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Enhanced nonlinear 3D Euler–Bernoulli beam with flying support

Hassan Zohoor · S. Mahdi Khorsandijou

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Abstract Using Hamilton's principle the coupled nonlinear partial differential motion equations of a flying 3D Euler–Bernoulli beam are derived. Stress is treated three dimensionally regardless of in-plane and out-of-plane warpings of cross-section. Tension, compression, twisting, and spatial deflections are nonlinearly coupled to each other. The flying support of the beam has three translational and three rotational degrees of freedom. The beam is made of a linearly elastic isotropic material and is dynamically modeled much more accurately than a nonlinear 3D Euler–Bernoulli beam. The accuracy is caused by two new elastic terms that are lost in the conventional nonlinear 3D Euler–Bernoulli beam theory by differentiation from the approximated strain field regarding negligible elastic orientation of cross-sectional frame. In this paper, the exact strain field concerning considerable elastic orientation of cross-sectional frame is used as a source in differentiations although the orientation of cross-section is negligible.

Keywords 3D Euler–Bernoulli beam theory

H. Zohoor (✉)
Center of Excellence in Design, Robotics, and Automation, School of Mechanical Engineering, Sharif University of Technology, Tehran, Iran
e-mail: zohoor@sharif.edu

S. M. Khorsandijou
School of Mechanical Engineering, Sharif University of Technology, Tehran, Iran

1 Introduction

Beams are one of the most important structural elements in engineering fields. To provide the structure with less weight makes designer choose more flexible beams. As the flexibility of the beam increases, its dynamic modeling becomes more complicated in return.

Karray et al. [1] have treated the two flexible links of space-based flexible manipulator as Euler–Bernoulli beams, free to deform transversely in the orbital plane. Hiller [2] has considered only three elastic degrees of freedom for each link as an Euler–Bernoulli beam. Shi et al. [3] have modeled a planar flexible link by an Euler–Bernoulli beam. Chen [4] has presented a linearized dynamic model for multilink planar flexible manipulators which can include an arbitrary number of flexible links. Flexible links are treated as Euler–Bernoulli beams and the rotary inertia and shear deformation are thus neglected. Bruno and Luigi [5] have modeled planar n-link flexible manipulators in accordance with Euler–Bernoulli beam. Jen et al. [6] have obtained dynamic model of a one-link flexible robot, using planar Euler–Bernoulli beam. Zohoor and Khorsandijou [7] have dynamically modeled a mobile robot with long and short spatially flexible links experiencing considerable and negligible elastic orientation in their cross-sectional frames. They have exposed the dynamic model of a flying manipulator with two highly flexible links [7]. The nonlinear 3D Euler–Bernoulli beam theory has been formulated for large elastic orientation of cross-section [7], and has been indirectly improved for negligible elastic orientation of the cross-section [7].

In this paper, partial differential equations of motion of a 3D Euler–Bernoulli beam with a six DOF flying support is obtained. The equations are more accurate than that of a conventional nonlinear 3D Euler–Bernoulli beam, because variation of strains and variation of elastic potential energy are derived from the exact strain field concerning considerable elastic orientation of the cross-sectional frame. The elastic orientation of the cross-sectional frame is negligible and the equations will be finally expressed by the approximated strain field. In the motion equations, two additional elastic terms have appeared that would perish in the conventional nonlinear 3D Euler Bernoulli beam theory. In this theory, variation of strains and variation of elastic potential energy are derived from approximated strain field regarding negligible elastic orientation of beam cross-sectional frame [8].

2 Flying support of the beam

The flying support of the beam has six degrees of freedom. In Fig. 1, the frame of the flying support of the beam is denoted by F_B. The inertial reference frame is shown by F_I, the direction of whose third axis is in the negative direction of the gravity. Position, variation of position, velocity, and acceleration of the flying support are projected onto F_I as expressions (1). F_B and F_I are orthogonal and right-handed coordinate reference frames and their axes are marked by 1, 2, and 3 in the figures to indicate, the first, second, and third axes, respectively.

$$b = [x \quad y \quad z]^{\mathrm{T}}, \quad \delta b = [\delta x \quad \delta y \quad \delta z]^{\mathrm{T}},$$
$$\underset{I}{V}^{B} = [\dot{x} \quad \dot{y} \quad \dot{z}]^{\mathrm{T}}, \quad \underset{I}{a}^{B} = [\ddot{x} \quad \ddot{y} \quad \ddot{z}]^{\mathrm{T}}. \tag{1}$$

Orientation of the flying support relative to inertial reference frame is described by three Euler angles as expression (2).

$$\underset{\mathrm{BI}}{R} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & \sin\psi \\ 0 & -\sin\psi & \cos\psi \end{bmatrix} \begin{bmatrix} \cos\phi & 0 & -\sin\phi \\ 0 & 1 & 0 \\ \sin\phi & 0 & \cos\phi \end{bmatrix}$$
$$\times \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{2}$$

Virtual rotations and angular velocity of the flying support that are respectively imperfect differentials and nonintegrable time-derivatives are given in

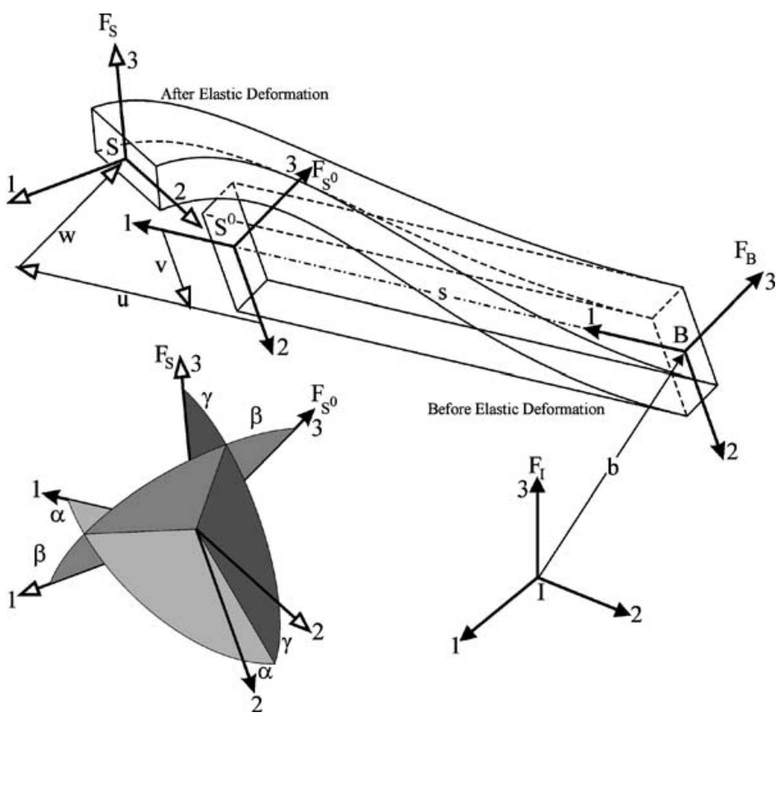


Fig. 1 Six dependent spatial elastic coordinates

expressions (3).

$$\delta\pi^B = \begin{bmatrix}\delta\pi_x\\ \delta\pi_y\\ \delta\pi_z\end{bmatrix} = \begin{bmatrix}1 & 0 & -\sin\phi\\ 0 & \cos\psi & \sin\psi\cos\phi\\ 0 & -\sin\psi & \cos\psi\cos\phi\end{bmatrix}\begin{bmatrix}\delta\psi\\ \delta\phi\\ \delta\theta\end{bmatrix},$$

$$\omega^B = \begin{bmatrix}\omega_x\\ \omega_y\\ \omega_z\end{bmatrix} = \begin{bmatrix}1 & 0 & -\sin\phi\\ 0 & \cos\psi & \sin\psi\cos\phi\\ 0 & -\sin\psi & \cos\psi\cos\phi\end{bmatrix}\begin{bmatrix}\dot\psi\\ \dot\phi\\ \dot\theta\end{bmatrix}. \quad (3)$$

Out of the nonphysical singularity, one has the set of Equation (4).

$$\begin{bmatrix}\dot\psi\\ \dot\phi\\ \dot\theta\end{bmatrix} = \begin{bmatrix}1 & \tan\phi\sin\psi & \tan\phi\cos\psi\\ 0 & \cos\psi & -\sin\psi\\ 0 & \sec\phi\sin\psi & \sec\phi\cos\psi\end{bmatrix}\begin{bmatrix}\omega_x\\ \omega_y\\ \omega_z\end{bmatrix}. \quad (4)$$

3 Beam

The beam is made from a linearly elastic isotropic material with uniform density and cross-sectional area. Density, cross-sectional area, and the length before elastic deformation are shown by ρ, A, and L, respectively. The beam is straight before elastic deformation and undergoes large elastic orientations in its cross-sectional frame.

Exact strain field is the strain field of a long beam that is obtained from displacement field with large elastic orientation of cross-sectional frame, and approximated strain field is the strain field of a short beam that is obtained from displacement field with small elastic orientation of cross-sectional frame. This paper tries to obtain motion equations of nonlinear 3D Euler–Bernoulli beam with small elastic orientation of cross-sectional frame and with a six DOF flying support. Rotational elastic degrees of freedom in the approximated strain field have been substituted with zero. Therefore, variation of strains cannot be derived from the approximated strain field. In this paper, variation of strains are derived from the exact strain field and then formulations are approximated for negligible elastic orientation of cross-sectional frame by substituting the rotational elastic degrees of freedom with zero. This provides us with motion equations much more accurate than that of the conventional nonlinear 3D Euler–Bernoulli beam theory that gains variation of strains and variation of elastic potential energy from the approximated strains indicating negligible elastic orientation in the cross-sectional frame. Two additional elastic terms have appeared that would perish in the nonlinear 3D Euler Bernoulli beam theory. Hence, the motion equations of this paper are more accurate than that of the nonlinear 3D Euler–Bernoulli beam. The two new elastic terms in the motion equations create this accuracy. This enhances the nonlinear 3D Euler–Bernoulli beam theory that derives variation of strains and consequently variation of elastic potential energy from the approximated strain field.

