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Bipartite synchronization in coupled delayed neural networks under pinning control

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Highlights (for review)

Highlights of the paper

The traditional synchronization problem for coupled delayed neural networks has been investigated under the common assumption that network interactions are described by *unsigned graphs*, where all the edges are positive. To the best of our knowledge, the bipartite synchronization of coupled delayed neural networks with *a signed graph topology*, which contains negative edges, has rarely been considered up to present.

In this paper, we study the bipartite synchronization in a network of delayed neural networks under a signed graph topology, which consists of both positive and negative edges, based on the pinning control approach. The main novelties of this paper can be highlighted as follows.

1) A distributed pinning control algorithm is proposed to achieve bipartite leader-following synchronization in a network of delayed neural networks under signed graph topology. Some remarks are provided to discuss how to effectively select the pinned nodes for the signed network.

2) Under some assumptions on the interaction graph and node dynamics, by developing some tools from M-matrix theory and stability of delayed systems, some novel criteria in terms of low-dimensional linear matrix inequalities (LMIs) are derived to reach bipartite leader-following synchronization in the network when the node delay is differentiable and bounded. Furthermore, a simple algebraic condition is given to estimate an upper bound for the node delay.

3) When the node delay is only bounded but may not be differentiable, some bipartite synchronization conditions are established based on the descriptor method and the reciprocally convex approach. Hence, the results of this paper also improve some existing results for the synchronization of coupled delayed neural networks with unsigned graphs where the node-delay is usually assumed to be differentiable or a constant.

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Bipartite synchronization in coupled delayed neural networks under pinning control

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Abstract

This paper considers the bipartite leader-following synchronization in a signed network composed by an array of coupled delayed neural networks by utilizing the pinning control strategy and M-matrix theory, where the communication links between neighboring nodes of the network can be either positive or negative. Under the assumption that the node-delay is bounded and differentiable, a sufficient condition in terms of a low-dimensional linear matrix inequality is derived for reaching bipartite leader-following synchronization in the signed network, based on which a simple algebraic formula is further given to estimate an upper bound of the node-delay. When the node-delay is bounded and non-differentiable, some criteria are established by using the descriptor method and the reciprocally convex approach such that the bipartite leader-following synchronization problem for the signed network can be successfully solved. Finally, numerical simulations are provided to illustrate the effectiveness of theoretical analysis.

Key words: Coupled delayed neural networks; bipartite synchronization; signed graph; pinning control; M-matrix.

1 Introduction

In the past few decades, the delayed neural networks (DNNs) have been successfully applied to solve many practical problems such as speech recognition (Waibel (1989)), image processing (Wöhler & Anlauf (1999)), optimization

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(Liu, Cao, & Xia (2005)), cryptography (Yu & Cao (2006)), and secure communication (Lakshmanan *et al.* (2018)). Since Pecora and Carroll laid the foundation for chaotic synchronization in the 1990s (Pecora & Carroll (1990)), the synchronization problem for a class of DNNs, including Cohen-Grossberg, cellular and memristive neural networks, has been one of the most active research topics and has been intensively investigated by lots of researchers via different control approaches (see Huang *et al.* (2014); Yang & Ho (2016); Wan *et al.* (2016); Zhang, Zhao, & Huang (2016); Li *et al.* (2018), and the references therein).

With the rapid development of communication technology and computer science, much effort has been devoted to the synchronization problem of coupled delayed neural networks (CDNNs), which is much more challenging than that of a single delayed neural network. Chen, Zhou, & Liu (2004) studied the synchronization in an array of symmetrically interconnected neural networks with a time delay. Lu, Ho, & Liu (2007) further considered the synchronization in CDNNs whose coupling matrices may not necessarily be symmetric. Yu, Cao, & Lü (2008) investigated the synchronization problem for CDNNs with a discrete communication delay. Cao, Chen, & Li (2008) and Zhang & Gao (2017) studied the synchronization in CDNNs whose communication delays include both discrete and distributed delays. Yang, Guo, & Wang (2017) explored the synchronization in CDNNs with impulsive interactions between neighboring nodes. Note that the synchronization in the above literature is reached via the interaction of network nodes without introducing any external forces to the network. More specifically, this kind of synchronization phenomenon can be called leaderless synchronization or self-synchronization.

In practical applications, some CDNNs may not achieve leaderless synchronization due to a number of reasons such as unconnected network structure and weak coupling. Hence, the leader-following synchronization problem for CDNNs has been considered by designing some appropriate controllers such that all the network nodes finally synchronize to the manifold of a prescribed leader. To reduce the control cost for reaching leader-following synchronization in large-scale networks, the pinning control approach is usually adopted by only placing controllers on some critical network nodes which are called pinned nodes (Wang & Chen (2002); Chen, Liu, & Lu (2007); Song & Cao (2010); Song *et al.* (2013); Wen *et al.* (2014)). The synchronization of networked systems under pinning control is called pinning synchronization for brevity. In the past few years, some progress has been made in the pinning synchronization of CDNNs. Lu, Ho, & Wang (2009) stabilized CDNNs subject to stochastic perturbations by pinning the fewest number of network nodes. Li & Cao (2011) considered the pinning-controlled cluster synchronization problem for CDNNs, where the nodes can be partitioned into a set of subgroups. Song, Cao, & Liu (2012) discussed the pinning synchronization in CDNNs based on the in and out degrees of network nodes. He, Qian, & Cao (2017) addressed the pinning synchronization of CDNNs by designing some distributed impulsive controllers.

In the coordination control of traditional networks, the interaction digraphs are all assumed to be unsigned where all the communication links are positive (Wang & Chen (2002); Chen, Liu, & Lu (2007); Yu, Cao, & Lü (2008); Song & Cao (2010); Wen *et al.* (2014); He, Qian, & Cao (2017)). In recent years, the coordination control of networks under signed interaction digraphs, where there may exist negative communication links, has received increasing attention (Altafini (2013); Hu & Zheng (2014); Fan, Zhang, & Wang (2014); Valcher & Misra (2014); Meng, Du, & Jia (2016); Zhai & Li (2016*a*,*b*); Meng (2017); Guo *et al.* (2018)). Different from the traditional synchronization of unsigned networks that all the nodes synchronize to a homogenous state (Yu, Cao, & Lü (2008); Song & Cao (2010); Wen *et al.* (2014); He, Qian, & Cao (2017)), signed networks may exhibit an interesting phenomenon named as bipartite synchronization, where a set of nodes synchronize to some manifold $s(t)$ and the remaining nodes synchronize to $-s(t)$ (Altafini (2013); Hu & Zheng (2014); Meng, Du, & Jia (2016); Zhai & Li (2016*a*,*b*)). The bipartite collective behaviors of signed networks with linear dynamics have been intensively studied (Altafini (2013); Hu & Zheng (2014);

Fan, Zhang, & Wang (2014); Valcher & Misra (2014); Meng, Du, & Jia (2016); Meng (2017); Guo *et al.* (2018)). Note that some progress has also been made in the bipartite synchronization of signed networks with nonlinear dynamics (Zhai & Li (2016*a*,*b*)). More recently, some researchers have analyzed the effect of time-delays on the bipartite synchronization of signed networks with first-order integrator dynamics (Guo *et al.* (2018)).

