Int. J. Fuzzy Syst.
https://doi.org/10.1007/s40815-017-0432-9

Hesitant Fuzzy Linguistic Term Set and Its Application in Decision Making: A State-of-the-Art Survey

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Received: 17 October 2017 / Accepted: 23 November 2017

Abstract The hesitant fuzzy linguistic term set (HFLTS) has gained great success as it can be used to represent several linguistic terms or comparative linguistic expressions together with some context-free grammars. This new approach has enabled the analysis and computing of linguistic expressions with uncertainties and opened the door for the possibility to develop more comprehensive and powerful decision theories and methods based on linguistic knowledge. Lots of new approaches and proposals for decision-making problems have been proposed to overcome the limitations of previous linguistic decision-making approaches. Now and in the future, decision-making methodologies and algorithms with hesitant fuzzy linguistic models would be a quite promising research line representing a high-quality breakthrough in this topic. To facilitate the study on HFLTS theory, this paper makes a state-of-the-art survey on HFLTSs based on the 134 selected papers from Web of Sciences published from January 2012 to October 2017. We justify the motivation, definitions, operations, comparison methods and measures of HFLTSs. We also summarize the different extensions of HFLTSs. The studies on multiple criteria decision making (MCDM) with HFLTSs in terms of aggregation operators and MCDM methods are clearly reviewed. We also conduce some overviews on decision making with hesitant fuzzy linguistic preference relations. The applications, research challenges and future directions are also given.

Keywords Hesitant fuzzy linguistic term set · Multiple criteria decision making · Qualitative decision making · Hesitant fuzzy linguistic preference relation · Linguistic expressions · Survey

1 Introduction

Even though it has been investigated for several decades, the decision-making process still faces many challenges as it may involve multiple alternatives (sometimes may be with huge amount), multiple objectives/criteria (sometimes may be conflicting) and multiple experts (sometimes may be out of mutual agreement). Even though scholars have proposed different methodologies for different circumstances of decision-making analyses, the most fundamental yet essential step is to represent information appropriately and objectively, especially in those ill-structured decision-making problems involving uncertainty, vagueness and incomplete information that cannot be identified by probabilistic models [4]. To formally formulate the uncertainty of linguistic descriptors in decision making, Zadeh [137] proposed the fuzzy linguistic model, which uses the linguistic variables, *whose values are not numbers but words or sentences in a natural or artificial language*, to represent the qualitative opinions of a person. In spite of being less precise than a number, the linguistic variables enhance the feasibility, flexibility and reliability of decision models and provide many useful applications in different fields [57]. The progress of analyzing linguistic variables has led to an active research area today named computing with words (CWW) [66]: *CWW is a methodology for reasoning,*

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computing and making decisions using information represented in natural language.

Different CWW models have been developed to tackle the linguistic qualitative decision-making problems with classical fuzzy linguistic variables [82], which mainly include the linguistic computing model based on membership functions [137], the symbolic linguistic computing model [127], the virtual linguistic computation model [123], the 2-tuple linguistic model [28], the linguistic model based on granular computing [5, 76], the personalized individual semantics in CWW [38] and the linguistic computational model based on discrete fuzzy numbers [64]. However, these models just use single and simple linguistic term to represent the value of a linguistic variable, and thus cannot represent the complex knowledge uncertainties or hesitations which are often found in the experts' assessments. For example, when evaluating a research proposal, an expert may say "*it is between good and excellent*" as he is uncertain or hesitant about his opinions. Furthermore, in group decision-making (GDM) problems, even each expert has no hesitation in his/her opinions, the group's assessments could be uncertain and hesitant if there are different opinions among the group. In addition, the CWW process with the above models may loss information as the information representation models are discrete in a continuous domain [82].

To overcome the inability of traditional CWW models in handling such uncertainties with complex linguistic expressions, Rodríguez et al. [83] developed a new approach by proposing the hesitant fuzzy linguistic term set (HFLTS) and the context-free grammars. Furthermore, to make it much easier to understand, Liao et al. [51] gave a mathematical definition of the HFLTS and introduced the hesitant fuzzy linguistic element (HFLE) to represent the hesitant fuzzy linguistic value of a linguistic variable. With the use of the HFLEs, the experts can provide their (uncertain) assessments by means of several linguistic terms or comparative linguistic expressions. The HFLTS increases the flexibility and capability of the elicitation of linguistic information by means of complex linguistic expressions. Therefore, it has attracted more and more scholars' attention and many fruitful achievements have been obtained (see Table 1). Table 1 classifies the papers related to HFLTSs into different categories. Note that the "Aggregation operators" category belongs to the "multiple criteria decision-making (MCDM)" category.

The aim of this paper is to make an overview on the state of the art of the researches on hesitant fuzzy linguistic theory during the period from January 2012 to October 2017. We select 134 journal papers from the well-known database, Web of Sciences, based on the systematic and structured method inspired by the guidelines of Webster and Watson [105] and Kitchenham [35], and the query we used for retrieving the papers is "hesitant fuzzy linguistic *" (see Fig. 1). The motivation, definitions and operations of HFLTSs are clearly summarized. As the comparison methods between HFLEs are essential for many decision-making methods, we make a detailed review on different comparison schemes. We also summarize the measures of HFLTSs. Over the past five years, many scholars have introduced different extensions of HFLTSs. We list all of these extensions. As MCDM is a very prevalent research topic, we make a survey on MCDM with HFLTSs in terms of aggregation operators and MCDM methods. We also conduct an overview on decision making with hesitant fuzzy linguistic preference relations (HFLPRs). The applications, research challenges and future directions are also given.

Table 1 Classification of the HFLTS papers

Topics	Papers
MCDM	[7, 8, 30, 47, 49, 52, 55, 68, 100, 108, 109, 134], [31, 40, 67, 70, 80, 97, 107, 113, 120, 132, 136], [19, 26, 27, 53, 56, 62, 69, 77, 88, 141, 150], [16, 18, 25, 43, 59, 61, 89, 102, 112, 119, 138], [3, 11, 23, 34, 78, 95, 98, 104, 111, 130, 154], [10, 21, 75, 79, 91, 96, 117, 121, 133, 135, 143], [17, 32, 63, 114, 131, 149]
Aggregation operators	[33, 37, 39, 52, 58, 93, 128, 129, 136, 146], [27, 31, 53, 62, 67, 69, 77, 107, 113, 132, 150], [26, 59, 75, 95, 98, 104, 111, 112, 119, 143, 154], [10, 32, 63, 114, 131, 133, 135, 149]
GDM	[84], [8, 13, 37, 43, 61, 72, 93, 107, 109, 110, 119], [3, 17, 21, 75, 78, 98, 117, 118, 121]
HFLPR	[54, 84, 91, 94, 118, 147, 153]
Clustering	[20]

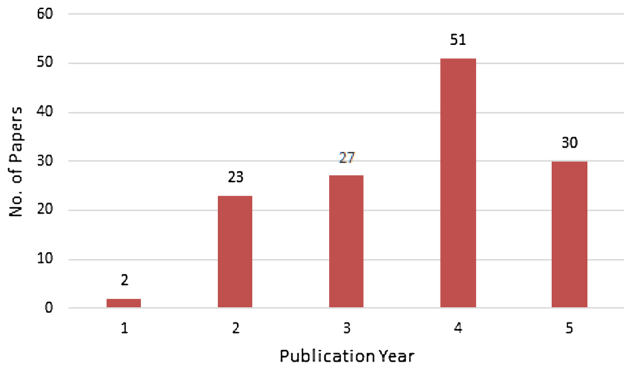

Fig. 1 Papers related to HFLTSs in Web of Sciences

The rest of this paper is organized as follows: The next section makes an overview on the HFLTSs, including the motivation, definitions, operations, comparison schemes and measures. Section 3 summarizes the different extensions of HFLTSs. Section 4 overviews the MCDM methods with HFLTSs. Section 5 discusses the HFLPR theory. Section 6 illustrates the applications of HFLTSs in different areas. Section 7 points out the current challenges and future research directions over HFLTSs. The paper ends with some concluding remarks in Sect. 8.

2 Hesitant Fuzzy Linguistic Term Set: Motivation, Definitions, Operations, Comparisons and Measures

In this section, we first clarify the motivation and definitions of HFLTS and then introduce the operations and comparison methods over HFLTSs. As the basis of some decision-making methods, we also review the distinct measures of HFLTSs.

