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# Random vibration analysis of multi-floor buildings using a distributed parameter model

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ABSTRACT

The objective of the current paper is to provide an accurate distributed parameter model for random vibration analysis of multi-floor buildings. The Hamilton's principle is employed to derive the equations governing the dynamic behavior of the system as well as the related kinematic and natural boundary conditions. The natural frequency and mode shapes of the developed model are then extracted analytically and validated using finite element simulations. It is also observed that the predictions of the proposed model for the natural frequencies of the system is far more accurate than those of that of the discrete model available in the literature. Using a single mode approximation in the Lagrange equation, the partial differential equations of the motion are reduced to a single ordinary differential equation. Assuming a band limited white noise for the acceleration of the support, the random response specifications (such as expected value, autocorrelation, spectral density and mean square) of the system is calculated by making use of the random vibration theory. The qualitative and quantitative nature of the response characteristic are also analyzed to reveal the effects of different design parameters on the system's response. The suggested modeling approach in this paper may be employed for prediction of the dynamic behavior of more complex structures to different types of deterministic or random excitations. Also the provided analytical method for the random response calculation of the system can be utilized to make more informed decisions in the design process.

## 1. Introduction

Protection of engineering structures against un-wanted vibrations have always been a challenge for civil and mechanical engineers. In building constructions, an earthquake can bring about severe unwelcomed vibrations of the system and produce large stresses which ultimately can lead to catastrophic collapse of the structure. So researchers have been trying to innovate new techniques to study the vibrational phenomena in such structures and also to develop new vibration suppression techniques to minimize their vibration level.

In some studies, the earthquakes have been modeled as harmonic support motion. For example Farshidianfar and Soheili [1] investigated the optimized parameters of tuned mass dampers for high rise structures considering soil structure interaction effects under harmonic base excitations. Park and Reed [2] examined the performance of uniformly and linearly distributed multiple mass dampers in suppressing the vibrations resulted from harmonic and earthquake excitations.

In practice, the nature of earthquake is not deterministic. So in many other studies, the vibrational response of buildings and other mechanical systems has been analyzed based on the hypothesis of random excitations. For example, Kiureghian and Neuenhofer [3] developed a new response spectrum method for seismic analysis of linear multi-degree-of-freedom, multiply supported structures subjected to spatially varying ground motions. Heredia-Zavoni and Vanmarcke [4] employed the random-vibration methodology to study the seismic random response analysis of linear multi support structural systems. While respecting the stationarity assumption, their method, simplified the random analysis by equalizing the response evaluation of the system to that of a series of linear one-degree systems. Wen [5] provided an overview of the major developments in modeling and response analysis of inelastic structures under random excitations. Kiureghian [6] developed a response spectrum method for stationary random vibration analysis of linear, multi-degree-of-freedom systems. His method was based on the assumption that the input excitation is a wide-band, stationary Gaussian excitation and the response. Radeva and Radev [7] proposed a method for simulation of seismic ground motions. They considered the random motion as a stationary filtered white noise with fuzzy parameters and proposed an analytical procedure to analyze the

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fuzzy random vibration of multi-degree-of-freedom hysteretic buildings. Coupling adjacent buildings with supplemental damping devices is a practical and effective approach to mitigating structural seismic response. Hao and Zhang [8] used the random vibration method to study earthquake ground motion spatial variation effects on relative linear elastic response of adjacent building structures. Ni et al. [9] developed a method for analyzing the random seismic response of a structural system consisting of two adjacent buildings interconnected by hysteretic damping devices. Effective elimination of the vibration of building structures due to earthquake and wind loading via passive, semi active and active structures have also been investigated in the prior art. Ikeda and Ibrahim [10] investigated the passive vibration control of an elastic structure carrying a rectangular tank partially filled with liquid, subjected to horizontal narrow band random ground excitation. Yang et al. [11] studied vibration suppression of structures using a semi-active mass damper under random base excitation. Gur and Mishra [12] presented the optimal stochastic performance of pure friction system supplemented with shape memory alloy assisted pure friction, based on a framework of multi-objective optimization. Ozbulut and Hurlebaus [13] proposed a new device which took advantages of both variable friction dampers as well as shape memory alloys to intelligently suppress the vibration of a building structure. Calise et al. [14] carried out a series of experiments to quantify the potential benefits of using robust control design methodology in active control of building structures.

Many researchers studied random vibration analysis of structures to non-stationary inputs. Alderucci and Muscolino [15] presented a closed-form solution for the evolutionary power spectral density of the response of linear classically damped structural systems subjected to fully non-stationary multi-correlated excitations. They utilized their method to study random vibration of a bridge, and validated their findings with Monte Carlo Simulations. Muscolino and Alderucci [16] presented a method to evaluate the evolutionary frequency response function of classically damped linear structural systems subjected to both separable and non-separable non-stationary excitations. Chakraborty and Basu [17] proposed an input-output relation for the non-stationary response of long-span bridges subjected to random differential support motions. Their developed methodology could evaluate non-stationarity in both the intensity and frequency content of the response statistics for spatially correlated multipoint random excitations.

Random vibration analysis of structures considering the nonlinearity of the structures have also received much attention. So far, many different approaches have been developed for nonlinear random vibration analysis, each of them having their own advantages and disadvantages. Among these technique, one can mention Markov vector approach, perturbation methods, equivalent linearization, stochastic linearization, equivalent nonlinearization, closure approximations, stochastic averaging and Mont Carlo simulation method [18–20]. The most interested approaches used for nonlinear random vibration analysis are based on the linearization of the nonlinear system. For example, Feldman [21] proposed a nonlinear technique based on the Hilbert transform for investigation of nonlinear systems. His suggested approach which involves some kind of linearization of the nonlinear system, enables direct extraction of linear and nonlinear system parameters from a measured time signal of input and output. His proposed strategy was mainly developed for deterministic nonlinear vibration analysis. However, he claimed that it can extract the instantaneous modal parameters of the equivalent linear system even if the excitation is a random signal. Fujimura and Kiureghian [19] developed a new, non-parametric linearization method for nonlinear random vibration analysis. They verified the accuracy of their technique via Mont Carlo simulations. Mishra et al. [22] studied the optimum performance of the shape memory–alloy-based rubber bearing for isolating the bridge deck against a random earthquake. The responses required for modeling the nonlinear random system were obtained by stochastic linearization of the cyclic nonlinear force-deformation behavior of the shape memory–alloy restrainers. Gur et al. [23] proposed the optimal parameters for the super-elastic damper by conducting systematic design optimization, in which, the stochastic response served as the objective function, evaluated through nonlinear random vibration analysis. They assumed the response processes to be Markovian and adopted linearization technique for nonlinear force-deformation hysteresis of the building frame and the dampers.