It should be noticed that most of the existing results for the synchronization of CDNNs are based on the assumption that the interaction graphs are unsigned, where the weights for the communication links between neighboring nodes are all positive. To the best of our knowledge, the bipartite synchronization of CDNNs with both positive and negative communication links has rarely been considered. Moreover, the coordination control of signed networks with both nonlinear dynamics and time delays has not received enough attention up to present. Therefore, it is imperative to address some challenging issues on the bipartite synchronization problem for a network of delayed neural networks with each node actually being a delayed nonlinear system, and to analyze the effects of the node-delay, node dynamics and network structure on the bipartite synchronization of the network.

The main novelties of this paper can be highlighted as follows. Firstly, a distributed pinning control algorithm is proposed to achieve bipartite leader-following synchronization in a network of delayed neural networks under signed graph topology. Some discussions are provided to address how to effectively select the pinned nodes for the signed network. Secondly, under some assumptions on the interaction graph and the node dynamics, by developing some tools from the property of M-matrix and the stability theory of delayed systems, some novel criteria in terms of low-dimensional linear matrix inequalities (LMIs) are derived to reach bipartite leader-following synchronization in the network when the node-delay is differentiable and bounded. Furthermore, a simple algebraic condition is given to estimate an upper bound for the node-delay. Thirdly, when the node-delay is only bounded but may not be differentiable, some bipartite synchronization criteria are established based on the descriptor method and the reciprocally convex approach. It is worth pointing out that the results of this paper also improve some existing results for the synchronization of coupled delayed neural networks with unsigned graphs where the node-delay is usually assumed to be differentiable or a constant.

The rest of this paper is organized as follows. Section 2 provides some mathematical preliminaries. Section 3 formulates the bipartite leader-following synchronization problem for a network of delayed neural networks and proposes a pinning algorithm for the network. Sections 4 and 5 study the pinning bipartite synchronization in the network with differentiable and non-differentiable node-delays, respectively. In Section 6, simulations are given to demonstrate the theoretical results. Finally, some conclusions and future trends are stated in Section 7.

2 Preliminaries

This section provides some mathematical preliminaries and supporting lemmas to derive the main results of the paper.

2.1 Notations

Let $\mathrm{Re}(z)$ denote the real part of a complex number z and I_n be the n-dimensional identity matrix. Let $\mathrm{sign}(\cdot)$ denote the standard sign function. For matrix $A \in \mathbb{R}^{n\times n}$, define $A_s \triangleq (A + A^T)/2$ as the symmetric part of A. The symbol $\otimes$ represents the Kronecker product (Horn & Johnson (1991)). Let $\begin{pmatrix} A & B \\ * & C \end{pmatrix}$ be a symmetric matrix where the asterisk '*' stands for the transpose of matrix B. Let $\mathrm{col}(x_1, \ldots, x_n) \triangleq (x_1^T, \ldots, x_n^T)^T$ be the column stack vector of $x_i \in \mathbb{R}^{n_i}, i = 1, \ldots, n$. For a Hermitian matrix X, write $X > 0$ ($X < 0$) if X is positive (negative) definite.

2.2 M-matrix theory

Some results related to M-matrix theory are useful to study the bipartite synchronization problem for coupled delayed neural networks.

Lemma 1 *(Horn & Johnson (1991)) For a nonsingular matrix $\Upsilon \in \mathbb{R}^{n\times n}$ with all off-diagonal elements being non-positive, the following statements are equivalent*

1) *Υ is a nonsingular M-matrix;*
2) *All the eigenvalues of Υ are located in the open right-half plane;*
3) *A diagonal matrix $\Xi = \mathrm{diag}(\xi_1, \cdots, \xi_n) > 0$ can be found such that $\Xi\Upsilon + \Upsilon^T\Xi > 0$ holds.*

2.3 Some supporting results

Lemma 2 (Jensen's inequality) *(Gu (2000)) For matrix $M \in \mathbb{R}^{n\times n} > 0$, scalars $\beta > \alpha$, and vector function $\omega : [\alpha, \beta] \to \mathbb{R}^n$ such that all the integrations in the following are well defined, then one has*

$$(\beta - \alpha)\int_\alpha^\beta \omega^T(\theta)M\omega(\theta)d\theta \geq \left(\int_\alpha^\beta \omega(\theta)d\theta\right)^T M \left(\int_\alpha^\beta \omega(\theta)d\theta\right).$$

Lemma 3 (Schur complement) *(Boyd* et al. *(1994)) The following linear matrix inequality (LMI)*

$$\begin{pmatrix} Q(x) & S(x) \\ S^T(x) & R(x) \end{pmatrix} > 0,$$

where $Q(x) = Q^T(x)$ and $R(x) = R^T(x)$, is equivalent to either of the following conditions:

1) *$Q(x) > 0,\ R(x) - S^T(x)Q^{-1}(x)S(x) > 0$;*
2) *$R(x) > 0,\ Q(x) - S(x)R^{-1}(x)S^T(x) > 0$.*

Lemma 4 (Reciprocally convex approach) *(Park, Ko, & Jeong (2011); Fridman (2014)) For any two real vectors y_1 and y_2, scalar $0 < \alpha < 1$, matrix $R > 0$ and any matrix S satisfying $\begin{pmatrix} R & S \\ S^T & R \end{pmatrix} \geq 0$, one has $\frac{1}{\alpha}y_1^T Ry_1 + \frac{1}{1-\alpha}y_2^T Ry_2 \geq y^T \begin{pmatrix} R & S \\ S^T & R \end{pmatrix} y$, where $y = \mathrm{col}(y_1, y_2)$.*

3 Problem formulation

Consider a signed network, whose interaction is denoted by the signed graph $\mathcal{G}^s$, composed by N identical nodes with each node being a delayed neural network described as follows:

$$\dot{x}_i(t) = -Cx_i(t) + Af(x_i(t)) + Bf(x_i(t-\tau(t))) - \sigma \sum_{j=1}^{N} |a_{ij}^s| \left(x_i(t) - \text{sign}(a_{ij}^s)x_j(t)\right) + u_i(t), \quad (1)$$

where $x_i(t) = (x_{i1}(t), \ldots, x_{in}(t))^T \in \mathbb{R}^n$ is the state variable of node i, $C = \text{diag}(c_1, \ldots, c_n) > 0$, $A, B \in \mathbb{R}^{n\times n}$ represent the weight and delayed weight matrices for the i-th neural-network node, respectively (Cao, Chen, & Li (2008); Yu, Cao, & Lü (2008)), $f(x_i(t)) = (f_1(x_{i1}(t)), \ldots, f_n(x_{in}(t)))^T \in \mathbb{R}^n$ is a continuous vector function, $\tau(t) \geq 0$ is called node-delay, $\sigma > 0$ is the coupling strength, a_{ij}^s is the (i,j)-th entry of the adjacency matrix $\mathcal{A}^s$ associated with the underlying signed graph $\mathcal{G}^s$ of network (1) defined as follows: $a_{ii}^s = 0$ for all $i = 1, \ldots, N$; for $i \neq j$, $a_{ij}^s \neq 0$ if there is a directed communication link from nodes j to i and $a_{ij}^s = 0$ otherwise (Altafini (2013); Hu & Zheng (2014)), and u_i is the control input to be designed.

The initial condition of network (1) is given by $x_i(t) = \phi_i(t),\ t \in [-r, 0],\ i = 1, \ldots, N$, where $r = \sup_{t\geq 0} \tau(t)$ and $\phi_i(t)$ belongs to the set of all continuous real-valued functions on the interval $[-r, 0]$ (Yu, Cao, & Lü (2008)).