2.1 Motivation

Experts always use linguistic term, such as "*medium*," "*high*" or "*a little high*," to represent the value of a linguistic variable and thus the fuzzy linguistic approach can be used to express and then compute over these single linguistic terms. For example, Xu [124] introduced a subscript-symmetric linguistic term set (LTS), shown as

$$S = \{s_t | t = -\tau, \ldots, 0, \cdots, \tau\}$$

where the middle term represents an assessment of "indifference." To facilitate the calculation process over the linguistic terms, the discrete LTS was further broadened to the continuous LTS $\bar{S} = \{s_t | t \in [-q, q]\}$, where $q (q > \tau)$ is a sufficiently large positive integer. This continuous LTS is known as virtual LTS [124] and only used in the calculation process. The virtual linguistic computation model is equivalent to the 2-tuple linguistic model [28] (for more discussion about these two models, please refer to [122]).

However, the traditional fuzzy linguistic approach cannot express complicated linguistic expressions, such as "*between medium and high*," "*at least a little high*," etc.

Torra [90] introduced the concept of hesitant fuzzy set (HFS) which uses a set of possible values to define the membership degree of an object to a given set. In analogous to the situations that are managed by HFSs in quantitative cases, if an individual hesitates among several values for a linguistic variable, it is adequate to introduce the HFLTS. The recent overview [81] and position paper [85] provided a snapshot on the HFSs, their applications and debates.

2.2 Definitions

Rodríguez et al. [83] introduced the concept of HFLTS, which can be used to elicit several linguistic terms or linguistic expressions for a linguistic variable.

Definition 2.1 [83] Let $S = \{s_0, s_1, \ldots, s_\tau\}$ be a LTS. A HFLTS, H_S, is an ordered finite subset of the consecutive linguistic terms of S.

Later, Liao et al. [51] redefined and formalized the HFLTS mathematically as follows, which is much easier to be understood.

Definition 2.2 [51] Let $x \in X$ be fixed and $S = \{s_t | t = -\tau, \ldots, 0, \ldots, \tau\}$ be a LTS. A HFLTS on X, H_S, is in mathematical form of

$$H_S = \{ <x, h_S(x)> | x \in X\} \tag{2.1}$$

where $h_S(x)$ is a set of some values in S and can be expressed as:

$$h_S(x) = \{s_{\phi_l}(x) | s_{\phi_l}(x) \in S; \quad l = 1, 2, \ldots, L(x); \\ \phi_l \in \{-\tau, \ldots, -1, 0, 1, \ldots, \tau\}\} \tag{2.2}$$

with $L(x)$ being the number of linguistic terms in $h_S(x)$ and $s_{\phi_l}(x)$ $(l = 1, 2, \ldots, L(x))$ in each $h_S(x)$ being the continuous terms in S. $h_S(x)$ denotes the possible degrees of the linguistic variable x to S. For convenience, $h_S(x)$ is called the hesitant fuzzy linguistic element (HFLE) and H_S being the set of all HFLEs.

Remark 2.1 Note that the LTS in Definition 2.1 is uniform, while the LTS in Definition 2.2 is subscript-symmetric. There are some other forms of labels on S ([14, 152]). This paper does not pay much attention on the distributions of S.

Remark 2.2 Note that in Definition 2.2, the linguistic terms are chosen in discrete form from S and the subscripts of $s_{\phi_l}(x)$, ϕ_l, belong to $\{-\tau, \ldots, -1, 0, 1, \ldots, \tau\}$. In order not to lose much information, we can extend it to continuous form, i.e., $\phi_l \in [-\tau, \tau]$, for the facility of calculation [49]. Of course, we can also extend it to 2-tuple form to avoid information loss of HFLTSs [107].

Definition 2.3 [107] Let $S = \{s_0, s_1, \ldots, s_\tau\}$ be a LTS and (s_l, α_l) be a 2-tuple on S, $\alpha_l \in [0, 0.5)$, $l = 1, 2, \ldots, L$. If $(s_l, \alpha_l) < (s_k, \alpha_k)$ for $l < k$, then $T_S = \{(s_l, \alpha_l) | s_l \in S, l = 1, 2, \ldots, L\}$ is a hesitant 2-tuple linguistic term set (H2TLTS) on S.

Remark 2.3 It is noted that, in Remark 2.2, we have extended the LTS associated to the HFLTS to the continuous form, i.e., $s_{\phi_l} \in [s_{-\tau}, s_\tau]$. Also note that the 2-tuple LTS is equivalent to the virtual LTS. Thus, the H2TLTS is actually equivalent to the HFLTS under the condition that the given LTS is in continuous form.

As the HFLTS is not similar to the human way of thinking and reasoning, the context-free grammar was proposed to generate simple but elaborated linguistic expressions that are much closer to the human expressions and can be easily represented by means of HFLTSs. The grammar G_H is a 4-tuple (V_N, V_T, I, P) where V_N is a set of nonterminal symbols, V_T is a set of terminal symbols, I is the starting symbol, and P is a set of production rules.

Definition 2.4 [83] Let S be a LTS, and G_H be a context-free grammar. The element of $G_H = (V_N, V_T, I, P)$ are defined as follows:

– $V_N = \{$ <primary term> , <composite term> , <unary relation> <binary relation> ,, <conjunction> $\}$;
– $V_T = \{$lower than, greater than, at least, at most, between, and, $s_{-\tau}, \ldots, s_{-1}, s_0, s_1, \ldots, s_\tau\}$;
– $I \in V_N$;
– $P = \{I ::=$ <primary term> | <composite term> ;
  <composite term> ::= <unary relation> <primary term> | <binary relation> <conjunction> <primary term> ;
  <primary term> ::= $s_{-\tau} | \cdots | s_{-1} | s_0 | s_1 | \cdots | s_\tau$;
  <unary relation> ::= lower than | greater than;
  <binary relation> ::= between;
  <conjunction> ::= and$\}$

Remark 2.4 The brackets in Definition 2.4 enclose optional elements and the symbol "|" indicates alternative elements.

Definition 2.5 [83] Given S being a LTS and S_{ll} being the expression domain generated by G_H, let $E_{G_H}: S_{ll} \rightarrow H_S$ be a function that transforms the linguistic expressions S_{ll} to the HFLTS H_S. The linguistic expression $ll \in S_{ll}$ is converted into the HFLE by means of the following transformations:

– $E_{G_H}(s_t) = \{s_t | s_t \in S\}$;
– $E_{G_H}(at\ most\ s_m) = \{s_t | s_t \in S\ and\ s_t \leq s_m\}$;
– $E_{G_H}(lower\ than\ s_m) = \{s_t | s_t \in S\ and\ s_t < s_m\}$;
– $E_{G_H}(at\ least\ s_m) = \{s_t | s_t \in S\ and\ s_t \geq s_m\}$;
– $E_{G_H}(great\ than\ s_m) = \{s_t | s_t \in S\ and\ s_t > s_m\}$;
– $E_{G_H}(between\ s_m\ and\ s_n) = \{s_t | s_t \in S\ and\ s_m \leq s_t \leq s_n\}$.

With the transformation function E_{G_H} given in Definition 2.5, it is easy to transform the initial linguistic expressions S_{ll} to the HFLTS H_S.

Remark 2.5 Some scholars also gave a graph representation of the linguistic expressions, which is another way to justify the HFLTS and can demonstrate the linguistic expressions intuitively. Please refer to Ref. [20] for details.

Example 2.1 Consider that a customer wants to buy a house and he evaluates the alternatives in terms of two criteria, i.e., x_1 (potential cost) and x_2 (comfort). Since the two criteria are qualitative, the customer gives his evaluation values in linguistic expressions. Different criteria are associated with different LTSs and different semantics. The LTSs for these two criteria are: $S_1 = \{s_{-3} = very\,expensive$, $s_{-2} = expensive, s_{-1} = a\,little\quad expensive, s_0 = medium$, $s_1 = a\,little\,cheap, s_2 = cheap, s_3 = very\,cheap\}, S_2 = \{s_{-3} = none, s_{-2} = very\,uncomfortable, s_{-1} = uncomfortable$, $s_0 = medium, s_1 = comfortable, s_2 = very\,comfortable, s_3 = perfect\}$,, respectively. With these LTSs and also the context-free grammar, the customer provides his evaluation values in linguistic expressions for a candidate house as: $ll_1 =$ *between cheap and very cheap*, $ll_2 =$*at least comfortable*. Using the transformation function E_{G_H}, a HFLTS is obtained as $H(x) = \{<x_1, h_{S_1}(x_1), <x_2, h_{S_2}(x_2)\}$ with $h_{S_1}(x_1) = \{s_2, s_3 | s_2, s_3 \in S_1\}$ and $h_{S_2}(x_2) = \{s_1, s_2, s_3 | s_1, s_2, s_3 \in S_2\}$ being two HFLEs.

Remark 2.6 Riera et al. [80] showed that the HFLTS can be interpreted as special cases of the linguistic computational model based on discrete fuzzy numbers [64] and the decision-making results would coincide the case where adequate aggregation functions and exploitation methods are employed.