From the provided brief literature review, one can conclude that vibration modeling of buildings under seismic activity of the ground have been well presented. However, some researchers have ignored the random nature of the earthquake excitation. Those who considered the stochastic character of the earth tremor, utilized a lumped parameter model for the multi-floor building. In practice, however, a building is a multi-body distributed parameter system constituted from some walls (which can be modeled using simple beams) interconnecting to some floors (which can be modeled using some rigid masses). As far as the authors know, such a model have not been yet reported in the literature. Accordingly, the distinct objectives and contributions of this paper are:

1. Proposing a novel multi-body distributed parameter model for a multi-floor building and deriving closed-form expressions for the natural frequencies and mode shapes of the system.
2. Analytical modeling of the random vibration response of the system to a band limited white noise excitation as the acceleration of the supporting ground.

To achieve these, Hamilton's principle is utilized to derive the partial differential equations governing the system's dynamic behavior. The normalized homogenous un-damped form of these equations are then solved for finding the natural frequencies and mode shapes of the system which are verified via finite element simulations. The suggested mode shapes are then utilized in an energy based approach to derive the temporal equations which are then modeled based on the random vibration theory and closed form expressions are derived for the statistical specifications of the response.

## 2. Mathematical modeling

The physical model of sample building with five floors is depicted in Fig. 1. As illustrated in this figure, each floor is considered as a concentrated line mass, while the walls are modeled as two beams, supporting the corresponded floor from the right and left. The structural damping of the system is taken into account via some concentrated dampers, which resist against the relative movements of the floors. The motion of the ground due to earthquake excitation is modeled with $\ddot{\hat{u}}_g$ and the resulting relative displacements in the structure is assumed to be small. It has to be noted that this assumption may not come true if the building experiences relatively large displacements. A distributed parameter model of this linear multi-body continuous system can be obtained using the Hamilton's principle.

Having long and slender geometry, uniform-thickness planar beams may be modeled using the Euler-Bernoulli beam theory. This theory assumes that plane cross-sections continue to remain plane and normal to the neutral axis after deformation [24] and has been successfully utilized to study the static, dynamic, and vibrational behavior of structures constituted from beams [25–27]. Assuming the Euler-Bernoulli assumptions hold for the problem under study, the strain energy stored in beams of Fig. 1 for the case of relatively small displacements can be expressed as [24]

$$\hat{\pi} = \sum_{i=1}^{5} 2\times\frac{1}{2}\int_0^{l_i} E_i I_i \left(\frac{\partial^2 \hat{w}_i(\hat{x}_i, \hat{t})}{\partial \hat{x}_i^2}\right)^2 d\hat{x}_i \tag{1}$$

where $\hat{\pi}$ is the strain energy of the system and $l_i$, $E_i$, $I_i$ and $\hat{w}_i$ are respectively the length, Young's modulus of elasticity, second area

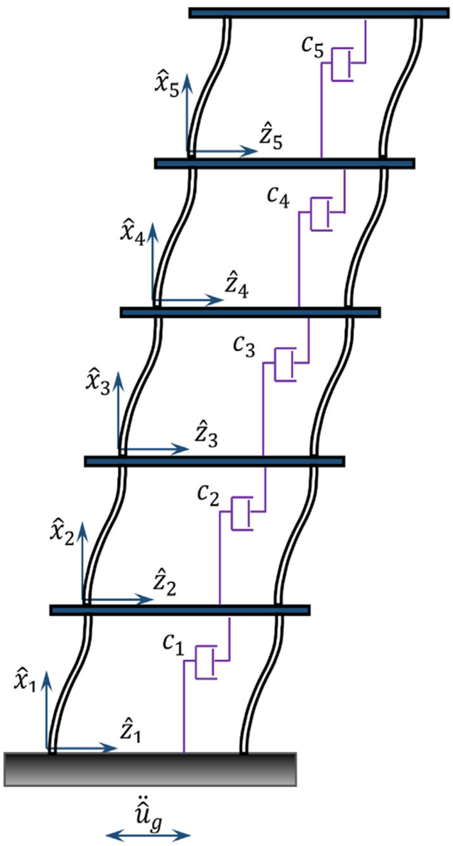


**Fig. 1.** Physical model of a five floor building.

moment of inertia of the beam's cross section around the neutral axis and relative deflection of the beam along the $\hat{z}_i$ direction.

Considering a support motion for the building, its kinetic energy, $\hat{T}$, can be easily expressed as

$$\hat{T} = \sum_{i=1}^{5}\left\{\int_0^{l_i}\rho_i A_i\left(\frac{d\hat{u}_g(\hat{t})}{d\hat{t}} + \frac{\partial\hat{w}_i(\hat{x}_i,\hat{t})}{\partial\hat{t}} + \left(\sum_{k=1}^{5}\frac{d\hat{w}_k(l_k,\hat{t})}{d\hat{t}} - \sum_{k=i}^{5}\frac{d\hat{w}_k(l_k,\hat{t})}{d\hat{t}}\right)\right)^2 d\hat{x}_i + \frac{1}{2}m_i\left(\frac{d\hat{u}_g(\hat{t})}{d\hat{t}} + \sum_{j=1}^{i}\frac{d\hat{w}_j(l_j,\hat{t})}{d\hat{t}}\right)^2\right\} \tag{2}$$

In this equation, $\rho_i$ and $A_i$ are the volumetric density of the beams material and the area cross section of the beams of the $i$'th floor respectively and $m_i$ is the mass of the $i$ th floor. Also $\hat{u}_g(\hat{t})$ is the displacement of the ground.