The leader node of network (1) is described by

$$\dot{s}(t) = -Cs(t) + Af(s(t)) + Bf(s(t-\tau(t))), \quad (2)$$

where $s(t) = (s_1(t), \ldots, s_n(t))^T \in \mathbb{R}^n$.

Remark 1 *It is necessary to provide some explanations for the coupling terms in the signed network (1). Let the pair (j,i) denote the directed link (or called edge) from nodes j to i in the signed graph $\mathcal{G}^s$. When $a_{ij}^s > 0$, the link (j,i) is positive and the coupling term is given by $a_{ij}^s(x_i(t) - x_j(t))$ which means that the interaction between nodes i and j is cooperative; when $a_{ij}^s < 0$, that is, the link (j,i) is negative, the coupling term is given by $-a_{ij}^s(x_i(t) + x_j(t))$ indicating that the nodes i and j has competitive relationship (Altafini (2013); Hu & Zheng (2014)). If the graph $\mathcal{G}^s$ is unsigned, that is, $a_{ij}^s \geq 0$ holds for any $i \neq j$, network (1) collapses into a traditional coupled neural network considered in the previous literature (Chen, Zhou, & Liu (2004); Cao, Chen, & Li (2008); Yu, Cao, & Lü (2008); Song, Cao, & Liu (2012)).*

To derive the main results of this paper, some assumptions are needed to be made for the interaction graph $\mathcal{G}^s$ and the node function $f(\cdot)$ of network (1).

Assumption 1 *The signed graph $\mathcal{G}^s$ of network (1) is structurally balanced. That is, the node set $\mathcal{V} = \{1, \ldots, N\}$ of signed graph $\mathcal{G}^s$ can be partitioned into two disjoint subsets $\mathcal{V}_1$ and $\mathcal{V}_2$ such that the induced subgraphs associated with $\mathcal{V}_1$ and $\mathcal{V}_2$ are both unsigned, and any link between these two unsigned subgraphs is always negative.*

Assumption 2 *In network (1), for any $k \in \{1, \ldots, n\}$, the node function $f_k(\cdot)$ is an odd function satisfying the Lipschitz condition, that is,*

$$f_k(-z) = -f_k(z), \forall z \in \mathbb{R},$$
$$|f_k(b) - f_k(a)| \leq \delta_k |b-a|, \delta_k > 0, \forall a, b \in \mathbb{R}.$$

Assumption 3 *The augmented graph composed by the leader node (2) and the nodes of network (1) contains a directed spanning tree with the leader node being the only root, that is, the leader node (2) has a directed path to every node of network (1).*

Under Assumption 1, define a diagonal matrix $W = \text{diag}(w_1, \ldots, w_N)$ where $w_i \in \{1, -1\}$. Then, network (1) is said to achieve **bipartite leader-following synchronization** if $\lim_{t\to\infty} \|x_i(t) - w_i s(t)\| = 0, i = 1, \ldots, N$. Obviously, if $W = I_N$, the bipartite leader-following synchronization reduces to the traditional leader-following synchronization.

One can apply the pinning control strategy to solve the bipartite leader-following synchronization problem for signed network (1). Let $\mathcal{V}_{\text{pin}} = \{i_1, \ldots, i_l\} \subset \mathcal{V}$ $(1 \leq l < N)$ be the set of pinned nodes. Under Assumption 1, consider a pinning control algorithm for network (1):

$$u_i(t) = -\sigma d_i(x_i(t) - w_i s(t)), \ i = 1, \ldots, N, \tag{3}$$

where $w_i = 1$ if $i \in \mathcal{V}_1$ and $w_i = -1$ if $i \in \mathcal{V}_2$, and d_i is the pinning feedback gain defined as follows: $d_i > 0$ if $i \in \mathcal{V}_{\text{pin}}$ and $d_i = 0$ if $i \notin \mathcal{V}_{\text{pin}}$.

Remark 2 *Letting the diagonal matrix* $W = I_N$*, the pinning control algorithm (3) can be used to solve the traditional leader-following synchronization of complex networks with unsigned interaction graphs (Wang & Chen (2002); Chen, Liu, & Lu (2007); Song & Cao (2010)). Hence, the pinning control algorithm (3) for signed network (1) generalizes the pinning control algorithms for unsigned networks. However, it is worth noting that when applying algorithm (3) to a signed network, one should consider the network structure. If Assumption 1 is not satisfied, the graph* $\mathcal{G}^s$ *is not structurally balanced and it will be quite difficult to investigate the bipartite leader-following synchronization of signed network (1). Actually, for a signed network with first-order integrator dynamics, when the network is not structurally balanced, Meng, Du, & Jia (2016) have shown that the network can not achieve bipartite synchronization and may only achieve interval bipartite synchronization.*

4 Bipartite leader-following synchronization of network with a differentiable node-delay

In this section, by using M-matrix theory, we study the bipartite leader-following synchronization problem for signed network (1) with a differentiable node-delay under pinning algorithm (3).

Assumption 4 *The node-delay* $\tau(t)$ *in signed network (1) is bounded and differentiable satisfying* $0 < \tau(t) \leq \bar{\tau}$ *and* $0 \leq \dot{\tau}(t) \leq \mu < 1$.

For signed network (1), based on the adjacency matrix $\mathcal{A}^s$, let $L^s = (l_{ij}^s)_{N\times N} = \mathcal{B} - \mathcal{A}^s$ be the Laplacian matrix of network (1), where $\mathcal{B} = \text{diag}(b_1, \ldots, b_N)$ with $b_i = \sum_{k=1,k\neq i}^{N} |a_{ik}^s|$ (Altafini (2013); Hu & Zheng (2014)), yielding $l_{ij}^s = -a_{ij}^s, i \neq j$ and $l_{ii}^s = \sum_{k=1,k\neq i}^{N} |a_{ik}^s|$. Obviously, if there exists some $k \in \{1, \ldots, N\}$ such that $a_{ik}^s < 0$ holds, L^s is *not* a zero-row-sum matrix due to $\sum_{k=1}^{N} l_{ik}^s \neq 0$, which is quite different from the diffusion property of the traditional Laplacian matrix for an unsigned network (see Wang & Chen (2002); Chen, Liu, & Lu (2007); Yu, Cao, & Lü (2008); Song & Cao (2010)).

Based on the adjacency matrix $\mathcal{A}^s$ of signed network (1), define $\mathcal{A}^u = (a_{ij}^u)_{N\times N} = \left(|a_{ij}^s|\right)_{N\times N}$. Obviously, all the elements of $\mathcal{A}^u$ are non-negative where all the diagonal entries are zeros. Let $\mathcal{G}^u$ and $L^u = (l_{ij}^u)$ be the graph and

Laplacian matrix associated with $\mathcal{A}^u$, respectively, where $l^u_{ij} = -|a^s_{ij}|$ for $i \neq j$ and $l^u_{ii} = \sum_{k=1,k\neq i}^{N} |a^s_{ik}|$. One can see that $\mathcal{G}^u$ is an unsigned graph and can easily obtain $L^u = \mathcal{B} - \mathcal{A}^u$. Note that L^u is a zero-row-sum matrix.

