncertain multi attributes group decision making.

Let $A = \{a_1, a_2, \ldots, a_m\}$ be the set of alternatives, $C = \{c_1, c_2, \ldots, c_n\}$ be the set of attributes, and $E = \{e_1, e_2, \ldots, e_q\}$ be the set of the ex perts. Assume that $\bar{d}_{ij}^k\quad \left[a_{ij}^k, b_{ij}^k\right]$ is the positive interval argument of the alternative a_i with respect to (w.r.t.) the attribute c_j given by the expert e_k. In other words, the evaluation of the alternative a_i w.r.t. the attribute c_j given by the expert e_k is a positive interval number $\bar{d}_{ij}^k\quad \left[a_{ij}^k, b_{ij}^k\right](i\quad 1, 2, \ldots, m)$; $j = 1, 2, \ldots, n$; $k = 1, 2, \ldots, q$. By $D^k\quad \left(\bar{d}_{ij}^k\right)_{m\times n}$, we denote the interval decision matrix given by the expert e_k ($k = 1, 2, \ldots, q$).

Based on the induced continuous Choquet integral operators, the main decision procedure to get the most desirable alternative (s) can be expressed in the following steps:

Step 1: Normalize the interval decision matrix $D^k\quad \left(\bar{d}_{ij}^k\right)_{m\times n}$ into $Q^k\quad \left(\bar{r}_{ij}^k\right)_{m\times n}$ ($k\quad 1, 2, \ldots, q$), where

$$r_{ij}^k = \begin{cases} \left[d_{ij}^k\ /\sum_{i\ 1}^{m} d_{ij}^{k+}, d_{ij}^{k+}/\sum_{i\ 1}^{m} d_{ij}^k\ \right] & \text{for benefit attribute } c_j \\ \left[1/d_{ij}^{k+}/\sum_{i\ 1}^{m} 1/d_{ij}^k\ , 1/d_{ij}^k\ /\sum_{i\ 1}^{m} 1/d_{ij}^{k+}\right] & \text{for cost attribute } c_j \end{cases}$$
$(i=1,2,\ldots,m;\ j=1,2,\ldots,n).$

Step 2: Assume that μ^E is the fuzzy measure on the expert set E, use the GSI ICCWA or GSI ICCGM operator to calculate the com prehensive matrix $H = (h_{ij})_{m\times n}$.

Step 3: Assume that μ^C is the fuzzy measure on experts set C, use the generalized Shapley index induced Choquet weighted averaging (GSI ICWA) operator

$$\text{GSI-ICWA}_{\varphi^{Sh}}(\langle u_1,a_1\rangle,\langle u_2,a_2\rangle,\ldots,\langle u_n,a_n\rangle)$$
$$\sum_{j\ 1}^{n}\left((\varphi^{Sh}(\mu,A_{\sigma(j)})\quad \varphi^{Sh}(\mu,A_{\sigma(j+1)}))a_{\sigma(j)}\right)$$

or the generalized Shapley index induced Choquet geometric mean (GSI ICGM) operator

$$\text{GSI-ICGM}_{\varphi^{Sh}}(\langle u_1,a_1\rangle,\langle u_2,a_2\rangle,\ldots,\langle u_n,a_n\rangle)$$
$$\prod_{j\ 1}^{n} a_{\sigma(j)}^{\varphi^{Sh}(\mu,A_{\sigma(j)})\quad \varphi^{Sh}(\mu,A_{\sigma(j+1)})}$$

to get comprehensive attribute values z_i($i = 1, 2, \ldots, m$).

Step 4: Rank these comprehensive attribute values z_i($i = 1, 2, \ldots, m$) in descending order, and select the biggest one (s). Then, we get the best choice (s).

Step 5: End.

The above decision steps are based on the assumption that the fuzzy measures on the attribute set and on the expert set are already known. As mentioned above, because of various kinds of reasons, it is difficult to obtain their weight vectors exactly. In most situations, we only have incomplete weight information.

Based on consistency principle (Chiclana et al., 2007) and TOPSIS method (Negi, 1989), we introduce the following models for the optimal fuzzy measures on the attribute set and on the expert set, respectively.

First, we introduce a possibility degree formula on interval numbers given by Xu and Da (2003). Let $\bar{\alpha} = [a_1, b_1]$ and $\bar{\beta} = [a_2, b_2]$ be any two positive interval numbers, then the degree of possibility of $\bar{\alpha} \geqslant \bar{\beta}$ is defined by (Xu & Da, 2003)

$$P(\alpha \geqslant \beta) = \max\left\{1 - \max\left\{\frac{b_2 - a_1}{b_1 + b_2 - a_1 - a_2}, 0\right\}, 0\right\}, \quad (25)$$

and the degree of possibility of $\bar{\beta} \geqslant \bar{\alpha}$ is equal to

$$P(\beta \geqslant \alpha) = 1 - P(\alpha \geqslant \beta). \quad (26)$$

Definition 10. Let $G = (g_{ij})_{n\times n}$ be a matrix. If $g_{ij} + g_{ji} = 1$ and $g_{ij} \in [0,1]$ for all $i, j = 1, 2, \ldots, n$, then matrix G is called a fuzzy preference relation or complementary matrix.