The virtual work done on the system can be classified into two parts: (a) the virtual work by the dampers, $\delta\hat{W}_{Ext}^{(D)}$, and (b) the virtual work done by the axial loads of the beams, $\delta\hat{W}_{Ext}^{(N)}$ (please refer to [24] regarding this virtual work). These virtual works may be mathematically expressed as

$$\delta\hat{W}_{Ext}^{(D)} = \sum_{i=1}^{5}\{-c_i\dot{\hat{w}}_i(l_i,\hat{t})\times\delta\hat{w}_i(l_i,\hat{t})\} \tag{3}$$

$$\delta\hat{W}_{Ext}^{(N)} = \sum_{i=1}^{5}\left\{-\frac{1}{2}\delta\int_0^{l_i}\sum_{j=i}^{5}m_j g\left(\frac{\partial\hat{w}_i(\hat{x}_i,\hat{t})}{\partial\hat{x}_i}\right)^2 d\hat{x}_i\right\} \tag{4}$$

where $\delta$ is the variation operator and $c_i$ s are the damping coefficients of the dampers shown in Fig. 1.

Obviously, the total virtual work, $\delta\hat{W}_{Ext}$ can be obtained as

$$\delta\hat{W}_{Ext} = \delta\hat{W}_{Ext}^{(D)} + \delta\hat{W}_{Ext}^{(N)} \tag{5}$$

Now the Hamilton's principle can be used to derive the governing equations as well as the corresponding boundary conditions of the multi-body system of Fig. 1. Based on this principle [24]

$$\delta\int_{\hat{t}_1}^{\hat{t}_2}(\hat{T} - \hat{\pi} + \hat{W}_{Ext})d\hat{t} = 0 \tag{6}$$

where $\hat{t}_1$ and $\hat{t}_2$ are any two arbitrary times.

Using the Hamilton's principle, the equations of motion of the system and the corresponded boundary conditions can be derived. To present these equations and boundary conditions more conveniently, the following normalized variables are introduced.

$$x_i = \frac{\hat{x}_i}{l_i}, \quad i = 1, 2, \ldots, 5 \tag{7}$$

$$w_i = \frac{\hat{w}_i}{l_i}, \quad i = 1, 2, \ldots, 5 \tag{8}$$

$$u_g = \frac{\hat{u}_g}{d} \tag{9}$$

$$\bar{t} = \frac{\hat{t}}{\mathbb{T}} \tag{10}$$

$$N_i = \frac{g l_i^2}{2E_i I_i}\sum_{j=i}^{5}m_j, \quad i = 1, 2, \ldots, 5 \tag{11}$$

$$\lambda_i = \frac{\rho_i}{\rho_1}\frac{A_i}{A_1}\frac{E_1}{E_i}\frac{I_1}{I_i}\left(\frac{l_i}{l_1}\right)^4, \quad i = 1, 2, \ldots, 5 \tag{12}$$

$$\mu_i = \frac{E_{i+1}}{E_i}\frac{I_{i+1}}{I_i}\left(\frac{l_i}{l_{i+1}}\right)^2, \quad i = 1, \ldots, 4 \tag{13}$$

$$\beta_i = \frac{m_i}{2\rho_1 A_1 l_1}\frac{E_1}{E_i}\frac{I_1}{I_i}\left(\frac{l_i}{l_1}\right)^3, \quad i = 1, 2, \ldots, 5 \tag{14}$$

$$r_i = \frac{c_i l_i}{2\sqrt{\rho_1 A_1}}\frac{\sqrt{E_1 I_1}}{E_i I_i}\left(\frac{l_i}{l_1}\right)^2, \quad i = 1, 2, \ldots, 5 \tag{15}$$

In these equations, $d$ is the horizontal distance between the vertical beams supporting the floors and $\mathbb{T}$ is a time scale defined by

$$\mathbb{T} = l_1^2\sqrt{\frac{\rho_1 A_1}{E_1 I_1}} \tag{16}$$

Using these normalized variables, the equations of motion of the system can be expressed in dimensionless form as

$$\frac{\partial^4 w_i(x_i,\bar{t})}{\partial x_i^4} + \lambda_i\left(\frac{d}{l_i}\ddot{u}_g(\bar{t}) + \ddot{w}_i(x_i,\bar{t}) + \sum_{k=1}^{5}\frac{l_k}{l_i}\ddot{w}_k(1,\bar{t}) - \sum_{k=i}^{5}\frac{l_k}{l_i}\ddot{w}_k(1,\bar{t})\right) - N_i\frac{\partial^2 w_i(x_i,\bar{t})}{\partial x_i^2} = 0, \quad i = 1, 2, \ldots, 5 \tag{17}$$

The $\dot{()}$ in Eq. (17) represents differentiation with respect to the normalized time $\bar{t}$.

Also, the normalizd boundary conditions will be as:

Normalized kinematic boundary conditions

$$w_i(0,\bar{t}) = 0, \quad i = 1, 2, \ldots, 5 \tag{18}$$

$$\frac{dw_i(0,\bar{t})}{dx_i} = 0, \quad i = 1, 2, \ldots, 5 \tag{19}$$

$$\frac{dw_i(l_i,\bar{t})}{dx_i} = 0, \quad i = 1, 2, \ldots, 5 \tag{20}$$

Normalized natural boundary conditions

$$\frac{d^3w_i(1,\bar{t})}{dx_i^3} - \mu_i\frac{d^3w_{i+1}(0,\bar{t})}{dx_{i+1}^3} - \beta_i(\frac{d}{l_i}\ddot{u}_g(\bar{t}) + \sum_{k=1}^{i}\frac{l_k}{l_i}\ddot{w}_k(1,\bar{t})) - r_i(\dot{w}_i(1,\bar{t}) - \frac{c_{i+1}l_{i+1}}{c_i l_i}\dot{w}_{i+1}(1,\bar{t})) = 0, \quad i = 1, 2, 3, 4 \tag{21}$$

$$\frac{d^3w_i(1,\bar{t})}{dx_i^3} - \beta_i\left(\frac{d}{l_i}\ddot{u}_g(\bar{t}) + \sum_{k=1}^{5}\frac{l_k}{l_5}\ddot{w}_k(1,\bar{t})\right) - r_i\dot{w}_i(1,\bar{t}) = 0, \qquad i = 5 \tag{22}$$

Each of these equations of motion and the corresponding boundary conditions has a physical/geometrical interpretation. Eq. (17) represents the equations of motion of the $i$'th beam in which the effects of the axial force and the support motion are taken into account. Eqs. (18)–(20) respectively reveals the fact that the beams in Fig. 1 have zero relative deflections at $x_i = 0$ and has zero slope at their beginning and ending points. Finally, the natural boundary conditions, Eqs. (21) and (22), reflect the equations of motion of the floors in horizontal direction.