Let

$$H = (h_{ij}) = L^u + D, \tag{4}$$

where $D = \mathrm{diag}(d_1, \ldots, d_N)$ is the matrix of pinning feedback gains in protocol (3).

The following two lemmas are important to study the bipartite leader-following synchronization of signed network (1) under pinning algorithm (3).

Lemma 5 *(Song* et al. *(2013)) Under Assumption 3, the matrix H defined in (4) is a nonsingular M-matrix.*

Lemma 6 *(Altafini (2013)) If a signed graph $\mathcal{G}^s$ is structurally balanced, a gauge transformation matrix $W = \mathrm{diag}(w_1, \ldots, w_N)$ with $w_i \in \{1, -1\}$ can be found such that $W\mathcal{A}^s W = \mathcal{A}^u$, where $\mathcal{A}^s = \left(a^s_{ij}\right)$ and $\mathcal{A}^u = \left(w_i a^s_{ij} w_j\right) = \left(\left|a^s_{ij}\right|\right)$ are the adjacency matrices for signed and unsigned graphs, respectively.*

The following theorem presents some conditions for reaching bipartite leader-following synchronization in the signed network (1) under algorithm (3).

Theorem 1 *Under Assumptions 1–4, the pinning control algorithm (3) can solve the bipartite leader-following synchronization problem for signed network (1) if there exist matrices $P \in \mathbb{R}^{n\times n} > 0$, $Q \in \mathbb{R}^{n\times n} > 0$ and $R \in \mathbb{R}^{n\times n} > 0$, and parameters $\beta > 0$, $\gamma > 0$ such that the following LMI condition holds:*

$$\Gamma = \begin{pmatrix} \Gamma_{11} & 0 & PA & PB & 0 \\ * & \Gamma_{22} & 0 & 0 & 0 \\ * & * & -\beta I_n & 0 & 0 \\ * & * & * & -\gamma I_n & 0 \\ * & * & * & * & -R \end{pmatrix} \leq 0, \tag{5}$$

where $\Gamma_{11} = -PC - C^T P - 2\alpha P + Q + \bar{\tau}^2 R + \beta\Delta^T\Delta$, $\Gamma_{22} = \gamma\Delta^T\Delta - (1-\mu)Q$, $\Delta = \mathrm{diag}(\delta_1, \ldots, \delta_n)$ describes the Lipschitz condition in Assumption 2, and α is a positive constant satisfying

$$0 < \alpha < \sigma \min_{1\leq i\leq N} \{\mathrm{Re}\,(\lambda_i)\}, \tag{6}$$

in which λ_i is the i-th eigenvalue of matrix H.

Proof. By the definition of the Laplacian matrix L^s, applying pinning control algorithm (3) to network (1) gives

$$\dot{x}_i(t) \;=\; -Cx_i(t) + Af(x_i(t)) + Bf(x_i(t-\tau(t))) - \sigma\sum_{j=1}^{N} l^s_{ij} x_j(t) - \sigma d_i \left(x_i(t) - w_i s(t)\right),\; i = 1, \ldots, N. \tag{7}$$

Recall that $L^s = \mathcal{B} - \mathcal{A}^s$ and $L^u = \mathcal{B} - \mathcal{A}^u$. Considering Assumption 1 and Lemma 6, it is easy to show that $L^u = WL^sW$ holds, indicating that $l_{ij}^u = w_i l_{ij}^s w_j = -|a_{ij}^s|$ for $i \neq j$ and $l_{ii}^u = \sum_{k=1,k\neq i}^N |a_{ik}^s|$.

One can use coordinate transform to analyze the bipartite synchronization in network (7). Let $\bar{x}_i(t) = w_i x_i, i = 1, \ldots, N$, which also gives $x_i = w_i \bar{x}_i(t)$. Then, it follows from (7) that

$$\dot{\bar{x}}_i(t) = -C\bar{x}_i(t) + Aw_i f(w_i\bar{x}_i(t)) + Bw_i f(w_i\bar{x}_i(t-\tau(t))) - \sigma \sum_{j=1}^N l_{ij}^u \bar{x}_j(t) - \sigma d_i \left(\bar{x}_i(t) - s(t)\right),\ i = 1, \ldots, N. \quad (8)$$

By Assumption 2, recall that $f_k(\cdot)$ $(k \in \{1, \ldots, n\})$ is an odd function. Since $w_i \in \{-1, 1\}$, $w_i f(w_i\bar{x}_i(t)) = f(\bar{x}_i(t))$ holds for any $i \in \{1, \ldots, N\}$. Then, from (8), one has

$$\dot{\bar{x}}_i(t) = -C\bar{x}_i(t) + Af(\bar{x}_i(t)) + Bf(\bar{x}_i(t-\tau(t))) - \sigma \sum_{j=1}^N l_{ij}^u \bar{x}_j(t) - \sigma d_i \left(\bar{x}_i(t) - s(t)\right),\ i = 1, \ldots, N. \quad (9)$$

For $i = 1, \ldots, N$, let

$$\begin{aligned} e_i(t) &= (e_{i1}(t), \ldots, e_{in}(t))^T \triangleq \bar{x}_i(t) - s(t) \\ \eta_i(t) &= (\eta_{i1}(t), \ldots, \eta_{in}(t))^T \triangleq f(\bar{x}_i(t)) - f(s(t)). \end{aligned} \quad (10)$$

Recall that L^u is a zero-row-sum matrix. From (2) and (9), one can obtain the following error system:

$$\dot{e}_i(t) = -Ce_i(t) + A\eta_i(t) + B\eta_i(t-\tau(t)) - \sigma \sum_{j=1}^N l_{ij}^u e_j(t) - \sigma d_i e_i(t),\ i = 1, \ldots, N. \quad (11)$$

If system (11) is asymptotically stable, $e_i(t)$ tends to zero as $t \to \infty$, that is, $w_i x_i(t) \to s(t),\ i = 1, \ldots, N$. Considering $w_i^2 = 1$, one has $x_i(t) \to w_i s(t), i = 1, \ldots, N$. Therefore, the bipartite leader-following synchronization problem of network (1) is transformed to the stability problem of the error system (11).

In view of the definition of matrix H in (4), one can rewrite (11) as

$$\dot{e}_i(t) = -Ce_i(t) + A\eta_i(t) + B\eta_i(t-\tau(t)) - \sigma \sum_{j=1}^N h_{ij} e_j(t),\ i = 1, \ldots, N. \quad (12)$$

By Lemma 5, $H = L^u + D$ is a nonsingular M-matrix under Assumption 3, ensuring $\mathrm{Re}(\lambda_i) > 0, i = 1, \ldots, N$. From condition (6) and Lemma 1, one can show that $\sigma H - \alpha I_N$ is a nonsingular M-matrix and there exists a matrix $\Xi = \mathrm{diag}(\xi_1, \cdots, \xi_N) > 0$ such that

$$\left[\Xi\left(\sigma H - \alpha I_N\right)\right]_s > 0. \quad (13)$$

From Assumption 2, it is easy to obtain

$$e_i^T(t')\Delta^T\Delta e_i(t') - \eta_i^T(t')\eta_i(t') \geq 0,\ \forall t' \in \mathbb{R}. \quad (14)$$

Construct the following Lyapunov-Krasovskii functional candidate:

$$V(t)=\sum_{i=1}^{N}\xi_i e_i^T(t)Pe_i(t)+\sum_{i=1}^{N}\xi_i\int_{t-\tau(t)}^{t}e_i^T(\zeta)Qe_i(\zeta)d\zeta+\sum_{i=1}^{N}\xi_i\bar{\tau}\int_{-\bar{\tau}}^{0}\int_{t+\theta}^{t}e_i^T(\zeta)Re_i(\zeta)d\zeta d\theta, \tag{15}$$

where matrices $P, Q, R > 0$ satisfy LMI condition (5).