As we know, the experts' knowledge, skills and experiences are different. It is unreasonable to give the equal weight of an expert w.r.t. different attributes. Further, if there exit interactive characteristics between experts, it is not suitable to give the weight vector of experts using additive measures. In the following, we introduce the model for the optimal fuzzy measure on the expert set, where every expert's importance is determined w.r.t. each attribute.

By d_k^j, we denote the jth column of the interval decision matrix $D^k = \left(\bar{d}_{ij}^k\right)_{m\times n}$ given by the expert e_k $(k = 1,2,\ldots,q)$. From Eqs. (25) and (26), we obtain the complementary matrix $P_k^j = \left(p_{hl}^{kj}\right)_{m\times m}$ w.r.t. the jth column d_k^j of the interval decision matrix $D^k = \left(\bar{d}_{ij}^k\right)_{m\times n}$. Using the method of constructing a consistent reciprocal fuzzy preference relation (Chiclana et al., 2007), we get the additive consistent complementary matrix $\widetilde{P}_k^j = \left(p_{hl}^{kj}\right)_{m\times m}$ on $A = \{a_1, a_2, \ldots, a_m\}$ from $m - 1$ preference values, where

$$\tilde{p}_{hl}^{kj} = \begin{cases} p_{hl}^{kj} & \text{if } h \leqslant l \leqslant h+1 \\ p_{hh+1}^{kj} + p_{h+1h+2}^{kj} + \cdots + p_{l-1l}^{kj} - \frac{j-(i+1)}{2} & \text{if } l \rangle h+1 \\ 1 - \tilde{p}_{lh}^{kj} & \text{if } h \rangle l \end{cases} \quad (27)$$

for all $h, l = 1, 2, \ldots, m$.

As Chiclana et al. (2007) noted, the matrix $\widetilde{P}_k^j$ maybe entirely do not in the interval [0, 1], but in an interval $\left[-a_j^k, 1 + a_j^k\right]$, where $a_j^k = |p_j^k|$ with $p_k^j = \min\left\{p_{hl}^{kj} : h, l = 1,2,\ldots,m\right\}$. In this situation, we adopt the transformation function $f(x) = \frac{x + a_j^k}{1 + 2a_j^k}$ given by Chiclana et al. (2007).

When we get the additive consistent complementary matrix $\widetilde{P}_k^j = \left(p_{hl}^{kj}\right)_{m\times m}$ w.r.t. the attribute c_j $(j = 1,2,\ldots,n)$ and the expert e_k $(k = 1,2,\ldots,q)$. Use the following consistent index

$$CI_k^j = \sqrt{\sum\nolimits_{l=h}^{m} \sum\nolimits_{h=1}^{m} \left(p_{hl}^{kj} - \tilde{p}_{hl}^{kj}\right)^2},$$

we get the consistent degree of the interval fuzzy preference relation given by the expert e_k $(k = 1,2,\ldots,q)$ w.r.t. the attribute c_j $(j = 1,2,\ldots,n)$.

According to the consistency principle, if the consistent index of an expert is small w.r.t. the attribute c_j $(j = 1,2,\ldots,n)$, it can provide useful information. Therefore, the expert w.r.t. the attribute c_j should be assigned a bigger weight; otherwise, such an expert w.r.t. the attribute c_j will be judged unimportant. In other words, such an expert w.r.t. the attribute c_j should be evaluated as a smaller weight. Further, the optimal fuzzy measure makes each alternative's optimal comprehensive value the bigger the better.

If the weight information of the experts is partly known, then we establish the following model for the optimal fuzzy measure on the expert set E w.r.t. the attribute c_j $(j = 1,2,\ldots,n)$:

$$\min \sum_{k=1}^{q} CI_k^j \varphi_k^j\left(\mu_j^E, E\right)$$
$$\text{s.t.} \begin{cases} \mu_j^E(E) = 1 \\ \mu_j^E(S) \leqslant \mu_j^E(T)\ \forall S, T \subseteq E \text{ s.t. } S \subseteq T \\ \mu_j^E(e_k) \in U_k^j,\ \mu_j^E(e_k) \geqslant 0,\ k = 1,2,\ldots,q \end{cases}, \quad (28)$$

where $\varphi_k^j\left(\mu_j^E, E\right)$ is the Shapley value (Shapley, 1953) of the expert e_k w.r.t. the attribute c_j, defined by

$$\varphi_k^j(\mu_j^E, E) = \sum_{S \subseteq E \setminus e_k} \frac{(q-s-1)!s!}{q!} \left(\mu_j^E(S, e_k) - \mu_j^E(S)\right),$$

with s being the number of experts in S, μ_j^E is the fuzzy measure on the expert set E w.r.t. the attribute c_j, and U_k^j is the range of the expert e_k w.r.t. the attribute c_j.