## 3. Eigen value analysis

The very first step in dynamic analysis of vibratory systems is solving the related eigen value-eigen function problem and derivation of the natural frequencies and mode shapes. The dynamic response of the system then can be approximated as a linear combination of these modes.

A mode shape of a vibratory system is defined as a state of movement of the free, un-damped vibration of that system in which all of its elements are vibrating with the same frequency and phase, but not necessarily the same amplitude. So if the system under study is vibrating in its $j$'th mode, then one can say

$$w_i^j(x_i,\bar{t}) = \varphi_i^j(x_i)\exp(I\omega_j\bar{t}), \qquad i = 1, 2, \ldots, 5, \quad j \in \mathbb{N} \tag{23}$$

where $I = \sqrt{-1}$, $\mathbb{N}$ is the set of natural numbers and $\varphi_i^j(x_i)$ is the $j$ th deflection shape (i.e. mode shape) of the $i$'th beam when the system is vibrating with its $j$'th normalized natural frequency $\omega_j$.

By substituting Eq. (23) into Eq. (17) and solving the resulting equations, one gets

$$\varphi_i^j(x_i) = c_{1,i}^j\cosh(s_{1,i}^j x_i) + c_{2,i}^j\sinh(s_{1,i}^j x_i) + c_{3,i}^j\cos(s_{2,i}^j x_i) + c_{4,i}^j\sin(s_{2,i}^j x_i) - \left(\sum_{k=1}^{5}\frac{l_k}{l_i}\varphi_k^j(1) - \sum_{k=i}^{5}\frac{l_k}{l_i}\varphi_k^j(1)\right), \qquad i = 1, 2, \ldots, 5, \quad j \in \mathbb{N} \tag{24}$$

where $c_{i,j}^k$ s are constants of integration and

$$s_{1,i}^j = \left(\frac{N_i}{2} + \sqrt{\left(\frac{N_i}{2}\right)^2 + \omega_j^2}\right)^{1/2}, \qquad i = 1, 2, \ldots, 5 \tag{25}$$

$$s_{2,i}^j = (-\frac{N_i}{2} + \sqrt{(\frac{N_i}{2})^2 + \omega_j^2})^{1/2}, \qquad i = 1, 2, \ldots, 5 \tag{26}$$

By employing Eq. (24) into Eqs. (18)–(22), removing the excitation and damping terms and canceling $\exp(I\omega_j\bar{t})$ from both sides, a set of linear, homogenous, algebraic equations in terms of $c_{m,n}^j$ are obtained as

$$[A(\omega_j)]_{12\times12}\vec{C_j} = \vec{0}_{12\times1} \tag{27}$$

in which

$$\vec{C_j} = [c_{1,1}^j, \quad c_{2,1}^j, \quad c_{3,1}^j, c_{4,1}^j, c_{1,2}^j \quad \ldots\ldots c_{4,5}^j]^T \tag{28}$$

The superscript $T$ in Eq. (28) denotes the transpose operator. The non-zero elements of the $[A(\omega_j)]$ are presented in Appendix A.

Eq. (27) has non-trivial solution if and only if the determinant of $[A(\omega_j)]$ is zero. By solving $|A(\omega_j)| = 0$, the natural frequencies of the systems are obtained. Then, by substituting $\omega_j$ into Eq. (27), the vector $\vec{C_j}$ and consequently the $j$ th mode shape of the system is derived analytically using Eq. (24).

**Table 1**
Physical properties of the floors [28].

| $i$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $m_i(\text{kg}) \times 10^{-3}$ | 6.417 | 5.514 | 5.514 | 5.514 | 5.514 |
| $c_i(\text{N. s/m}) \times 10^{-3}$ | 67 | 58 | 57 | 50 | 38 |

**Table 2**
Physical and geometrical properties of the constitutive beams.

| Parameter | $l$ | $E$ | $I$ | $A$ | $\rho$ |
|---|---|---|---|---|---|
| Value | 3 | 200 | $186.1\times10^{-6}$ | $8130\times10^{-6}$ | 7860 |
| Unit | m | Gpa | $\text{m}^4$ | $\text{m}^2$ | $\text{kg/m}^3$ |

In order to verify the accuracy of the proposed analytical technique for finding the mode shapes of the system, a typical five floor building is considered. The physical properties of the floors are as provided in Table 1, while the physical and geometrical properties of the constitutive beams are compiled in Table 2. Also, the profile of the beam's cross sections are assumed to be S380 × 64.

The building has been analyzed using the proposed analytical technique as well as the discrete analysis approach already presented in the literature (please see chapter six of [29] as an example). To verify the accuracy of the proposed solutions, finite element (FE) simulations were also carried out in the commercial FE software ABAQUS. In FE analysis, the fixity condition of the beams and the floors were chosen to be tie. A tie constraint ties separate surfaces together, so that there is no relative motion between them. This type of constraint allows to fuse together two regions even though the meshes created on the surfaces of the regions may be dissimilar. 2D modeling space for both beams and floors was considered, while wire was picked for their base feature. To account for the rigidity of the floors, they were modeled as rigid wire option available in the ABAQUS. Moreover, via trial and error approach, the number of elements for each beam/floor was taken to be 3000 to guaranty a converged results.

In Table 3, the first three natural frequencies of the system are presented. It is observed that the analytical results of the proposed continuous model well agree with the findings of FE simulations, while the results of the discrete model (commonly used in the literature for analysis of buildings) are not sufficiently accurate.