Let $e(t) = \mathrm{col}(e_1(t), \ldots, e_N(t))$ and $\eta(t) = \mathrm{col}(\eta_1(t), \ldots, \eta_N(t))$. Considering Assumption 4, Jensen's inequality in Lemma 2, inequalities (13) and (14), calculate $\dot{V}(t)$ along the trajectory of system (12) as follows:

$$\begin{aligned}
\dot{V}(t) &= -2\sum_{i=1}^{N}\xi_i e_i^T(t)PCe_i(t)+2\sum_{i=1}^{N}\xi_i e_i^T(t)PA\eta_i(t)+2\sum_{i=1}^{N}\xi_i e_i^T(t)PB\eta_i(t-\tau(t))\\
&\quad -2\sigma\sum_{i=1}^{N}\xi_i e_i^T(t)\sum_{j=1}^{N}h_{ij}Pe_j(t)+\sum_{i=1}^{N}\xi_i e_i^T(t)Qe_i(t)-(1-\dot{\tau}(t))\sum_{i=1}^{N}\xi_i e_i^T(t-\tau(t))Qe_i(t-\tau(t))\\
&\quad +\sum_{i=1}^{N}\xi_i\bar{\tau}^2 e_i^T(t)Re_i(t)-\sum_{i=1}^{N}\xi_i\bar{\tau}\int_{t-\bar{\tau}}^{t}e_i^T(\zeta)Re_i(\zeta)d\zeta\\
&\le -2e^T(t)\left(\Xi\otimes(PC)\right)e(t)+2e^T(t)\left(\Xi\otimes(PA)\right)\eta(t)+2e^T(t)\left(\Xi\otimes(PB)\right)\eta(t-\tau(t))\\
&\quad -2e^T(t)\left[\sigma\left(\Xi H\right)\otimes P\right]e(t)+2\alpha e^T(t)\left(\Xi\otimes P\right)e(t)-2\alpha e^T(t)\left(\Xi\otimes P\right)e(t)\\
&\quad +e^T(t)\left(\Xi\otimes Q\right)e(t)-(1-\mu)e^T(t-\tau(t))\left(\Xi\otimes Q\right)e^T(t-\tau(t))\\
&\quad +\bar{\tau}^2 e^T(t)\left(\Xi\otimes R\right)e(t)-\sum_{i=1}^{N}\xi_i\left(\int_{t-\bar{\tau}}^{t}e_i(\zeta)d\zeta\right)^T R\left(\int_{t-\bar{\tau}}^{t}e_i(\zeta)d\zeta\right)\\
&\quad +\beta\sum_{i=1}^{N}\xi_i\left[e_i^T(t)\Delta^T\Delta e_i(t)-\eta_i^T(t)\eta_i(t)\right]\\
&\quad +\gamma\sum_{i=1}^{N}\xi_i\left[e_i^T(t-\tau(t))\Delta^T\Delta e_i(t-\tau(t))-\eta_i^T(t-\tau(t))\eta_i(t-\tau(t))\right]\\
&= \sum_{i=1}^{N}\xi_i y_i^T(t)\Gamma y_i-2e^T(t)\left(\left[\Xi\left(\sigma H-\alpha I_N\right)\right]_s\otimes P\right)e(t),
\end{aligned} \tag{16}$$

where $y_i(t) = \mathrm{col}\left(e_i, e_i(t-\tau(t)), \eta_i(t), \eta_i(t-\tau(t)), \int_{t-\bar{\tau}}^{t} e_i(\zeta)d\zeta\right)$.

Then, it follows from LMI condition (5) and inequality (13) that the error system (12) is globally asymptotically stable. Therefore, the pinning control algorithm (3) solves the bipartite leader-following synchronization problem of signed network (1). This completes the proof. □

Remark 3 *From the proof of Theorem 1, one can see the Assumption 1 on the network topology and Assumption 2 on the node-function are necessary to investigate the bipartite leader-following synchronization of signed network (1).*

Remark 4 *By using the property of M-matrix, Theorem 1 provides a low-dimensional LMI condition (5) whose dimension is $5n \times 5n$ determined by that of a single network node rather than by the size of the network, that is, $5n << N$ holds for a large-scale network.*

Remark 5 *For LMI condition (5), let $P = R = I_n$, $Q = (1/(1-\mu))\delta_m^2 I_n$, and $\beta = \gamma = 1$, where $\delta_m = \max_{1\le k\le n}\{\delta_k\}$. Let $M = 2C+2\alpha I_n-(1/(1-\mu))\delta_m^2 I_n-\Delta^T\Delta-AA^T-BB^T$, in which $0<\alpha<\sigma\min_{1\le i\le N}\{\mathrm{Re}\left(\lambda_i\right)\}$.*

Considering Schur complement in Lemma 3, one sees that condition (5) is equivalent to $\bar{\tau}^2 I_n - M \leq 0$. If α is large enough such that $\bar{\tau}^2 I_n - M \leq 0$ holds, LMI condition (5) is satisfied, which gives $\bar{\tau} \leq \sqrt{\rho_{\min}}$ with $\rho_{\min}$ being the minimum eigenvalue of matrix M, that is, $\sqrt{\rho_{\min}}$ is an upper bound for the node-delay $\tau(t)$. Then, an important conclusion can be drawn as follows: if the coupling strength σ is sufficiently large, a parameter α can always be found such that $\bar{\tau}^2 I_n - M \leq 0$ holds, indicating that pinning control algorithm (3) solves the bipartite leader-following synchronization problem of signed network (1) under Assumptions 1–4.

Remark 6 *By Assumption 3 and the proof of Theorem 1, one can select the pinned nodes and design the pinning feedback gains based on the unsigned graph associated with the adjacency matrix $\mathcal{A}^u$, which can be easily carried out by using the pinning control strategies for the traditional unsigned networked systems (Wang & Chen (2002); Chen, Liu, & Lu (2007); Song & Cao (2010); Song* et al. *(2013); Wen* et al. *(2014)).*

5 Bipartite leader-following synchronization of network with a non-differentiable node-delay

In the previous section, we have studied the bipartite leader-following synchronization in signed network with a differentiable node-delay. However, in some practical cases, the node-delay may not be differentiable. In this section, we relax the restriction on the node-delay to investigate the bipartite synchronization problem of signed network with a bounded node-delay.

Assumption 5 *The node-delay $\tau(t)$ in signed network (1) is non-differentiable and bounded satisfying $0 \leq \tau(t) \leq \bar{\tau}$.*

The following result addresses the bipartite leader-following synchronization of signed network (1) with a non-differentiable and bounded delay.