In Fig. 1, F_S is the cross-sectional frame after elastic deformation which is a curvilinear orthogonal right-handed coordinate frame whose first axis is tangent to the curve created by cross-sectional area centers. Its axes are marked by 1, 2, and 3 in the figures to indicate, the first, second, and the third axes, respectively. Cross-sectional frame before elastic deformation is shown by F_{S^0}. Center of cross-sectional area before and after elastic deformation is shown by S^0 and S, respectively. Center of cross-sectional area before elastic deformation, S^0, is located in distance s from B. The spatial independent variable s denotes the distance of S from B before deformation. It is a Lagrangian and not an Eulerian coordinate.

Figure 1 simply expresses spatial elastic deformation of F_S by six dependent coordinates of which only four are independent. Elastic displacement vector of S from B projected onto F_B is shown by d, and elastic rotation transformation matrix projecting a vector from F_B onto F_S is shown by R_{SB} in expressions (5)

$$d = \begin{bmatrix}u+s\\ v\\ w\end{bmatrix},$$

$$\underset{\mathrm{SB}}{R} = \begin{bmatrix}1 & 0 & 0\\ 0 & \cos\gamma & \sin\gamma\\ 0 & -\sin\gamma & \cos\gamma\end{bmatrix}\begin{bmatrix}\cos\beta & 0 & -\sin\beta\\ 0 & 1 & 0\\ \sin\beta & 0 & \cos\beta\end{bmatrix} \times \begin{bmatrix}\cos\alpha & \sin\alpha & 0\\ -\sin\alpha & \cos\alpha & 0\\ 0 & 0 & 1\end{bmatrix}. \quad (5)$$

Elastic deformation at S along the first axis of F_B due to extension or compression is shown by u. Elastic bending deflections at S along the second and the third axes of F_B are shown by v and w, respectively. The first and second Euler angles, α and β, are the so-called elastic bending rotation angles at S about the third axis of F_B and about the second axis of the updated F_B by α, respectively. The third Euler angle, γ, is the so-called elastic twisting angle at S about the first axis of F_S [7]. In some contexts these three Euler angles are called Bryant angles [2].

To avoid lengthy expressions, the beams with circular and square cross-sections are considered only. It implies that F_S is a principal frame for the beam cross-section and the two moments of cross-sectional area about the second and third axes are equal as shown in expressions (6). In Fig. 2, σ is a general point of the beam cross-section. Position vector of σ from S projected onto F_S is shown by p. It is a constant vector in 3D Euler–Bernoulli beam theory and in this paper to neglect in-plane and out-of-plane warpings

$$\int_A p^{\mathrm{T}} p\, dA = 2J,$$

$$[J_S] = \int_A -\tilde{p}\tilde{p}\, dA = J \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$p = \begin{bmatrix} 0 \\ \hat{y} \\ \hat{z} \end{bmatrix}, \quad \tilde{p} = \begin{bmatrix} 0 & -\hat{z} & \hat{y} \\ \hat{z} & 0 & 0 \\ -\hat{y} & 0 & 0 \end{bmatrix}. \tag{6}$$

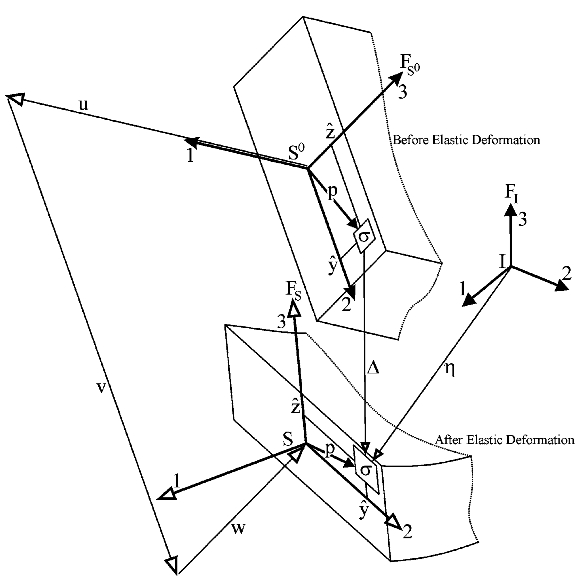


Fig. 2 Displacement field

3.1 Beam structural constraints

Figure 3 displays two holonomic constraints among the six simply created elastic coordinates. By the application of these constraints, the two excess coordinates are eliminated. In expressions (9), the axial strain is denoted by e. Length of a beam element is Δs before and is $(1+e)\Delta s$ after elastic deformation. In order to study the deformation relative to beam structure the origins of F_{S^0} and F_S coincide via translation in the lower part of Fig. 3. Position and orientation of beam cross-section facing S in the element after elastic deformation relative to beam cross-section facing S^0 in the element before deformation are related to Δu, Δv, Δw, α, β and γ. Position and orientation of F_S relative to F_{S^0} are related to u, v, w, α, β, and γ. In the following equations, $[\,]'$ refers to $\frac{\partial}{\partial s}[\,]$ and $[\,]^{\cdot}$ to $\frac{\partial}{\partial t}[\,]$. For the two triangles in lower part of Fig. 3, one can write expressions (7)

$$\alpha = \lim_{\Delta s \to 0} \left[\tan^{-1} \frac{\Delta v}{\Delta s + \Delta u} \right],$$

$$\beta = \lim_{\Delta s \to 0} \left[\tan^{-1} \frac{-\Delta w}{\sqrt{(\Delta s + \Delta u)^2 + \Delta v^2}} \right]. \tag{7}$$

Hence [7],

$$\alpha = \tan^{-1}(v'/h), \quad \beta = \tan^{-1}(-w'/r) \tag{8}$$

where

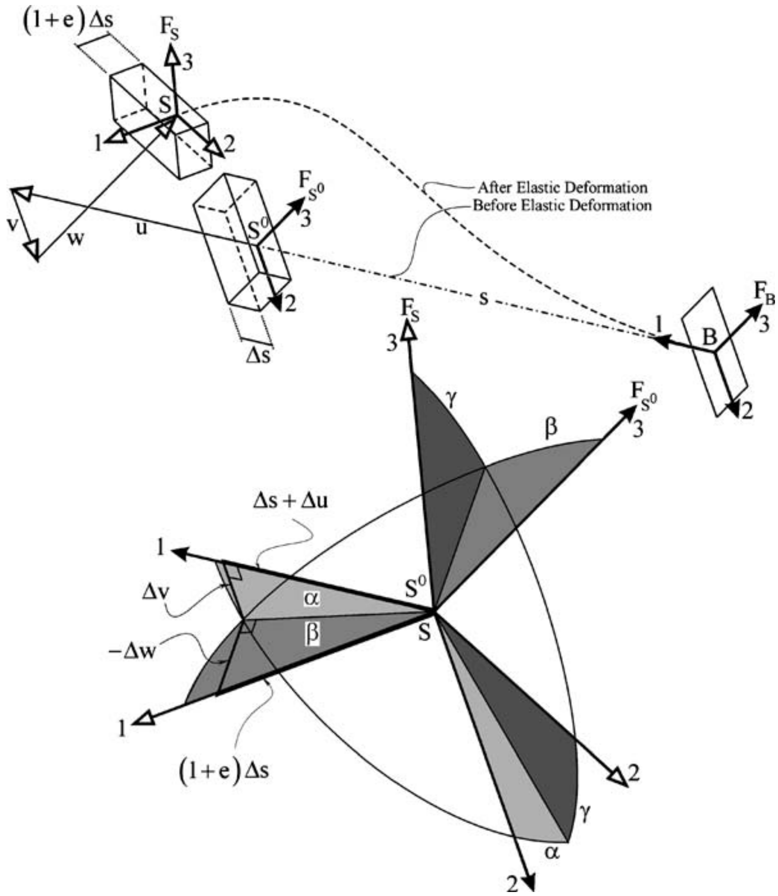
$$\underset{\mathrm{SB}}{R}[(\Delta s + \Delta u) \quad \Delta v \quad \Delta w]^{\mathrm{T}} = [(1+e)\Delta s \quad 0 \quad 0]^{\mathrm{T}},$$

$$e = \sqrt{r^2 + w'^2} - 1, \quad r = \sqrt{h^2 + v'^2}, \quad h = 1 + u'. \tag{9}$$

Since Equation (8) are nondifferential equations, their differentials are evidently perfect and integrable. Therefore the two-beam structural constraint Equation (8) belongs to the class of holonomic constraints [9, 10] and can eliminate two excess or superfluous coordinates namely α and β from the six elastic coordinates u, v, w, α, β, and γ. Hence, the nonlinear 3D Euler–Bernoulli beam is a holonomic system with u, v, w, and γ as its independent elastic degrees of freedom.