There are some special HFLEs, such as [83]:

1. Empty HFLE: $h_S = \{\}$;
2. Full HFLE: $h_S = S$.

2.3 Operations

Rodríguez et al. [83] introduced the following operations for HFLEs.

Definition 2.6 [83] Let h_S, h_S^1, h_S^2 be three HFLEs. Then

1. Upper bound: $h_S^+ = max(s_i) = s_j$, $s_i \in h_S$ and $s_i \leq s_j$, $\forall i$;
2. Lower bound: $h_S^- = min(s_i) = s_j$, $s_i \in h_S$ and $s_i \geq s_j$, $\forall i$;
3. Complement: $h_S^c = S - h_S = \{s_i | s_i \in S,\ s_i \in h_S\}$;
4. Union: $h_S^1 \cup h_S^2 = \{s_i | s_i \in h_S^1 \text{ or } s_i \in h_S^2\}$;
5. Intersection: $h_S^1 \cap h_S^2 = \{s_i | s_i \in h_S^1 \text{ and } s_i \in h_S^2\}$.

Wei et al. [109] further introduced the following operations for HFLEs:

Definition 2.7 [109] Let $S = \{s_t | t = -\tau, \ldots, 0, \cdots, \tau\}$ be a LTS and h_S, h_S^1, h_S^2 be three HFLEs on S. Then

1. Negation: $\bar{h_S} = \{s_{2\tau+1-i} | i \in I(h_S)\}$;
2. Max-union: $h_S^1 \vee h_S^2 = \{max\{s_i, s_j\} | s_i \in h_S^1, s_j \in h_S^2\}$;
3. Min-intersection: $h_S^1 \wedge h_S^2 = \{min\{s_i, s_j\} | s_i \in h_S^1, s_j \in h_S^2\}$.

where $I(h_S)$ is the set of subscripts of h_S.

Considering that different HFLEs usually have different numbers of linguistic terms, to operate correctly, we can extend the shorter HFLE to make it have the same length as the larger one by adding some linguistic terms. Suppose that the linguistic terms in each HFLE are arranged in ascending order. The shorter HFLE h_S can be extended by adding the linguistic term $\bar{s} = \frac{1}{2}(h_S^+ \oplus h_S^-)$ where h_S^+ and h_S^- are the upper bound and the lower bound of the HFLE h_S, respectively. For more information about the extension rule, please refer to Ref. [49].

Definition 2.8 [153] Let S be a LTS, and $h_S^1 = \{s_{\phi_l}^1 | s_{\phi_l}^1 \in S, l = 1, 2, \ldots, L\}$ and $h_S^2 = \{s_{\phi_l}^2 | s_{\phi_l}^2 \in S, l = 1, 2, \ldots, L\}$ be two HFLEs on S. Then

1. $h_S^1 \oplus h_S^2 = \cup_{s_{\phi_l}^1 \in h_S^1, s_{\phi_l}^2 \in h_S^2} \{s_{\phi_l}^1 \oplus s_{\phi_l}^2\}$;
2. $h_S^1 \oplus h_S^2 = \cup_{s_{\phi_{(l)}}^1 \in h_S^1, s_{\phi_{(l)}}^2 \in h_S^2} \{s_{\phi_{(l)}}^1 \oplus s_{\phi_{(l)}}^2\}$,

where $s_{\phi_{(l)}}^1$ and $s_{\phi_{(l)}}^2$ are the lth linguistic terms in h_S^1 and h_S^2, respectively.

Remark 2.7 The difference between the operation laws in Definition 2.8 is the ordered positions of the operated linguistic terms. By (1), the number of linguistic terms would be increased exponentially, while by (2), the number of linguistic terms is always the same. This property is the same as the operations of hesitant fuzzy elements [48]. Normally, (2) is much more popular in calculation, especially in the calculation over HFLPRs.

To extend the operations on HFSs to those on HFLTSs, Gou et al. [23] introduced the equivalent transformation functions between the HFLE and its associate HFE.

Definition 2.9 [23] Let $S = \{s_t | t = -\tau, \ldots, 0, \cdots, \tau\}$ be a LTS, $h_S = \{s_{\phi_l} | s_{\phi_l} \in S; l = 1, 2, \ldots, L; \phi_l \in [-\tau, \tau]\}$ be a HFLE with $L(\times)$ being the number of linguistic terms in h_S, and $h_\sigma = \{\sigma_l | \sigma_l \in [0, 1]; l = 1, 2, \ldots, L\}$ be a HFE. Then, the subscript ϕ_l and the membership degree σ_l can be transformed to each other by the functions g and g^{-1} given as:

$$g : [-\tau, \tau] \to [0, 1], g(\phi_l) = \frac{\phi_l + \tau}{2\tau} = \sigma_l \tag{2.3}$$

$$g^{-1} : [0, 1] \to [-\tau, \tau], g^{-1}(\sigma_l) = (2\sigma_l - 1)\tau = \phi_l \tag{2.4}$$

Based on Definition 2.9, the HFLE h_S and the HFE h_σ can be transformed to each other:

$$\begin{aligned} G &: \mathrm{H}_S \to \Theta, G(h_S) = G\big(\{s_{\phi_l} | s_{\phi_l} \in S; \\ &l = 1, 2, \ldots, L; \phi_l \in [-\tau, \tau]\}\big) = \{\sigma_l | \sigma_l = g(\phi_l)\} = h_\sigma \end{aligned} \tag{2.5}$$

$$\begin{aligned} &G^{-1} : \Theta \to \mathrm{H}_S, G^{-1}(h_\sigma) = G^{-1}(\{\sigma_l | \sigma_l \in [0, 1]; \\ &l = 1, 2, \ldots, L\}) = \{s_{\phi_l} | \phi_l = g^{-1}(\sigma_l)\} = h_S \end{aligned} \tag{2.6}$$

With the aid of the above transformation functions, the operations on HFSs can be extended to HFLTSs. For details, please refer to Refs. [23, 24].

Remark 2.8 Although we can do the transformation of linguistic terms into real numbers mathematically via the functions in Definition 2.9, some useful information may be lost as the derived numerical values do not reflect the original properties of vague evaluations. More discussion related to this topic is about the linguistic scale functions [98].

2.4 Schemes to Compare HFLEs

How to rank the HFLEs is the foundation of many decision-making methods with HFLTSs. Thus, it is very important to determine a reasonable and convincing comparison scheme for HFLEs. Many scholars have introduced quite a lot of methods to compare any two HFLEs. This subsection gives a detailed overview on these distinct comparison schemes.

Definition 2.10 [83] Given a HFLE h_S, its envelope, $\mathrm{env}(h_S)$, is defined as

$$\mathrm{env}(h_S) = [h_S^-, h_S^+] \tag{2.7}$$

Remark 2.9 The envelop of the HFLE h_S, $\mathrm{env}(h_S)$, is actually an interval-valued linguistic term [124]. Given that the linguistic interval may lose the initial fuzzy representation, Liu and Rodríguez [55] introduced a new representation of fuzzy envelope of the HFLE, which is in the form of trapezoidal fuzzy set (for details, please refer to Ref. [55]).

Scheme 2.1 [83] *The comparison between h_S^1 and h_S^2 based on the envelop of HFLE is defined as*:

1. $h_S^1 > h_S^2$ *iff* $\mathrm{env}(h_S^1) > env(h_S^2)$;
2. $h_S^1 = h_S^2$ *iff* $\mathrm{env}(h_S^1) = env(h_S^2)$.

where the comparison between the envelopes can be conducted by the possibility formula between intervals, which is

$$p(\mathrm{env}(h_S^1) > \mathrm{env}(h_S^2)) = \frac{\max(0, h_S^{1+} - h_S^{2-}) - \max(0, h_S^{1-} - h_S^{2+})}{(h_S^{1+} - h_S^{1-}) + (h_S^{2+} - h_S^{2-})} \tag{2.8}$$

Scheme 2.1 is quite easy, but it has a significant defect that it cannot distinguish two HFLEs which have different envelopes but common linguistic terms.

Liao et al. [50] introduced the score function and the variance function of the HFLEs.

Definition 2.11 [50] For a HFLE $h_S = \{s_{\phi_l} | s_{\phi_l} \in S; l = 1, 2, \cdots, L\}$,

$$\varsigma_1(h_S) = \frac{1}{L}\sum_{l=1}^{L} s_{\phi_l} = s_{\frac{1}{L}\sum_{l=1}^{L}\phi_l} \tag{2.9}$$

is called the score function of h_S.