Theorem 2 *Suppose that Assumptions 1–3 and 5 are satisfied. Under algorithm (3), the bipartite leader-following synchronization can be achieved in signed network (1) if there exist matrices $P_k \in \mathbb{R}^{Nn\times Nn} > 0, k = 1,2,3$, $S \in \mathbb{R}^{Nn\times Nn} > 0$, $M \in \mathbb{R}^{Nn\times Nn} > 0$, $T \in \mathbb{R}^{Nn\times Nn}$, $G = \mathrm{diag}(g_1,\ldots,g_N) > 0$ and $K = \mathrm{diag}(k_1,\ldots,k_N) > 0$ such that*

$$\Omega = \begin{pmatrix} \Omega_{11} & \Omega_{12} & \Omega_{13} & T & P_2A_1 & P_2A_2 \\ * & \Omega_{22} & 0 & 0 & P_3A_1 & P_3A_2 \\ * & * & \Omega_{33} & S-T & 0 & 0 \\ * & * & * & -S-M & 0 & 0 \\ * & * & * & * & -G\otimes I_n & 0 \\ * & * & * & * & * & -K\otimes I_n \end{pmatrix} < 0, \tag{17}$$

and

$$\begin{pmatrix} S & T \\ T^T & S \end{pmatrix} \geq 0, \tag{18}$$

where

$$
\begin{aligned}
&\Omega_{11} = P_2A_0 + A_0^TP_2 - S + M + G \otimes (\Delta^T\Delta),\\
&\Omega_{12} = P_1 - P_2 + A_0^TP_3,\ \Omega_{13} = S - T,\\
&\Omega_{22} = -2P_3 + \bar{\tau}^2S,\ \Omega_{33} = -2S + T + T^T + K \otimes (\Delta^T\Delta),\\
&A_0 = -I_N \otimes C - \sigma(H \otimes I_n),\ A_1 = I_N \otimes A, \quad A_2 = I_N \otimes B.
\end{aligned} \tag{19}
$$

Proof. Letting $e(t) = \mathrm{col}(e_1(t), \dots, e_N(t))$ and $\eta(t) = \mathrm{col}(\eta_1(t), \dots, \eta_N(t))$, rewrite the error system (12) in the proof of Theorem 1 as follows:

$$
\dot{e}(t) = A_0e(t) + A_1\eta(t) + A_2\eta(t - \tau(t)), \tag{20}
$$

where A_0, A_1 and A_2 are defined in (19).

Consider the following Lyapunov-Krasovskii functional candidate

$$
V(t) = e^T(t)P_1e(t) + \int_{t-\bar{\tau}}^{t} e^T(\zeta)Me(\zeta)d\zeta + \bar{\tau}\int_{-\bar{\tau}}^{0}\int_{t+\theta}^{t} \dot{e}^T(\zeta)S\dot{e}(\zeta)d\zeta d\theta. \tag{21}
$$

where matrices P_1, M and S satisfy LMI condition (17).

In view of the descriptor method (Fridman (2014)), let $y(t) = \mathrm{col}(y_1(t), \dots, y_N(t)) \triangleq \dot{e}(t)$ where $y_i(t) = \dot{e}_i(t)$. Then, from (20), one has

$$
A_0e(t) + A_1\eta(t) + A_2\eta(t - \tau(t)) - y(t) = 0. \tag{22}
$$

By inequality (14) in the proof of Theorem 1, it is easy to obtain

$$
\begin{aligned}
&e^T(t)\left(G \otimes (\Delta^T\Delta)\right)e(t) - \eta^T(t)\left(G \otimes I_n\right)\eta(t) \geq 0,\\
&e^T(t-\tau(t))\left(K \otimes (\Delta^T\Delta)\right)e(t-\tau(t)) - \eta^T(t-\tau(t))\left(K \otimes I_n\right)\eta(t-\tau(t)) \geq 0,
\end{aligned} \tag{23}
$$

where $G = \mathrm{diag}(g_1, \dots, g_N) > 0$ and $K = \mathrm{diag}(k_1, \dots, k_N) > 0$.

Considering (22)–(23) and $y(t) = \dot{e}(t)$, calculate the derivative of $V(t)$ along the trajectory of system (20):

$$
\begin{aligned}
\dot{V}(t) =\ & 2e^T(t)P_1\dot{e}(t) + e^T(t)Me(t) - e^T(t-\bar{\tau})Me(t-\bar{\tau}) + \bar{\tau}^2\dot{e}^T(t)S\dot{e}(t) - \bar{\tau}\int_{t-\bar{\tau}}^{t}\dot{e}^T(\zeta)S\dot{e}(\zeta)d\zeta\\
\leq\ & 2e^T(t)P_1y(t) + e^T(t)Me(t) - e^T(t-\bar{\tau})Me(t-\bar{\tau}) + \bar{\tau}^2y^T(t)Sy(t) - \bar{\tau}\int_{t-\bar{\tau}}^{t}\dot{e}^T(\zeta)S\dot{e}(\zeta)d\zeta\\
& +2\left[e^T(t)P_2 + y^T(t)P_3\right]\left[A_0e(t) + A_1\eta(t) + A_2\eta(t-\tau(t)) - y(t)\right]\\
& +e^T(t)\left(G \otimes (\Delta^T\Delta)\right)e(t) - \eta^T(t)\left(G \otimes I_n\right)\eta(t)\\
& +e^T(t-\tau(t))\left(K \otimes (\Delta^T\Delta)\right)e(t-\tau(t)) - \eta^T(t-\tau(t))\left(K \otimes I_n\right)\eta(t-\tau(t)).
\end{aligned} \tag{24}
$$

For any $\tau(t) \in (0, \bar{\tau})$, it follows from Jensen's inequality and the reciprocally convex approach in Lemma 4 that

$$
\begin{aligned}
&- \bar{\tau} \int_{t-\bar{\tau}}^{t} \dot{e}^T(\zeta) S \dot{e}(\zeta) d\zeta \\
&= -\bar{\tau} \int_{t-\tau(t)}^{t} \dot{e}^T(\zeta) S \dot{e}(\zeta) d\zeta - \bar{\tau} \int_{t-\bar{\tau}}^{t-\tau(t)} \dot{e}^T(\zeta) S \dot{e}(\zeta) d\zeta \\
&\leq -\frac{\bar{\tau}}{\tau(t)} z_1^T(t) S z_1(t) - \frac{\bar{\tau}}{\bar{\tau} - \tau(t)} z_2^T(t) S z_2(t) \\
&\leq -z^T(t) \begin{pmatrix} S & T \\ T^T & S \end{pmatrix} z(t),
\end{aligned} \tag{25}
$$

where $z_1(t) = e(t) - e(t - \tau(t))$, $z_2(t) = e(t - \tau(t)) - e(t - \bar{\tau})$, $z(t) = \mathrm{col}(z_1(t), z_2(t))$, and matrices S and T satisfy LMI condition (18). Note that $z_1(t) = 0$ when $\tau(t) = 0$ and $z_2(t) = 0$ when $\tau(t) = \bar{\tau}$. By using Jensen's inequality, it is easy to show that $-\bar{\tau} \int_{t-\bar{\tau}}^{t} \dot{e}^T(\zeta) S \dot{e}(\zeta) d\zeta \leq -z^T(t) \begin{pmatrix} S & T \\ T^T & S \end{pmatrix} z(t)$ still holds for $\tau(t) = 0$ or $\tau(t) = \bar{\tau}$.