Solve the model (28), we get the optimal fuzzy measure on the expert set E w.r.t. each attribute c_j $(j = 1,2,\ldots,n)$. Then, we can use the introduced aggregation operators to get the comprehensive matrix $H = (h_{ij})_{m\times n}$.

Remark 3. In order to overall reflect the inter dependent characteristics between experts, in the model (28) we use their Shapley values as their weights.

From the comprehensive matrix $H = (h_{ij})_{m\times n}$, let $h^+ = \{h_1^+, h_2^+, \ldots, h_n^+\}$ and $h^- = \{h_1^-, h_2^-, \ldots, h_n^-\}$, where $h_j^+ = \max_{1\leqslant i\leqslant m}\{h_{ij}\}$ and $h_j^- = \min_{1\leqslant i\leqslant m}\{h_{ij}\}$ for all $j = 1,2,\ldots,n$.

Let

$$d_{ij} = \frac{d_{ij}^+}{d_{ij}^+ + d_{ij}^-},$$

where $d_{ij}^+ = |h_{ij} - h_j^+|$ and $d_{ij}^- = |h_{ij} - h_j^-|$.

Similar to the analysis about the model for the optimal fuzzy measure on the expert set, the optimal fuzzy measure makes bigger comprehensive value for each alternative preferable. If the information about the weights of attributes is partly known, then we build the following model for the optimal fuzzy measure on the attribute set C w.r.t. the alternative a_i $(i = 1,2,\ldots,m)$ based on TOPSIS method.

$$\min \sum_{j=1}^{n} d_{ij} \varphi_j(\mu^C, C)$$
$$\text{s.t.} \begin{cases} \mu^C(C) = 1 \\ \mu^C(S) \leqslant \mu^C(T)\ \forall S, T \subseteq C \text{ s.t. } S \subseteq T \\ \mu^C(c_j) \in U_j,\ \mu^C(c_j) \geqslant 0,\ j = 1,2,\ldots,n \end{cases},$$

where $\varphi_j(\mu^C, C)$ is the Shapley value of the attribute c_j given as in the model (28), μ^C is the fuzzy measure on the attribute set C, and U_j is the range of the attribute c_j.

Since all alternatives are non inferior, we build the following model for the optimal fuzzy measure on the attribute set C by using TOPSIS method.

$$\min \sum_{i=1}^{m} \sum_{j=1}^{n} d_{ij} \varphi_j(\mu^C, C)$$
$$\text{s.t.} \begin{cases} \mu^C(C) = 1 \\ \mu^C(S) \leqslant \mu^C(T)\ \forall S, T \subseteq C \text{ s.t. } S \subseteq T \\ \mu^C(c_j) \in U_j,\ \mu^C(c_j) \geqslant 0,\ j = 1,2,\ldots,n \end{cases}. \quad (29)$$

Solve the model (29), we get the optimal fuzzy measure on the attribute set *C*. Then, we can use the GSI ICWA or GSI ICGM oper ator to get the collective attribute values.

6. An illustrative example

Let us suppose a bid inviting process through which the em ployer or investor is trying to find out the optimal bidding scheme (Zhou & Chen, 2011). In order to keep pace with the development of modern iron and steel industry as well as to improve the envi ronmental equality of the city, Steel and Iron Works wants to con struct a pelletizing plant in his primary producing area of iron ore where the production capacity reaches 1.20 million tons per year. According to the characteristics of the project construction, the construction is divided into four bid packages including construc tion project, installation project, etc., between which the construc tion project is the principal part of the civil works. Considering the regulations of the project, the investor will invite bidding for the construction project and select from four bidders according to the following five attributes:

(1) c_1 is the project quotation;
(2) c_2 is the construction period;
(3) c_3 is the quality of construction project;
(4) c_4 is the construction technology;
(5) c_5 is the business reputation.

Suppose that the uncertain weight information of the attributes is given by $U = ([0.2, 0.3], [0.1, 0.25], [0.2, 0.3], [0.15, 0.25][0.1, 0.2])$. There are four construction organizations ($\{a_1, a_2, a_3, a_4\}$) are se lected as possible alternatives. Four experts ($\{e_1, e_2, e_3, e_4\}$) evaluate the four alternatives by using the interval arguments with scores of centesimal system according to the above five attributes. The uncertain weight information of the experts w.r.t. each attribute is given by

$$W_1 \quad ([0.2, 0.3], [0.15, 0.2], [0.25, 0.3], [0.1, 0.15], [0.25, 0.35]),$$
$$W_2 \quad ([0.15, 0.35], [0.15, 0.25], [0.2, 0.3], [0.2, 0.35], [0.15, 0.3]),$$
$$W_3 \quad ([0.15, 0.25], [0.2, 0.25], [0.15, 0.3], [0.15, 0.3], [0.2, 0.3]),$$
$$W_4 \quad ([0.25, 0.3], [0.2, 0.4], [0.2, 0.3], [0.2, 0.35], [0.2, 0.35]).$$

The decision matrix $D^k \quad \left(\tilde{d}_{ij}^k\right)_{m\times n}$ given by the expert e_k ($k = 1, 2, 3, 4$) as listed in Tables 1 4.