Remark 2.10 We should note that Zhang and Wu [146] defined the score function of a HFLE as:

$$\varsigma_2(h_S) = \frac{1}{L}\sum_{l=1}^{L}\left(\frac{\phi_l}{2\tau}\right) \tag{2.10}$$

Definition 2.12 [50] For a HFLE $h_S = \{s_{\phi_l} | s_{\phi_l} \in S; l = 1, 2, \cdots, L\}$,

$$\sigma(h_S) = \frac{1}{L}\sqrt{\sum_{l,k=1}^{L}(s_{\phi_l} - s_{\phi_k})^2} = s_{\frac{1}{L}\sqrt{\sum_{l,k=1}^{L}(\phi_l - \phi_k)^2}} \tag{2.11}$$

is called the variance function of h_S.

Scheme 2.2 [50] *The comparison between h_S^1 and h_S^2 based on the score function and the variance function of the HFLE is defined as*:

1. *if* $\varsigma(h_S^1) > \varsigma(h_S^2)$, *then* $h_S^1 > h_S^2$;
2. *if* $\varsigma(h_S^1) = \varsigma(h_S^2)$, *then*
   (a) *if* $\sigma(h_S^1) < \sigma(h_S^2)$, *then* $h_S^1 > h_S^2$;
   (b) *if* $\sigma(h_S^1) = \sigma(h_S^2)$, *then* $h_S^1 = h_S^2$.

Schemes 2.1 and 2.2 have some drawbacks as they are unreasonable to distinguish the HFLEs which have common element(s) [109]. For example, for two HFLEs $h_S^1 = \{s_3, s_4\}$ and $h_S^2 = \{s_4, s_5\}$, h_S^2 is absolutely greater than h_S^1 according to Schemes 2.1 and 2.2. However, s_4 is the possible term of a linguistic variable both for h_S^1 and h_S^2; thus, it is not convincing to say that h_S^2 is absolutely greater than h_S^1. For this reason, Wei et al. [109] proposed a comparison method for HFLEs based on the probability theory.

Definition 2.13 [109] Let h_S^1 and h_S^2 be two HFLEs on S and extend the HFLEs to h_S^{1*} and h_S^{2*}, respectively, by adding any linguistic terms in them to make them have equal length. Let $|h_{S(1,2)}^*|$ be the number of common terms in h_S^1 and h_S^2, and let $|h_{h_S^{1*} > h_S^{2*}}|$ be the number of terms in h_S^{1*} that are larger than the corresponding terms in h_S^{2*}. Then, the possibility degree of h_S^1 being not less than h_S^2 is defined as:

$$p\left(h_S^1 \geq h_S^2\right) = \frac{0.5|h_{S(1,2)}^*| + |h_{h_S^{1*} > h_S^{2*}}|}{|h_S^{1*}|} \tag{2.12}$$

where $|h_S^{1*}|$ is the cardinal number of h_S^{1*}.

Scheme 2.3 [109] *If $p\left(h_S^1 \geq h_S^2\right) > p\left(h_S^2 \geq h_S^1\right)$, then h_S^1 is superior to h_S^2 with the degree of $p\left(h_S^1 \geq h_S^2\right)$. If $p\left(h_S^1 \geq h_S^2\right) = 1$, then h_S^1 is absolutely superior to h_S^2. If $p\left(h_S^1 \geq h_S^2\right) = 0.5$, then h_S^1 is indifferent to h_S^2.*

Remark 2.11 It should be noted that some scholars also proposed other possibility degree formulas. For other, please refer to Refs. [19, 88].

After observing that Scheme 2.3 cannot distinguish the HFLEs whose average values are the same, Wei et al. [108] then introduced a new score function of HFLE, which takes into account the average linguistic term and the hesitant degree.

Definition 2.14 [108] Let $S = \{s_{-\tau}, \ldots, s_0, \ldots, s_\tau\}$ be a LTS. For a HFLE $h_S = \{s_{\phi_l} | s_{\phi_l} \in S; l = 1, 2, \cdots, L\}$, a score function of h_S is defined as:

$$\varsigma_3(h_S) = \bar{\delta} - \frac{\frac{1}{L}\sum_{l=1}^{L}(\delta_l - \bar{\delta})^2}{var(\tau)} \tag{2.13}$$

where $\bar{\delta} = \frac{1}{L}\sum_{l=1}^{L}\delta_l$ and $var(\tau) = \frac{(-\tau-\tau)^2 + \cdots + (\tau-1-\tau)^2 + (\tau-\tau)^2}{2\tau+1}$.

Scheme 2.4 [108] *The comparison between h_S^1 and h_S^2 based on the score function $\varsigma_3(h_S)$ is defined as*:

1. *if* $\varsigma_3(h_S^1) > \varsigma_3(h_S^2)$, *then* $h_S^1 > h_S^2$;
2. *if* $\varsigma_3(h_S^1) = \varsigma_3(h_S^2)$, *then* $h_S^1 = h_S^2$.

For two HFLEs h_S^1 and h_S^2, Hesamian and Shams [29] defined the preference degree P_d that h_S^1 is "*greater than*" h_S^2 as

$$P_d\left(h_S^1 \geq h_S^2\right) = \frac{\Delta_{h_S^1,h_S^2}}{\Delta_{h_S^1,h_S^2} + \Delta_{h_S^2,h_S^1}} \tag{2.14}$$

where $\Delta_{h_S^1,h_S^2} = \sum_{s_i \in h_S^1} ||\gamma_{s_i,h_S^2}||$, and $\gamma_{s_i,h_S^2} = \{s_j \in h_S^2 | s_i \geq s_j\}$ with $||A||$ being the cardinality of the set A.

Scheme 2.5 [29] *The HFLEs* h_S^1 and h_S^2 *can be compared that* $h_S^1 >_{P_d} h_S^2$ if $P_d\left(h_S^1 \geq h_S^2\right) \geq 0.5$.

There is another comparison method which is based on the pairwise distance matrix $D(h_S^1, h_S^2)$ between the different linguistic terms in h_S^1 and h_S^2. $D(h_S^1, h_S^2)$ is constructed as:

$$D(h_S^1, h_S^2) = \left(d(s_i, s_j)\right)_{L_1 \times L_2}, \quad s_i \in h_S^1,\ s_j \in h_S^2 \tag{2.15}$$

where $d(s_i, s_j) = i - j$, and L_1 and L_2 are the numbers of elements in h_S^1 and h_S^2, respectively.

Scheme 2.6 [30] *The HFLEs can be compared in analogous to Scheme 2.3 , but the preference degree is defined as:*

$$\begin{cases} p\left(h_S^1 > h_S^2\right) = \dfrac{\left|\sum_{d_{ij} > 0} d_{ij}\right|}{l_{d_{ij}=0} + \sum |d_{ij}|} \\ p\left(h_S^1 = h_S^2\right) = \dfrac{l_{d_{ij}=0}}{l_{d_{ij}=0} + \sum |d_{ij}|} \\ p\left(h_S^1 < h_S^2\right) = \dfrac{\left|\sum_{d_{ij} < 0} d_{ij}\right|}{l_{d_{ij}=0} + \sum |d_{ij}|} \end{cases} \tag{2.16}$$

Remark 2.12 Scheme 2.6 does not require that the HFLEs have equal length. It is observed that Dong et al. [14] also proposed some similar possibility degree formulas as given in Scheme 2.6.

Lee and Chen [37] introduced the concept of the 1-cut of a HFLE, which is quite similar to the concept of the envelope of a HFLE. The 1-cut of a HFLE is actually an interval whose lower and upper bounds are the subscripts of the envelope, respectively. Based on the 1-cut of each HFLE and motivated by the possibility degree of intervals, Lee and Chen [37] defined the likelihood-based comparison relation $p(h_S^1 \geq h_S^2)$ between two HFLEs h_S^1 and h_S^2 as:

$$p(h_S^1 \geq h_S^2) = \max\left(1 - \max\left(\frac{I(h_S^{2+}) - I(h_S^{1-})}{(I(h_S^{1+}) - I(h_S^{1-})) + (I(h_S^{2+}) - I(h_S^{2-}))}, 0\right), 0\right) \tag{2.17}$$

where $I(h_S)$ represents the subscripts of h_S.

Scheme 2.7 [37] *The HFLEs* h_S^i $(i = 1, 2, \ldots, n)$ *can be ranked according to the descending order of the values of* $p(h_S^i \geq S)$, $(i = 1, 2, \ldots, n)$.

Remark 2.13 Since Scheme 2.7 is based on the intervals corresponding to the HFLEs, it has the same defect as that of Scheme 2.1, i.e., it cannot distinguish the HFLEs which have common linguistic terms. In addition, both Scheme 2.1 and Scheme 2.7 cannot be used the compare the extended HFLEs which are composed of nonconsecutive linguistic terms because, in this case, we cannot obtain those intervals. Furthermore, for two HFLEs which have the same mean but different deviations, such as $h_S^1 = \{s_2, s_3, s_4\}$ and $h_S^2 = \{s_3\}$, by Scheme 2.7, these two HFLEs would be taken as indifferent. It is of course unconvincing.

