Chaos Synchronization of Uncertain Fractional-Order Chaotic Systems With Time Delay Based on Adaptive Fuzzy Sliding Mode Control

Tsung-Chih Lin, *Senior Member, IEEE*, and Tun-Yuan Lee

***Abstract*—This paper proposes an adaptive fuzzy sliding mode control (AFSMC) to synchronize two different uncertain fractional-order time-delay chaotic systems, which are infinite dimensional in nature, and time delay is a source of instability. Because modeling the behavior of dynamical systems by fractional-order differential equations has more advantages than integer-order modeling, the adaptive time-delay fuzzy-logic system is constructed to approximate the unknown fractional-order time-delay-system functions. By using Lyapunov stability criterion, the free parameters of the adaptive fuzzy controller can be tuned online by output-feedback-control law and adaptive law. The sliding mode design procedure not only guarantees the stability and robustness of the proposed AFSMC, but it also guarantees that the external disturbance on the synchronization error can be attenuated. The simulation example is included to confirm validity and synchronization performance of the advocated design methodology.**

***Index Terms*—Adaptive fuzzy, chaos synchronization, fractional order, Lyapunov criterion, sliding-mode control (SMC), time delay.**

I. Introduction

TIME delays are often present in many control systems, such as aircraft and chemical or process control systems either in the state, the control input, or the measurements. The existence of pure time delay, regardless of its presence in a control and/or state, is often the cause of poor performance, undesirable system transient response, and instability. The stabilization problem of time-delay systems is a true challenge and has received considerable attention [1]–[3]. Over the past decade, various methods have been developed in the analysis and synthesis of uncertain systems with time delay. Based on the Lyapunov theory of stability, the sliding-mode control (SMC) has been extensively used and various results have been obtained, because it offers fast response, good transient response, and it is also insensitive to uncertainty in the system [4]. Some works deal with the control problem of time-delay systems via a predictor-based sliding mode [1], [5]–[8].

In recent years, fractional calculus deals with derivatives and integrations of arbitrary order [9]–[11] and has found many applications in the fields of physics, applied mathematics, and engineering. It is observed that the description of some systems is more accurate when the fractional derivative is used. For instance, electrochemical processes and flexible structures are modeled by fractional-order models [12], the behavior of some biological systems is explored using fractional calculus [13], and the dielectric polarization, electromagnetic waves, and viscoelastic systems are described by fractional-order differential equations [14], [15]. Nowadays, many fractional-order differential systems behave chaotically, such as the fractional-order Chua's system [16], [17], the fractional-order Duffing system [18], [19], the fractional-order Lu system [20], the fractional-order Chen's system [21], [22], the fractional-order cellular neural network [23], [24], and the fractional-order neural network [25].

Recently, due to its potential applications in secure communication and control processing [26], study of chaos synchronization in fractional-order dynamical systems and related phenomena is receiving increasing attention [27], [28], [45]–[48]. The synchronization problem of fractional-order chaotic systems is first investigated by Deng and Li [20], who carried out synchronization in case of the fractional Lü system. Afterward, they studied chaos synchronization of the Chen system with a fractional order in a different manner [29]–[31].

SMC is a well-known robust nonlinear control technique [4], which guarantees the stability and robustness of the resulting system. This control strategy makes use of the desired sliding surface in the state space and produces the switched control settings based on the observed plant input–output behavior and on considerations concerning the boundary of modeling uncertainties and unknown disturbances [4], [32]. However, there exists chattering phenomena while implementing an SMC, which may excite high-frequency dynamics. In order to eliminate chattering, Palm [33] noted the similarity between fuzzy controller and sliding-mode controller with a boundary layer and provided a fuzzy-sliding-mode-design approach. This design can lead to a stable closed-loop system that avoids the chattering problem in the SMC.

Unfortunately, not many contributions are available for the problem of the SMC of fractional-order systems with time delays. In [49], some results are obtained without using a fractional sliding manifold. In this paper, we develop new results on SMC of fractional-order systems with time delays. In this paper, the core of innovation is based on the fact that fractional-order expression of chaotic systems is very compact in comparison with conventional mathematics. This makes the fractional calculus

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The authors are with the Department of Electronic Engineering, Feng-Chia University, Taichung 40724, Taiwan (e-mail: tclin@fcu.edu.tw).

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easier to find an appropriate function for stability analysis. We incorporate adaptive fuzzy-control scheme with SMC approach to synchronize two nonlinear fractional-order Duffing–Holmes chaotic systems with time delay. In our design procedure, both the drive- and response-system dynamics are represented by the Takagi–Sugeno (T–S) fuzzy-neural-network (FNN) model, which expresses the local dynamics of each fuzzy rule by linear combination of all system states.

This paper is organized as follows. In Section II, an introduction to fractional derivative and its relation to the approximation solution are addressed. A brief description of the T–S FNN is presented in Section III. Section IV proposes in a general way the employment of the adaptive fuzzy SMC to synchronize the fractional-order chaotic system with time delay in the presence of uncertainty and its stability analysis. In Section V, application of the proposed method to fractional-order expression of Duffing–Holmes chaotic system with time delay is investigated. Finally, the simulation results and conclusion are presented in Section VI.

II. Basic Definition and Preliminaries for Fractional-Order Systems

Fractional calculus is a mathematical topic for more than 300 years. It is a generalization of integration and differentiation to noninteger-order fundamental operator, which is denoted by ${}_aD_t^q$, where a and t are the limits of the operator. This operator is a notation for taking both the fractional integral and functional derivative in a single expression, which is defined as

$$ {}_aD_t^q = \begin{cases} \dfrac{d^q}{dt^q}, & q > 0 \\ 1, & q = 0 \\ \int_a^t (d\tau)^{-q}, & q < 0. \end{cases} \tag{1}$$

There are some basic definitions of the general fractional integration and differentiation. The commonly used definitions are Grunwald–Letnikov and Riemann–Liouville. The Grunwald–Letnikov definition is expressed as

$$ {}_aD_t^q f(t) = \lim_{h\to 0} \sum_{j=0}^{[t-a/h]} (-1)^j \binom{q}{j} f(t-jh) \tag{2}$$

where $[\cdot]$ is the integer part. The simplest and easiest definition is Riemann–Liouville definition, which is given as

$$ {}_aD_t^q f(t) = \frac{1}{\Gamma(n-q)} \frac{d^n}{dt^n} \int_0^t \frac{f(\tau)}{(t-\tau)^{q-n+1}} d\tau \tag{3}$$

where n is the first integer, which is not less q, i.e., $n-1 < q < n$, and Γ is the Gamma function.

The numerical simulation of a fractional differential equation is not simple like an ordinary differential equation. In this paper, the algorithm, which is an improved version of Adams–Bashforth–Moulton algorithm [34]–[36] to find an approximation for fractional-order systems based on predictor–corrector [36], [37], is given. Let us consider the following differential equation:

$$ {}_0D_t^q y(t) = r(y(t), t), \qquad 0 \le t \le T$$

and

$$ y^{(k)}(0) = y_0^{(k)}, \qquad k = 0, 1, 2, \ldots, m-1 \tag{4}$$

where

$$ {}_0D_t^q y(t) = \begin{cases} \dfrac{1}{\Gamma(m-q)} \displaystyle\int_0^t \frac{f^{(m)}(\tau)}{(t-\tau)^{q-m+1}} d\tau, & m-1 < q < m \\ \dfrac{d^m}{dt^m} y(t), & q = m \end{cases} \tag{5}$$

and is the first integer larger than q. The solution of (4) is equivalent to Volterra integral equation [38], which is described as

$$ y(t) = \sum_{k=0}^{[q]-1} y_0^{(k)} \frac{t^k}{k!} + \frac{1}{\Gamma(q)} \int_0^t (t-\lambda)^{q-1} r(y(\lambda), \lambda) d\lambda. \tag{6}$$

Letting $h = T/N, t_n = nh, n = 0, 1, 2, \ldots, N$, (6) can then be discretized as follows:

$$ y_h(t_{n+1}) = \sum_{k=0}^{[q]-1} y_0^{(k)} \frac{t_{n+1}^k}{k!} + \frac{h^q}{\Gamma(q+2)} r(y_h^p(t_{n+1}), t_{n+1}) + \frac{h^q}{\Gamma(q+2)} \sum_{j=0}^{n} a_{j,n+1} r(y_h(t_j), t_j) \tag{7}$$

where predict value $y_h^p(t_{n+1})$ is determined by

$$ y_h^p(t_{n+1}) = \sum_{k=0}^{[q]-1} y_0^{(k)} \frac{t_{n+1}^k}{k!} + \frac{h^q}{\Gamma(q)} \sum_{j=0}^{n} b_{j,n+1} r(y_h(t_j), t_j) \tag{8}$$

and

$$ a_{j,n+1} = \begin{cases} n^{q+1} - (n-q)(n+1)^q, & j = 0 \\ (n-j+2)^{q+1} + (n-j)^{q+1} - 2(n-j+1)^{q+1}, & 1 \le j \le n \\ 1, & j = n+1 \end{cases} \tag{9}$$

$$ b_{j,n+1} = \frac{h^q}{q}((n+1-j)^q - (n-j)^q). \tag{10}$$

The approximation error is given as

$$ \max_{j=0,1,2,\ldots,N} |y(t_j) - y_h(t_j)| = \mathrm{O}(h^p) \tag{11}$$

where $p = \min(2, 1+q)$. Therefore, the numerical solution of a fractional-order system can be obtained by applying the aforementioned algorithm.