The first three mode shapes of the building with characteristics given in Tables 1 and 2 are depicted in Fig. 2. In order to make these modes comparable, they have been all normalized so that the normalized absolute displacement of the highest floor to be unity. It is evident that the modes predicted by the continuous model as well as those obtained based on a discrete model, both agree well with the those of the FE simulations. However, the discrete model has no idea about the deflection shapes of the beams and can only be used to predict the movement of the floors.

As Fig. 2 suggests, there are small discrepancies between the FE findings and those of the continuous model. This inconsistency can be improved by refining the meshes used for the simulations which albeit severely increases the computational cost. Moreover, it worth noting that the linear Euler-Bernoulli beam assumptions employed in the

**Table 3**
Natural frequencies of the building (in terms of Hz), comparison of the results of the distributed and lumped parameter models with those of FEA.

| | Finite Element Method | Continuous Model | Error % | Discrete Model | Error % |
|---|---|---|---|---|---|
| First mode | 3.1609 | 3.4107 | 7.9028 | 2.4754 | 21.6869 |
| Second mode | 9.2416 | 9.8267 | 6.3312 | 7.1165 | 22.9949 |
| Third mode | 14.608 | 15.3697 | 5.2143 | 11.0938 | 24.0567 |

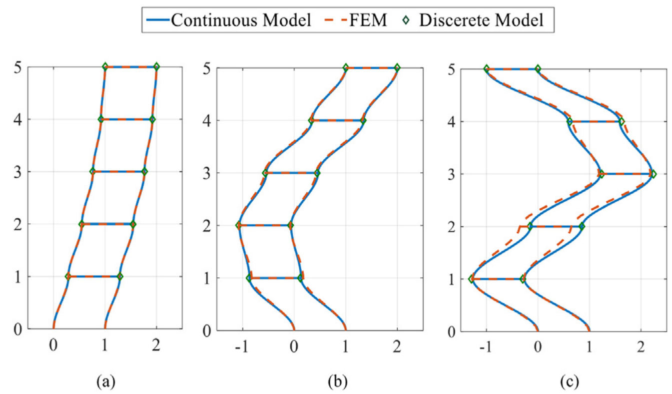


**Fig. 2.** First three mode shapes of the building under study, comparison of the results of the lumped and distributed parameter models with those of FEA (a) first mode (b) second mode (c) third mode.

analytical modeling, does not take into account the axial displacements of the mid-plane of the beams. In the FEA, however, the nodes are allowed to displace in both axial and transverse directions. So the FE approach will be able to model the elasto-kinematic and load stiffening effects which in presence of axial loads (resulted from weight of the floors), become increasingly important [27]. This can ultimately leads to appreciable difference between the FE and exact results.

## 4. Random vibration modeling

A common well respected approach in modeling linear and nonlinear vibrations of continuous systems is to assume that the vibrational response is composed of the linear combinations of the mode shapes of the system with some time dependent coefficients [24–26,30–35]. It has been shown in the prior arts that if the natural frequencies of the system are well-separated, a considerable percentage of the energy of the system is injected to the first vibrational mode [25,26]. In such a case, the first mode shape becomes the dominant mode and can be used to precisely approximate the dynamic response of the system. Using a single mode approximation for the system under study, one can say $\hat{w}_i(\hat{x}_i,\hat{t}) = \varphi_i^{(1)}(\hat{x}_i/l_i)\hat{q}(\hat{t})$. By defining a non-dimensional $q$ as $q(\bar{t}) = \hat{q}(\hat{t})/l_1$, the normalized deflection of the beams can be stated as

$$w_i(x_i,\bar{t}) = \frac{l_1}{l_i}\varphi_i(x_i)q(\bar{t}), \quad i = 1, 2, ..., 5 \tag{29}$$

where $\varphi_i(x_i)$ is the deflection shape of the $i$'th beam when the system is vibrating in its first mode. Substituting Eq. (29) into Eq. (1) while considering the normalization scheme for $w(x,\bar{t})$ (Eq. (8)), the potential energy of the system is obtained as

$$\hat{\pi} = \frac{1}{2}Kq(\bar{t})^2 \tag{30}$$

in which

$$K = 2\sum_{i=1}^{5}\frac{E_iI_il_1^2}{l_i^3}\int_0^1\left(\frac{d^2\varphi_i(x_i)}{dx_i^2}\right)^2 dx_i \tag{31}$$

Also by employing Eq. (29) into Eq. (2) and simplifying the outcome, the kinetic energy of the system is re-derived as

$$\hat{T} = \frac{1}{2}M_1\dot{q}^2 + M_2\dot{q}\dot{u}_g + \frac{1}{2}M_3\dot{u}_g^2 \tag{32}$$

where the parameters $M_i$, $i = 1,2,3$ are defined as

$$M_1 = \sum_{i=1}^{5}\left\{\int_0^1 \frac{2\rho_iA_il_i^3l_1^2}{\mathbb{T}^2}\left(\frac{\varphi_i(x_i)}{l_i} + \left(\sum_{k=1}^{5}\frac{\varphi_k(1)}{l_k} - \sum_{k=i}^{5}\frac{\varphi_k(1)}{l_k}\right)\right)^2 dx_i + \frac{m_il_i^2l_1^2}{\mathbb{T}^2}\sum_{j=1}^{i}\frac{\varphi_j(1)^2}{l_j^2}\right\} \tag{33}$$

$$M_2 = \sum_{i=1}^{5}\left\{\int_0^1 \frac{2\rho_iA_il_i^2dl_1}{\mathbb{T}^2}\left(\frac{\varphi_i(x_i)}{l_i} + \left(\sum_{k=1}^{5}\frac{\varphi_k(1)}{l_k} - \sum_{k=i}^{5}\frac{\varphi_k(1)}{l_k}\right)\right)dx_i + \frac{m_il_idl_1}{\mathbb{T}^2}\sum_{j=1}^{i}\frac{\varphi_j(1)}{l_j}\right\} \tag{34}$$

$$M_3 = \frac{d^2}{\mathbb{T}^2}\sum_{i=1}^{5}(2\rho_iA_il_i + m_i) \tag{35}$$

Finally, the virtual work expression, Eq. (5), can also be expressed in terms of the first mode shape as

$$\delta\widehat{W}_{Ext} = (-Q_1\dot{q}(\bar{t}) - Q_2q(\bar{t}))\delta q(\bar{t}) = \frac{1}{l_1}(-Q_1\dot{q}(\bar{t}) - Q_2q(\bar{t}))\delta\hat{q}(\bar{t}) \tag{36}$$

in which $Q_i$ s are defined as

$$Q_1 = \frac{1}{\mathbb{T}}\sum_{i=1}^{5}l_i^2c_i\varphi_i(1)^2 \tag{37}$$

$$Q_2 = \sum_{i=1}^{5}\left\{\int_0^1\sum_{j=i}^{5}m_jgl_i\left(\frac{d\varphi_i(x_i)}{dx_i}\right)^2dx_i\right\} \tag{38}$$