To overcome these drawbacks, Tian et al. [89] introduced a new formula to measure the likelihood preference between two HFLEs h_S^1 and h_S^2, which is defined as:

$$L(h_S^1 \geq h_S^2) = \begin{cases} 0, & h_S^{1+} < h_S^{2-} \\ \dfrac{1}{L_1 L_2}\left(\displaystyle\sum_{l_1=2}^{L_1}\sum_{l_2=1}^{L_2} \frac{\phi_{l_1}}{\phi_{l_1} + \phi_{l_2} + \frac{1}{2}}\right), & h_S^{1-} \leq h_S^{2+}, h_S^{2-} \leq h_S^{1+} \text{ and } h_S^{1-} = h_S^{2-} = s_0 \\ \dfrac{1}{L_1 L_2}\displaystyle\sum_{l_1=1}^{L_1}\sum_{l_2=1}^{L_2} \frac{\phi_{l_1}}{\phi_{l_1} + \phi_{l_2}}, & h_S^{1-} \leq h_S^{2+}, h_S^{2-} \leq h_S^{1+} \text{ and } h_S^{1-} \neq s_0 \text{ or } h_S^{2-} \neq s_0 \\ 1, & h_S^{2+} < h_S^{1-} \end{cases} \tag{2.18}$$

Scheme 2.8 [89] *The HFLEs* h_S^1 *and* h_S^2 *are compared that*

1. *If* $L(h_S^1 \geq h_S^2) > 0.5$, *then* $h_S^1 > h_S^2$;
2. *If* $L(h_S^1 \geq h_S^2) < 0.5$, *then* $h_S^1 < h_S^2$;
3. *If* $L(h_S^1 \geq h_S^2) = 0.5$, *then* $h_S^1 = h_S^2$;

2.5 Measures of HFLTS

Information measures, such as the distance measures, the similarity measures, the correlation measures and the entropy measures, are used to identify the relationships between different hesitant fuzzy linguistic variables. They are the basis of many decision-making methods, and thus are essential and significantly important in forming an integral framework of hesitant fuzzy linguistic decision-making theory. Many fruitful achievements have been obtained since 2012. This subsection mainly describes the advances in this research direction.

Liao et al. [49] firstly defined the axioms of distance and similarity measures for HFLTSs. They revealed the relationship between the distance measure $d(h_S^1, h_S^2)$ and the similarity measure $\rho(h_S^1, h_S^2)$, which is

$$\rho(h_S^1, h_S^2) = 1 - d(h_S^1, h_S^2) \tag{2.19}$$

Based on the axioms, Liao et al. [49] introduced a family of distance measures between HFLTSs, including the Hamming distances, the Euclidean distances, the generalized distances, the generalized Hausdorff distances, the hybrid distances, the weighted distances, the ordered weighted distances, the continuous weighted distances, etc. All these distance measures are based on the algebra distance measures. Furthermore, Liao and Xu [47] defined a sort of cosine distance and similarity measures and their weighted forms from the geometric point of view. Observing that Liao et al. [49] failed to take into account the different numbers of values in HFLTSs, Tan et al. [88] proposed some new distance measures between HFLTSs by including the hesitant degrees of HFLTSs. After introducing the methods to construct the distance measures between HFLTSs, Zhao et al. [148] investigated the properties of different distance measures and then applied the distance measures to solve the MCDM problems.

Beg and Rashid [2] defined the distance between two HFLEs directly by the subscripts of the envelopes associated to the HFLEs. For two HFLEs h_S^1 and h_S^2, whose envelopes are $\mathrm{env}(h_S^1) = [s_p^1, s_q^1]$ and $\mathrm{env}(h_S^2) = [s_p^2, s_q^2]$, respectively, their distance is defined as:

$$d(h_S^1, h_S^2) = |q^2 - q^1| + |p^2 - p^1| \tag{2.20}$$

After that, Wang et al. [100] extended the above Hamming distance to the Euclidean distance form.

Relying on the subscripts of HFLEs, Zhu and Xu [153] gave a formula to calculate the distance between two HFLEs. Based on this formula, Xu et al. [120] developed the hesitant fuzzy linguistic weighted distance (HFLWD) operator and the hesitant fuzzy linguistic order weighted distance (HFLOWD) operator, and then investigated their properties according to the different values of the parameters. Meng and Chen [67] also defined a distance measure based on the subscripts of the HFLEs, and then developed some weighted distance measures, including the generalized hesitant fuzzy linguistic weighted distance (GHFLWD) measure and the generalized hesitant fuzzy linguistic additive Shapley weighted distance (GHFLASWD) measure. García-Lapresta and PérezRomán [20] defined the geodesic distance between HFLTSs and then proposed an agglomerative hierarchical clustering process based on the geodesic distance-based consensus measure.

Huang and Yang [30] defined another distance measure between HFLEs as the average value of the pairwise distance matrix $D(h_S^1, h_S^2)$ given by Eq. (2.15), that is,

$$d(h_S^1, h_S^2) = \frac{1}{L_1 \times L_2} \sum_{i=1}^{L_1} \sum_{j=1}^{L_2} d_{ij} \tag{2.21}$$

Based on the intersection and connected union of HFLTSs, Montserrat-Adell et al. [72] developed a lattice structure of HFLTS and then introduced two distance measures for GDM with HFLTSs. Similar to Montserrat-Adell et al. [72]'s idea, Dong et al. [13] also introduced a simple formula to calculate the number of different linguistic terms between two HFLE h_S^1 and h_S^1, where

$$d(h_S^1, h_S^2) = l_{h_S^1 \cup h_S^2} - l_{h_S^1 \cap h_S^2} \tag{2.22}$$

Based on the linguistic scale function, Wang et al. [97] introduced the hesitant directional Hausdorff distance and the generalized hesitant directional Hausdorff distance between HFLEs.

Based on Eq. (2.17), Lee and Chen [37] defined the degree of similarity $\mathrm{sim}(h_S^1, h_S^2)$ between two HFLEs as:

$$\mathrm{sim}(h_S^1, h_S^2) = 1 - |p(h_S^1 \geq S) - p(h_S^2 \geq S)| \tag{2.23}$$

where S is the given LTS.

Based on the union and intersection of HFLTSs, Hesamian and Shams [29] introduced a similarity measure between HFLTSs and investigated its properties.

Correlation measures are another type of measures to identify the relationships between different variables [42]. Liao et al. [51] proposed a sort of correlation measures of HFLTSs from the information energy point of view.

Entropy measures identify the degree of fuzziness and uncertainty of a set. After giving the properties of entropy measures of HFLTSs, Farhadinia [18] introduced some entropy measures based on the systematic transformation from the distance and similarity measures between HFLTSs, and also proposed some new entropy measures for HFLTSs on the basis of the existing entropy measures. Gou et al. [25] proposed some entropy and cross-entropy measures for HFLTSs based on the equivalent transformation function between HFS and HFLTS. They also investigated the relationships between the entropy measures and the similarity measures of HFLTSs.

3 Extensions of the HFLTS

Since the HFLTS was introduced in 2012, many scholars have proposed quite a lot of extended forms related to the hesitant linguistic model. This section tries to demonstrate these extensions in details and justifies their differences.

3.1 EHFLTS (HFLS-I) & MHFLTS

In GDM problems, each individual may hesitate among different linguistic terms when evaluating candidate alternatives. By directly combining all possible linguistic terms together in a set, Wang [93] extended the HFLTS to the so-called extended HFLTS (EHFLTS). It is observed that Zhang and Wu [146] also proposed the same concept as the EHFLTS, but they named it as the hesitant fuzzy linguistic set (HFLS-I. Here we add "I" after the "HFLS" to distinguish it from the same name of "HFLS" that was proposed by Lin et al. [52] in Definition 3.11).

Definition 3.1 [93, 146] Let $S = \{s_0, s_1, \ldots, s_g\}$ be a LTS and X be a reference set. An EHFLTS (HFLS-I) on X is a function H_S^E and

$$H_S^E = \{ <x, h_S^E(x)> | x \in X\} \tag{3.1}$$

where $h_S^E(x)$ is an ordered subset of some linguistic terms of S, represented as $h_S^E(x) = \{s_\alpha | s_\alpha \in S\}$.