III. Brief Description of the Takagi–Sugeno Fuzzy-Neural-Network Systems

Fuzzy-logic systems address the imprecision of the input and output variables directly by defining them with fuzzy numbers (and fuzzy sets) that can be expressed in linguistic terms (e.g., *small*, *medium*, and *large*) [39]–[43], [50]. The basic configuration of the T–S FNN system includes a fuzzy rule base, which consists of a collection of fuzzy IF–THEN rules in the following form:

$$R^{(l)}: \text{IF } x_1 \text{ is } F_1^l, \text{ and } \ldots, \text{and } x_n \text{ is } F_n^l,$$
$$\text{THEN } y_l = q_0^l + q_1^l x_1 + \cdots + q_n^l x_n = \underline{\theta}_l^T [1\; \underline{x}^T]^T \quad (12)$$

where F_i^l are input fuzzy sets, $i = 1 - n$, $l = 1 - M$, n and *M* are the number of the input variables and fuzzy IF–THEN rules, respectively, $\underline{\theta}_l^T = [q_0^l, q_1^l, \ldots, q_n^l]$ is a vector of the adjustable factors of the consequence part of the fuzzy rule, and y_l is a crisp value. Moreover, a fuzzy-inference engine combines the fuzzy IF–THEN rules in the fuzzy rule base to generate an output variable $y \in R$ from an input linguistic vector $\underline{x}^T = [x_1, x_2, \ldots, x_n] \in R^n$. The output of the fuzzy-logic systems with central-average defuzzifier, product inference, and singleton fuzzifier can be expressed as

$$y(\underline{x}) = \frac{\sum_{l=1}^{M} \upsilon^l \cdot y_l}{\sum_{l=1}^{M} \upsilon^l} = \frac{\sum_{l=1}^{M} \upsilon^l \cdot \underline{\theta}_l^T [1 \quad \underline{x}^T]}{\sum_{l=1}^{M} \upsilon^l} \quad (13)$$

where $\mu_{F_i^l}(x_i)$ is the membership function value of the fuzzy variable x_i, and $\upsilon^l = \prod_{i=1}^{n} \mu_{F_i^l(x_i)}$ is the truth value of the *l*th implication. Equation (13) can be rewritten as

$$y(\underline{x}) = \underline{\theta}^T \underline{\psi}(\underline{x}) \quad (14)$$

where $\underline{\theta}^T = \left[\underline{\theta}_1^T \quad \underline{\theta}_2^T \quad \cdots \quad \underline{\theta}_M^T\right]$ is an adjustable parameter vector, and $\underline{\psi}^T(\underline{x}) = [\psi^1(\underline{x}), \psi^2(\underline{x}), \ldots, \psi^M(\underline{x})]$ is a fuzzy-basis function vector defined as

$$\psi^l(\underline{x}) = \frac{\upsilon^l [1 \quad \underline{x}^T]}{\sum_{l=1}^{M} \upsilon^l}. \quad (15)$$

When the inputs are fed into the T–S FNN, the truth value υ^l of the *l*th implication is computed. Applying the common defuzzification strategy, the output of the neural networks, which is expressed as (13), is pumped out. The overall configuration of the T–S FNN is shown in Fig. 1 [50].

Based on the universal approximation theorem [44], the above fuzzy-logic system is capable of uniformly approximating any well-defined nonlinear function over a compact set U_c to any degree of accuracy. In addition, it is straightforward to show that a multioutput system can always be approximated by a group of single-output approximation systems.

IV. Adaptive Fuzzy-Sliding-Mode Synchronization of Fractional-Order Chaotic Systems With Time Delay

Let us consider drive and response with fractional-order-derivative time-delay chaotic systems as follows.

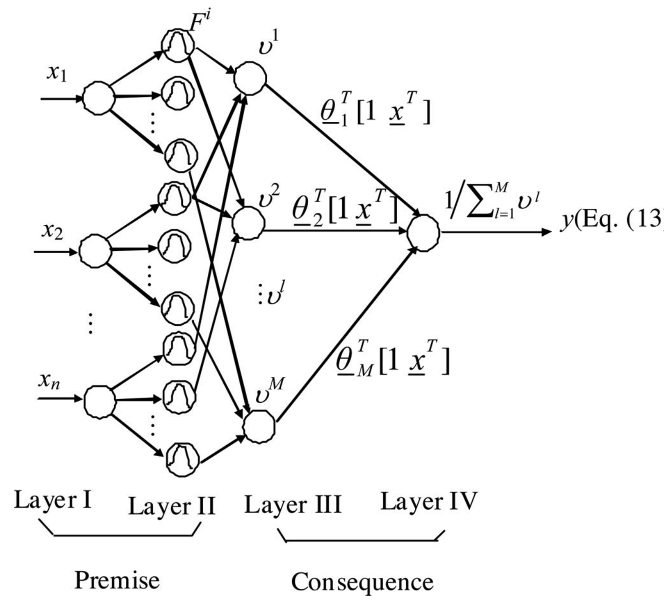

Fig. 1. Configuration of the T–S fuzzy-neural network.

1) Drive system: This is given by

$$D^q x_1 = x_2$$
$$D^q x_2 = f(x, x(t-\tau_1), \ldots, x(t-\tau_r)). \quad (16)$$

2) Response system: This is given by

$$D^q y_1 = y_2$$
$$D^q y_2 = g(y, y(t-\tau_1), \ldots, y(t-\tau_r)) + u(t) + d(t) \quad (17)$$

where $0 < q < 1$ is fractional derivative order, x_1, x_2, y_1, and y_2 are the state variables, $\tau_i, \quad i = 1, 2, \ldots, r$, are time delays, $f(x, x(t-\tau_1), \ldots, x(t-\tau_r))$ and $g(y, y(t-\tau_1), \ldots, y(t-\tau_r))$ are unknown, but bounded nonlinear functions, which express system dynamics, $d(t)$ is the external bounded disturbance, and *u*(*t*) is the control input of the response system. The control objective is to synchronize both drive and response systems by designing a nonlinear controller that obtains signals from drive system to tune behavior of the response system.

Let the synchronization error vector be

$$e = [e_1, e_2] \quad (18)$$

where $e_i = y_i - x_i, \quad i = 1 \text{ and } 2$. Then, sliding surface in the space of the synchronization error can be defined as

$$S(t) = k_1 e_1 + k_2 e_2 \quad (19)$$

where k_1 and k_2 are arbitrary constants, which are chosen such that dynamics of the sliding surface vanished quickly. The process can be classified into two phases: approaching phase with $S(t) \neq 0$ and sliding phase with $S(t) = 0$. A sufficient condition to guarantee that the trajectory of the synchronization-error vector *e* will move from approaching phase to sliding phase is to design the control effort such that sliding condition

$$S(t)\dot{S}(t) \le -\eta\,|S(t)|\,, \qquad \eta > 0 \tag{20}$$

is satisfied. During the sliding phase, we have $S(t) = 0$, and $\dot{S}(t) = 0$. If $f(x, x(t-\tau_1), \ldots, x(t-\tau_r))$ and $g(y, y(t-\tau_1), \ldots, y(t-\tau_r))$ are known and free of external disturbance, i.e., $d(t) = 0$, the corresponding equivalent control effort $u_{\rm eq}$ to force the system dynamics to stay on the sliding surface can be obtained from $\dot{S}(t) = 0$

$$\dot{S}(t) = D^{1-q}(D^q(S(t))) = 0 \rightarrow D^q(S(t)) = 0. \tag{21}$$

Substituting (19) into (21), we have

$$\begin{aligned} D^q(S(t)) &= D^q(k_1 e_1 + k_2 e_2) = k_1 D^q e_1 + k_2 D^q e_2 \\ &= k_1 D^q(y_1 - x_1) + k_2 D^q(y_2 - x_2) \\ &= k_1(y_2 - x_2) + k_2[g(y, y(t-\tau_1), \ldots, y(t-\tau_r)) \\ &\quad - f(x, x(t-\tau_1), \ldots, x(t-\tau_r)) + u_{\rm eq}(t)] \\ &= k_1 e_2 + k_2[g(y, y(t-\tau_1), \ldots, y(t-\tau_r)) \\ &\quad - f(x, x(t-\tau_1), \ldots, x(t-\tau_r)) + u_{\rm eq}(t)] = 0. \end{aligned}$$

Then, the equivalent control effort can be obtained as

$$\begin{aligned} u_{\rm eq}(t) = &-\frac{k_1}{k_2} e_2 + f(x, x(t-\tau_1), \ldots, x(t-\tau_r)) \\ &- g(y, y(t-\tau_1), \ldots, y(t-\tau_r)). \end{aligned} \tag{22}$$

In the approaching phase, in order to satisfy the sliding condition (20) and improve the robustness against system uncertainties and external disturbances such that all system states stay on the sliding surface, a switching control action $u_{\rm sw} = \eta_{\rm sw} D^{q-1}(\text{sgn}(S(t))$ must be added. The complete sliding control can be expressed as

$$\begin{aligned} u(t) &= u_{\rm eq}(t) - \eta_{\rm sw} D^{q-1}(\text{sgn}(S(t)) \\ &= -\frac{k_1}{k_2} e_2 + f(x, x(t-\tau_1), \ldots, x(t-\tau_r)) \\ &\quad - g(y, y(t-\tau_1), \ldots, y(t-\tau_r)) - \eta_{\rm sw} D^{q-1}(\text{sgn}(S(t)) \end{aligned} \tag{23}$$

where $\eta_{\rm sw}$ is a positive constant such that the approaching condition can be guaranteed for

$$\eta_{\rm sw} > \left|D^{1-q} d(t)\right| + \left|D^{1-q}\omega_{\rm total}\right| \tag{24}$$

where $\omega_{\rm total}$ is the total minimum approximation error, and $\left|D^{1-q} d(t)\right|$ and $\left|D^{1-q}\omega_{\rm total}\right|$ are assumed to be bounded.

However, as $f(x, x(t-\tau_1), \ldots, x(t-\tau_r))$ and $g(y, y(t-\tau_1), \ldots, y(t-\tau_r))$ are unknown and external disturbance $d(t) \neq 0$, the ideal control effort (23) cannot be implemented. We replace $f(x, x(t-\tau_1), \ldots, x(t-\tau_r))$ and $g(y, y(t-\tau_1), \ldots, y(t-\tau_r))$ by the fuzzy-logic system $f(x, \tau|\,\underline{\theta}_f, \underline{m}, \underline{\sigma})$ and $g(y, \tau|\,\underline{\theta}_g, \underline{m}, \underline{\sigma})$ in specified form as (14), i.e.,

$$\begin{aligned} f(x, \tau|\,\underline{\theta}_f, \underline{m}, \underline{\sigma}) &= \underline{\theta}_f^T \xi(x, \tau, \underline{m}, \underline{\sigma}) \\ g(y, \tau|\,\underline{\theta}_g, \underline{m}, \underline{\sigma}) &= \underline{\theta}_g^T \xi(y, \tau, \underline{m}, \underline{\sigma}) \end{aligned} \tag{25}$$

Here, the fuzzy basis functions $\xi(x, \tau, m, \sigma) = \xi(x, x(t-\tau_1), \ldots, x(t-\tau_r), m, \sigma)$ and $\xi(y, \tau, m, \sigma) = \xi(y, y(t-\tau_1), \ldots, y(t-\tau_r), m, \sigma)$ depend on the fuzzy-membership functions and are supposed to be fixed, while $\underline{\theta}_f^T$ and $\underline{\theta}_g^T$ are adjusted by adaptive laws based on Lyapunov stability criterion. Especially, mean (*m*) and deviation (σ) are updated simultaneously. Therefore, the resulting control effort can be obtained as

$$\begin{aligned} u(t) = &-\frac{k_1}{k_2} e_2 + f(x, \tau|\,\underline{\theta}_f, \underline{m}, \underline{\sigma}) - g(y, \tau|\,\underline{\theta}_g, \underline{m}, \underline{\sigma}) \\ &- \eta_{\rm sw} D^{q-1}(\text{sgn}(S(t)). \end{aligned} \tag{26}$$

Following the preceding consideration, the following theorem can be obtained.