Now by considering Eqs. (30), (32) and (36) and using the Lagrange equation, the differential equation governing the $q(\bar{t})$ is obtained as

$$M_1\ddot{q} + Q_1\dot{q} + (K + Q_2)q = -M_2\ddot{u}_g \tag{39}$$

To further simplify the notations, the normalized time $\bar{t}$ is re-normalized using $t = \omega_n\bar{t}$ where $\omega_n$ is defined as

$$\omega_n = \sqrt{\frac{K + Q_2}{M_1}} \tag{40}$$

Using the new time scale $t$, Eq. (39) may be re-expressed as

$$q'' + 2\zeta\omega_nq' + q = -\frac{M_2}{M_1}u_g'' \tag{41}$$

where the prime denotes differentiation with respect to $t$, and $\zeta$ is defined as

$$\zeta = \frac{Q_1}{2M_1\omega_n^2} \tag{42}$$

Considering $u_g''$ as the input of the dynamic system and $q(t)$ as its output, the complex frequency response of the system, $H(\omega)$ is readily derived as equation. The magnitude and the phase of $H(\omega)$ for a building with characteristics given in Tables 1 and 2 are shown in Fig. 3.

$$H(\omega) = \frac{M_2}{M_1}\left(\frac{1}{\omega^2 - 2i\zeta\omega_n\omega - 1}\right) \tag{43}$$

$$\angle H(\omega) = \tan^{-1}\left(\frac{2i\zeta\omega_n\omega}{\omega^2 - 1}\right) \tag{44}$$

Also, assuming small damping (i.e. $\zeta < 1$), the impulse response of the system with the mentioned input and output will be as

$$h(t) = -\frac{M_2}{M_1\omega_n\sqrt{1-\zeta^2}}e^{-\zeta\omega_nt}\sin(\omega_n\sqrt{1-\zeta^2}t)U(t) \tag{45}$$

in which $U(t)$ is the unit step function. The impulse response function of the sample building under study is graphically depicted in Fig. 4.

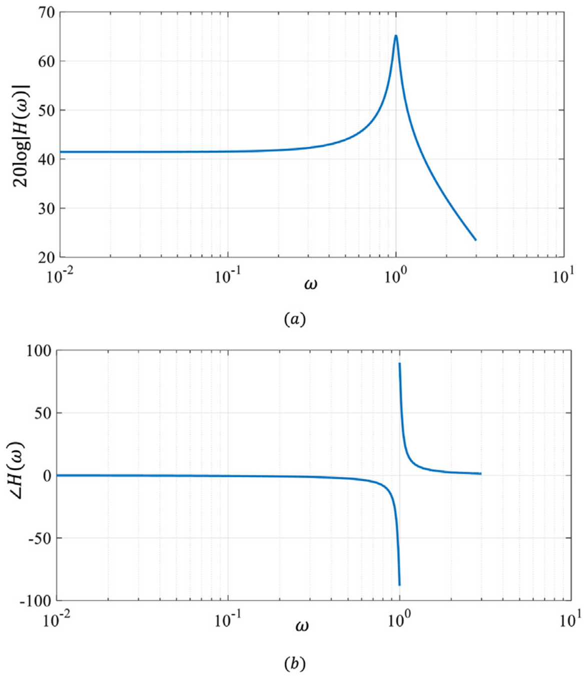


**Fig. 3.** Frequency response of the building under study to the base acceleration input in a logarithmic scale; (a) The magnitude part (in decibel) (b) The phase part (in degree).

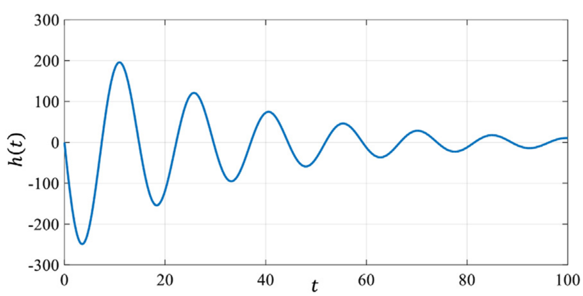


**Fig. 4.** Unit impulse response of the system due to the base acceleration input.

It has to be noted that white noise is a reasonable representation of earthquake motion. Bycroft [36] showed that a white-noise source used in conjunction with an analog computer is a convenient method of analyzing structures subjected to complex ground motions and of assigning probabilities to the deformations arising. He suggested to represent standard large earthquakes with white ground accelerations having a flat spectral density. Many other researchers used the white noise assumption to simulate random vibration of civil structures. For example, Abdel-Rahman and Ahmadi [37] studied the stability analysis of multi-degree-of-freedom elastic frames subjected to a white noise earthquake excitation. By comparing the frequency content of stationary random white noise and a measured ground acceleration time history, Meinhardt et al. [38] showed that ordinary earthquake excitation can be approximated with sufficient accuracy by a stationary white noise stochastic process. If the rock bed effects are not negligible, the linear Kanai–Tajimi filter [39–41] or its generalized format proposed by Fan and Ahmadi [42] which also consider the non-stationarity of the ground motion, shall be employed.

As discussed above, in the absence of rock bed effects, white noise is a suitable choice for modeling the ground acceleration in an earthquake. Moreover, the white noise assumption greatly simplifies the formulations. So here in this paper, the ground acceleration $u_g''$ is modeled with a band limited white noise whose spectral density is depicted in Fig. 5.