In Equations (10) and (11) elastic rotation transformation matrix projecting a vector from F_B onto F_S and orthogonal virtual elastic rotation vector of F_S relative to F_B projected onto F_S are exposed in terms of elastic

Fig. 3 Two holonomic constraints among elastic coordinates [7]



degrees of freedom after elimination of the excess coordinates by the constraints.

$$\underset{\mathrm{SB}}{R} = \begin{bmatrix} h/(e+1) & v'/(e+1) & w'/(e+1) \\ -v'\cos\gamma/r - w'h\sin\gamma/r(e+1) & h\cos\gamma/r - w'v'\sin\gamma/r(e+1) & r\sin\gamma/(e+1) \\ v'\sin\gamma/r - w'h\cos\gamma/r(e+1) & -h\sin\gamma/r - w'v'\cos\gamma/r(e+1) & r\cos\gamma/(e+1) \end{bmatrix} \tag{10}$$

$$\delta\Pi^S = \begin{bmatrix} \delta\Pi_x \\ \delta\Pi_y \\ \delta\Pi_z \end{bmatrix} = \begin{bmatrix} 1 & 0 & w'/(e+1) \\ 0 & \cos\gamma & r\sin\gamma/(e+1) \\ 0 & -\sin\gamma & r\cos\gamma/(e+1) \end{bmatrix} \begin{bmatrix} \delta\gamma \\ \delta\beta \\ \delta\alpha \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \delta\gamma + C \begin{bmatrix} \delta u' \\ \delta v' \\ \delta w' \end{bmatrix} \tag{11}$$

where

$$C = \begin{bmatrix} -w'v'/r^2(e+1) & w'h/r^2(e+1) & 0 \\ [-v'(e+1)\sin\gamma + w'h\cos\gamma]/r(e+1)^2 & [h(e+1)\sin\gamma + v'w'\cos\gamma]/r(e+1)^2 & -r\cos\gamma/(e+1)^2 \\ [-v'(e+1)\cos\gamma - w'h\sin\gamma]/r(e+1)^2 & [h(e+1)\cos\gamma - v'w'\sin\gamma]/r(e+1)^2 & r\sin\gamma/(e+1)^2 \end{bmatrix}. \tag{12}$$

In expressions (13), Kirchhoff's kinetic analogy can be seen between elastic angular velocity and elastic normalized curvature vectors [7].

$$\Omega^S = [1 \quad 0 \quad 0]^{\mathrm{T}}\dot{\gamma} + C[\dot{u}' \quad \dot{v}' \quad \dot{w}']^{\mathrm{T}},$$
$$\kappa^S = [1 \quad 0 \quad 0]^{\mathrm{T}}\gamma' + C[u'' \quad v'' \quad w'']^{\mathrm{T}} \tag{13}$$

Elimination of the excess coordinates produces a too long formulation for the elastic angular acceleration and variation of elastic curvature. So, it is an obligation to expose them as Equations (14) and (15) [7].

$$\dot{\Omega}^S = \begin{bmatrix} 1 & 0 & w'/(e+1) \\ 0 & \cos\gamma & r\sin\gamma/(e+1) \\ 0 & -\sin\gamma & r\cos\gamma/(e+1) \end{bmatrix} \begin{bmatrix} \ddot{\gamma} \\ \ddot{\beta} \\ \ddot{\alpha} \end{bmatrix} + \begin{bmatrix} -r/(e+1) & 0 & 0 \\ w'\sin\gamma/(e+1) & r\cos\gamma/(e+1) & -\sin\gamma \\ w'\cos\gamma/(e+1) & -r\sin\gamma/(e+1) & -\cos\gamma \end{bmatrix} \times \begin{bmatrix} \dot{\alpha}\dot{\beta} \\ \dot{\alpha}\dot{\gamma} \\ \dot{\beta}\dot{\gamma} \end{bmatrix} \tag{14}$$

$$\delta\kappa^S = \begin{bmatrix} 1 & 0 & w'/(e+1) \\ 0 & \cos\gamma & r\sin\gamma/(e+1) \\ 0 & -\sin\gamma & r\cos\gamma/(e+1) \end{bmatrix} \begin{bmatrix} \delta\gamma' \\ \delta\beta' \\ \delta\alpha' \end{bmatrix} + \begin{bmatrix} -r/(e+1) & 0 & 0 \\ w'\sin\gamma/(e+1) & r\cos\gamma/(e+1) & -\sin\gamma \\ w'\cos\gamma/(e+1) & -r\sin\gamma/(e+1) & -\cos\gamma \end{bmatrix} \times \begin{bmatrix} \alpha'\delta\beta \\ \alpha'\delta\gamma \\ \beta'\delta\gamma \end{bmatrix}. \tag{15}$$

4 Variation of kinetic energy

Time integration of variation of kinetic energy is

$$\int_0^t \delta T\,dt = \int_0^t \int_0^L \int_A \left\{-\rho\left[\underset{B}{a^{\sigma}}\right]^{\mathrm{T}}\left[\underset{\text{BI}}{R}\,\delta\eta\right]\right\} dA\,ds\,dt$$
$$= -\rho_n \int_0^t \int_0^L \int_A \left\{\left(\left[\underset{I}{a^B}\right]^{\mathrm{T}} \underset{\text{IB}}{R} + \ddot{d}^{\mathrm{T}} + p^{\mathrm{T}}[\widetilde{\Omega}^S\widetilde{\Omega}^S - \dot{\widetilde{\Omega}}^S]\underset{\text{SB}}{R} - 2\left[\dot{d}^{\mathrm{T}} - p^{\mathrm{T}}\widetilde{\Omega}^S\underset{\text{SB}}{R}\right]\widetilde{\omega}^B + d^{\mathrm{T}}[\widetilde{\omega}^B\widetilde{\omega}^B - \dot{\widetilde{\omega}}^B] + p^{\mathrm{T}}\underset{\text{SB}}{R}[\widetilde{\omega}^B\widetilde{\omega}^B - \dot{\widetilde{\omega}}^B]\right) \times \left(\underset{\text{BI}}{R}\,\delta b + \delta d - \underset{\text{BS}}{R}\,\widetilde{p}\,\delta\Pi^S + \delta\widetilde{\pi}^B\left[d + \underset{\text{BS}}{R}\,p\right]\right)\right\} dA\,ds\,dt$$

where δd, $\dot{d}$, and $\ddot{d}$ are $[\delta u\,\delta v\,\delta w]^{\mathrm{T}}$, $[\dot{u}\,\dot{v}\,\dot{w}]^{\mathrm{T}}$, and $[\ddot{u}\,\ddot{v}\,\ddot{w}]^{\mathrm{T}}$ respectively.

Position, velocity, acceleration, and variation of position of σ projected onto F_B are

$$\underset{\text{BI}}{R}\,\eta = \underset{\text{BI}}{R}\,b + \xi, \quad \underset{B}{V^{\sigma}} = \underset{\text{BI}}{R}\,\dot{\eta} = \underset{\text{BI}}{R}\,\underset{I}{V^B} + \dot{\xi} + \widetilde{\omega}^B\xi,$$
$$\underset{B}{a^{\sigma}} = \underset{\text{BI}}{R}\,\ddot{\eta} = \underset{\text{BI}}{R}\,\underset{I}{a^B} + \ddot{\xi} + 2\widetilde{\omega}^B\dot{\xi} + \widetilde{\omega}^B\widetilde{\omega}^B\xi + \dot{\widetilde{\omega}}^B\xi$$
$$\underset{\text{BI}}{R}\,\delta\eta = \underset{\text{BI}}{R}\,\delta b + \delta d - \underset{\text{BS}}{R}\,\widetilde{p}\,\delta\Pi^S + \delta\widetilde{\pi}^B\left(d + \underset{\text{BS}}{R}\,p\right)$$

where apparent position, velocity, acceleration, and variation of position of σ in F_B are

$$\xi = d + \underset{\text{BS}}{R}\,p, \quad \dot{\xi} = \dot{d} + \underset{\text{BS}}{R}\,\widetilde{\Omega}^S p,$$
$$\ddot{\xi} = \ddot{d} + \underset{\text{BS}}{R}\,[\widetilde{\Omega}^S\widetilde{\Omega}^S + \dot{\widetilde{\Omega}}^S]p,$$
$$\delta\xi = \delta d - \underset{\text{BS}}{R}\,\widetilde{p}\,\delta\Pi^S = \begin{bmatrix} \delta u \\ \delta v \\ \delta w \end{bmatrix} + \underset{\text{BS}}{R} \begin{bmatrix} \hat{z}\delta\Pi_y - \hat{y}\delta\Pi_z \\ -\hat{z}\delta\Pi_x \\ \hat{y}\delta\Pi_x. \end{bmatrix}$$

Considering the following equations

$$\underset{S}{\delta\pi^B} = \underset{\text{SB}}{R}\,\delta\pi^B, \quad \underset{S}{\delta\widetilde{\pi}^B} = \underset{\text{SB}}{R}\,\delta\widetilde{\pi}^B\,\underset{\text{BS}}{R},$$
$$\underset{\text{SB}}{R}[\widetilde{\omega}^B \quad \widetilde{\omega}^B \quad -\dot{\widetilde{\omega}}^B]\delta\widetilde{\pi}^B$$
$$\underset{\text{BS}}{R} = \left[\underset{S}{\widetilde{\omega}^B} \quad \underset{S}{\widetilde{\omega}^B} \quad -\underset{S}{\dot{\widetilde{\omega}}^B}\right]\underset{S}{\delta\widetilde{\pi}^B}$$

and the fact that S is the center of cross-sectional area, one can write the following:

$$\int_0^t \delta T\,dt = -\rho\int_0^t\int_0^L \left\{A\left(\left[\underset{I}{a^B}\right]^{\mathrm{T}}\underset{\text{IB}}{R} + \ddot{d}^{\mathrm{T}} - 2\dot{d}^{\mathrm{T}}\widetilde{\omega}^B + d^{\mathrm{T}}[\widetilde{\omega}^B \quad \widetilde{\omega}^B \quad -\dot{\widetilde{\omega}}^B]\right)\right.$$

$$\times\left(\underset{\mathrm{BI}}{R}\,\delta b+\delta d-\widetilde{d}\delta\pi^{B}\right)$$
$$+\left(\left(\int_A p^{\mathrm{T}}\widetilde{\dot{\Omega}}^{S}\widetilde{p}dA-\int_A p^{\mathrm{T}}\widetilde{\Omega}^{S}\widetilde{\Omega}^{S}\widetilde{p}dA\right.\right.$$
$$-\int_A 2p^{\mathrm{T}}\widetilde{\Omega}^{S}\underset{S}{\widetilde{\omega}^{B}}\widetilde{p}dA-\int_A p^{\mathrm{T}}\underset{S}{\widetilde{\omega}^{B}}\underset{S}{\widetilde{\omega}^{B}}\widetilde{p}dA$$
$$\left.+\int_A p^{\mathrm{T}}\underset{S}{\widetilde{\dot{\omega}}^{B}}\widetilde{p}dA\right)\left(\delta\Pi^{S}+\underset{S}{\delta\pi^{B}}\right)\Bigg\}dsdt.$$