In this problem, all attributes are measured with the same dimension units by scores ranging from 0 to 100, thus the decision matrices D^k ($k = 1, 2, 3, 4$) have no need to be normalized. Based on above analysis, we give the following steps to obtain the optimal bidding scheme.

Step 1: Use Eqs. (25) and (26) to calculate the complementary matrix$P_j^k \quad \left(p_{hl}^{kj}\right)_{m\times m}$ given by the expert e_k ($k = 1, 2, 3, 4$) w.r.t. each attribute c_j ($j = 1, 2, 3, 4, 5$), take $k = j = 1$ for example,

$$P_1^1 \quad \begin{pmatrix} 0.5 & 0 & 1 & 1 \\ 1 & 0.5 & 1 & 1 \\ 0 & 0 & 0.5 & 0.57 \\ 0 & 0 & 0.43 & 0.5 \end{pmatrix}.$$

Step 2: According to matrix $P_j^k \quad \left(p_{hl}^{kj}\right)_{m\times m}$ (k = 1, 2, 3, 4; $j = 1, 2, 3, 4, 5$), utilize Eq. (27) to construct the additive consistent complementary matrix $\widetilde{P}_j^k \quad \left(p_{hl}^{kj}\right)_{m\times m}$, take $k = j = 1$ for example,

$$\widetilde{P}_1^1 \quad \begin{pmatrix} 0.5 & 0.0614 & 0.5 & 0.5614 \\ 0.9386 & 0.5 & 0.9386 & 1 \\ 0.5 & 0.0614 & 0.5 & 0.5614 \\ 0.4386 & 0 & 0.4386 & 0.5 \end{pmatrix}.$$

Step 3: According to the model (28), we get the following linear pro gramming for the optimal fuzzy measure on the expert set *E* w.r.t. the attribute c_1.

$$\begin{aligned} \min \quad & 0.175(\mu_1^E(e_1) \quad \mu_1^E(e_2, e_3, e_4)) \\ & + 0.164(\mu_1^E(e_2) \quad \mu_1^E(e_1, e_3, e_4)) \quad 0.185(\mu_1^E(e_3) \quad \mu_1^E(e_1, e_2, e_4)) \\ & \quad 0.154(\mu_1^E(e_4) \quad \mu_1^E(e_1, e_2, e_3)) + 0.169(\mu_1^E(e_1, e_2) \quad \mu_1^E(e_3, e_4)) \\ & \quad 0.005(\mu_1^E(e_1, e_3) \quad \mu_1^E(e_2, e_4)) + 0.011(\mu_1^E(e_1, e_4) \quad \mu_1^E(e_2, e_3)) \\ & + 1.075 \end{aligned}$$

$$s.t. \begin{cases} \mu_1^E(E) \quad 1 \mu_1^E(S) \leqslant \mu_1^E(T) \forall S, T \subseteq \{e_1, e_2, e_3, e_4\} s.t. S \subseteq T \\ \mu_1^E(e_1) \in [0.2, 0.3], \mu_1^E(e_2) \in [0.15, 0.35] \mu_1^E(e_3) \\ \in [0.15, 0.25], \mu_1^E(e_4) \in [0.25, 0.3] \end{cases}.$$

Solve the above model, we have

$$\mu_1^E(e_2) \quad 0.15, \mu_1^E(e_1) \quad \mu_1^E(e_1, e_2) \quad 0.2, \mu_1^E(e_3, e_4) \quad \mu_1^E(e_1, e_3, e_4) \quad \mu_1^E(e_2, e_3, e_4) \quad \mu_1^E(E) \quad 1,$$

$$\mu_1^E(e_3) \quad \mu_1^E(e_4) \quad \mu_1^E(e_1, e_3) \quad \mu_1^E(e_1, e_4) \quad \mu_1^E(e_2, e_3) \quad \mu_1^E(e_2, e_4) \quad \mu_1^E(e_1, e_2, e_3) \quad \mu_1^E(e_1, e_2, e_4) \quad 0.25.$$

Similarly, we obtain

$$\mu_2^E(e_1) \quad \mu_2^E(e_3) \quad \mu_2^E(e_4) \quad \mu_2^E(e_1, e_3) \quad \mu_2^E(e_1, e_4) \quad \mu_2^E(e_3, e_4) \quad \mu_2^E(e_1, e_3, e_4) \quad 0.2,$$

Table 1
The interval decision matrix D^1 given by the expert e_1.