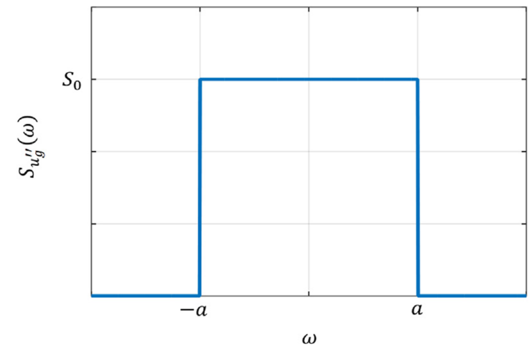


**Fig. 5.** Spectral density of the earthquake excitation modeled as a band limited white noise.

Now, the autocorrelation function for the stochastic excitation $u_g''$ can be simply derived as [43]

$$R_{u''}(\tau) = \int_{-\infty}^{\infty} S_{u''}(\omega) e^{i\omega\tau} d\omega = \frac{2S_0 \sin(a\tau)}{\tau} \tag{46}$$

where $a$ is the parameter characterizing the frequency content of the system as illustrated in Fig. 5.

Knowing the statistical properties of the input excitation, one can use the random vibration theory to derive closed-form expressions for the statistical properties of the response. For example, in order to find the expected value of the movement of the floors, one first needs to find $E[q(t)]$ which can be simply obtained based on the complex frequency response of the system as [43,44]

$$E[q(t)] = E[u_g''(t)]H(0) = -\frac{M_2}{M_1}E[u_g''(t)] \tag{47}$$

So, as the expectation operator is linear, one can conclude that

$$E[w_i(x_i, t)] = -\frac{l_1}{l_i}\frac{M_2}{M_1}\varphi_i(x_i)E[u_g''(t)] \tag{48}$$

The scaled expected values of the relative deflections $w_i(x_i,t)$ at the floors are listed in Table 4. As it is observed, the relative mean deflection of the higher floors are less than those of the lower ones. However, it has to be noted that the mean deflection of each floor with respect to the ground is obtained by summing the relative deflections up to that specific floor.

Utilizing the random vibration theory for a system governed by Eq. (41), the autocorrelation function for $q(t)$ can be derived as

$$\begin{aligned} R_q(\tau) &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} h(\theta_1)h(\theta_2)R_{u_g''}(\tau+\theta_1-\theta_2)d\theta_1 d\theta_2 \\ &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} h(\theta_1)h(\theta_2)\frac{\sin a(\tau+\theta_1-\theta_2)}{\tau+\theta_1-\theta_2}d\theta_1 d\theta_2 \end{aligned} \tag{49}$$

in which $h(\theta)$ is the impulse response of the system given in Eq. (45). Fig. 6 depicts the autocorrelation $R_q(\tau) = E[q(t)q(t+\tau)]$. This figure shows that $R_q(\tau)$ is practically dampened out at large values of $\tau$.

Knowing $R_q(\tau)$ along with employing Eq. (29), $R_{w_i}(x, \tau)$ can also be

**Table 4**
Expected scaled relative deflection ($E[w_i(1, t)]/E[u_g''(t)]$) of different floors.

| First floor | Second floor | Third floor | Fourth floor | Fifth floor |
|---|---|---|---|---|
| 1.0331 | 0.938 | 0.7637 | 0.5665 | 0.2691 |

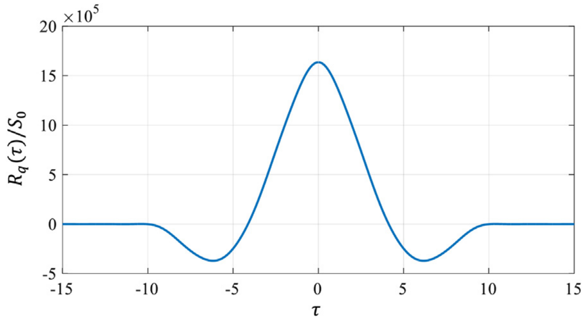


**Fig. 6.** Scaled values of the autocorrelation function of $q(t)$, $R_q(\tau)$.

derived as

$$R_{w_i}(x,\tau) = E[w_i(x,t)w_i(x,t+\tau)] = \frac{l_1^2}{l_i^2}\varphi_i^2(x)R_q(\tau) \quad , \quad i = 1, 2, \ldots, 5 \tag{50}$$

The spectral density $s_q(\omega)$ can be simply expressed in terms of $s_{u''}(\omega)$ as

$$S_q(\omega) = |H(\omega)|^2 S_{u_g''}(\omega) = \begin{cases} 0 & |\omega| > a \\ \frac{S_0 M_2^2}{M_1^2[(\omega^2-1)^2+4\zeta^2\omega_n^2\omega^2]} & |\omega| \le a \end{cases} \tag{51}$$

Then considering Eq. (29), the spectral densities $S_{w_i}(x,\omega)$ become

$$S_{w_i}(x,\omega) = \begin{cases} 0 & |\omega| > a \\ S_0\frac{l_1^2}{l_i^2}\frac{M_2^2}{M_1^2}\frac{\varphi_i(x)^2}{[(\omega^2-1)^2+4\zeta^2\omega_n^2\omega^2]} & |\omega| \le a \end{cases} \quad , \quad i = 1, 2, \ldots, 5 \tag{52}$$

The spectral density $S_q(\omega)$ of the building with characteristics given in Tables 1 and 2 is graphically displayed in Fig. 7 in a logarithmic scale. Considering the similarities between Eqs. (51) and (52), the spectral densities of the floors movements will be similar to that of the $q(t)$.