Considering the following vector identities

$$p^{\mathrm{T}}\widetilde{\dot{\Omega}}^{S}\widetilde{p}\Pi=[\dot{\Omega}^{S}]^{\mathrm{T}}[-\widetilde{p}\widetilde{p}]\,\Pi,$$
$$p^{\mathrm{T}}\widetilde{\Omega}^{S}\underset{S}{\widetilde{\omega}^{B}}\widetilde{p}\,\Pi$$
$$=\left\{\left[\underset{S}{\omega^{B}}\right]^{\mathrm{T}}[-\widetilde{p}\,\widetilde{p}]\widetilde{\Omega}^{S}-(p^{\mathrm{T}}p)\left[\underset{S}{\omega^{B}}\right]^{\mathrm{T}}\widetilde{\Omega}^{S}\right\}\Pi$$

one can write the following:

$$p^{\mathrm{T}}\widetilde{\dot{\Omega}}^{S}\widetilde{p}\left[\delta\Pi^{S}+\underset{S}{\delta\pi^{B}}\right]=[\dot{\Omega}^{S}]^{\mathrm{T}}\left[-\widetilde{p}\widetilde{p}\right]\left[\delta\Pi^{S}+\underset{S}{\delta\pi^{B}}\right]$$
$$p^{\mathrm{T}}\underset{S}{\widetilde{\dot{\omega}}^{B}}\widetilde{p}\left[\delta\Pi^{S}+\underset{S}{\delta\pi^{B}}\right]=\left[\underset{S}{\dot{\omega}^{B}}\right]^{\mathrm{T}}\left[-\widetilde{p}\widetilde{p}\right]\left[\delta\Pi^{S}+\underset{S}{\delta\pi^{B}}\right]$$
$$p^{\mathrm{T}}\underset{S}{\widetilde{\omega}^{B}}\underset{S}{\widetilde{\omega}^{B}}\widetilde{p}\left[\delta\Pi^{S}+\underset{S}{\delta\pi^{B}}\right]$$
$$=\left[\underset{S}{\omega^{B}}\right]^{\mathrm{T}}\left[-\widetilde{p}\widetilde{p}\right]\underset{S}{\widetilde{\omega}^{B}}\left[\delta\Pi^{S}+\underset{S}{\delta\pi^{B}}\right]$$
$$p^{\mathrm{T}}\,\widetilde{\Omega}^{S}\widetilde{\Omega}^{S}\,\widetilde{p}\left[\delta\Pi^{S}+\underset{S}{\delta\pi^{B}}\right]$$
$$=[\Omega^{S}]^{\mathrm{T}}\left[-\widetilde{p}\widetilde{p}\right]\widetilde{\Omega}^{S}\left[\delta\Pi^{S}+\underset{S}{\delta\pi^{B}}\right]$$
$$p^{\mathrm{T}}\widetilde{\Omega}^{S}\underset{S}{\widetilde{\omega}^{B}}\widetilde{p}\left[\delta\Pi^{S}+\underset{S}{\delta\pi^{B}}\right]$$
$$=\left\{\left[\underset{S}{\omega^{B}}\right]^{\mathrm{T}}\left[-\widetilde{p}\,\widetilde{p}\right]\widetilde{\Omega}^{S}-(p^{\mathrm{T}}p)\left[\underset{S}{\omega^{B}}\right]^{\mathrm{T}}\widetilde{\Omega}^{S}\right\}$$
$$\times\left[\delta\Pi^{S}+\underset{S}{\delta\pi^{B}}\right].$$

Hence,

$$\int_0^t\delta T\,dt=-\rho\int_0^t\int_0^L\left[\varphi^{\mathrm{T}}\left(\delta d+\underset{\mathrm{BI}}{R}\,\delta b-\widetilde{d}\delta\pi^{B}\right)\right.$$
$$\left.+\varphi^{R}\left(\delta\Pi^{S}+\underset{\mathrm{SB}}{R}\,\delta\pi^{B}\right)\right]dsdt \tag{16}$$

where

$$\varphi^{R}=[\dot{\Omega}^{S}]^{\mathrm{T}}[J_S]-[\Omega^{S}]^{\mathrm{T}}[J_S]\widetilde{\Omega}^{S}-2[\omega^{B}]^{\mathrm{T}}\underset{\mathrm{BS}}{R}[J_S]\widetilde{\Omega}^{S}$$
$$+2J_{S_p}[\omega^{B}]^{\mathrm{T}}\underset{\mathrm{BS}}{R}\,\widetilde{\Omega}^{S}-[\omega^{B}]^{\mathrm{T}}\underset{\mathrm{BS}}{R}[J_S]\underset{\mathrm{SB}}{R}\,\widetilde{\omega}^{B}\underset{\mathrm{BS}}{R}$$
$$+[\dot{\omega}^{B}]^{\mathrm{T}}\underset{\mathrm{BS}}{R}[J_S]$$
$$\varphi^{\mathrm{T}}=A\left(\left[\underset{I}{a^{B}}\right]^{\mathrm{T}}\underset{\mathrm{IB}}{R}+\ddot{d}^{\mathrm{T}}-2\dot{d}^{\mathrm{T}}\widetilde{\omega}^{B}+d^{\mathrm{T}}[\widetilde{\omega}^{B}\widetilde{\omega}^{B}-\widetilde{\dot{\omega}}^{B}]\right). \tag{17}$$

The most applicable form of time integration of variation of kinetic energy is obtained by substitution of Equation (11) into Equation (16) and integration by part with respect to s [7]:

$$\int_0^t\delta T\,dt=-\rho\int_0^t\int_0^L\left\{\varphi^{\mathrm{T}}\underset{\mathrm{BI}}{R}\,\delta b\right.$$
$$+\left(\varphi^{R}\underset{\mathrm{SB}}{R}-\varphi^{\mathrm{T}}\widetilde{d}\right)\delta\pi^{B}+\varphi^{R}\begin{bmatrix}1\\0\\0\end{bmatrix}\delta\gamma$$
$$\left.+[\varphi^{\mathrm{T}}-(\varphi^{R}C)']\begin{bmatrix}\delta u\\\delta v\\\delta w\end{bmatrix}\right\}ds\,dt$$
$$-\rho\int_0^t\varphi^{R}C\begin{bmatrix}\delta u\\\delta v\\\delta w\end{bmatrix}\Big|_{s=0}^{s=L}.dt. \tag{18}$$

5 Variation of gravitational potential energy

Variation of gravitational potential energy is given by the following equation [7]:

$$\delta U^{g}=\int_0^L\int_A\{[0\quad 0\quad 1]\delta\eta\}g\rho dA\,ds$$
$$=g\rho[0\quad 0\quad 1]\int_0^L\int_A\left\{\delta b+\underset{\mathrm{IB}}{R}\left[\delta d+\underset{\mathrm{BS}}{R}\,\delta\widetilde{\Pi}^{S}p\right]\right.$$
$$\left.+\underset{\mathrm{IB}}{R}\,\delta\widetilde{\pi}^{B}\left[d+\underset{\mathrm{BS}}{R}\,p\right]\right\}dA\,ds$$
$$=g\rho[0\quad 0\quad 1]\int_0^L\left(\delta b\int_A dA+\underset{\mathrm{IB}}{R}\,\delta d\int_A dA\right.$$
$$+\underset{\mathrm{IB}}{R}\,\underset{\mathrm{BS}}{R}\,\delta\widetilde{\Pi}^{S}\int_A pdA+\underset{\mathrm{IB}}{R}\,\delta\widetilde{\pi}^{B}d\int_A dA$$
$$\left.+\underset{\mathrm{IB}}{R}\,\delta\widetilde{\pi}^{B}\underset{\mathrm{BS}}{R}\int_A pdA\right)ds.$$

Since S_n is the center of cross-sectional area, it is simplified as follows:

$$\delta U^g = g\rho A[0 \quad 0 \quad 1]\int_0^L \left\{\delta b + \underset{\text{IB}}{R}\,\delta d - \underset{\text{IB}}{R}\,\widetilde{d}\delta\pi^B\right\} ds \tag{19}$$

6 Variation of elastic potential energy

In Fig. 2, σ is the general point of the beam's medium. Vector p is assumed to be constant, implying the lack of in-plane and out-of-plane cross-section warping in the beams [7]. The displacement field is

$$\Delta = \begin{bmatrix}\Delta_x\\ \Delta_y\\ \Delta_z\end{bmatrix} = \xi - \xi|_{\substack{\text{before}\\ \text{elastic}\\ \text{deformation}}}$$
$$= \left\{\begin{bmatrix}u+s\\ v\\ w\end{bmatrix} + \underset{\text{BS}}{R}\,p\right\} - \left\{\begin{bmatrix}s\\ 0\\ 0\end{bmatrix} + \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} p\right\}$$
$$= \begin{bmatrix}u\\ v\\ w\end{bmatrix} + \underset{\text{BS}}{R}\begin{bmatrix}0\\ \hat{y}\\ \hat{z}\end{bmatrix} - \begin{bmatrix}0\\ \hat{y}\\ \hat{z}\end{bmatrix}.$$

Linear part of Green–Lagrange geometric strain tensor is taken into consideration as follows:

$$\hat{\varepsilon}_{ij} = \frac{1}{2}\left(\frac{\partial\Delta_i}{\partial x_j} + \frac{\partial\Delta_j}{\partial x_i}\right),$$
$$\frac{\partial}{\partial s}\begin{bmatrix}\Delta_x\\ \Delta_y\\ \Delta_z\end{bmatrix} = \begin{bmatrix}u'\\ v'\\ w'\end{bmatrix} + \underset{\text{BS}}{R}\begin{bmatrix}-\hat{y}\kappa_z + \hat{z}\kappa_y\\ -\hat{z}\kappa_x\\ \hat{y}\kappa_x\end{bmatrix},$$
$$\frac{\partial}{\partial y}\begin{bmatrix}\Delta_x\\ \Delta_y\\ \Delta_z\end{bmatrix} = \frac{\partial}{\partial z}\begin{bmatrix}\Delta_x\\ \Delta_y\\ \Delta_z\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix}.$$