Combining (24)–(25), some tedious calculations give

$$
\dot{V}(t) \leq \vartheta^T(t) \Omega \vartheta(t), \tag{26}
$$

where $\vartheta(t) = \mathrm{col}(e(t), y(t), e(t - \tau(t)), e(t - \bar{\tau}), \eta(t), \eta(t - \tau(t)))$. It follows from LMI condition (17) that $\dot{V}(t) < 0$ holds for any $\vartheta(t) \neq 0$. Then, the error system (20) is asymptotically stable at the origin. Hence, algorithm (3) solves the bipartite leader-following synchronization problem for signed network (1). The proof is finished. □

Remark 7 *In this section, the node-delay for signed network (1) is only assumed to be bounded. It is worth mentioning that in the literature on the synchronization of coupled delayed neural networks with unsigned graphs, the node-delay is usually assumed to be a differentiable function or a constant (Lu, Ho, & Wang (2009); Song, Cao, & Liu (2012); He, Qian, & Cao (2017)).*

Remark 8 *It is necessary to discuss the feasibility of LMI conditions (17) and (18) in Theorem 2. Let* λ_i *be the i-th eigenvalue of matrix* H*. By Lemma 5, under Assumptions 1 and 3, one has* $\mathrm{Re}(\lambda_i) > 0$, $i = 1, \ldots, N$, *and can see that the eigenvalue-set of matrix* $A_0 = -I_N \otimes C - \sigma(H \otimes I_n)$ *is given by* $\{-\sigma\lambda_i - c_k, i = 1, \ldots, N, k = 1, \ldots, n\}$*, which indicates that* A_0 *is a Hurwitz matrix. Then, a positive definite matrix* $P_2 > 0$ *and a parameter* $\beta > 0$ *can be found to satisfy* $P_2 A_0 + A_0^T P_2 < -\beta I_{Nn}$*. Let* $P_1 = P_2$, $P_3 = \bar{\tau}(I_N \otimes (\Delta^T \Delta))$, $S = P_3/\bar{\tau}^2$, $T = 0$, $M = S/2$, $G = I_N/(2\bar{\tau})$ *and* $K = I_N/(3\bar{\tau})$*. Obviously, for* $S > 0$ *and* $T = 0$*, condition (18) always holds. Applying Schur complement, one can show that LMI (17) is feasible if* $\bar{\tau}$ *is relatively small and* $\Delta^T \Delta \leq \beta \bar{\tau} I_n$ *holds.*

Remark 9 *Due to different assumptions on the node-delay* $\tau(t)$ *for signed network (1), note that the dimension of LMI condition (17) in Theorem 2 is much higher than that of LMI condition (5) in Theorem 1. Since the node-delay* $\tau(t)$ *in Theorem 2 is only assumed to be bounded, it is difficult to construct an appropriate Lyapunov-Krasovskii functional including a term explicitly related to* $\tau(t)$*. How to establish some lower-dimensional LMI conditions for reaching bipartite synchronization in signed network (1) with a non-differentiable node-delay is quite challenging and deserves to be further investigated in our future work.*

Remark 10 *This paper focuses on the leader-following bipartite synchronization of signed network (1). Letting* $u_i(t) = 0, i = 1, \ldots, N$*, signed network (1) is said to reach* **bipartite leaderless synchronization** *if* $\lim_{t\to\infty} \|w_i x_i(t) - w_j x_j(t)\| = 0$ *holds for any* $i, j = 1, \ldots, N (i \neq j)$ *where* $w_i \in \{1, -1\}$*. Under Assumptions 1–2, considering (9) in the proof of Theorem 1, one can obtain* $\dot{\bar{x}}_i(t) = -C\bar{x}_i(t) + Af(\bar{x}_i(t)) + Bf(\bar{x}_i(t - \tau(t))) - \sigma \sum_{j=1}^{N} l^u_{ij}\bar{x}_j(t),\ i = 1, \ldots, N,$ *where* $\bar{x}_i(t) = w_i x_i(t)$*. Let* $e_i(t) = \bar{x}_i(t) - \bar{x}_1(t)$ *and* $\eta_i(t) = f(\bar{x}_i(t)) - f(\bar{x}_1(t)),\ i = 2, \ldots, N$*. Recall that* $\sum_{j=1}^{N} l^u_{ij} = 0$ *holds for all* $i = 1, \ldots, N$*. By some simple calculations, one has the following error system*

$$\dot{e}_i(t) = -Ce_i(t) + A\eta_i(t) + B\eta_i(t - \tau(t)) - \sigma \sum_{j=2}^{N} \left(l^u_{ij} - l^u_{1j}\right) e_j(t). \tag{27}$$

Define $\bar{L}^u \triangleq \left(\bar{l}^u_{pq}\right) \in \mathbb{R}^{(N-1)\times(N-1)}$ *where* $\bar{l}^u_{pq} = l^u_{(p+1)(q+1)} - l^u_{1(q+1)}, p, q = 1, \ldots, N-1$*. Letting* $e(t) = \mathrm{col}(e_2(t), \ldots, e_N(t))$ *and* $\eta(t) = \mathrm{col}(\eta_2(t), \ldots, \eta_N(t))$*, write the error system (27) in compact matrix form:*

$$\dot{e}(t) = A_0 e(t) + A_1 \eta(t) + A_2 \eta(t - \tau(t)), \tag{28}$$

where $A_0 = -I_{N-1} \otimes C - \sigma(\bar{L}^u \otimes I_n),\ A_1 = I_{N-1} \otimes A$ *and* $A_2 = I_{N-1} \otimes B$*. It is easy to show that the signed network (1) achieves bipartite leaderless synchronization if the error system (28) is asymptotically stable. By Lemma 1 in (Zhang & Tian (2009)), all the eigenvalues of matrix* $\bar{L}^u$ *have positive real parts if the graph of* $\mathcal{G}^u$ *contains a directed spanning tree, where* $\mathcal{G}^u$ *is the unsigned graph associated with the adjacency matrix* $\mathcal{A}^u$*. Following the similar line in Theorem 2, one can analyze the stability of system (28). The related details are omitted due to space limit.*

6 Numerical results

This section presents some simulations to study the bipartite leader-following synchronization of network (1) under pinning control algorithm (3).

Consider the signed network (1) composed by seven delayed neural networks (Lu (2002)) shown in Fig. 1, where $\sigma = 25$, $f(x_i(t)) = (\tanh(x_{i1}(t)), \tanh(x_{i2}(t)))^T,\ i = 1, \ldots, 7$, and the parameters are give by as follows (Lu (2002))

$$C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, A = \begin{bmatrix} 2.0 & -0.1 \\ -5.0 & 1.5 \end{bmatrix}, B = \begin{bmatrix} -1.5 & -0.1 \\ -0.2 & -1 \end{bmatrix}. \tag{29}$$

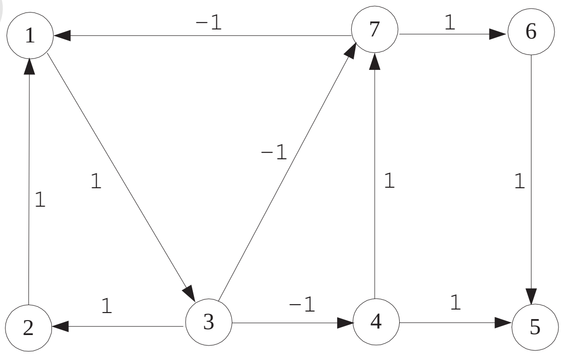

Fig. 1. A signed graph with seven nodes.