	c_1	c_2	c_3	c_4	c_5
a_1	[85,88]	[86,90]	[72,78]	[86,90]	[72,76]
a_2	[88,92]	[70,75]	[90,92]	[65,75]	[85,90]
a_3	[75,80]	[77,82]	[81,85]	[87,90]	[75,82]
a_4	[76,78]	[86,88]	[90,93]	[88,90]	[85,88]

Table 2
The interval decision matrix D^2 given by the expert e_2.

	c_1	c_2	c_3	c_4	c_5
a_1	[75,79]	[81,83]	[78,85]	[67,71]	[85,88]
a_2	[76,81]	[82,85]	[75,79]	[71,75]	[81,84]
a_3	[65,70]	[75,82]	[68,72]	[89,90]	[92,95]
a_4	[68,75]	[82,86]	[76,80]	[75,77]	[84,88]

Table 3
The interval decision matrix D^3 given by the expert e_3.

	c_1	c_2	c_3	c_4	c_5
a_1	[80,90]	[85,90]	[81,85]	[70,75]	[70,74]
a_2	[95,97]	[72,76]	[71,75]	[85,91]	[85,89]
a_3	[92,96]	[80,85]	[85,90]	[80,86]	[91,95]
a_4	[90,93]	[62,68]	[75,80]	[76,80]	[68,72]

Table 4
The interval decision matrix D^4 given by the expert e_4.

	c_1	c_2	c_3	c_4	c_5
a_1	[74,77]	[95,98]	[71,75]	[66,71]	[90,95]
a_2	[89,91]	[75,78]	[94,97]	[67,71]	[68,72]
a_3	[89,93]	[95,98]	[90,95]	[81,87]	[75,81]
a_4	[75,81]	[86,92]	[71,76]	[92,98]	[68,75]

Table 5
The Shapley values of the experts w.r.t. each attribute.

	c_1	c_2	c_3	c_4	c_5
$\varphi_1^j\left(\mu_j^E, E\right)$	0.054	0.175	0.138	0.388	0.525
$\varphi_2^j\left(\mu_j^E, E\right)$	0.038	0.596	0.108	0.054	0.375
$\varphi_3^j\left(\mu_j^E, E\right)$	0.454	0.05	0.096	0.054	0.05
$\varphi_4^j\left(\mu_j^E, E\right)$	0.454	0.175	0.654	0.504	0.05

$$\mu_2^E(e_2) \quad \mu_2^E(e_2,e_3) \quad 0.25, \mu_2^E(e_1,e_2) \quad \mu_2^E(e_2,e_4) \quad \mu_2^E(e_1,e_2,e_3) \quad \mu_2^E(e_1,e_2,e_4) \quad \mu_2^E(e_2,e_3,e_4) \quad \mu_2^E(E) \quad 1;$$

$$\mu_3^E(e_3) \quad 0.15, \mu_3^E(e_2) \quad \mu_3^E(e_2,e_3) \quad 0.2, \mu_3^E(e_4) \quad 0.3, \mu_3^E(e_1) \quad \mu_3^E(e_1,e_2) \quad \mu_3^E(e_1,e_3) \quad \mu_3^E(e_1,e_2,e_3) \quad 0.25, \mu_3^E(e_1,e_4) \quad \mu_3^E(e_2,e_4) \quad \mu_3^E(e_3,e_4) \quad \mu_3^E(e_1,e_2,e_4) \quad \mu_3^E(e_1,e_3,e_4) \quad \mu_3^E(e_2,e_3,e_4) \quad \mu_3^E(E) \quad 1;$$

$$\mu_4^E(e_1) \quad 0.15, \mu_4^E(e_2) \quad \mu_4^E(e_4) \quad \mu_4^E(e_1,e_2) \quad \mu_4^E(e_1,e_4) \quad \mu_4^E(e_2,e_4) \quad \mu_4^E(e_1,e_2,e_4) \quad 0.2, \mu_4^E(e_3) \quad \mu_4^E(e_2,e_3) \quad \mu_4^E(e_3,e_4) \quad \mu_4^E(e_2,e_3,e_4) \quad 0.3, \mu_4^E(e_1,e_3) \quad \mu_4^E(e_1,e_2,e_3) \quad \mu_4^E(e_1,e_3,e_4) \quad \mu_4^E(E) \quad 1;$$

$$\mu_5^E(e_2) \quad \mu_5^E(e_3) \quad \mu_5^E(e_4) \quad \mu_5^E(e_2,e_3) \quad \mu_5^E(e_2,e_4) \quad \mu_5^E(e_3,e_4) \quad \mu_5^E(e_2,e_3,e_4) \quad 0.2, \quad \mu_5^E(e_1) \quad \mu_5^E(e_1,e_3) \quad \mu_5^E(e_1,e_4) \quad \mu_5^E(e_1,e_3,e_4) \quad 0.35, \mu_5^E(e_1,e_2) \quad \mu_5^E(e_1,e_2,e_3) \quad \mu_5^E(e_1,e_2,e_4) \quad \mu_5^E(E) \quad 1.$$

Step 4: From the fuzzy measures on the expert set E, we get the Shapley values of the experts w.r.t. each attribute as listed in Table 5.