Remark 3.1 We should note that even though the EHFLTS (HFLS-I) reduces to the HFLTS when the included linguistic terms are consecutive, the EHFLTS (HFLS-I) is totally different from the HFLTS in terms of their physical meanings. The EHFLTS (HFLS-I) does not make any sense to represent an individual's cognition if the terms are not consecutive. The EHFLTS (HFLS-I) does not provide linguistic expressions similar to the common language but only can be used to handle multiple linguistic terms given by different experts [82]. Anyway, as we can see, the results of the operations over the HFLTSs are sometimes not consecutive anymore and they reduce to the EHFLTS (HFLS-I). That is to say, the EHFLTS (HFLS-I) can be used as an aiding tool in the calculation process of HFLTSs, which is just in analogous to the relationship between LTS and virtual LTS [124]. Strictly speaking, the EHFLTS (HFLS-I) cannot be taken as an extension of HFLTSs, but be a concept combining the ideas of HFS and LTS.

Zhang and Wu [146] introduced the score function of the HFLS-I to compare the HFLS-Is and then defined the operations of the HFLS-Is. Wei et al. [110] defined some operations and distance measures over the EHFLTSs. Afterward, Wei et al. [106] defined two entropy measures of the EHFLTSs. Wang and Xu [95] discussed the total orders of the EHFLTSs.

It is observed that some scholars [98, 99] introduced the concept of multi-HFLTS (MHFLTS) in which $h_S^E(x)$ is a multi-subset of S. The operations on MHFLTSs are quite similar to those of EHFLTSs. The main difference between the MHFLTS and the EHFLTS is the former permits repeated linguistic terms for each linguistic variable. Wang et al. [96] introduced the likelihood function of MHFLTEs.

3.2 HFULS-I & IVHFLTS

Besides the HFLS-I, Zhang and Wu [146] also defined the hesitant fuzzy uncertain linguistic set (HFULS-I. Here we add "I" after the "HFULS" to distinguish it from the same name of "HFULS" that was proposed by Li et al. [39] in Definition 3.14).

Definition 3.2 [146] Let X be a reference set and S be a LTS. A HFULS-I on X is in terms of a function $\tilde{H}$, and

$$\tilde{H} = \{ <x, \tilde{h}_S(x)> | x \in X\} \tag{3.2}$$

where $\tilde{h}_S(x)$ is characterized by a set of some uncertain linguistic variables in S. For convenience, we call $\tilde{h}_S(x)$ the hesitant fuzzy uncertain linguistic element (HFULE-I).

The HFULS-I can be taken as the interval-valued form of HFLS-I. For example, $\tilde{h}_S = \{[s_5, s_6], [s_4, s_6], [s_1, s_3]\}$ is a HFULE. The operations on HFULE-Is can be found in Ref. [146].

In analogous, Zhu and Xu [153] introduced the interval-valued HFLTS (IVHFLTS) by assigning the interval-valued linguistic terms to the HFLTS. Later, Liang et al. [45] redefined the IVHFLTS as follows:

Definition 3.3 [45, 153] Let $\bar{S} = \{s_t | t \in [-\tau, \tau]\}$ be a LTS. An IVHFLTS is defined as $\tilde{H}_{\bar{S}} = \{ <x, \tilde{h}_{\bar{S}}(x)> | x \in X\}$ where $\tilde{h}_{\bar{S}}(x) = \{\tilde{s}_{\phi_l} | l = 1, 2, \ldots, L\}$, where $\tilde{s}_{\phi_l} = [(s_{\phi_l})^u, (s_{\phi_l})^v]$ is an interval-valued linguistic term satisfying $(s_{\phi_l})^u, (s_{\phi_l})^v \in \bar{S}$, and $(s_{\phi_l})^u \le (s_{\phi_l})^v$.

Remark 3.2 Since an uncertain linguistic variable is in fact an interval-valued linguistic term, the HFULS-I is somehow equivalent to the IVHFLTS; however, there is still a slight difference between the HFULS-I and the IVHFLTS. The HFULS-I is defined on the LTS S, while the IVHFLTS is defined on the extended LTS $\bar{S}$. That is to say, the IVHFLTS is much more general than the HFULS-I. For example, $\tilde{h}_{\bar{S}} = \{[s_{0.5}, s_{1.5}], [s_1, s_{2.5}], [s_2, s_{3.5}]\}$ is an interval-valued HFLE (IVHFLE) but not a HFULE-I, while the HFULE-I $\tilde{h}_S$ given above is also an IVHFLE.

3.3 DHHFLTS

It is observed that the traditional LTS is composed by single or simple linguistic terms. These terms cannot be utilized to represent some complex or detailed linguistic preferences such as "*a little fast*," "*90% high*" and so forth. To describe the vagueness clearly, Gou et al. [23] defined the double hierarchy LTS (DHLTS).

Definition 3.4 [23] Let $S = \{s_t | t = -\tau, \ldots, 0, \cdots, \tau\}$ and $O = \{o_k | k = -\xi, \ldots, 0, \cdots, \xi\}$ be the first and second hierarchy LTSs, respectively, and they are fully independent. A DHLTS, S_O, is defined as:

$$S_O = \{s_{t<o_k>} | t = -\tau, \ldots, 0, \ldots, \tau; k = -\xi, \ldots, 0, \ldots, \xi\} \tag{3.3}$$

where o_k represents the second hierarchy linguistic term in case the first hierarchy linguistic term is s_t. $s_{t<o_k>}$ is called the double hierarchy linguistic term (DHLT).

Motivated by the HFLTS, if someone hesitates among several DHLTs, then we can introduce the double hierarchy HFLTS (DHHFLTS).

Definition 3.5 [23] Let S_O be a DHLTS. A DHHFLTS on the reference set X, H_{S_O}, is defined as:

$$H_{S_O} = \{<x, h_{S_O}(x)> | x \in X\} \tag{3.4}$$

where $h_{S_O}(x) = \{s_{\phi_l<o_{\varphi_l}>}(x) | s_{\phi_l<o_{\varphi_l}>} \in S_O; l = 1, 2, \ldots, L; \phi_l \in \{-\tau, \ldots, 0, \cdots, \tau\}; \varphi_l \in \{-\xi, \ldots, 0, \ldots, \xi\}\}$ is a set of continuous values in S_O, denoting the possible degrees of the linguistic variable x to S_O.

Gou et al. [23] further proposed some operations on DHHFLTSs.

3.4 LDA, PDHFLTS, PHFLTS and PLTS

The HFLTS and the EHFLTS (HFLS-I) provide the possible linguistic terms of a given objective; however, they do not give the support degree of each linguistic term. Thus, some scholars have extended the HFLTS by adding another parameter to describe the intensities of the possible linguistic terms. Below we illustrate these extensions one by one.

3.4.1 LDA

Zhang et al. [142] introduced the concept of linguistic distribution assessment (LDA).

Definition 3.6 [142] Let $S = \{s_0, s_1, \ldots, s_g\}$ be a LTS, and β_k be the symbolic proportion of s_k, where $s_k \in S, \beta_k \geq 0, k = 0, 1, \ldots, g$ and $\sum_{k=0}^{g} \beta_k = 1$. The assessment $m = \{<s_k, \beta_k> | k = 0, 1, \ldots, g\}$ is called a LDA of S.

The expectation of m is defined as $E(m) = \sum_{k=0}^{g} \beta_k s_k$. For the LDAs m_1 and m_2, if $E(m_1) > E(m_2)$, then $m_1 > m_2$.

3.4.2 PDHFLTS

Assuming that each linguistic term in the HFLE has equal possibility, Wu and Xu [119] introduced the concept of possibility distribution for HFLTS (PDHFLTS).

Definition 3.7 [119] Let $S = \{s_0, s_1, \ldots, s_g\}$ be a LTS, $h_S = \{s_L, s_{L+1}, \cdots, s_U\}$ be a HFLE. The possibility distribution for h_S is represented by $P = \{p_0, p_1, \ldots, p_l, \ldots, p_g\}$, where p_l is given by the following:

$$p_l = \begin{cases} 0 & l = 0, 1, \ldots, L-1 \\ 1/(U-L+1) & l = l, l+1, \ldots, U \\ 0, & l = U, \ldots, g \end{cases} \tag{3.5}$$

and p_l denotes the possibility of the linguistic term s_l, such that $\sum_{l=0}^{g} p_l = 1$ and $0 \leq p_l \leq 1$ $(l = 0, 1, \ldots, g)$.

Remark 3.3 It is pointed out that the prevalent characteristic of the PDHFLTS is that each linguistic term included has equal possibility. From this point of view, the PDHFLTS is in fact the HFLTS but with a different form. It does not add more information than the original HFLTS. To further enhance this point, Chen et al. [9] proposed different methods to generate the possibility distribution for the PDHFLTS, for example, by using the normal distribution, the exponential distribution and their inverse forms.

3.4.3 PHFLTS

Based on Definitions 3.6 and 3.7, Chen et al. [8] proposed the concept of proportional HFLTS (PHFLTS).