Theorem 1: Let us consider the two fractional-order chaotic time-delay systems, i.e., drive system (16) and response system (17); the control effort of the response system is given in (26), and the fuzzy-based adaptive laws are chosen as

$$\begin{aligned} (D^{-q}\underline{\theta}_f) &= r_1(\xi(x, \tau, \underline{m}, \underline{\sigma}) - m\xi_m(x, \tau, \underline{m}, \underline{\sigma}) \\ &\quad - \sigma\xi_\sigma(x, \tau, \underline{m}, \underline{\sigma}))S(t) \end{aligned} \tag{27}$$

$$\begin{aligned} (D^{-q}\underline{\theta}_g) &= -r_2(\xi(y, \tau, \underline{m}, \underline{\sigma}) - m\xi_m(y, \tau, \underline{m}, \underline{\sigma}) \\ &\quad - \sigma\xi_\sigma(y, \tau, \underline{m}, \underline{\sigma}))S(t) \end{aligned} \tag{28}$$

$$\begin{aligned} \dot{m} &= -r_3[-(D^{1-q}\theta_g)\xi_m^T(y, \tau, \underline{m}, \underline{\sigma}) \\ &\quad + (D^{1-q}\theta_f)\xi_m^T(x, \tau, \underline{m}, \underline{\sigma}))]S(t) \end{aligned} \tag{29}$$

$$\begin{aligned} \dot{\sigma} &= -r_4[-(D^{1-q}\theta_g)\xi_\sigma^T(y, \tau, \underline{m}, \underline{\sigma}) \\ &\quad + (D^{1-q}\theta_f)\xi_\sigma^T(x, \tau, \underline{m}, \underline{\sigma}))]S(t). \end{aligned} \tag{30}$$

Then, the overall adaptive scheme guarantees the global stability of the resulting closed-loop system in the sense that all signals involved are uniformly bounded and the synchronization error will converge to zero asymptotically. The proof of Theorem 1 is given in Appendix A.

To summarize the above analysis, the design algorithm for the proposed adaptive fuzzy-sliding-model control (AFSMC) is given as follows.

Step 1) Specify the desired coefficients k_1 and k_2 such that dynamic of the sliding surface vanished quickly.

Step 2) Define the membership function $\mu_{F_i^l}(\underline{x})$ and $\mu_{F_i^l}(\underline{y})$ for $i = 1, 2, \cdots, M$, and compute the fuzzy basis functions $\xi(x, \tau, m, \sigma)$ and $\xi(y, \tau, m, \sigma)$, respectively. Then, fuzzy-logic control systems are constructed as

$$\begin{aligned} f(x, \tau|\,\underline{\theta}_f, \underline{m}, \underline{\sigma}) &= \underline{\theta}_f^T \xi(x, \tau, \underline{m}, \underline{\sigma}) \\ g(y, \tau|\,\underline{\theta}_g, \underline{m}, \underline{\sigma}) &= \underline{\theta}_g^T \xi(y, \tau, \underline{m}, \underline{\sigma}). \end{aligned}$$

Step 3) Specify all appropriate adaptation parameters to obtain the adaptive laws in (27)–(30) and to adjust the parameter vectors $\underline{\theta}_f$, $\underline{\theta}_g$, m, and σ.

Step 4) Select suitable switching parameter $\eta_{\rm sw}$. Obtain the control in (26) and apply to the plant.

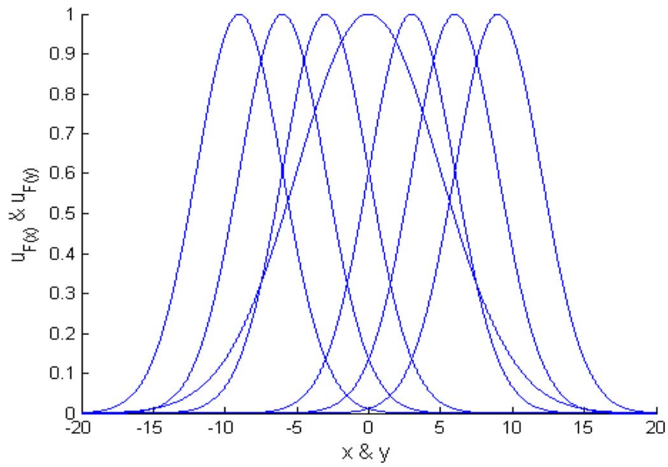

Fig. 2. Membership function for x_i and $y_i . i = 1$ and 2.

V. Simulation Example

In this section, we will apply our AFSMC to synchronize two different uncertain fractional-order Duffing–Holmes chaotic time-delay systems.

Let us consider two different uncertain fractional-order Duffing–Holmes chaotic time-delay systems as follows.

1) *Drive system:* This is given by

$$\begin{cases} D^q x_1 = 2.5x_2 \\ D^q x_2 = -\left(\dfrac{1}{2.5}x_1\right)^3 - \dfrac{1}{2.5}x_1 - 0.1x_2 + 0.01x_1(t-0.001) \\ \quad +0.01x_1^2(t-0.001) + 0.01x_2(t-0.001) + 25\cos(1.29t). \end{cases}$$

2) Response system: This is given by

$$\begin{cases} D^q y_1 = 2.2y_2 \\ D^q y_2 = -\left(\dfrac{1}{2.0}y_1\right)^3 - \dfrac{1}{1.8}y_1 - 0.1y_2 + 0.01y_1(t-0.001) \\ \qquad +0.01y_1^2(t-0.001) + 0.01y_2(t-0.001) \\ \qquad +25\cos(1.29t) + d(t) + u(t) \end{cases}$$

where $d(t) = 0.7\sin t$ is external bounded disturbance, and $u(t)$ is control input of the response system. The main objective of synchronization is to force the trajectory of the response system to become identical to that of the drive system. The initial conditions of drive and response systems are chosen as $\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, respectively. Two different values of $q = 0.98$ and $q = 0.94$ are considered. All design constants are specified as $k_1 = k_2 = 1$, $r_1 = 50$, $r_2 = 5$, $r_3 = 1.5$, $r_4 = 1.5$, $\eta_{\text{sw}} = 1$, and step size $h = 0.001$.

The initial membership functions for x_i and y_i, $i = 1$ and 2 are shown in Fig. 2 and are selected as follows:

$$\mu_{F_1^i}(x_i) = \exp\left[-\frac{1}{2}\left(\frac{x_i+9}{3}\right)^2\right]$$

$$\mu_{F_2^i}(x_i) = \exp\left[-\frac{1}{2}\left(\frac{x_i+6}{3}\right)^2\right]$$

$$\mu_{F_3^i}(x_i) = \exp\left[-\frac{1}{2}\left(\frac{x_i+3}{3}\right)^2\right]$$

$$\mu_{F_4^i}(x_i) = \exp\left[-\frac{1}{2}\left(\frac{x_i}{5}\right)^2\right]$$

$$\mu_{F_5^i}(x_i) = \exp\left[-\frac{1}{2}\left(\frac{x_i-3}{3}\right)^2\right]$$

$$\mu_{F_6^i}(x_i) = \exp\left[-\frac{1}{2}\left(\frac{x_i-6}{3}\right)^2\right]$$

$$\mu_{F_7^i}(x_i) = \exp\left[-\frac{1}{2}\left(\frac{x_i-9}{3}\right)^2\right],$$

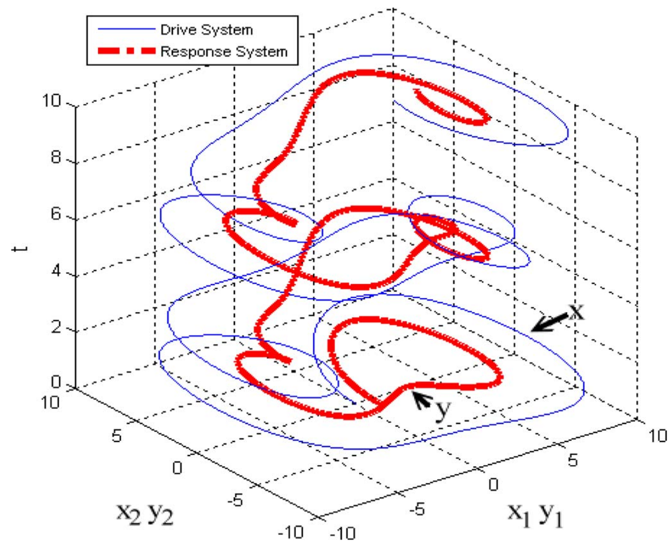

Fig. 3. Three-dimensional phase portrait of chaotic drive and response systems.

A. *q = 0.98*

For $q = 0.98$ and free of control input, the 3-D phase portrait of the drive and response systems is shown in Fig. 3.

Computing the adaptive laws (27)–(30), the control effort of the response can be obtained as

$$u(t) = -\frac{k_1}{k_2}e_2 + f(x,\tau|\underline{\theta}_f,\underline{m},\underline{\sigma}) - g(y,\tau|\underline{\theta}_g,\underline{m},\underline{\sigma}) \\ - \eta_{\text{sw}} D^{q-1}(\text{sgn}(S(t)).$$

Figs. 4 and 5 show the trajectories of the states x_1, y_1 and x_2, y_2, respectively. Control-effort trajectory is shown in Fig. 6 and trajectory of the sliding surface $S(t)$ is shown in Fig. 7, which shows that the chattering can be eliminated. The 3-D phase portrait, i.e., synchronization performance, of the drive and response systems is shown in Fig. 8. As one can see, the designed controller is effectively able to synchronize two fractional-order chaotic systems with time delay, i.e., a fast synchronization can be achieved. Fig. 9 shows the graph of $\dot{V}, (t)$, which is always negatively defined and consequently, is stable.

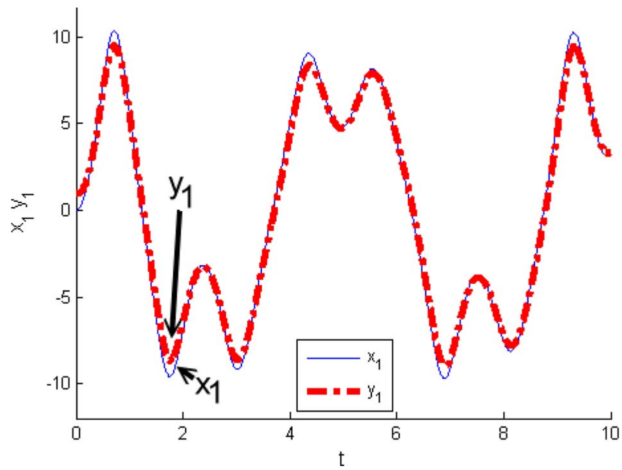

Fig. 4. Trajectories of the states x_1 and y_1.