Let $u_k \quad \varphi_k^j\left(\mu_j^E, E\right)$ ($k \quad 1,2,3,4$). When there exist more than one expert's Shapley value is equal, we rearrange them according to the index in ascending order. Further, we take$Q(y) = y$, then $\lambda = 1/2$. Use the GSI ICCWA operator to aggregate the interval decision matrices D^k ($k = 1,2,3,4$), for all i, j, e.g., $i = 1$, $j = 1$,

$$h_{11} \quad \text{GSI-ICCWA}_{\varphi^{Sh}}\left(\langle\varphi_1^1(\mu_1^E,E),\left[a_{11}^1,b_{11}^1\right]\rangle,\langle\varphi_2^1(\mu_1^E,E),\left[a_{11}^2,b_{11}^2\right]\rangle,\left\langle\varphi_3^1(\mu_1^E,E),\left[a_{11}^3,b_{11}^3\right]\right\rangle,\langle\varphi_4^1(\mu_1^E,E),\left[a_{11}^4,b_{11}^4\right]\rangle\right)$$
$$(\varphi^{Sh}(\mu_1^E,E) \quad \varphi^{Sh}(\mu_1^E,E\setminus e_3))F_Q\left(\left[a_{11}^3,b_{11}^3\right]\right) + (\varphi^{Sh}(\mu_1^E,E\setminus e_3) \quad \varphi^{Sh}(\mu_1^E,\{e_1,e_2\}))F_Q\left(\left[a_{11}^4,b_{11}^4\right]\right)$$
$$+ (\varphi^{Sh}(\mu_1^E,\{e_1,e_2\}) \quad \varphi^{Sh}(\mu_1^E,\{e_2\}))F_Q\left(\left[a_{11}^1,b_{11}^1\right]\right) + (\varphi^{Sh}(\mu_1^E,\{e_2\}) \quad \varphi^{Sh}(\mu_1^E,\emptyset))F_Q\left(\left[a_{11}^2,b_{11}^2\right]\right) \quad 80.44.$$

Similar to the calculation of h_{11}, we get the following comprehensive matrix:

$$H \quad \begin{pmatrix} 80.44 & 86.03 & 76.03 & 76.26 & 76.23 \\ 92.56 & 78.92 & 90.47 & 70.8 & 85.14 \\ 91.45 & 81.49 & 86.37 & 86 & 84.78 \\ 84.49 & 84.61 & 79.43 & 89.75 & 83.27 \end{pmatrix}.$$

Step 5: From the comprehensive matrix $H = (h_{ij})_{4\times 5}$, we get the following relative distance matrix:

$$D \quad \begin{pmatrix} 0 & 0 & 1 & 0.71 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 0.09 & 0.64 & 0.28 & 0.2 & 0.04 \\ 0.67 & 0.2 & 0.76 & 0 & 0.21 \end{pmatrix}.$$

According to the model (29), we get the following linear programming for the optimal fuzzy measure on the attribute set C.

$$\begin{aligned} \min \quad & 0.001(\mu^C(c_1) \quad \mu^C(C\setminus c_1)) + 0.02(\mu^C(c_2) \quad \mu^C(C\setminus c_2)) \\ & + 0.072(\mu^C(c_3) \quad \mu^C(C\setminus c_3)) + 0.037(\mu^C(c_4) \quad \mu^C(C\setminus c_4)) \\ & 0.128(\mu^C(c_5) \quad \mu^C(C\setminus c_5)) + 0.006(\mu^C(c_1,c_2) \\ & \mu^C(C\setminus\{c_1,c_2\})) + 0.024(\mu^C(c_1,c_3) \quad \mu^C(C\setminus\{c_1,c_3\})) \\ & + 0.012(\mu^C(c_1,c_4) \quad \mu^C(C\setminus\{c_1,c_4\})) \quad 0.043(\mu^C(c_1,c_5)) \\ & \mu^C(C\setminus\{c_1,c_5\}) + 0.031(\mu^C(c_2,c_3) \quad \mu^C(C\setminus\{c_2,c_3\})) \\ & + 0.019(\mu^C(c_2,c_4)) \quad \mu^C(C\setminus\{c_2,c_4\}) \quad 0.036(\mu^C(c_2,c_5)) \\ & \mu^C(C\setminus\{c_2,c_5\}) + 0.036(\mu^C(c_3,c_4)) \quad \mu^C(C\setminus\{c_3,c_4\}) \\ & 0.019(\mu^C(c_3,c_5)) \quad \mu^C(C\setminus\{c_3,c_5\}) \quad 0.03(\mu^C(c_4,c_5)) \\ & \mu^C(C\setminus\{c_4,c_5\}) + 1.762 \end{aligned}$$

$$s.t. \begin{cases} \mu^C(S) \leqslant \mu^C(T)\forall S, \quad T\subseteq\{c_1,c_2,c_3,c_4,c_5\} s.t.\ S\subseteq T \\ \mu^C(c_1)\in[0.2,0.3], \quad \mu^C(c_2)\in[0.1,0.25] \\ \mu^C(c_3)\in[0.2,0.3], \quad \mu^C(c_4)\in[0.15,0.25] \\ \mu^C(c_5)\in[0.1,0.2], \quad \mu^C(C) \quad 1 \end{cases}.$$