Definition 3.8 [8] Let $S = \{s_0, s_1, \ldots, s_g\}$ be a LTS, h_S^k $(k = 1, 2, \ldots, n)$ be n HFLEs given by an expert group. A PHFLTS formed by the union of h_S^k $(k = 1, 2, \ldots, n)$, namely, P_{H_S}, is a set of ordered finite proportional linguistic pairs:

$$P_{H_S} = \{(s_i, p_i) | s_i \in S, i = 0, 1, \ldots, g\}$$

where $P = (p_0, p_1, \ldots, p_g)^T$ is a proportional vector and p_i denotes the possibility that the alternative carries an assessment value s_i provided by the expert group, such that $\sum_{i=0}^{g} p_i = 1$ and $0 \leq p_i \leq 1$ $(i = 0, 1, \ldots, g)$.

Remark 3.4 As we can see, the PHFLTS can be taken as the union of HFLEs and it can be used in the GDM context. If individuals furnish their assessments by HFLEs, then the group assessment can be taken as a PHFLTS.

Remark 3.5 Compared with these extensions, the HFLTS is used to represent an expert's hesitancy among several possible terms for a linguistic variable; the EHFLTS is used to represent a group of experts' collective opinions; the PHFLTS can be used either by an individual or a group of experts.

After proposing the PHFLTS, Chen et al. [8] then introduced some operation laws over the PHFLTSs.

3.4.4 PLTS

Pang et al. [75] defined the probabilistic linguistic term set (PLTS), which is quite similar to the PHFLTS.

Definition 3.9 [75] Let $S = \{s_0, s_1, \ldots, s_g\}$ be a LTS. A PLTS is defined as

$$L(p) = \{L^{(k)}(p^{(k)}) | L^{(k)} \in S, p^{(k)} \geq 0, k = 1, 2, \ldots, \#L(p), \sum_{k=1}^{\#L(p)} p^{(k)} \leq 1\} \tag{3.6}$$

where $L^{(k)}(p^{(k)})$ is the linguistic term $L^{(k)}$ with the probability $p^{(k)}$, and $\#L(p)$ is the number of all different terms in $L(p)$.

Remark 3.6 Comparing Definitions 3.8 and 3.9, we can find that the only difference between the PHFLTS and the PLTS is that the combination of the possibilities is "$= 1$" or "≤ 1".

Pang et al. [75] further proposed some operations, comparison schemes and aggregation operators for PLTSs. Some novel comparison methods for PLTSs can be found in Ref. [1]. Afterward, Zhai et al. [138] defined the probabilistic linguistic vector-term sets (PLVTSs) as a vector formula of PLTS, which considers not only the score but also the change degree of each linguistic term. Some operations of the PLVTSs were also introduced for calculation and application.

3.5 LHFS & LIHFS

Since the HFLTS cannot reflect the possible membership degrees of each linguistic term and the LDA requires to give crisp proportion value for each linguistic term, combining the HFS and the HFLTS, Meng et al. [68] developed the concept of linguistic HFS (LHFS), which presents the possible linguistic terms for a linguistic variable and also gives the possible membership degrees of each linguistic term.

Definition 3.10 [68] Let $S = \{s_0, s_1, \ldots, s_g\}$ be a LTS. A LHFS in S is a set that when applied to the linguistic terms of S it returns a subset of S with several values in [0, 1], denoted by $LH = \{(s_{\phi_l}, lh(s_{\phi_l}) | s_{\phi_l} \in S)\}$, where $lh(s_{\phi_l}) = \{r_1, r_2, \ldots, r_{m_l}\}$ is a set with m_l values in [0, 1] denoting the possible membership degrees of the element $s_{\phi_l} \in S$ to the set LH.

One example of the LHFS is given as $LH = \{(s_1, 0.1, 0.2), (s_2, 0.4, 0.5), (s_3, 0.3, 0.4)\}$, which implies that the DM hesitates to give the value 0.1 or 0.2 for the linguistic term s_1, the value 0.4 or 0.5 for the linguistic term s_2, and the value 0.3 or 0.4 for the linguistic term s_3.

Remark 3.7 The LHFS is different from the LDA as the possible membership degrees come from the hesitancy and ambiguity of the DM's cognition, while the proportion in LDA is given as percentage values raising from the DM's self-belief. In addition, the LHFS contains a set of possible membership values for each linguistic term, while the LDA only has one value for each linguistic term. We should also note that if the membership degrees of each linguistic term reduce to 1, then the LHFS reduces to a HFLE.

If $lh(s_{\phi_l})$ is represented by several intervals, then the linguistic interval HFS (LIHFS) is obtained. Meng et al. [69] gave the comparison method and operations over the LIHFSs.

3.6 HFLS-II & IVHFLS & HIFLS (HILFS or LHIFS) & DHFLS & DHFTLS & HFULS-II

Strictly speaking, the information representation forms described in this subsection do not belong to the extensions of HFLTS, because there is only one linguistic term or one linguistic interval in each form for an object.

3.6.1 HFLS-II

It is observed that Lin et al. [52] introduced the concept of hesitant fuzzy linguistic set (HFLS-II. Here we add "II" after the "HFLS" to distinguish it from the same name of "HFLS" that was proposed by Zhang and Wu [146] in Definition 3.1), which can be taken as the combination of LTS and HFS.

Definition 3.11 [52] Let S be a LTS and X be a fixed set. A HFLS-II on X is a set

$$\mathrm{HF} = \{ <x, s_{\theta(x)}, h_S(x)> | x \in X, s_{\theta(x)} \in S\} \tag{3.7}$$

where $h_S(x)$ is a set of finite numbers in [0, 1], denoting the possible membership degrees that x belongs to $s_{\theta(x)}$.

Remark 3.8 We should note that the LHFS and the HFLS-II are quite different concepts. The LHFS can be taken as the extension of HFLTS, but HFLS-II is just a simple extension of LTS with possible intensities of each terms. For example, $A = \{<x_1, s_1, \{0.1, 0.2\}>, <x_2, s_3, \{0.4, 0.6\}>\}$ is a HFLS-II, representing the possible membership degrees to which x_1 belongs to s_1 and the possible membership degrees to which x_2 belongs to s_3. For detailed comparison between LHFS and HFLS-II, please refer to Ref. [115].

Lin et al. [52] gave the operations of the HFLS-IIs, and then Wu et al. [115] proposed some new operations to overcome the limitations of those in Lin et al. [52].

3.6.2 IVHFLS

In addition, Wang et al. [101] introduced the interval-valued HFLS (IVHFLS):

Definition 3.12 [101] Let X be a reference set and S be a LTS. An IVHFLS is in mathematical form as

$$\mathrm{IVHF} = \{ < x, s_{\theta(x)}, \tilde{h}_S(x) > | x \in X, s_{\theta(x)} \in S \} \tag{3.8}$$

where the possible membership degrees that x belongs to $s_{\theta(x)}$ are represented by a set of intervals belonging to [0, 1].

After giving the definition of IVHFLS, Wang et al. [101] further introduced the operations and comparison rule for the IVHFLSs.

3.6.3 HIFLS (HILFS or LHIFS) & HIVIFLS

Observe that in Definition 3.11, $h_S(x)$ is actually a HFE. If $h_S(x)$ is represented by an intuitionistic fuzzy number [125], then the hesitant intuitionistic fuzzy linguistic set (HIFLS) is introduced. Liu et al. [58] gave the definition of HIFLS and some operations on it. Yang et al. [129] named it as the hesitant intuitionistic linguistic fuzzy set (HILFS), but the formula is actually the same as that given by Liu et al. [58]. In addition, Yang et al. [132] named it as linguistic hesitant intuitionistic fuzzy term set (LHIFS). Rashid et al. [79] also gave the definition of HIFLS and defined the generalized distance measures between two HIFLSs.

Furthermore, Yang et al. [133] defined the hesitant interval-valued intuitionistic fuzzy linguistic set (HIFLS) in which $h_S(x)$ is a interval-valued intuitionistic fuzzy number.

3.6.4 DHFLS & DHFTLS & IVDHFLS

Yang and Ju [128] introduced the dual HFLS (DHFLS):

Definition 3.13 [128] Let S be a LTS and X be a fixed set. A DHFLS on X is a set

$$\mathrm{DHF} = \{ < x, s_{\theta(x)}, h_S(x), g_S(x) > | x \in X, s_{\theta(x)} \in S \} \tag{3.9}$$

where $h_S(x)$ and $g_S(x)$ are two sets of finite numbers in [0, 1], denoting the possible membership degrees and nonmembership degrees that x belongs to $s_{\theta(x)}$, respectively, with the conditions:

$$0 \le \gamma, \zeta \le 1,\ 0 \le \gamma^+, \zeta^+ \le 1$$

where $\gamma \in h_S(x)$, $\zeta \in g_S(x)$, $\gamma^+ = \max_{\gamma \in h_S(x)} \gamma$ and $\zeta^+ = \max_{\zeta \in g_S(x)} \zeta$ for all X. For convenience, $< s_\theta, h_S, g_S >$ is called a dual hesitant fuzzy linguistic element (DHFLE).