Strains concerning large elastic orientation of cross-sectional frame are referred to as exact strains in this paper and are presented in Equation (20) as follows:

$$\hat{\varepsilon}_{xx} = u' + \hat{y}\left[\frac{v'\kappa_x}{r}\sin\gamma - \frac{h}{1+e}\left(\kappa_z + \frac{w'\kappa_x}{r}\cos\gamma\right)\right] + \hat{z}\left[\frac{v'\kappa_x}{r}\cos\gamma + \frac{h}{1+e}\left(\kappa_y + \frac{w'\kappa_x}{r}\sin\gamma\right)\right]$$
$$\hat{\varepsilon}_{xy} = \frac{v'}{2} - \frac{\hat{y}}{2}\left[\frac{h\kappa_x}{r}\sin\gamma + \frac{v'}{1+e}\left(\kappa_z + \frac{w'\kappa_x}{r}\cos\gamma\right)\right] - \frac{\hat{z}}{2}\left[\frac{h\kappa_x}{r}\cos\gamma - \frac{v'}{1+e}\left(\kappa_y + \frac{w'\kappa_x}{r}\sin\gamma\right)\right]$$
$$\hat{\varepsilon}_{xz} = \frac{w'}{2} + \frac{\hat{y}}{2}\left[\frac{-w'\kappa_z + r\kappa_x\cos\gamma}{1+e}\right] + \frac{\hat{z}}{2}\left[\frac{w'\kappa_y - r\kappa_x\sin\gamma}{1+e}\right]$$
$$\hat{\varepsilon}_{yy} = \hat{\varepsilon}_{zz} = \hat{\varepsilon}_{yz} = 0. \tag{20}$$

The motion equations will be approximated for a beam with negligible elastic orientation in its cross-sectional frame, but variation of strains used in variation of elastic potential energy must be derived from the strains regarding large elastic orientation of cross-sectional frame. This is an enhancement for the conventional nonlinear 3D Euler–Bernoulli beam theory.

Using Hook's law for a linearly elastic isotropic beam gives the stress tensor as follows:

$$\tau_{ij} = 2\mu\hat{\varepsilon}_{ij} + \lambda(\hat{\varepsilon}_{xx} + \hat{\varepsilon}_{yy} + \hat{\varepsilon}_{zz})\delta_{ij},$$
$$\mu = G = \frac{E}{2(1+\nu)},$$
$$\lambda = \frac{G(2G-E)}{E-3G} = \frac{\nu E}{(1+\nu)(1-2\nu)}. \tag{21}$$

Variation of elastic potential energy is

$$\delta U^e = \int_0^L\int_A \{\tau_{xx}\delta\hat{\varepsilon}_{xx} + \tau_{yy}\delta\hat{\varepsilon}_{yy} + \tau_{zz}\delta\hat{\varepsilon}_{zz} + 2\tau_{xy}\delta\hat{\varepsilon}_{xy} + 2\tau_{xz}\delta\hat{\varepsilon}_{xz} + 2\tau_{yz}\delta\hat{\varepsilon}_{yz}\}dA\,ds$$
$$= \int_0^L\int_A \{\tau_{xx}\delta\hat{\varepsilon}_{xx} + 2\tau_{xy}\delta\hat{\varepsilon}_{xy} + 2\tau_{xz}\delta\hat{\varepsilon}_{xz}\}dA\,ds. \tag{22}$$

The variation of elastic potential energy of expression (22) is formulated by performing mathematical operations such as substitution of stresses from (21), substitution of variation of strains, simplifications for beams with circular or square cross-sections and substitution of all variations with the variations of independent elastic degrees of freedom. This formulation is too long to be exposed [7]. Since the beam experiences negligible elastic orientation in its cross-sectional frame, the limit of the formulation is exposed when α, β, and γ tend toward zero [7]. Hence, one may write the following:

$$v' \approx 0, \quad w' \approx 0, \quad r \approx 1+u', \quad e \approx u',$$
$$\delta e \approx \delta u', \quad \kappa_x \approx \gamma', \quad \kappa_y \approx \frac{-w''}{h}, \quad \kappa_z \approx \frac{v''}{h}$$
$$\delta\kappa_x \approx \frac{v''}{h^2}\delta w' + \delta\gamma',$$
$$\delta\kappa_y \approx \frac{v''}{h}\delta\gamma + \frac{w''}{h^2}\delta u' + \frac{u''}{h^2}\delta w' - \frac{1}{h}\delta w'',$$
$$\delta\kappa_z \approx \frac{w''}{h}\delta\gamma - \frac{v''}{h^2}\delta u' - \frac{u''}{h^2}\delta v' + \frac{1}{h}\delta v''$$
$$\underset{\mathrm{BS}}{R} \approx \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$
$$\delta \underset{\mathrm{BS}}{R} \approx \begin{bmatrix} 0 & -\delta\Pi_z & \delta\Pi_y \\ \delta\Pi_z & 0 & -\delta\Pi_x \\ -\delta\Pi_y & \delta\Pi_x & 0 \end{bmatrix}, \quad \delta\Pi^S \approx \begin{bmatrix} \delta\gamma \\ \delta\beta \\ \delta\alpha \end{bmatrix}.$$

Equation (23) is a correct formulation for variation of elastic potential energy of a beam with circular or square cross-section [7]:

$$\begin{aligned}\delta U^e = \int_0^L \Bigg\{ & \frac{(1-\nu)E}{(1+\nu)(1-2\nu)}\left[Au' - J\frac{v''^2 + w''^2}{h^3}\right]\delta u' \\ & + 2GJ\gamma'\delta\gamma' + J\frac{(1-\nu)E}{(1+\nu)(1-2\nu)}\frac{v''}{h^2}\delta v'' \\ & + J\frac{(1-\nu)E}{(1+\nu)(1-2\nu)}\frac{w''}{h^2}\delta w'' \\ & - J\left[\frac{(1-\nu)E}{(1+\nu)(1-2\nu)}\left(\frac{u''v''}{h} + \gamma' w''\right)\right. \\ & \left. - G\gamma' w''\right]\frac{1}{h^2}\delta v' + J\left[\frac{(1-\nu)E}{(1+\nu)(1-2\nu)}\right. \\ & \left. \times\left(\gamma' v'' - \frac{u''w''}{h}\right) + G\gamma' v''\right]\frac{1}{h^2}\delta w'\Bigg\} ds. \end{aligned} \tag{23}$$

Equation (23) can also be rewritten as Equation (24) for easier comparison with variation of elastic potential energy in the nonlinear 3D Euler–Bernoulli beam theory (25).

$$\begin{aligned}\delta U^e = \int_0^L \Bigg\{ & \frac{(1-\nu)E}{(1+\nu)(1-2\nu)}[Ae\delta e + J\kappa_z\delta\kappa_z \\ & + J\kappa_y\delta\kappa_y] + G2J\kappa_x\delta\kappa_x \\ & + J\left[\frac{(1-\nu)E}{(1+\nu)(1-2\nu)} - G\right]\frac{\kappa_x\kappa_y}{r}\delta v' \\ & + J\left[\frac{(1-\nu)E}{(1+\nu)(1-2\nu)} - G\right]\frac{\kappa_x\kappa_z}{r}\delta w'\Bigg\} ds. \end{aligned} \tag{24}$$

In nonlinear 3D Euler–Bernoulli beam theory, the variation of strains are derived from the approximated strain field concerning negligible elastic orientation of the cross-section frame, whereas it should have been derived from the exact strains corresponding to large elastic orientation of the cross-section frame. As a consequence, the nonlinear 3D Euler–Bernoulli beam theory loses two terms of the variation of elastic potential energy, Equation (25), compared to Equation [7]. In nonlinear 3D Euler–Bernoulli beam theory [8], one has

$$\hat\varepsilon_{xx} = e - \hat y\kappa_z + \hat z\kappa_y, \quad \hat\varepsilon_{xy} = -\frac{1}{2}\kappa_x\hat z,$$
$$\hat\varepsilon_{xz} = \frac{1}{2}\kappa_x\hat y, \quad \hat\varepsilon_{yy} = \hat\varepsilon_{zz} = \hat\varepsilon_{yz} = 0$$
$$\tau_{xx} = \frac{(1-\nu)E}{(1+\nu)(1-2\nu)}(e - \hat y\kappa_z + \hat z\kappa_y),$$
$$\tau_{yy} = \tau_{zz} = \frac{\nu E}{(1+\nu)(1-2\nu)}(e - \hat y\kappa_z + \hat z\kappa_y)$$
$$\tau_{xy} = -G\,\kappa_x\hat z, \quad \tau_{xz} = G\,\kappa_x\hat y, \quad \tau_{yz} = 0$$

and the incorrect variation of strains [8]:

$$\delta\hat\varepsilon_{xx} = \delta e - \hat y\delta\kappa_z + \hat z\delta\kappa_y, \quad \delta\hat\varepsilon_{xy} = -\frac{1}{2}\hat z\delta\kappa_x,$$
$$\delta\hat\varepsilon_{xz} = \frac{1}{2}\hat y\delta\kappa_x, \quad \delta\hat\varepsilon_{yy} = \delta\hat\varepsilon_{zz} = \delta\hat\varepsilon_{yz} = 0$$

and finally Equation (25), the incorrect variation of elastic potential energy of a beams with circular or square cross-section:

$$\begin{aligned}\delta U^e = \int_0^L \Bigg\{ & \frac{(1-\nu)E}{(1+\nu)(1-2\nu)}[Ae\delta e + J\kappa_z\delta\kappa_z \\ & + J\kappa_y\delta\kappa_y] + G\,2J\kappa_x\delta\kappa_x\Bigg\} ds. \end{aligned} \tag{25}$$

Using Equation (26) that show integrations by part with respect to s, Equation (23) has been reformulated for a uniform beam with circular or square cross-section. One can seek it in the coming section under the title of Motion equations.

$$\int F\delta u' ds = F\delta u - \int F'\delta u\, ds,$$
$$\int F\delta u'' ds = F\delta u' - F'\delta u + \int F''\delta u\, ds. \tag{26}$$