Solve the above model, we obtain

$$\mu^C(c_2) \quad 0.1, \mu^C(c_4) \quad \mu^C(c_2,c_4) \quad 0.15, \mu^C(c_1) \quad \mu^C(c_3) \quad \mu^C(c_5) \quad \mu^C(c_1,c_2) \quad \mu^C(c_1,c_3) \quad \mu^C(c_1,c_4) \quad \mu^C(c_2,c_3) \quad \mu^C(c_2,c_5) \quad (\mu^C(c_3,c_4)) \quad \mu^C(c_3,c_5) \quad \mu^C(c_4,c_5) \quad \mu^C(C\setminus\{c_4,c_5\}) \quad \mu^C(C\setminus\{c_2,c_5\}) \quad \mu^C(C\setminus\{c_1,c_5\}) \quad \mu^C(C\setminus\{c_1,c_4\}) \quad \mu^C(C\setminus\{c_1,c_3\}) \quad \mu^C(C\setminus\{c_1,c_2\}) \quad \mu^C(C\setminus c_1) \quad 0.2, \mu^C(C\setminus\{c_3,c_5\}) \quad \mu^C(C\setminus c_5) \quad 0.4,$$

$$\mu^C(c_1,c_5) \quad \mu^C(C\setminus\{c_3,c_4\}) \quad \mu^C(C\setminus\{c_2,c_4\}) \quad \mu^C(C\setminus\{c_2,c_3\}) \quad \mu^C(C\setminus c_2) \quad \mu^C(C\setminus c_3) \quad \mu^C(C\setminus c_4) \quad \mu^C(C) \quad 1.$$

Step 6: From the fuzzy measure on the attribute set C, we get the Shapley values of the attributes

$$\varphi_1(\mu^C,C) \quad 0.46, \quad \varphi_2(\mu^C,C) \quad 0.037, \quad \varphi_3(\mu^C,C) \quad 0.05, \quad \varphi_4(\mu^C,C) \quad 0.049, \quad \varphi_5(\mu^C,C) \quad 0.41.$$

Let $u_j = \varphi_j(\mu^C,C)$ ($j = 1,2,3,4,5$), utilize the GSI ICWA operator to calculate the collective attribute value z_i for all i = 1, 2, 3, 4, e.g., i = 1

$$\begin{aligned} z_1 \quad & GSI \quad ICWA_{\varphi^{Sh}}(\langle\varphi_1(\mu^C,C),h_{11}\rangle,\langle\varphi_2(\mu^C,C),h_{12}\rangle, \\ & \langle\varphi_3(\mu^C,C),h_{13}\rangle,\langle\varphi_4(\mu^C,C),h_{14}\rangle, \quad \langle\varphi_5(\mu^C,C),h_{15}\rangle) \\ & (\varphi^{Sh}(\mu^C,C) \quad \varphi^{Sh}(\mu^C,C\setminus c_1))h_{11} + (\varphi^{Sh}(\mu^C,C\setminus c_1) \\ & \varphi^{Sh}(\mu^C,C\setminus\{c_1,c_5\}))h_{15} \quad + (\varphi^{Sh}(\mu^C,C\setminus\{c_1,c_5\}) \\ & \varphi^{Sh}(\mu^C,\{c_2,c_4\}))h_{13} + (\varphi^{Sh}(\mu^C,\{c_2,c_4\}) \quad \varphi^{Sh}(\mu^C,\{c_2\}))h_{14} \\ & + (\varphi^{Sh}(\mu^C,\{c_2\}) \quad \varphi^{Sh}(\mu^C,\emptyset))h_{12} \\ & 0.5\times 80.44 + 0.4\times 86.03 + 0.029\times 76.03 + 0.034 \\ & \times 76.26 + 0.037\times 76.23 \quad 78.69. \end{aligned}$$

Similarly, we obtain

$$Z_2 \quad 88.28, \quad Z_3 \quad 88.08, \quad Z_4 \quad 84.04.$$

Thus,

Table 6
Ranking orders for the continuous Shapley operators (30) and (31).

The continuous Shapley weighted operator	Ranking order
The CSA operator	$a_3 > a_4 > a_2 > a_1$
The CSGM operator	$a_3 > a_4 > a_2 > a_1$

$Z_2 > Z_3 > Z_4 > Z_1$.