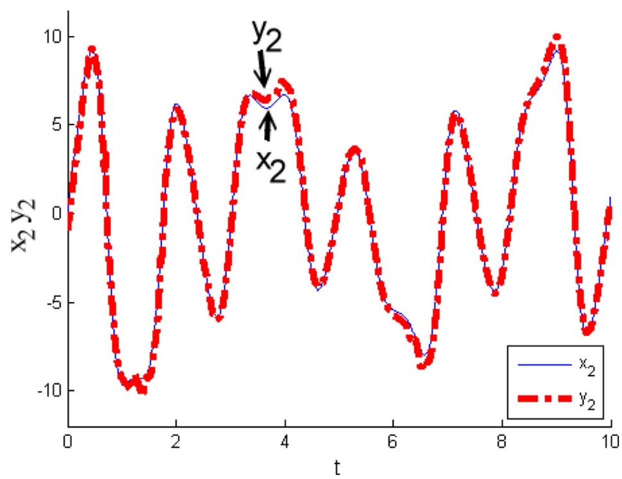

Fig. 5. Trajectories of the states x_2 and y_2.

In order to show the robustness of the proposed AFSMC, the control effort is activated at $t = 5$ s. The 3-D phase portrait, i.e., synchronization performance, of the drive and response systems is shown in Fig. 10. Figs. 11 and 12 show the trajectories of the states x_1, y_1 and x_2, y_2, respectively. We can see that a fast synchronization of drive and response is achieved as the control effort is activated. Control-effort trajectory is shown in Fig. 13, and trajectory of the sliding surface $S(t)$ is shown as in Fig. 14.

B. $q = 0.94$

For $q = 0.94$ and free of control input, the 3-D phase portrait of the drive and response systems is shown in Fig. 15.

Figs. 16 and 17 show the trajectories of the states x_1, y_1 and x_2, y_2, respectively. Control-effort trajectory is shown in Fig. 18 and trajectory of the sliding surface $S(t)$ is shown in Fig. 19, which shows that the chattering can be eliminated. The 3-D phase portrait, i.e., synchronization performance, of the drive and response systems is shown in Fig. 20. As one can see, the designed controller is effectively able to synchronize two fractional-order chaotic systems with time delay, i.e., a fast

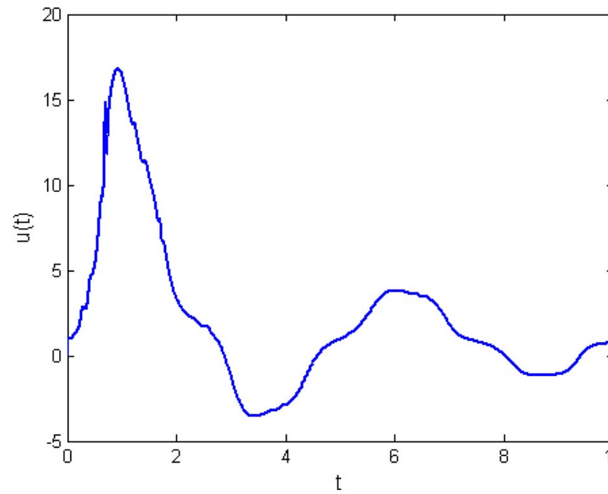

Fig. 6. Trajectory of the control effort.

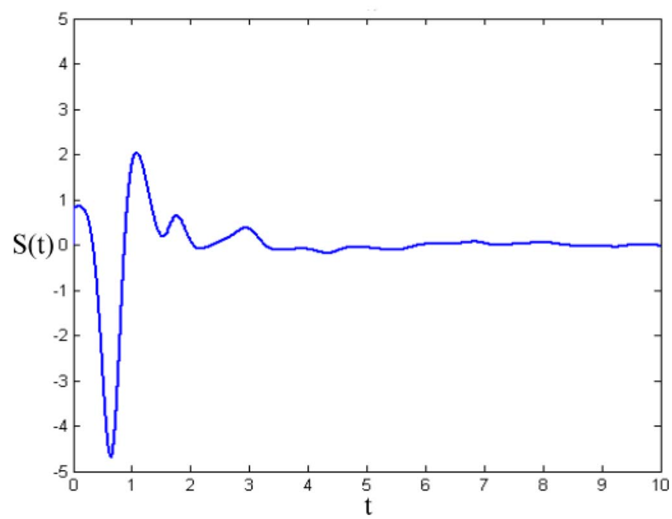

Fig. 7. Trajectory of the sliding surface $S(t)$.

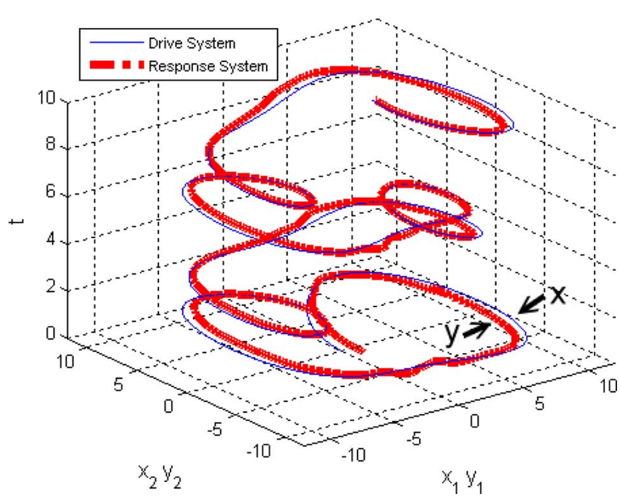

Fig. 8. Three-dimensional phase portrait, i.e., synchronization performance, of the drive and response systems.

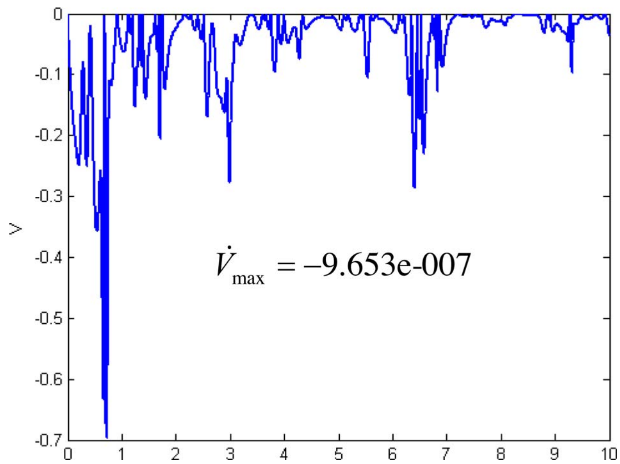

Fig. 9. Graph of $\dot{V}(t)$.

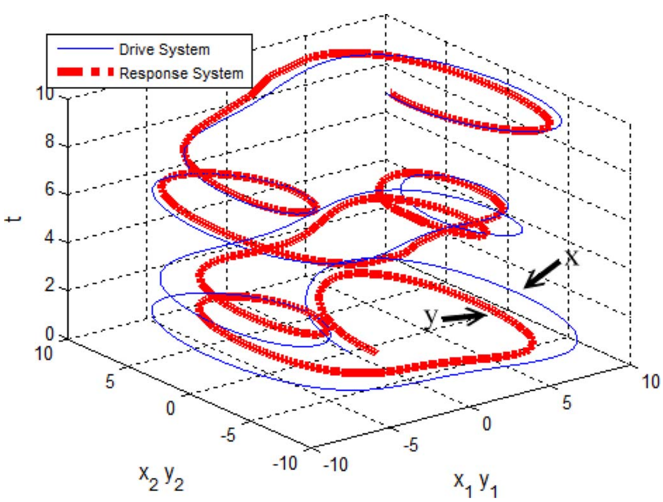

Fig. 10. Three-dimensional phase portrait, i.e., synchronization performance, of the drive and response systems.

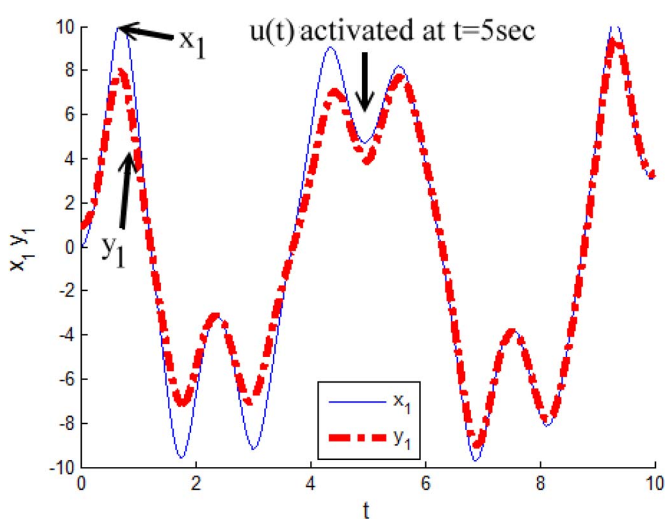

Fig. 11. Trajectories of the states x_1 and y_1.

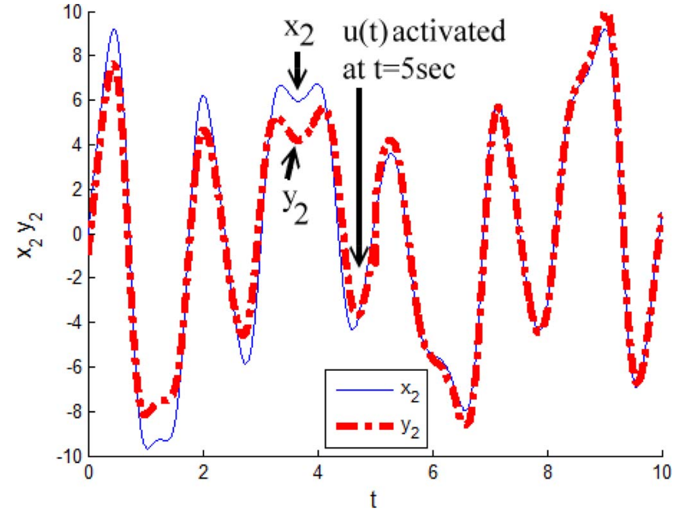

Fig. 12. Trajectories of the states x_2 and y_2.

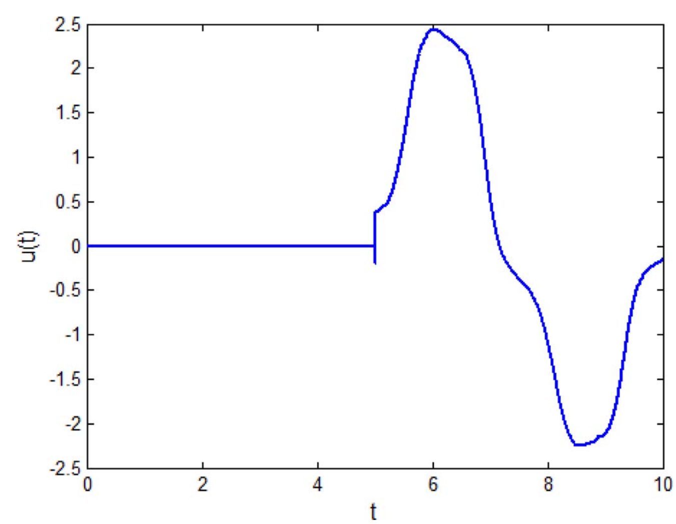

Fig. 13. Trajectory of the control effort.