7 Motion equations

Nonlinear partial differential equations of motion of a beam with negligible elastic orientation of its cross-sectional frame are obtained via Hamilton's principle. The beam can tolerate compression, tension, torsion, and two spatial bendings. Regardless of nonconservative forces and moments that produce damping and exciting terms, one may write the Hamilton's principle as Equation (27).

$$\int_0^t (\delta T - \delta U^g - \delta U^e) dt = 0. \tag{27}$$

Therefore, by substitution of Equations (17), (18), (19), and (23) into Equation (27) and using Equation (26), one can write:

$$\int_0^t \Big\{ \int_0^L \big(A_1\delta x + A_2\delta y + A_3\delta z + A_4\delta\pi_x + A_5\delta\pi_y + A_6\delta\pi_z + A_7\delta\gamma + A_8\delta u + A_9\delta v + A_{10}\delta w)\, ds + (B_7\delta\gamma + B_8\delta u + B_9\delta v + B_{10}\delta w + \bar{B}_8\delta u' + \bar{B}_9\delta v' + \bar{B}_{10}\delta w')|_{s=0}^{s=L} \Big\} dt = 0$$

that is simplified as Equation (28).

$$\left(\int_0^L A_1\, ds\right)\delta x + \left(\int_0^L A_2\, ds\right)\delta y + \left(\int_0^L A_3\, ds\right)\delta z + \left(\int_0^L A_4\, ds\right)\delta\pi_x + \left(\int_0^L A_5\, ds\right)\delta\pi_y + \left(\int_0^L A_6\, ds\right)\delta\pi_z + \int_0^L (A_7\delta\gamma + A_8\delta u + A_9\delta v + A_{10}\delta w)\, ds + (B_7\delta\gamma + B_8\delta u + B_9\delta v + B_{10}\delta w + \bar{B}_8\delta u' + \bar{B}_9\delta v' + \bar{B}_{10}\delta w')|_{s=0}^{s=L} = 0 \tag{28}$$

This equation indicates that four partial differential Equations (29) should be solved under boundary conditions (30) and should satisfy six partial differential Equations (31).

$$A_7 = 0, \quad A_8 = 0, \quad A_9 = 0, \quad A_{10} = 0 \tag{29}$$

$$(B_7\delta\gamma + B_8\delta u + B_9\delta v + B_{10}\delta w + \bar{B}_8\delta u' + \bar{B}_9\delta v' + \bar{B}_{10}\delta w')|_{s=0}^{s=L} = 0 \tag{30}$$

$$\int_0^L A_1\, ds = 0, \quad \int_0^L A_2\, ds = 0, \quad \int_0^L A_3\, ds = 0, \quad \int_0^L A_4\, ds = 0, \quad \int_0^L A_5\, ds = 0, \quad \int_0^L A_6\, ds = 0. \tag{31}$$

The agent variables $A_7, \ldots, A_{10}, A_1, \ldots, A_6, B_7, \ldots, B_{10}, \bar{B}_8, \ldots, \bar{B}_{10}$ are exposed as follows:

$$A_7 = 2GJ\gamma'' - \rho\Big([\dot{\Omega}^S]^{\mathrm{T}}[J_S] - [\Omega^S]^{\mathrm{T}}[J_S]\widetilde{\Omega}^S - 2[\omega^B]^{\mathrm{T}} \underset{\mathrm{BS}}{R}[J_S]\widetilde{\Omega}^S + 2J_{S_P}[\omega^B]^{\mathrm{T}} \underset{\mathrm{BS}}{R}\, \widetilde{\Omega}^S - [\omega^B]^{\mathrm{T}} \underset{\mathrm{BS}}{R}[J_S] \underset{\mathrm{SB}}{R}\, \widetilde{\omega}^B \underset{\mathrm{BS}}{R} + [\dot{\omega}^B]^{\mathrm{T}} \underset{\mathrm{BS}}{R}[J_S]\Big)[1 \quad 0 \quad 0]^{\mathrm{T}} = 0$$

$$[A_8 \quad A_9 \quad A_{10}] = -g\rho A[0 \quad 0 \quad 1] \underset{\mathrm{IB}}{R} - \rho\Big\{A\Big(\big[\underset{I}{a}^B\big]^{\mathrm{T}} \underset{\mathrm{IB}}{R} + \ddot{d}^{\mathrm{T}} - 2\dot{d}^{\mathrm{T}}\widetilde{\omega}^B + d^{\mathrm{T}}\big[\widetilde{\omega}^B \quad \widetilde{\omega}^B \quad -\widetilde{\dot{\omega}}^B\big]\Big) - \Big\langle\Big(([\dot{\Omega}^S]^{\mathrm{T}}[J_S] - [\Omega^S]^{\mathrm{T}}[J_S]\widetilde{\Omega}^S - 2[\omega^B]^{\mathrm{T}} \underset{\mathrm{BS}}{R}[J_S]\widetilde{\Omega}^S + 2J_{S_P}[\omega^B]^{\mathrm{T}} \underset{\mathrm{BS}}{R}\, \widetilde{\Omega}^S - [\omega^B]^{\mathrm{T}} \underset{\mathrm{BS}}{R}[J_S] \underset{\mathrm{SB}}{R}\, \widetilde{\omega}^B \underset{\mathrm{BS}}{R} + [\dot{\omega}^B]^{\mathrm{T}} \underset{\mathrm{BS}}{R}[J_S]\Big)C\Big\rangle'\Big\}$$

$$+ \frac{(1-\nu)E}{(1+\nu)(1-2\nu)}\left(Au'' - J\left\langle\frac{v''^2 + w''^2}{(1+u')^3}\right\rangle'\right)[1 \quad 0 \quad 0]$$

$$
\begin{aligned}
&+\left\{-\frac{(1-\nu)EJ}{(1+\nu)(1-2\nu)}\left(\left\langle\frac{v''}{(1+u')^2}\right\rangle''\right.\right.\\
&\left.+\left\langle\frac{u''v''}{(1+u')^3}+\frac{\gamma'w''}{(1+u')^2}\right\rangle'\right)\\
&\left.+GJ\left\langle\frac{\gamma'w''}{(1+u')^2}\right\rangle'\right\}\begin{bmatrix}0 & 1 & 0\end{bmatrix}\\
&+\left\{-\frac{(1-\nu)EJ}{(1+\nu)(1-2\nu)}\left(\left\langle\frac{w''}{(1+u')^2}\right\rangle''\right.\right.\\
&\left.+\left\langle\frac{u''w''}{(1+u')^3}-\frac{\gamma'v''}{(1+u')^2}\right\rangle'\right)\\
&\left.+GJ\left\langle\frac{\gamma'v''}{(1+u')^2}\right\rangle'\right\}\begin{bmatrix}0 & 0 & 1\end{bmatrix}=\begin{bmatrix}0 & 0 & 0\end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
&\left[\int_0^L A_1\,ds \quad \int_0^L A_2\,ds \quad \int_0^L A_3\,ds\right]\\
&=-\rho A\int_0^L\left\{\left[\underset{I}{a}^B\right]^{\mathrm{T}}\right.\\
&\left.+(\ddot{d}^{\mathrm{T}}-2\dot{d}^{\mathrm{T}}\widetilde{\omega}^B+d^{\mathrm{T}}[\widetilde{\omega}^B\quad\widetilde{\omega}^B\quad-\widetilde{\dot{\omega}}^B])\underset{\mathrm{BI}}{R}\right\}ds\\
&-g\rho AL\begin{bmatrix}0 & 0 & 1\end{bmatrix}=\begin{bmatrix}0 & 0 & 0\end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
&\left[\int_0^L A_4\,ds \quad \int_0^L A_5\,ds \quad \int_0^L A_6\,ds\right]\\
&=g\rho A\begin{bmatrix}0 & 0 & 1\end{bmatrix}\underset{\mathrm{IB}}{R}\int_0^L\widetilde{d}\,ds+\\
&-\rho\int_0^L\left\langle-A\left(\left[\underset{I}{a}^B\right]^{\mathrm{T}}\underset{\mathrm{IB}}{R}+\ddot{d}^{\mathrm{T}}-2\dot{d}^{\mathrm{T}}\widetilde{\omega}^B\right.\right.\\
&\left.+d^{\mathrm{T}}[\widetilde{\omega}^B\quad\widetilde{\omega}^B\quad-\widetilde{\dot{\omega}}^B]\right)\widetilde{d}\\
&+\left([\dot{\Omega}^S]^{\mathrm{T}}[J_S]-[\Omega^S]^{\mathrm{T}}[J_S]\widetilde{\Omega}^S-2[\omega^B]^{\mathrm{T}}\underset{\mathrm{BS}}{R}[J_S]\widetilde{\Omega}^S\right.\\
&+2J_{S_p}[\omega^B]^{\mathrm{T}}\underset{\mathrm{BS}}{R}\widetilde{\Omega}^S-[\omega^B]^{\mathrm{T}}\underset{\mathrm{BS}}{R}[J_S]\underset{\mathrm{SB}}{R}\widetilde{\omega}^B\underset{\mathrm{BS}}{R}\\
&\left.\left.+[\dot{\omega}^B]^{\mathrm{T}}\underset{\mathrm{BS}}{R}[J_S]\right)\underset{\mathrm{SB}}{R}\right\rangle ds=\begin{bmatrix}0 & 0 & 0\end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
B_7&=-2GJ\gamma'\\
B_8&=-\rho\left([\dot{\Omega}^S]^{\mathrm{T}}[J_S]-[\Omega^S]^{\mathrm{T}}[J_S]\widetilde{\Omega}^S\right.\\
&\quad-2[\omega^B]^{\mathrm{T}}\underset{\mathrm{BS}}{R}[J_S]\widetilde{\Omega}^S+2J_{S_p}[\omega^B]^{\mathrm{T}}\underset{\mathrm{BS}}{R}\widetilde{\Omega}^S\\
&\quad-[\omega^B]^{\mathrm{T}}\underset{\mathrm{BS}}{R}[J_S]\underset{\mathrm{SB}}{R}\widetilde{\omega}^B\underset{\mathrm{BS}}{R}\\
&\quad\left.+[\dot{\omega}^B]^{\mathrm{T}}\underset{\mathrm{BS}}{R}[J_S]\right)C\begin{bmatrix}1\\0\\0\end{bmatrix}\\
&\quad-\frac{(1-\nu)E}{(1+\nu)(1-2\nu)}\left[Au'-J\frac{v''^2+w''^2}{(1+u')^3}\right]\\
B_9&=-\rho\left([\dot{\Omega}^S]^{\mathrm{T}}[J_S]-[\Omega^S]^{\mathrm{T}}[J_S]\widetilde{\Omega}^S\right.\\
&\quad-2[\omega^B]^{\mathrm{T}}\underset{\mathrm{BS}}{R}[J_S]\widetilde{\Omega}^S+2J_{S_p}[\omega^B]^{\mathrm{T}}\underset{\mathrm{BS}}{R}\widetilde{\Omega}^S\\
&\quad\left.-[\omega^B]^{\mathrm{T}}\underset{\mathrm{BS}}{R}[J_S]\underset{\mathrm{SB}}{R}\widetilde{\omega}^B\underset{\mathrm{BS}}{R}+[\dot{\omega}^B]^{\mathrm{T}}\underset{\mathrm{BS}}{R}[J_S]\right)C\begin{bmatrix}1\\0\\0\end{bmatrix}\\
&\quad+\frac{(1-\nu)EJ}{(1+\nu)(1-2\nu)}\left(\frac{\gamma'w''}{(1+u')^2}+\left\langle\frac{v''}{(1+u')^2}\right\rangle'\right.\\
&\quad\left.+\frac{u''v''}{(1+u')^3}\right)-GJ\frac{\gamma'w''}{(1+u')^2}\\
B_{10}&=-\rho\left([\dot{\Omega}^S]^{\mathrm{T}}[J_S]-[\Omega^S]^{\mathrm{T}}[J_S]\widetilde{\Omega}^S\right.\\
&\quad-2[\omega^B]^{\mathrm{T}}\underset{\mathrm{BS}}{R}[J_S]\widetilde{\Omega}^S+2J_{S_p}[\omega^B]^{\mathrm{T}}\underset{\mathrm{BS}}{R}\widetilde{\Omega}^S\\
&\quad-[\omega^B]^{\mathrm{T}}\underset{\mathrm{BS}}{R}[J_S]\underset{\mathrm{SB}}{R}\widetilde{\omega}^B\underset{\mathrm{BS}}{R}\\
&\quad\left.+[\dot{\omega}^B]^{\mathrm{T}}\underset{\mathrm{BS}}{R}[J_S]\right)C\begin{bmatrix}0\\0\\1\end{bmatrix}-\frac{(1-\nu)EJ}{(1+\nu)(1-2\nu)}\\
&\quad\times\left(\frac{\gamma'v''}{(1+u')^2}-\left\langle\frac{w''}{(1+u')^2}\right\rangle'-\frac{u''w''}{(1+u')^3}\right)\\
&\quad-GJ\frac{\gamma'v''}{(1+u')^2}\\
\bar{B}_8&=0,\quad \bar{B}_9=-\frac{(1-\nu)EJ}{(1+\nu)(1-2\nu)}\frac{v''}{(1+u')^2},\\
\bar{B}_{10}&=-\frac{(1-\nu)EJ}{(1+\nu)(1-2\nu)}\frac{w''}{(1+u')^2}.
\end{aligned}
$$