Step 7: From Step 6, we know the second construction organization a_2 is the best choice, which is different to the ranking result got by Chen and Zhou (2011).

Remark 4. In this example, we only use the GSI ICCWA and GSI ICWA operators to making decision. Similarly, we can adopt the GSI ICCGM and GSI ICGM operators to obtain the best choice (s).

Based on the Shapley function, Zhang, Xu, and Yu (2011) defined the so called Shapely value based intuitionistic fuzzy aggregation (SIFA) operator on IFSs. We here restrict the domain of IFSs in the setting of positive interval numbers and get the following continuous Shapley averaging (CSA) operator

$$\mathrm{CSA}(\alpha_1, \alpha_2, \ldots, \alpha_n) \quad \sum_{i\;\;1}^{n} \varphi_{\alpha_i}(\mu, A) F_Q(\alpha_i), \tag{30}$$

where $\varphi_{\alpha_i}(\mu, A)$ is the Shapley value with respect to the fuzzy measure μ on $A \quad \{\bar{\alpha}_i\}_{i\in N}$ for $\bar{\alpha}_i \quad [a_i, b_i](i \quad 1, 2, \ldots, n)$, and $F_Q(\bar{\alpha}_i)$ as given in Eq. (1).

Remark 5. From Eq. (30), we know that the SIFA operator is based on the Shapley function and degenerates to the weighted averaging (WA) operator if there is no interaction between elements. While the GSI ICCWA operator is based on the generalized Shapley function and Choquet integral and reduces to the induced weighted averaging (I WA) operator or the induced ordered weighted averaging (I OWA) operator if there is no correlation between elements. Their main difference is that the SIFA operator considers the elements' importance and interactions, while the GSI ICCWA operator gives these two aspects by considering their ordered positions.

Similar to the induced continuous Choquet geometric mean (ICCGM) operator, we define the following continuous Shapley geometric mean (CSGM) operator

$$\mathrm{CSGM}(\alpha_1, \alpha_2, \ldots, \alpha_n) \quad \prod_{i\;\;1}^{n} G_Q(\alpha_i)^{\varphi_{\alpha_i}(\mu, A)} \tag{31}$$

where the notations as given in Eq. (30), and $G_Q(\bar{\alpha}_i)$ as shown in Eq. (2).

The difference between the CSGM operator and the GSI ICCGM operator is similar to that between the SIFA operator and the GSI ICCWA operator.

Let $Q(y) = y$, for the comparative convenience, the ranking results with respect to the continuous Shapley operators (30) and (31) are obtained in Table 6.

Although the CSA and CSGM operators can globally reflect the interactions between elements in a set, these two operators neither consider the significance of the elements' ordered positions nor reflect the correlations between them.

The numerical results show that different optimal alternatives may be yielded by using different aggregation operators, and thus, the decision maker can properly select the desirable alternative according to his interest and the actual needs.

In this study, we only select one practical example in project bidding to show the concrete practicality and validity of the proposed method. Besides its application in this field, we can also use the introduced Choquet integral operators and the models for the optimal fuzzy measures in other fields, such as industrial engineering, expert systems, neural networks, digital image processing, and uncertain systems and controls.

7. Conclusions

We have researched some probabilistic generalized semivalue inducing continuous Choquet integral operators, which globally consider the interactions between elements in a set. If there is no correlation between elements in a set, the introduced operators degenerate to be the corresponding induced continuous operators based on additive measures. Meantime, some desirable properties, such as *monotonicity*, *idempotency*, *boundary*, and *linearity*, are studied to provide assurance in applications. Due to the complexity and uncertainty of real world decision making problems and the inherent subjective nature of human thinking, the information about weight vector is usually partly known. To address this situation, the models for the optimal weight vectors on the attribute set and on the expert set based on the Shapley function, consistency principle, and TOPSIS method are built, respectively. Consequently, it has developed a procedure to uncertain multi attribute group decision making with incomplete weight information and interactive conditions, which is new and different to any existing method.

Fuzzy measures and fuzzy integrals, as powerful tools to reflect interactions and to aggregate fuzzy information, give us a new viewpoint to study decision making problems. Although the fuzzy measure is a powerful tool to reflect the interactions between elements in a set, it is defined on the power set. Thus, it is not easy to obtain the fuzzy measure of each combination in a set when it is large. It will be interesting to research the interactions between elements in a set by using some special fuzzy measures, which will largely simplify the complexity of solving a fuzzy measure. Further, we here only consider the Choquet integral operators, and it will be interesting to research aggregation operators based on other fuzzy integrals.

It is worth pointing out that this paper only research the application of the defined aggregation operators and the building models for the weight vectors in uncertain multi attribute group decision making. In a similar way, we can also use them in some other fields, such as education, medical care, military, engineering, social sciences, and economics.

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