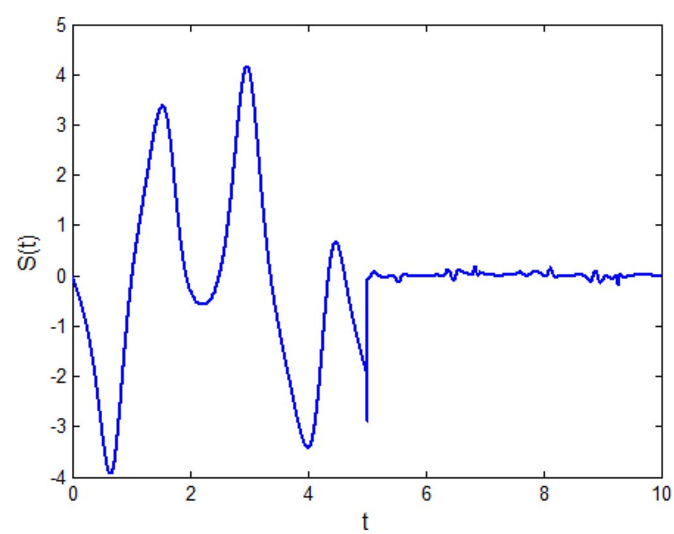

Fig. 14. Trajectory of the sliding surface $S(t)$.

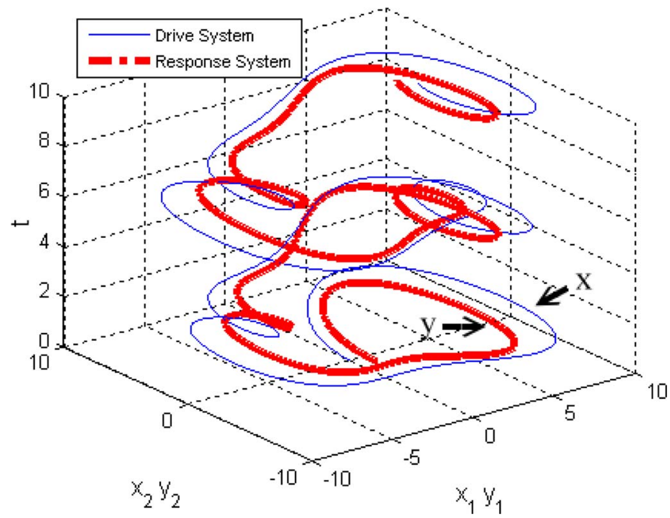

Fig. 15. Three-dimensional phase portrait of chaotic drive and response systems.

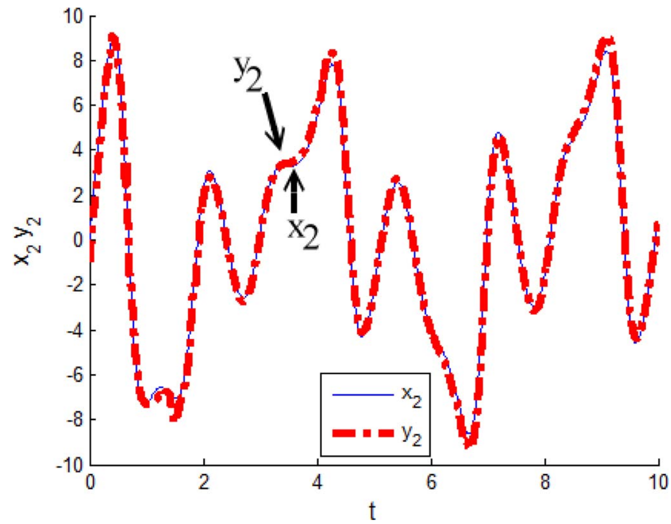

Fig. 17. Trajectories of the states x_2 and y_2.

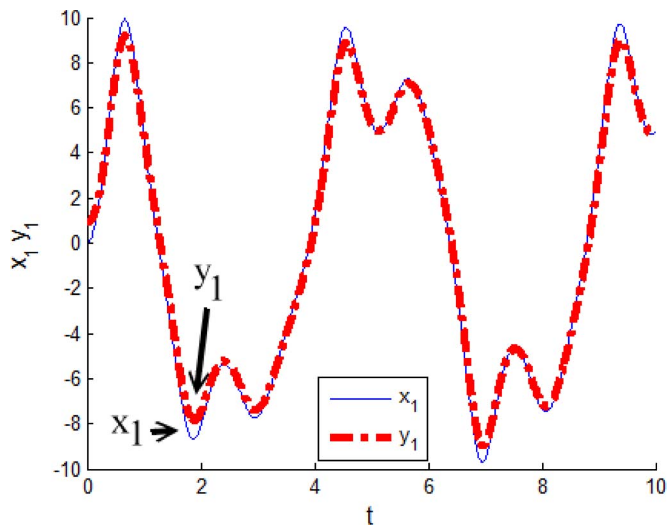

Fig. 16. Trajectories of the states x_1 and y_1.

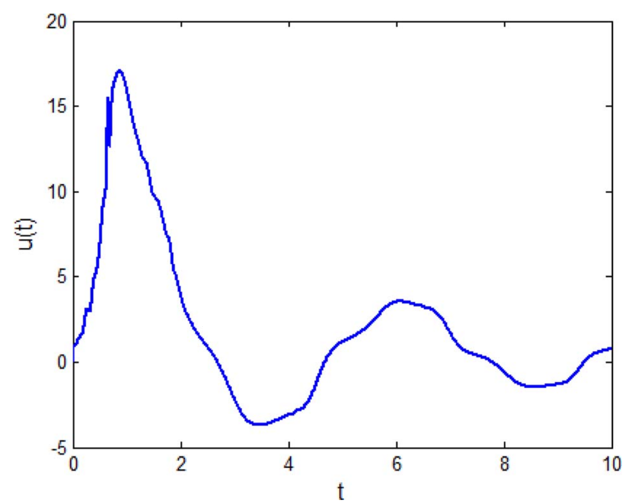

Fig. 18. Trajectory of the control effort.

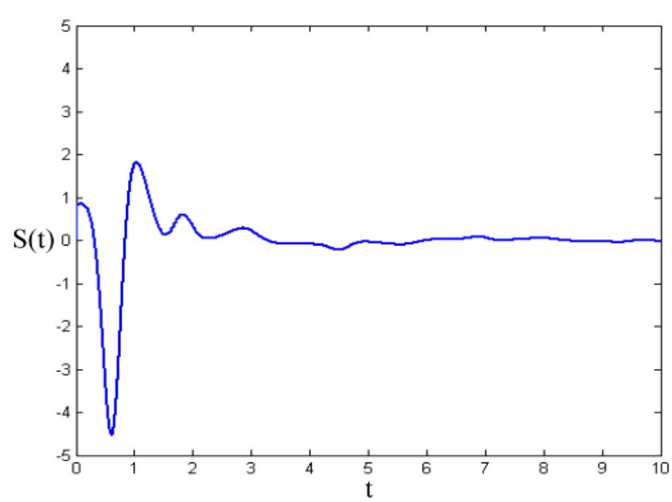

Fig. 19. Trajectory of the sliding surface $S(t)$.

synchronization can be achieved. Fig. 21 shows the graph of $\dot{V}(t)$, which is always negatively defined and, consequently, is stable.

In order to show the robustness of the proposed AFSMC, the control effort is activated at $t = 5$ s. The 3-D phase portrait, i.e., synchronization performance, of the drive and response systems is shown in Fig. 22. Figs. 23 and 24 show the trajectories of the states x_1, y_1 and x_2, y_2, respectively. We can see that a fast synchronization of drive and response is achieved as the control effort is activated. Control-effort trajectory is shown in Fig. 25, and trajectory of the sliding surface $S(t)$ is shown in Fig. 26.

For different q, i.e., $q = 0.98$ and $q = 0.94$, control efforts are activated at $t = 0$ and $t = 5$ s, after 10-s simulation time, and the final mean (m) and deviation (σ) of the Gaussian membership functions are given in Table I.

As regards the synchronization performance, mean square errors (MSEs) of $\mathrm{MSE1} = y_1 - x_1$ and $\mathrm{MSE2} = y_2 - x_2$, for

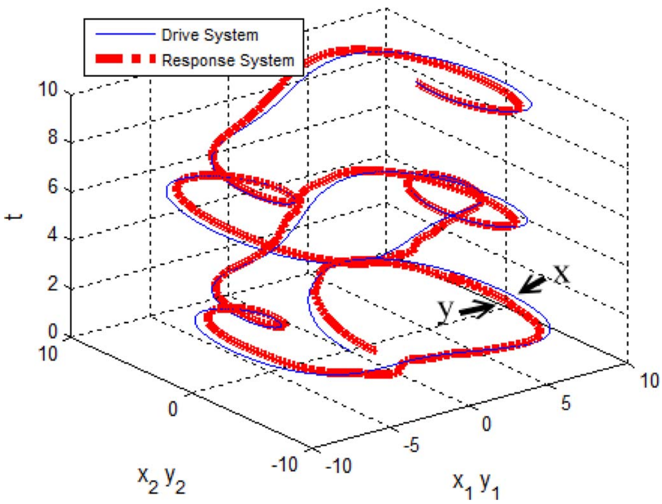

Fig. 20. Three-dimensional phase portrait, i.e., synchronization performance, of the drive and response systems.

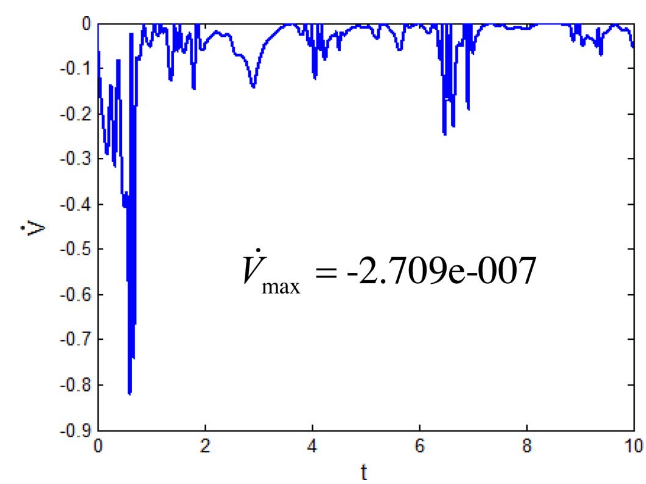

Fig. 21. Graph of $\dot{V}(t)$.