The spectral density function $S_{w_i}(x,\omega)$ is important in several ways. First it implicitly indicates the frequency content of the $w_i(x,t)$ along with the contribution of each frequency in this random response. Most importantly, the area under the $S_{w_i}(x,\omega)$ curve represents the mean square value of the response, which is an important statistical parameter in design of the buildings against the random loads.

$$E[w_i^2(1,t)] = \int_{-\infty}^{+\infty} S_{w_i}(1,\omega)d\omega, \quad i = 1, 2, \ldots, 5 \tag{53}$$

Table 5 lists these mean square deflections for different floors at different extents of the frequency bands $a$, and different damping ratios $\zeta$. Based on the information provided in this figure, one can conclude that increasing the damping ratio leads to a decrement in the mean square response, while increasing the extent of the frequency limit of

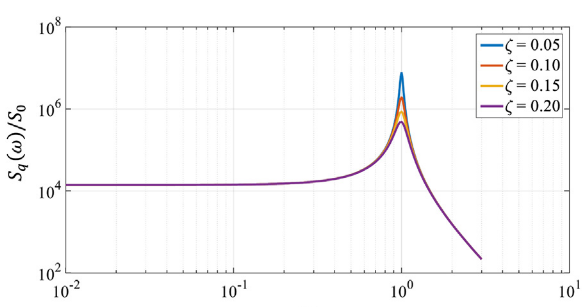


**Fig. 7.** Spectral density function $S_q(\omega)$ at different damping values in a logarithmic scale.

**Table 5**
Scaled mean square deflection $(E[w_i^2(1,t)]/(\sqrt{S_0}\varphi_i(1) \times l_1/l_i)^2 \times 10^{-6})$ at different values of $\zeta$ and $a$.

| | | $\zeta$ | | | |
|---|---|---|---|---|---|
| | | **0.05** | **0.10** | **0.15** | **0.20** |
| ***a*** | **2** | 1.0382 | 0.5183 | 0.3450 | 0.2583 |
| | **3** | 1.0394 | 0.5195 | 0.3462 | 0.2596 |
| | **4** | 1.0397 | 0.5198 | 0.3465 | 0.2598 |
| | **5** | 1.0398 | 0.5198 | 0.3465 | 0.2599 |

the input acceleration yields higher values for the mean square response.

## 5. Conclusion

Investigation of the dynamic response of the buildings under earthquake loads is an important step in the design of these constructions. Such a study can be complicated in different ways. First, the nature of the earthquake excitations is random which intricates the simulations. Also, buildings are multi-body systems consisting of rigid masses (representing the floors) interconnected to some flexible beams (representing the walls). As far as the authors knew, the researchers had not yet provided an accurate multi-body distributed parameter model for the vibratory behavior of such structures, but they usually considered a simple lumped parameter model for the system. So, the objective of the current paper was to provide a more comprehensive and accurate model for dynamic analysis of buildings under the effect of random loads. The equations of motion and the corresponding boundary conditions were derived based on Hamilton's principle. The exact eigen value problem were solved and analytical expressions were derived for mode shapes of the system which were validated via FEA. Then utilizing a single mode assumption, the response of the building to stochastic motion of the support were also simulated based on the random vibration theory and closed-form expressions were provided for different statistical parameters of the response in terms of those of the excitation. Specifically, the mean, autocorrelation, spectral density and the mean square of the relative deflection of the floors were obtained and the effect of different design parameters on the random response of the structure were discussed.

The current study provided a new frame work for improving the dynamic models of complex structures in response to random earthquake excitations. This new frame work which takes the distribution of mass and stiffness of the system into account can still be improved in several ways. For example, in this paper, a linear elasticity theory was exploited for the analysis. However, when a structure experience earthquake, the system may go to the nonlinear regime and a nonlinear distributed parameter structural modeling followed by a subsequent nonlinear random vibration analysis seems to be necessary. Moreover, considering the rock-bed effects and possible non-stationarity of the inputs may ameliorate this research. Despite all these, the developed multi-body continuous model and the qualitative and quantitative analysis provided for the stochastic simulation of the building in this paper can be employed to better address the design requirements of these constructions in a more accurate way.

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## Appendix A

$$A_{4i-3,4i-3} = 1 \quad i = 1, 2, ..., 5 \tag{A1}$$

$$A_{4i-3,4i-1} = 1 \quad i = 1, 2, ..., 5 \tag{A2}$$

$$A_{4i-2,4i-2} = s_{1,i} i = 1, 2, ..., 5 \tag{A3}$$

$$A_{4i-2,4i} = s_{2,i} i = 1, 2, ..., 5 \tag{A4}$$

$$A_{4i-1,4i-3} = s_{1,i}\sinh(s_{1,i}) i = 1, 2, ..., 5 \tag{A5}$$

$$A_{4i-1,4i-2} = s_{1,i}\cosh(s_{1,i}) i = 1, 2, ..., 5 \tag{A6}$$

$$A_{4i-1,4i-1} = -s_{2,i}\sin(s_{2,i}) i = 1, 2, ..., 5 \tag{A7}$$

$$A_{4i-1,4i} = s_{2,i}\cos(s_{2,i}) i = 1, 2, ..., 5 \tag{A8}$$

$$A_{4i-1,4i-3} = s_{1,i}^3\sinh(s_{1,i}) + \beta_i\omega^2\cosh(s_{1,i}) i = 1, 2, ..., 5 \tag{A9}$$

$$A_{4i-1,4i-2} = s_{1,i}^3\cosh(s_{1,i}) + \beta_i\omega^2\sinh(s_{1,i}) i = 1, 2, ..., 5 \tag{A10}$$

$$A_{4i,4i-1} = s_{2,i}^3\sin(s_{2,i}) + \beta_i\omega^2\cos(s_{2,i}) i = 1, 2, ..., 5 \tag{A11}$$

$$A_{4i,4i} = -s_{2,i}^3\cos(s_{2,i}) + \beta_i\omega^2\sin(s_{2,i}) i = 1, 2, ..., 5 \tag{A12}$$

$$A_{4i,4i+2} = -\mu_i s_{1,i+1}^3 i = 1, 2, ..., 4 \tag{A13}$$

$$A_{4i,4i+4} = \mu_i s_{2,i+1}^3 i = 1, 2, ..., 4 \tag{A14}$$

$$A_{4i+1,4i-3} = -l_i\cosh(s_{1,i})/l_{i+1} i = 1, 2, ..., 4 \tag{A15}$$

$$A_{4i+1,4i-2} = -l_i\sinh(s_{1,i})/l_{i+1} i = 1, 2, ..., 4 \tag{A16}$$

$$A_{4i+1,4i-1} = -l_i\cos(s_{2,i})/l_{i+1} i = 1, 2, ..., 4 \tag{A17}$$

$$A_{4i+1,4i} = -l_i\sin(s_{2,i})/l_{i+1} i = 1, 2, ..., 4 \tag{A18}$$

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