8 Verification of the motion equations

The validity of Equations (29) and (31) has been verified for three simpler cases.

8.1 Case I: Flying rigid body under gravitational force

For negligible elastic deformation, the motion Equations (29) and (31) are simplified to the motion equations of a flying rigid prism with circular or square cross-section. Since the elastic degrees of freedom, u, v, w, γ, and their derivatives are zero, one has:

$$d=\begin{bmatrix}s\\0\\0\end{bmatrix},\quad \widetilde{d}=\begin{bmatrix}0&0&0\\0&0&-s\\0&s&0\end{bmatrix},$$

$$\underset{\mathrm{SB}}{R}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix},\quad \Omega^S=\dot{\Omega}^S=\begin{bmatrix}0\\0\\0\end{bmatrix},$$

$$C=\begin{bmatrix}0&0&0\\0&0&-1\\0&1&0\end{bmatrix},\quad e=0,\quad r=h=1.$$

Using Equations (1)–(3), and (6), the motion Equations (29) and (31) are simplified as follows:

$$\begin{aligned}\int_0^L A_1\,ds &= \rho AL\big[-\ddot{x}+(L/2)\big(\omega_z^2+\omega_y^2\big)\\&\quad\times\cos\phi\cos\theta-(L/2)(\omega_x\omega_y+\dot{\omega}_z)\\&\quad\times(-\cos\psi\sin\theta+\sin\psi\sin\phi\cos\theta)\\&\quad+(L/2)(-\omega_x\omega_z+\dot{\omega}_y)\\&\quad\times(\sin\psi\sin\theta+\cos\psi\sin\phi\cos\theta)]\\&=0\end{aligned}$$

$$\begin{aligned}\int_0^L A_2\,ds &= \rho AL[-\ddot{y}+(L/2)\big(\omega_z^2+\omega_y^2\big)\\&\quad\times\cos\phi\sin\theta-(L/2)(\omega_x\omega_y+\dot{\omega}_z)\\&\quad\times(\cos\psi\cos\theta+\sin\psi\sin\phi\sin\theta)\\&\quad+(L/2)(-\omega_x\omega_z+\dot{\omega}_y)\\&\quad\times(-\sin\psi\cos\theta+\cos\psi\sin\phi\sin\theta)]\\&=0\end{aligned}$$

$$\begin{aligned}\int_0^L A_3\,ds &= \rho AL[-(\ddot{z}+g)-(L/2)\big(\omega_z^2+\omega_y^2\big)\sin\phi\\&\quad-(L/2)(\omega_x\omega_y+\dot{\omega}_z)\sin\psi\cos\phi+\\&\quad+(L/2)(-\omega_x\omega_z+\dot{\omega}_y)\cos\psi\cos\phi\big]\\&=0\end{aligned}$$

$$\int_0^L A_4\,ds=-\rho JL2\dot{\omega}_x=0$$

$$\begin{aligned}\int_0^L A_5\,ds &= \rho AL\,(L/2)\\&\quad\times[\ddot{x}(\sin\psi\sin\theta+\cos\psi\sin\phi\cos\theta)\\&\quad+\ddot{y}(-\sin\psi\cos\theta+\cos\psi\sin\phi\sin\theta)\\&\quad+(\ddot{z}+g)\cos\psi\cos\phi]-\rho AL(L^2/3)\\&\quad\times(\dot{\omega}_y-\omega_x\omega_z)-\rho LJ(\dot{\omega}_y+\omega_x\omega_z)\\&=0\end{aligned}$$

$$\begin{aligned}\int_0^L A_6\,ds &= \rho AL\frac{L}{2}\\&\quad\times[\ddot{x}\,(+\cos\psi\sin\theta-\sin\psi\sin\phi\cos\theta)\\&\quad+\ddot{y}(-\cos\psi\cos\theta-\sin\psi\sin\phi\sin\theta)\\&\quad-(\ddot{z}+g)\sin\psi\cos\phi]-\rho AL(L^2/3)\\&\quad\times(\dot{\omega}_z+\omega_x\omega_y)-\rho LJ(\dot{\omega}_z-\omega_x\omega_y)\\&=0.\end{aligned}$$

These equations are the motion equations of a flying rigid prism with circular or square cross-section. They are the same as what obtained from rigid body dynamics Equations (32)

$$\begin{aligned}F_{\mathrm{ext}}&=m\underset{I}{a}^{\overline{C}},\\M_{\mathrm{ext\,B}}&=\dot{H}_B+m\,\overline{r}\times\underset{B}{a}^{B}=\dot{H}_{\overline{C}}+m\,\overline{r}\underset{B}{a}^{\overline{C}}\\&=[I_{\overline{C}}]\dot{\omega}^B+\widetilde{\omega}^B[I_{\overline{C}}]\omega^B+m\,\overline{r}\times\underset{\mathrm{BI}}{R}\,\underset{I}{a}^{\overline{C}}\end{aligned}\tag{32}$$

where

$$m=\rho AL,\quad \overline{r}=\frac{L}{2}\begin{bmatrix}1\\0\\0\end{bmatrix},\quad F_{\mathrm{ext}}=-mg\begin{bmatrix}0\\0\\1\end{bmatrix},$$

$$[I_{\overline{C}}]=\begin{bmatrix}m2J/A&0&0\\0&mJ/A+mL^2/12&0\\0&0&mJ/A+mL^2/12\end{bmatrix}$$

$$\underset{\mathrm{BI}}{R}=\begin{bmatrix}\mathrm{c}\phi\mathrm{c}\theta&\mathrm{c}\phi\mathrm{s}\theta&-\mathrm{s}\phi\\-\mathrm{c}\psi\mathrm{s}\theta+\mathrm{s}\psi\mathrm{s}\phi\mathrm{c}\theta&\mathrm{c}\psi\mathrm{c}\theta+\mathrm{s}\psi\mathrm{s}\phi\mathrm{s}\theta&\mathrm{s}\psi\mathrm{c}\phi\\\mathrm{s}\psi\mathrm{s}\theta+\mathrm{c}\psi\mathrm{s}\phi\mathrm{c}\theta&-\mathrm{s}\psi\mathrm{c}\theta+\mathrm{c}\psi\mathrm{s}\phi\mathrm{s}\theta&\mathrm{c}\psi\mathrm{c}\phi\end{bmatrix},$$