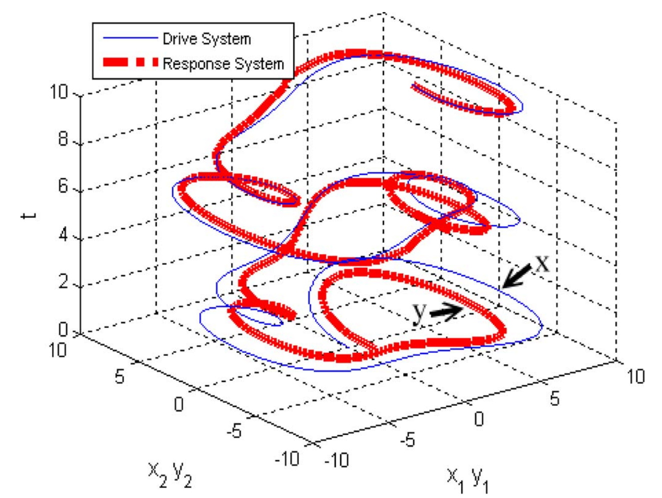

Fig. 22. Three-dimensional phase portrait, i.e., synchronization performance, of the drive and response systems.

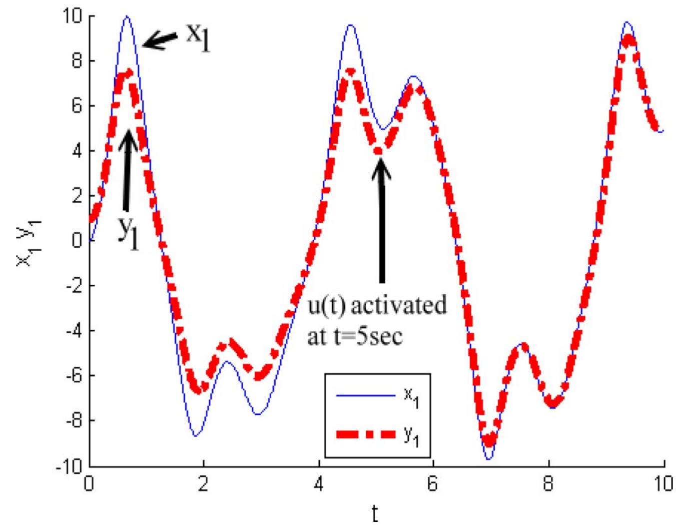

Fig. 23. Trajectories of the states x_1 and y_1.

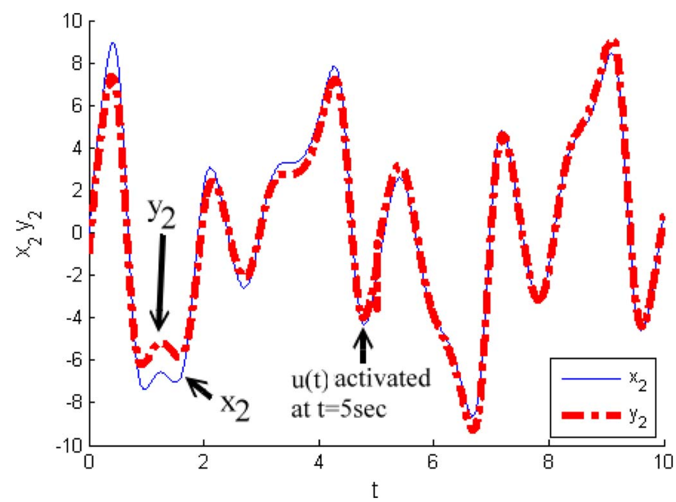

Fig. 24. Trajectories of the states x_2 and y_2.

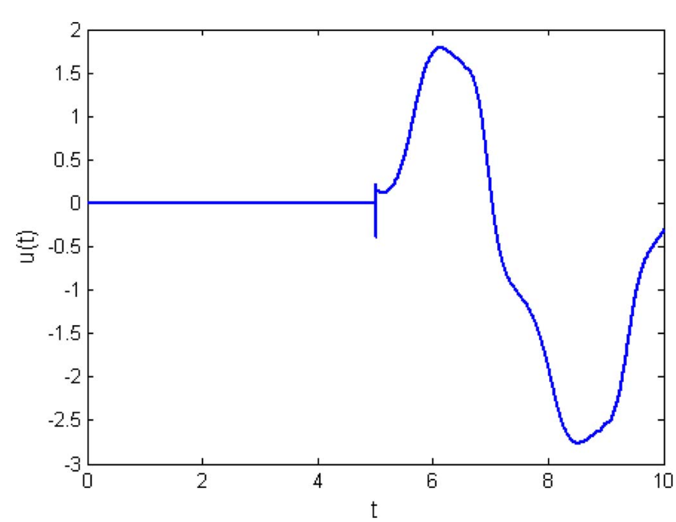

Fig. 25. Trajectory of the control effort.

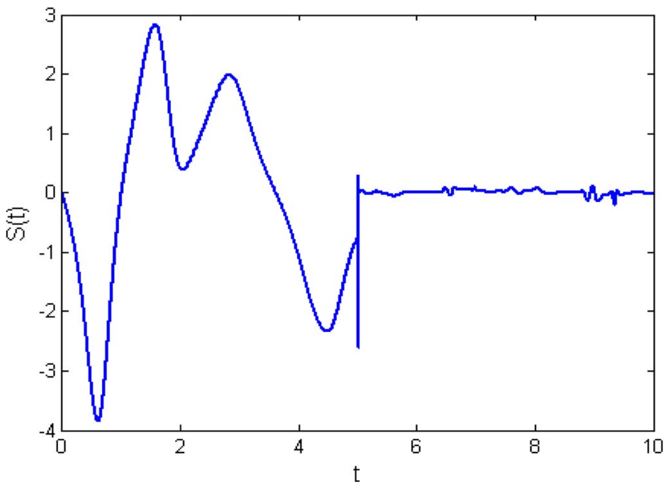

Fig. 26. Trajectory of the sliding surface $S(t)$.

TABLE I
FINAL MEAN (m) AND DERIVATION (σ) OF THE MEMBERSHIP FUNCTIONS

		Gauss function $e^{-\frac{1}{2}\left(\frac{x_i-m_i}{\sigma_i}\right)^2}$							
		i	1	2	3	4	5	6	7
	Initial	mean: m_i	9.00	6.00	3.00	0.00	-3.00	-6.00	-9.00
		deviation: σ_i	3.00	3.00	3.00	5.00	3.00	3.00	3.00
q =0.98 $u(t)$ activated at	t=0 sec final	mean: m_i	8.95	5.98	2.95	0.07	-3.14	-6.04	-9.02
		deviation: σ_i	3.09	2.92	2.84	5.01	3.39	2.64	2.99
	t=5sec final	mean: m_i	8.98	5.99	3.00	0.01	-3.00	-5.99	-8.99
		deviation: σ_i	3.01	3.02	3.00	4.99	3.01	3.01	3.00
q=0.94 $u(t)$ activated at	t=0 sec final	mean: m_i	8.97	5.99	2.96	0.03	-3.06	-6.03	-8.97
		deviation: σ_i	3.04	2.99	2.87	5.01	3.12	2.87	3.03
	t=5sec final	mean: m_i	8.99	5.98	3.00	0.01	-3.01	-6.00	-8.99
		deviation: σ_i	3.01	3.02	3.01	4.98	3.01	2.98	3.01

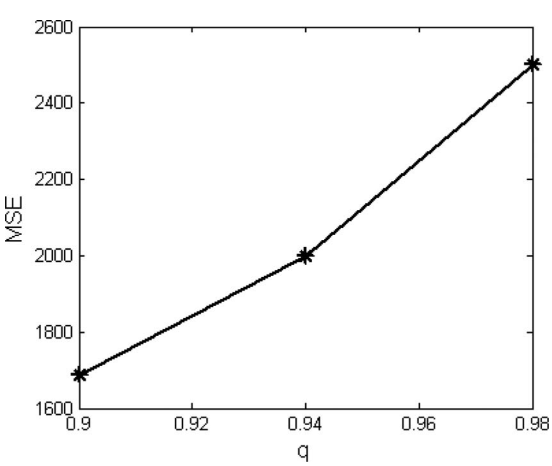

Fig. 27. MSEs of MSE1 $= y_1 - x_1$.

different values of q are shown in Figs. 27 and 28, respectively. Moreover, when control is activated at $t = 5$ s, the synchronization performance, i.e., MSEs of MSE3 $= y_1 - x_1$ and MSE4 $= y_2 - x_2$, for different values of q are shown in Figs. 29 and 30, respectively.

It is obvious that if q is reduced, the chaos appears to be reduced, i.e., the synchronization error is reduced, accordingly.

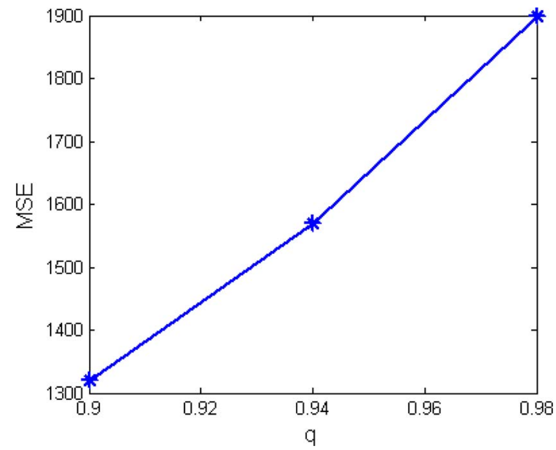

Fig. 28 MSEs of MSE2 $= y_2 - x_2$.

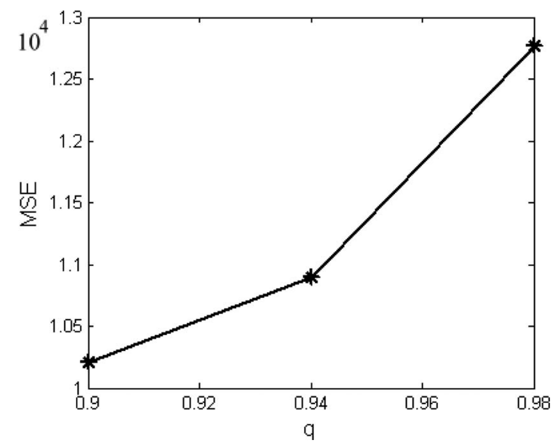

Fig. 29. MSEs of MSE3 $= y_1 - x_1$ when control effort is activated at $t = 5$ s.