By the results in (Lu (2002)), the leader node (2) with the parameters in (29) may exhibit rich dynamics including chaotic behaviors. For example, letting the node-delay $\tau(t) = 1.0$ and the initial condition $s(t) = (0.4, 0.6)^T, \forall t \in [-1, 0]$, Fig. 2 plots a single-scroll-like chaotic attractor of the leader node (2).

From Fig. 1, one can see that the topology of the signed network (1) is structurally balanced by simply letting $\mathcal{V}_1 = \{1, 2, 3\}$ and $\mathcal{V}_2 = \{4, 5, 6, 7\}$, which indicates that Assumption 1 holds with $W = \text{diag}(1, 1, 1, -1, -1, -1, -1)$.

Note that the node function $f_k(\cdot)(k \in \{1, 2\})$ of network (1) is an odd function satisfying the Lipschitz condition with $\Delta = \text{diag}(1, 1)$. Hence, Assumption 2 is satisfied.

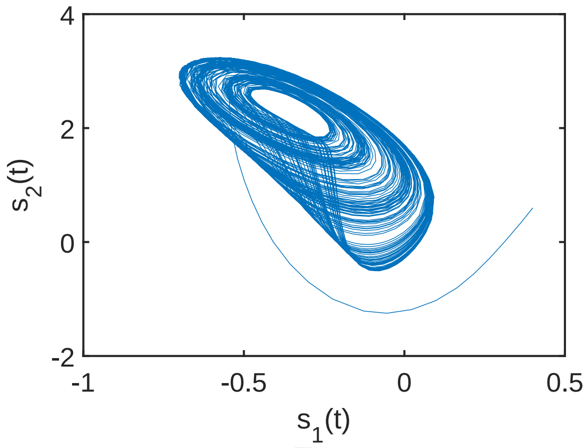

Fig. 2. Chaotic trajectory of leader node (2).

Choose node 3 in Fig. 1 as pinned node with the pinning feedback gain $d_3 = 20$ such that Assumption 3 holds. Some simple calculations give $\min_{i=1}^{7}\{\text{Re}(\lambda_i)\} = 0.9088$, where λ_i is the i-th eigenvalue of matrix H defined in (4).

We now investigate the bipartite synchronization of network (1) under pinning control algorithm (3) with differentiable and non-differentiable node-delays, respectively. For $t \in [-\bar{\tau}, 0]$, let $s(t) = (0.4, 0.6)^T$ be the initial condition of leader node (2), and choose the initial condition of each node in signed network (1) from $[-0.2, 0.2] \times [-0.2, 0.2]$, where $\bar{\tau}$ is the bound for the node-delay to be given.

Case 1: Bipartite synchronization of network with a differentiable node-delay

Let $\tau(t) = 1/(1+e^{-t})$. It is easy to obtain $0 < \tau(t) \leq 1$ and $\dot{\tau}(t) = e^{-t}/(1+e^{-t})^2 \leq \mu = 0.25$. According to condition (6) in Theorem 1, choose $\alpha = 18 < 25 \times \min_{i=1}^{7} \text{Re}(\lambda_i) = 22.72$. By using MATLAB's LMI toolbox, one can verify that LMI condition (5) is satisfied. Applying pinning control to node 3, the state evolutions of signed network (1) and the leader node (2) are depicted in Fig. 3, where the dotted line denotes the variations of leader node (2). From Fig. 3, one can clearly see that the signed network (1) achieves bipartite leader-following synchronization.

Remark 11 *One can estimate an upper bound for the node-delay by Remark 5. Recall that* $\alpha = 18$ *and* $\delta_m = \max_{1\leq k\leq 2}\{\delta_k\} = 1$. *Some simple calculations yield* $M = 2C + 2\alpha I_2 - (1/(1-\mu))\delta_m^2 I_2 - \Delta^T\Delta - AA^T - BB^T = \begin{bmatrix} 29.3967 & 9.7500 \\ 9.7500 & 7.3767 \end{bmatrix}$, *whose eigenvalues are given by* 3.6801 *and* 33.0932. *Then, an upper bound of the node-delay is*

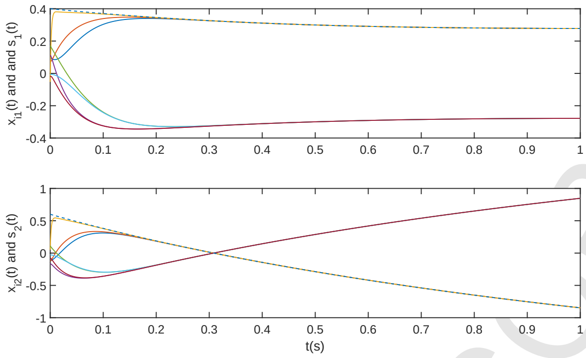

Fig. 3. Bipartite leader-following synchronization of signed network with a differentiable node-delay.

estimated to be $\bar{\tau} = \sqrt{3.6801} = 1.9184$*, which means that the bipartite leader-following synchronization of signed network (1) with algorithm (3) can be reached.*

Case 2: Bipartite synchronization of network with a non-differentiable node-delay

Let $\tau(t) = 0.5 + 0.5|\sin(t)|$, whose upper bound is 1.0. However, $\tau(t)$ is not differentiable with respect to time t. Considering Theorem 2, one can verify that LMI conditions (17) and (18) hold simultaneously with the help of MATLAB's LMI toolbox. Fig. 4 presents the variations of the states of signed network (1) and leader node (2), indicating that the bipartite leader-following synchronization problem for signed network (1) is successfully solved by pinning control algorithm (3).

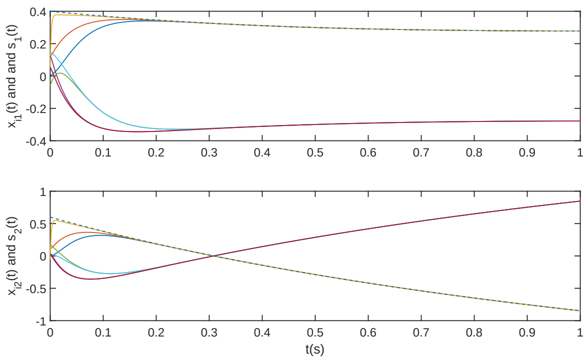

Fig. 4. Bipartite leader-following synchronization of signed network with a non-differentiable node-delay.

7 Conclusions

In this paper, we have studied the bipartite leader-following synchronization in a network of delayed neural networks under signed graph based on pinning control strategy, where only a subset of nodes can access the information of the leader. By using the property of M-matrix and the theory of algebraic graphs, we have established some conditions to ensure that the bipartite leader-following synchronization problem of the signed network can be successfully solved. Both differentiable and non-differentiable cases for the node-delay in the network have been considered by using some techniques from delayed systems such as Jensen's inequality and the reciprocally convex approach. Moreover, when the node-delay is bounded and differentiable, a simple algebraic approach is given to estimate an upper bound of the node-delay. Simulation results have been provided to validate the effectiveness of our theoretical analysis. It is worth mentioning that the bipartite leaderless synchronization of the signed network has also been briefly discussed.

For the coupled delayed neural networks in this paper, the signed interaction graph is assumed to be fixed and the communication delay between neighboring nodes has not been considered. It would be of interest to investigate the bipartite synchronization of coupled delayed neural networks with switching network topology and hybrid delayed coupling in the near future.

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