$$\underset{\mathrm{IB}}{R} = \begin{bmatrix} \mathrm{c}\phi\mathrm{c}\theta & -\mathrm{c}\psi\mathrm{s}\theta + \mathrm{s}\psi\mathrm{s}\phi\mathrm{c}\theta & \mathrm{s}\psi\mathrm{s}\theta + \mathrm{c}\psi\mathrm{s}\phi\mathrm{c}\theta \\ \mathrm{c}\phi\mathrm{s}\theta & \mathrm{c}\psi\mathrm{c}\theta + \mathrm{s}\psi\mathrm{s}\phi\mathrm{s}\theta & -\mathrm{s}\psi\mathrm{c}\theta + \mathrm{c}\psi\mathrm{s}\phi\mathrm{s}\theta \\ -\mathrm{s}\phi & \mathrm{s}\psi\mathrm{c}\phi & \mathrm{c}\psi\mathrm{c}\phi \end{bmatrix}$$

$$\mathrm{s}\psi = \sin\psi, \quad \mathrm{s}\phi = \sin\phi, \quad \mathrm{s}\theta = \sin\theta, \quad \mathrm{c}\psi = \cos\psi, \quad \mathrm{c}\phi = \cos\phi, \quad \mathrm{c}\theta = \cos\theta$$

$$\underset{I}{a}^{\overline{C}} = \underset{I}{a}^{B} + \underset{\mathrm{IB}}{R}(\widetilde{\omega}^B\widetilde{\omega}^B + \widetilde{\dot{\omega}}^B)\overline{r} = \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} + \frac{L}{2}\underset{\mathrm{IB}}{R}\begin{bmatrix} -\left(\omega_z^2 + \omega_y^2\right) \\ \omega_x\omega_y + \dot{\omega}_z \\ \omega_x\omega_z - \dot{\omega}_y \end{bmatrix},$$

$$M_{\mathrm{ext\,B}} = \overline{r} \times \underset{\mathrm{BI}}{R}\begin{bmatrix} 0 \\ 0 \\ -mg \end{bmatrix} = mg\frac{L}{2}\begin{bmatrix} 0 \\ \mathrm{c}\psi\mathrm{c}\phi \\ -\mathrm{s}\psi\mathrm{c}\phi \end{bmatrix}$$

$$[I_{\overline{C}}]\dot{\omega}^B + \widetilde{\omega}^B[I_{\overline{C}}]\omega^B = m\begin{bmatrix} 2J/A\dot{\omega}_x \\ (J/A + L^2/12)\dot{\omega}_y + (J/A - L^2/12)\omega_x\omega_z \\ (J/A + L^2/12)\dot{\omega}_z - (J/A - L^2/12)\omega_x\omega_y \end{bmatrix}$$

$$\overline{r} \times \underset{\mathrm{BI}}{R}\,\underset{I}{a}^{\overline{C}} = \frac{L}{2}\begin{bmatrix} 0 \\ -\ddot{x}\,(\mathrm{s}\psi\mathrm{s}\theta + \mathrm{c}\psi\mathrm{s}\phi\mathrm{c}\theta) - \ddot{y}\,(-\mathrm{s}\psi\mathrm{c}\theta + \mathrm{c}\psi\mathrm{s}\phi\mathrm{s}\theta) - \ddot{z}\mathrm{c}\psi\mathrm{c}\phi - (L/2)(\omega_x\omega_z - \dot{\omega}_y) \\ \ddot{x}(-\mathrm{c}\psi\mathrm{s}\theta + \mathrm{s}\psi\mathrm{s}\phi\mathrm{c}\theta) + \ddot{y}\,(\mathrm{c}\psi\mathrm{c}\theta + \mathrm{s}\psi\mathrm{s}\phi\mathrm{s}\theta) + \ddot{z}\mathrm{s}\psi\mathrm{c}\phi + (L/2)(\omega_x\omega_y + \dot{\omega}_z) \end{bmatrix}.$$

8.2 Case II: Enhanced nonlinear 3D short Euler–Bernoulli beam with a fixed support

Substitution of Equation (33) into Equations (29) and (31) fixes the flying support of the beam and gives Equations (34)–(37). Equations (34)–(37) are the motion equations of enhanced nonlinear 3D short Euler–Bernoulli beam with a fixed support. The term short is to indicate that the elastic orientation of the beam cross-sectional frame is negligible.

$$\delta b = \underset{I}{V}^B = \underset{I}{a}^B = \delta\pi^B = \omega^B = \dot{\omega}^B = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^{\mathrm{T}}, \quad \dot{\psi} = \dot{\phi} = \dot{\theta} = \ddot{\psi} = \ddot{\phi} = \ddot{\theta} = 0 \tag{33}$$

$$A_7 = 2GJ\gamma'' - \rho([\dot{\Omega}^S]^{\mathrm{T}}[J_S] - [\Omega^S]^{\mathrm{T}}[J_S]\widetilde{\Omega}^S)\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^{\mathrm{T}} = 0 \tag{34}$$

$$A_8 = -\rho A\ddot{u} + \rho\langle([\dot{\Omega}^S]^{\mathrm{T}}[J_S] - [\Omega^S]^{\mathrm{T}}[J_S]\widetilde{\Omega}^S)C\rangle'\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \frac{(1-\nu)E}{(1+\nu)(1-2\nu)}\left(Au'' - J\left\langle\frac{v''^2 + w''^2}{(1+u')^3}\right\rangle'\right) = 0 \tag{35}$$

$$\begin{aligned} A_9 = {} & -\rho A\ddot{v} + \rho\langle\left([\dot{\Omega}^S]^{\mathrm{T}}[J_S] - [\Omega^S]^{\mathrm{T}}[J_S]\widetilde{\Omega}^S\right)C\rangle'[0 \quad 1 \quad 0]^{\mathrm{T}} \\ & - \frac{(1-\nu)EJ}{(1+\nu)(1-2\nu)}\left(\left\langle\frac{v''}{(1+u')^2}\right\rangle'' + \left\langle\frac{u''v''}{(1+u')^3} + \frac{\gamma'w''}{(1+u')^2}\right\rangle'\right) \\ & + GJ\left\langle\frac{\gamma'w''}{(1+u')^2}\right\rangle' = 0 \end{aligned} \tag{36}$$

$$\begin{aligned} A_{10} = {} & -g\rho A - \rho A\ddot{w} + \rho\left\langle\left([\dot{\Omega}^S]^{\mathrm{T}}[J_S] - [\Omega^S]^{\mathrm{T}}[J_S]\widetilde{\Omega}^S\right)C\right\rangle'\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^{\mathrm{T}} + \\ & - \frac{(1-\nu)EJ}{(1+\nu)(1-2\nu)}\left(\left\langle\frac{w''}{(1+u')^2}\right\rangle'' + \left\langle\frac{u''w''}{(1+u')^3} - \frac{\gamma'v''}{(1+u')^2}\right\rangle'\right) \\ & + GJ\left\langle\frac{\gamma'v''}{(1+u')^2}\right\rangle' = 0 \end{aligned} \tag{37}$$

Equation (24) has two more elastic terms than Equation (25) that is the variation of elastic potential energy of the nonlinear 3D Euler–Bernoulli beam theory [8]. Consequently, the additional elastic terms in Equations (36) and (37) are the terms (38) and (39), respectively. These two terms improve the nonlinear 3D

Euler–Bernoulli beam theory.

$$+\left\langle J\left[-\frac{(1-\nu)E}{(1+\nu)(1-2\nu)}+G\right]\frac{\gamma' w''}{(1+u')^2}\right\rangle' \tag{38}$$

$$+\left\langle J\left[\frac{(1-\nu)E}{(1+\nu)(1-2\nu)}-G\right]\frac{\gamma' v''}{(1+u')^2}\right\rangle'. \tag{39}$$

8.3 Case III: Linear 3D short Euler–Bernoulli beam with a fixed support

Motion equations of a linear 3D short Euler–Bernoulli beam with a fixed support are obtained by neglecting Poisson's effect and nonlinear terms in Equations (34)–(37). Therefore, one has:

$$\begin{aligned}
&(\dot{\Omega}^{\mathrm{T}}[J_S]-\Omega^{\mathrm{T}}[J_S]\widetilde{\Omega})[1\quad 0\quad 0]^{\mathrm{T}}\approx 2J\ddot{\gamma},\\
&A_7 = 2GJ\gamma'' - \rho 2J\ddot{\gamma} = 0,\\
&A_8 = -\rho A\ddot{u} + EAu'' = 0,\\
&A_9 = -\rho A\ddot{v} - EJ\langle v''\rangle'' = 0,\\
&A_{10} = -g\rho A - \rho A\ddot{w} - EJ\langle w''\rangle'' = 0
\end{aligned}$$

that can be simplified as follows:

$$\begin{aligned}
&\frac{G}{\rho}\frac{\partial^2}{\partial s^2}\gamma = \frac{\partial^2}{\partial t^2}\gamma, \quad \frac{E}{\rho}\frac{\partial^2}{\partial s^2}u = \frac{\partial^2}{\partial t^2}u,\\
&\frac{EJ}{\rho A}\frac{\partial^4}{\partial s^4}v + \frac{\partial^2}{\partial t^2}v = 0,\\
&\frac{EJ}{\rho A}\frac{\partial^4}{\partial s^4}w + \frac{\partial^2}{\partial t^2}w = -g.
\end{aligned} \tag{40}$$

The linear partial differential Equations (40) are the well-known equations for torsional, longitudinal, and two transverse vibrations of a uniform beam.

9 Conclusions

Nonlinear partial differential equations of motion of a 3D Euler–Bernoulli beam with six-DOF flying support are exposed. The beam experiences compression, tension, torsion, and two spatial bendings elastically while negligible elastic orientation is produced in its cross-sectional frame. The motion equations involve two new elastic terms that had been lost in the traditional nonlinear 3D Euler–Bernoulli beam theory by differentiation from the approximated strain field regarding negligible elastic orientation of cross-sectional frame. The differentiation should have been derived from the exact strain field concerning large elastic orientation of the cross-sectional frame. These two elastic terms have given the resulting motion equations a higher accuracy than that of the nonlinear 3D Euler–Bernoulli beam theory. They account for the enhancement of the nonlinear 3D Euler–Bernoulli beam theory.

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