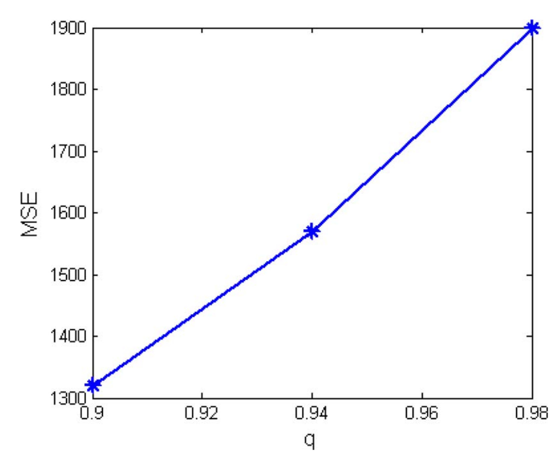

Fig. 30. MSEs of MSE4 $= y_2 - x_2$ when control effort is activated at $t = 5$ s.

1) Remark:

1) From Figs. 3 and 15, we can see that for free of control input, output of the response system cannot track the output of the drive system well. After control effort is added into response system, a fast synchronization of the drive and response systems can be achieved, as shown in Figs 8, 10, 20, and 22.

2) For different q, i.e., 0.98 and 0.94, the initial membership functions are shown in Fig. 3. In order to handle linguistic uncertainties, the adaptive laws for centers and widths of the membership functions are given in (35)–(36), and after 10-s simulation time, the final membership functions are shown in Table I.

VI. Conclusion

In this paper, a novel AFSMC is proposed to deal with chaos synchronization between two different uncertain fractional-order time-delay chaotic systems. Based on the Lyapunov synthesis approach, free parameters of the adaptive fuzzy controller can be tuned online by output feedback-control law and adaptive law, and the chattering phenomena in the control efforts can be reduced. The simulation example, i.e., chaos synchronization of two different fractional-order time-delay Duffing–Holmes chaotic systems, is given to demonstrate the effectiveness of the proposed methodology. The significance of the AFSMC in the simulations for different values of q is demonstrated. Simulation results show that a fast synchronization of drive and response can be achieved, and as q is reduced, the chaos appears to be reduced, i.e., the synchronization error is reduced, accordingly.

Appendix A
Proof of Theorem 1

Proof: The optimal-parameter estimations, i.e., $\underline{\theta}_f^*$, $\underline{\theta}_g^*$ $\underline{m}^*$, and $\underline{\sigma}^*$, are defined as

$$\underline{\theta}_f^* = \arg\min_{\underline{\theta}_f \in \Omega_f} [\sup_{x\in\Omega_x} |f(x,\tau|\,\underline{\theta}_f, \underline{m}, \underline{\sigma}) - f(x, x(t-\tau_1), \ldots, x(t-\tau_r))|] \quad (31)$$

$$\underline{\theta}_g^* = \arg\min_{\underline{\theta}_g \in \Omega g} [\sup_{y\in\Omega_y} |g(y,\tau|\,\underline{\theta}_g, \underline{m}, \underline{\sigma}) - g(y, y(t-\tau_1), \ldots, y(t-\tau_r))|] \quad (32)$$

$$\underline{m}^* = \arg\min_{\underline{m} \in \Omega_m} [\sup_{x\in\Omega_x \cup y \in \Omega_y} |f(x,\tau|\,\underline{\theta}_f, \underline{m}, \underline{\sigma}) - f(x, x(t-\tau_1), \ldots, x(t-\tau_r))| + |g(y,\tau|\,\underline{\theta}_g, \underline{m}, \underline{\sigma}) - g(y, y(t-\tau_1), \ldots, y(t-\tau_r))|] \quad (33)$$

$$\underline{\sigma}^* = \arg\min_{\underline{m} \in \Omega_m} [\sup_{x\in\Omega_x \cup y \in \Omega_y} |f(x,\tau|\,\underline{\theta}_f, \underline{m}, \underline{\sigma}) - f(x, x(t-\tau_1), \ldots, x(t-\tau_r))| + |g(y,\tau|\,\underline{\theta}_g, \underline{m}, \underline{\sigma}) - g(y, y(t-\tau_1), \ldots, y(t-\tau_r))|] \quad (34)$$

where Ω_f, Ω_g, Ω_y, and Ω_x are constraint sets of suitable bounds on $\underline{\theta}_f$, $\underline{\theta}_g$, y, and x, respectively, and they are defined as $\Omega_f = \{\underline{\theta}_f | \, |\underline{\theta}_f| \le M_f\}$, $\Omega_g = \{\underline{\theta}_g | \, |\underline{\theta}_g| \le M_g\}$, $\Omega_y = \{y|\,|y| \le M_y\}$, and $\Omega_x = \{x|\,|x| \le M_x\}$, where M_f, M_g, M_y, and M_x are positive constants.

Then, we have

$$\begin{aligned} D^q(S(t)) &= D^q(k_1 e_1 + k_2 e_2) = k_1 D^q e_1 + k_2 D^q e_2 \\ &= k_1 e_2 + k_2 [g(y, y(t-\tau_1), \ldots, y(t-\tau_r)) \\ &\quad - f(x, x(t-\tau_1), \ldots, x(t-\tau_r) + u(t) + d(t)] \\ &= k_1 e_2 + k_2 \{[g(y, y(t-\tau_1), \ldots, y(t-\tau_r)) \\ &\quad - f(x, x(t-\tau_1), \ldots, x(t-\tau_r)] \\ &\quad + \left[-\frac{k_1}{k_2} e_2 + f(x,\tau|\,\underline{\theta}_f, \underline{m}, \underline{\sigma}) - g(y,\tau|\,\underline{\theta}_g, \underline{m}, \underline{\sigma}) \right. \\ &\quad \left. -\eta_{\rm sw} D^{q-1}(\mathrm{sgn}(S(t))\right] + d(t)\} \\ &= k_2 \{[g(y, y(t-\tau_1), \ldots, y(t-\tau_r)) \\ &\quad - g(y,\tau|\underline{\theta}_g, \underline{m}, \underline{\sigma})] \\ &\quad + [f(x,\tau|\,\underline{\theta}_f, \underline{m}, \underline{\sigma}) \\ &\quad - f(x, x(t-\tau_1), \ldots, x(t-\tau_r)] \\ &\quad + d(t) - \eta_{\rm sw} D^{q-1}(\mathrm{sgn}(S(t))\} \\ &= k_2 \{[g(y,\tau|\,\underline{\theta}_g^*, \underline{m}^*, \underline{\sigma}^*) - g(y,\tau|\,\underline{\theta}_g, \underline{m}, \underline{\sigma})] \\ &\quad + [f(x,\tau|\,\underline{\theta}_f, \underline{m}, \underline{\sigma}) - f(x,\tau|\,\underline{\theta}_f^*, \underline{m}^*, \underline{\sigma}^*)] \\ &\quad + d(t) - \eta_{\rm sw} D^{q-1}(\mathrm{sgn}(S(t)) + \omega_1\} \end{aligned} \quad (35)$$

where the minimum approximation errors is defined as

$$\omega_1 = g(y, y(t-\tau_1), \ldots, y(t-\tau_r)) - g(y,\tau|\,\underline{\theta}_g^*, \underline{m}^*, \underline{\sigma}^*) + f(x,\tau|\,\underline{\theta}_f^*, \underline{m}^*, \underline{\sigma}^*) - f(x, x(t-\tau_1), \ldots, x(t-\tau_r). \quad (36)$$

Let

$$\begin{aligned} f(x,\tau|\,\underline{\theta}_f, \underline{m}, \underline{\sigma}) - f(x,\tau|\,\underline{\theta}_f^*, \underline{m}^*, \underline{\sigma}^*) &= \tilde{\theta}_f^T [\xi(x,\tau,\underline{m},\underline{\sigma}) \\ &\quad - m\xi_m(x,\tau,\underline{m},\underline{\sigma}) - \sigma\xi_\sigma(x,\tau,\underline{m},\underline{\sigma})] \\ &\quad + \theta_f^T [\tilde{m}\xi_m(x,\tau,\underline{m},\underline{\sigma}) + \tilde{\sigma}\xi_\sigma(x,\tau,\underline{m},\underline{\sigma})] \\ &\quad + \tilde{\theta}_f^T [m^*\xi_m(x,\tau,\underline{m},\underline{\sigma}) + \sigma^*\xi_\sigma(x,\tau,\underline{m},\underline{\sigma})] \end{aligned} \quad (37)$$

$$\begin{aligned} g(y\ ,\tau|\,\underline{\theta}_g, \underline{m}, \underline{\sigma}) - g(y,\tau|\,\underline{\theta}_g^*, \underline{m}^*, \underline{\sigma}^*) &= \tilde{\theta}_g^T [\xi(y,\tau,\underline{m},\underline{\sigma}) \\ &\quad - m\xi_m(y,\tau,\underline{m},\underline{\sigma}) - \sigma\xi_\sigma(y,\tau,\underline{m},\underline{\sigma})] \\ &\quad + \theta_g^T [\tilde{m}\xi_m(y,\tau,\underline{m},\underline{\sigma}) \\ &\quad + \tilde{\sigma}\xi_\sigma(x,\tau,\underline{m},\underline{\sigma})] + \tilde{\theta}_g^T [m^*\xi_m(y,\tau,\underline{m},\underline{\sigma}) \\ &\quad + \sigma^*\xi_\sigma(y,\tau,\underline{m},\underline{\sigma})] \end{aligned} \quad (38)$$

where $\tilde{\underline{\theta}}_f = \underline{\theta}_f^* - \underline{\theta}_f, \tilde{\underline{\theta}}_g = \underline{\theta}_g^* - \underline{\theta}_g$, $\underline{\tilde{m}} = \underline{m} - \underline{m}^*$, and $\underline{\tilde{\sigma}} = \underline{\sigma} - \underline{\sigma}^*$. In addition, $\xi_m(y,\tau,\underline{m},\underline{\sigma})$ and $\xi_\sigma(y,\tau,m,\sigma)$ are partial derivatives of $\xi(y,\tau,\underline{m},\underline{\sigma})$ with respect to m and σ, respectively, and $\xi_m(x,\tau,\underline{m},\underline{\sigma})$ and $\xi_\sigma(x,\tau,\underline{m},\underline{\sigma})$ are partial derivatives of $\xi(x,\tau,\underline{m},\underline{\sigma})$ with respect to m and $\underline{\sigma}$, respectively.

Equation (35) can be rewritten as

$$\begin{aligned} D^q(S(t)) = k_2\{-[\tilde{\theta}_g^T(\xi(y,\tau,\underline{m},\underline{\sigma}) - m\xi_m(y,\tau,\underline{m},\underline{\sigma}) \\ - \sigma\xi_\sigma(y,\tau,\underline{m},\underline{\sigma})) + \theta_g^T(\tilde{m}\xi_m(y,\tau,\underline{m},\underline{\sigma}) \end{aligned}$$

$$+ \tilde{\sigma}\xi_\sigma(x,\tau,\underline{m},\underline{\sigma})) + \tilde{\theta}_g^T(m^*\xi_m(y,\tau,\underline{m},\underline{\sigma}) + \sigma^*\xi_\sigma(y,\tau,\underline{m},\underline{\sigma}))] + [\tilde{\theta}_f^T(\xi(x,\tau,\underline{m},\underline{\sigma}) - m\xi_m(x,\tau,\underline{m},\underline{\sigma}) - \sigma\xi_\sigma(x,\tau,\underline{m},\underline{\sigma})) + \theta_f^T(\tilde{m}\xi_m(x,\tau,\underline{m},\underline{\sigma}) + \tilde{\sigma}\xi_\sigma(x,\tau,\underline{m},\underline{\sigma})) + \tilde{\theta}_f^T(m^*\xi_m(y,\tau,\underline{m},\underline{\sigma}) + \sigma^*\xi_\sigma(y,\tau,\underline{m},\underline{\sigma}))] + d(t) - \eta_{\rm sw} D^{q-1}(\operatorname{sgn}(S(t)) + \omega_1\}. \tag{39}$$

Hence, the derivative of the sliding surface can be presented as

$$\begin{aligned}\dot{S}(t) &= D^{1-q}(D^q(S(t))) \\ &= k_2\{-[(D^{1-q}\tilde{\theta}_g^T)(\xi(y,\tau,\underline{m},\underline{\sigma}) - \underline{m}\xi_m(y,\tau,\underline{m},\underline{\sigma}) \\ &\quad - \sigma\xi_\sigma(y,\tau,\underline{m},\underline{\sigma})) \\ &\quad + (D^{1-q}\theta_g^T)(\underline{\tilde{m}}\xi_m(y,\tau,\underline{m},\underline{\sigma}) + \tilde{\sigma}\xi_\sigma(x,\tau,\underline{m},\underline{\sigma}))] \\ &\quad + [(D^{1-q}\theta_f^T)(\tilde{m}\xi_m(x,\tau,\underline{m},\underline{\sigma}) + \tilde{\sigma}\xi_\sigma(x,\tau,\underline{m},\underline{\sigma})) \\ &\quad + (D^{1-q}\tilde{\theta}_f^T)(\xi(x,\tau,\underline{m},\underline{\sigma}) - m\xi_m(x,\tau,\underline{m},\underline{\sigma}) \\ &\quad - \sigma\xi_\sigma(x,\tau,\underline{m},\underline{\sigma}))]\} \\ &\quad - k_2\eta_{\rm sw}(\operatorname{sgn}(S(t)) + k_2 D^{1-q} d(t) + k_2 D^{1-q}\omega_{\rm total}\end{aligned} \tag{40}$$

where the total minimum approximation errors $\omega_{\rm total}$ is given as

$$\omega_{\rm total} = \omega_1 + \tilde{\theta}_g^T(\underline{m}^*\xi_m(y,\tau,\underline{m},\underline{\sigma}) + \sigma^*\xi_\sigma(y,\tau,\underline{m},\underline{\sigma})) + \tilde{\theta}_f^T(m^*\xi_m(y,\tau,\underline{m},\underline{\sigma}) + \sigma^*\xi_\sigma(y,\tau,\underline{m},\underline{\sigma})).$$

Now, consider the Lyapunov-function candidate

$$V = \frac{1}{2}S^2(t) + \frac{k_2}{2r_1}(D^{-q}\tilde{\underline{\theta}}_f^T)(D^{-q}\tilde{\underline{\theta}}_f) + \frac{k_2}{2r_2}(D^{-q}\tilde{\underline{\theta}}_g^T)(D^{-q}\tilde{\underline{\theta}}_g) + \frac{k_2}{2r_3}\operatorname{tr}(\tilde{m}^T\tilde{m}) + \frac{k_2}{2r_4}\operatorname{tr}(\tilde{\sigma}^T\tilde{\sigma}) \tag{41}$$

where r_1, r_2, r_3, and r_4 are positive constants. Taking the derivative of (41) with respect to time, we get

$$\begin{aligned}\dot{V} &= S(t)\dot{S}(t) + \frac{k_2}{r_1}(D^{1-q}\tilde{\underline{\theta}}_f^T)(D^{-q}\tilde{\underline{\theta}}_f) + \frac{k_2}{r_2}(D^{1-q}\tilde{\underline{\theta}}_g^T)(D^{-q}\tilde{\underline{\theta}}_g) \\ &\quad + \frac{k_2}{2r_3}\operatorname{tr}(\underline{\tilde{m}}^T\dot{\underline{\tilde{m}}}) + \frac{k_2}{2r_4}\operatorname{tr}(\underline{\tilde{\sigma}}^T\dot{\underline{\tilde{\sigma}}}) \\ &= k_2 S(t)\{-[(D^{1-q}\tilde{\theta}_g^T)(\xi(y,\tau,\underline{m},\underline{\sigma}) - m\xi_m(y,\tau,\underline{m},\underline{\sigma}) \\ &\quad - \sigma\xi_\sigma(y,\tau,\underline{m},\underline{\sigma})) \\ &\quad + (D^{1-q}\theta_g^T)(\underline{\tilde{m}}\xi_m(y,\tau,\underline{m},\underline{\sigma}) + \tilde{\sigma}\xi_\sigma(y,\tau,\underline{m},\underline{\sigma}))] \\ &\quad + [(D^{1-q}\theta_f^T)(\underline{\tilde{m}}\xi_m(x,\tau,\underline{m},\underline{\sigma}) + \underline{\tilde{\sigma}}\xi_\sigma(x,\tau,\underline{m},\underline{\sigma})) \\ &\quad + (D^{1-q}\tilde{\underline{\theta}}_f^T)(\xi(x,\tau,\underline{m},\underline{\sigma}) - \underline{m}\xi_m(x,\tau,\underline{m},\underline{\sigma}) \\ &\quad - \sigma\xi_\sigma(x,\tau,\underline{m},\underline{\sigma}))]\} \\ &\quad - k_2\eta_{\rm sw}|S(t)| + k_2 S(t) D^{1-q} d(t) + k_2 S(t) D^{1-q}\omega_{\rm total} \\ &\quad + \frac{k_2}{r_1}(D^{1-q}\tilde{\underline{\theta}}_f^T)(D^{-q}\tilde{\underline{\theta}}_f) \\ &\quad + \frac{k_2}{r_2}(D^{1-q}\tilde{\underline{\theta}}_g^T)(D^{-q}\tilde{\underline{\theta}}_g) + \frac{k_2}{r_3}\operatorname{tr}(\underline{\tilde{m}}^T\dot{\underline{\tilde{m}}}) + \frac{k_2}{r_4}\operatorname{tr}(\underline{\tilde{\sigma}}^T\dot{\underline{\tilde{\sigma}}})\end{aligned} \tag{42}$$

$$\begin{aligned}&= k_2\left\{(D^{1-q}\tilde{\underline{\theta}}_g^T)\left[\frac{1}{r_2}(D^{-q}\tilde{\underline{\theta}}_g) - (\xi(y,\tau,\underline{m},\underline{\sigma}) - m\xi_m(y,\tau,\underline{m},\underline{\sigma}) - \sigma\xi_\sigma(y,\tau,\underline{m},\underline{\sigma}))S(t)\right]\right. \\ &\quad + (D^{1-q}\tilde{\underline{\theta}}_f^T)\left[\frac{1}{r_1}(D^{-q}\tilde{\underline{\theta}}_f) + (\xi(x,\tau,\underline{m},\underline{\sigma}) - m\xi_m(x,\tau,\underline{m},\underline{\sigma}) - \sigma\xi_\sigma(x,\tau,\underline{m},\underline{\sigma}))S(t)\right] \\ &\quad + \left[\frac{1}{r_3}\operatorname{tr}(\underline{\tilde{m}}^T\dot{\underline{\tilde{m}}}) + (-(D^{1-q}\theta_g^T)\underline{\tilde{m}}\xi_m(y,\tau,\underline{m},\underline{\sigma}) + (D^{1-q}\theta_f^T)\underline{\tilde{m}}\xi_m(x,\tau,\underline{m},\underline{\sigma}))S(t)\right] \\ &\quad \left. + \left[\frac{1}{r_4}\operatorname{tr}(\underline{\tilde{\sigma}}^T\dot{\underline{\tilde{\sigma}}}) + (-(D^{1-q}\underline{\theta}_g^T)\tilde{\sigma}\xi_\sigma(y,\tau,\underline{m},\underline{\sigma}) + (D^{1-q}\underline{\theta}_f^T)\tilde{\sigma}\xi_\sigma(x,\tau,\underline{m},\underline{\sigma}))S(t)\right]\right\} \\ &\quad - k_2\eta_{\rm sw}|S(t)| + k_2 S(t) D^{1-q} d(t) + k_2 S(t) D^{1-q}\omega_{\rm total}.\end{aligned} \tag{43}$$

Substituting adaptive laws (27)–(30) into (43), we have

$$\begin{aligned}\dot{V}(t) &= k_2\eta_{\rm sw}|S(t)| + k_2 S(t) D^{1-q} d(t) + k_2 S(t) D^{1-q}\omega_{\rm total} \\ &\le -k_2\{\eta_{\rm sw}|S(t)| - |S(t)|\,\|D^{1-q}d(t)\| - |S(t)|\,|D^{1-q}\omega_{\rm total}|\} \\ &= -k_2|S(t)|\{\eta_{\rm sw} - |D^{1-q}d(t)| - |D^{1-q}\omega_{\rm total}|\} \\ &\le 0.\end{aligned} \tag{44}$$

Based on condition (24), the existence of adaptive fuzzy-sliding-mode dynamics is confirmed by (44), and the closed-loop system is globally asymptotically stable. Thus, the proof is completed.

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Tsung-Chih Lin (M’11–SM’11) was born in Changhua, Taiwan, in 1961. He received the B.S. degree in electronic engineering from the Feng-Chia University, Taichung, Taiwan, in 1984, the M.S. degree in control engineering from the National Chiao Tung University, Hsinchu, Taiwan, in 1986, and the Ph.D. degree from the School of Microelectronic Engineering, Griffith University, Brisbane, Qld., Australia, in 2002.

He is currently an Associate Professor with the Department of Electronic Engineering, Feng Chia University. His current research interests include adaptive control, fuzzy-neural networks, robust control, and analog circuit testing. He is member of the Editorial Boards for several national and international journals. He is currently a Reviewer for many international journals and conferences. He participated in many international conferences as a Session Chair and a member of the International Program Committee.

Tun-Yuan Lee was born in Taichung, Taiwan, in 1988. He received the B.S. degree in electronic engineering from the Feng-Chia University, Taichung, Taiwan, in 2010. He is currently working toward the M.S. degree in electronic engineering with the Electronic Engineering University, Taichung.

His current research interests include adaptive control and fuzzy-neural networks.