Yang and Ju [128] gave some operations and comparison method for the DHFLEs. An example of DHFLS is $DHF = \{dhf_1, dhf_2\}$, where $dhf_1 = < s_1, \{0.3, 0.6\}, \{0.2, 0.4\} >$ and $dhf_2 = < s_4, \{0.6, 0.7\}, \{0.1, 0.2\} >$ are two DHFLEs.

Moreover, if $s_{\theta(x)}$ is replaced by the triangular linguistic variable $[s_{\theta(x)}, s_{\pi(x)}, s_{\chi(x)}]$, then the dual hesitant fuzzy triangular linguistic set (DHFTLS) is obtained [33]. If $h_S(x)$ and $g_S(x)$ are represented by intervals, then the interval-valued dual hesitant fuzzy linguistic set (IVDHFLS) is determined [113]. Both Wei et al. [113] and Qi et al. [78] introduced the definition of IVDHFLS and gave some operations on IVDHFLSs.

3.6.5 HFULS-II & HFULZN & HIFULS & DHFULS & IVHFULS

Moreover, Li et al. [39] defined the hesitant fuzzy uncertain linguistic set (HFULS-II. Here we add "II" after the "HFULS" to distinguish it from the same name of "HFULS" that was proposed by Zhang and Wu [146] in Definition 3.2).

Definition 3.14 [39] Let X be fixed and $\tilde{S}$ be an uncertain LTS. A HFULS-II is in mathematical form as

$$\mathrm{HFU} = \{ < x, \tilde{s}_{\theta(x)}, h_{\tilde{S}}(x) > | x \in X, \tilde{s}_{\theta(x)} \in \tilde{S} \} \tag{3.10}$$

where $h_{\tilde{S}}(x)$ is a set of possible membership degrees of x to the uncertain linguistic term $\tilde{s}_{\theta(x)} = [s^L_{\theta(x)}, s^U_{\theta(x)}]$.

When $s^L_{\theta(x)} = s^U_{\theta(x)} = s_{\theta(x)}$, then the HFULS-II reduces to the HFLS-II. Li et al. [39] further introduced some operations and comparison method for the HFULS-IIs. Note that Peng and Wang [77] introduced the concept of hesitant uncertain linguistic Z-number (HFLZN), which is equivalent to the HFULS-II even though the originalities of them are a little different.

Given that $h_{\tilde{S}}(x)$ is actually a HFS, if it is represented by the intuitionistic fuzzy set, then the hesitant intuitionistic fuzzy uncertain linguistic set (HIFULS) is defined [58]. Lu and Wei [62] defined the dual hesitant fuzzy uncertain linguistic term set (DHFULTS) in which $h_{\tilde{S}}(x)$ in Definition 3.14 is replaced by two membership functions. It is noted that the HIFULS is slightly different from the DHFULTS. If $h_{\tilde{S}}(x)$ in Definition 3.14 is replaced by interval-valued HFS, then the interval-valued HFULS (IVHFULS) is obtained [59, 111].

3.7 HFLRS

To represent the linguistic information efficiently, Zhang et al. [140] introduced the hesitant fuzzy linguistic rough set (HFLRS).

Definition 3.15 [140] Let S be a LTS and X, Y be two fixed set, $H \in \mathrm{HFLR}(X \times Y)$ be a HFL relation. The pair (X, Y, H) is a HFL approximation space. For any $A \in \mathrm{HFL}(Y)$, the lower and upper approximations of A with respect to (X, Y, H) are HFLTSs, shown respectively as:

$$\underline{H}(A) = \{ < x, h_{\underline{H}(A)}(x) > | x \in X \}, \tag{3.11}$$

$$\overline{H}(A) = \{ < x, h_{\overline{H}(A)}(x) > | x \in X \}, \tag{3.12}$$

where

$$h_{\underline{H}(A)}(x) = \wedge_{y \in Y} h_{H^c}(x, y) \vee h_A(y),$$
$$h_{\overline{H}(A)}(x) = \vee_{y \in Y} h_H(x, y) \wedge h_A(y).$$

The pair $(\underline{H}(A), \overline{H}(A))$ is called the HFLRS over two universes of A associated to the HFL relation H with $\underline{H}(A)$ and $\overline{H}(A)$ being the lower and upper HFL rough approximations and represented as HFLTSs.

Zhang et al. [140] further investigated the properties of HFLRSs.

4 Multiple Criteria Decision Making with HFLTSs

MCDM is a very important research issue in the field of decision analysis. For a MCDM problem, there are m alternatives $A = \{a_1, a_2, \ldots, a_m\}$, which are evaluated over n criteria $C = \{c_1, c_2, \ldots, c_n\}$ whose weights are represented as $w = \{w_1, w_2, \ldots, w_n\}$ with $w_j \geq 0$ $(j = 1, 2, \ldots, n)$, and $\sum_{j=1}^{n} w_j = 1$. The assessments of each alternative are represented by linguistic expressions, which then can be transformed to the HFLEs based on the transformation function given as Definition 2.5.

As we can see, quite a lot of MCDM methods have been extended to the HFL context. Basically, the MCDM approaches include two categories, i.e., the multi-attribute value theory (MAVT) and the outranking methods. This section mainly reviews these distinct MCDM approaches with HFLEs.

4.1 Multi-attribute Value Theory

MAVT is characterized by weighting the criteria and evaluating alternatives with respect to criteria, and then obtaining the ranking by calculating the weights and alternative assessments. Basically, we can roughly divide the methods related to MAVT into the following subtypes:

4.1.1 Hesitant Fuzzy Linguistic Aggregation Operator Based Methods

Rodríguez et al. [83] pointed out that the MCDM model with HFLTSs consists three phases, which are (1) the transformation phase, (2) the aggregation phase and (3) the exploitation phase. For the phase (1), all the linguistic expressions can be transformed to the HFLEs based on the context-free grammar and the transformation function E_{G_H}. For the phase (2), many scholars have proposed many different aggregation operators. While for the phase (3), the comparison schemes (see Sect. 2.4) can be used to rank the alternatives. As we can see, in this kind of MCDM model, the aggregation operators play a very important role and many scholars have engaged in this research direction. Thus, below we review the latest achievements related to hesitant fuzzy aggregation operators.

(1) Aggregation operators for HFLEs

To get the main information of an alternative, motivated by the pessimistic and optimistic rules, Rodríguez et al. [83] introduced the min_upper and max_lower operators as an aggregation method and finally obtained a linguistic interval for each alternative. The interval comparison law was used to exploit the final ranking in their method. Considering the pessimistic attitude and the optimistic attitude of the decision makers, Chen and Hong [7] proposed a MCDM method to aggregate the fuzzy sets in each HFLTS to a fuzzy set and then used α-cut operations to get intervals for each alternative. The likelihood method was used to rank the priorities between the obtained intervals to get the ranking of the alternatives.

Based on the convex combination of two linguistic terms, Wei et al. [109] introduced the convex combination of two HFLEs, based on which, the hesitant fuzzy linguistic weighted averaging (HFLWA) operator and the hesitant fuzzy linguistic ordered weighted averaging (HFLOWA) operator were proposed. They then applied these two operators to MCDM and GDM, respectively.

Zhang and Wu [146] introduced some aggregation operators for HFLTSs, such as the hesitant fuzzy linguistic averaging (HFLA) operator and the hesitant fuzzy linguistic geometric (HFLG) operator. Gou et al. [26] developed some hesitant fuzzy linguistic Bonferroni means operators based on the equivalent transformation functions between HFS and HFLTS.

Based on the likelihood-based comparison relation between each HFLE and the given LTS S defined as Eq. (2.17), Lee and Chen [37] introduced the HFLWA, HFLWG, HFOWA and HFLOWG operators for the

HFLEs. Based on these aggregation operators, they then developed some algorithms for MCDM and GDM with HFLEs.

Based on the proposed numerical scale model, Dong et al. [14] introduced the hesitant linguistic weighted aggregation (HLWA) operator and the hesitant linguistic ordered weighted aggregation (HLOWA) operator to fuse the hesitant fuzzy unbalanced linguistic information.

(2) Aggregation operators for the extensions of HFLEs

Scholars have also proposed some aggregation operators for the extensions of HFLTSs.

After giving the extension principle to export the operators on LTSs to EHFLTSs, Wang [93] introduced some aggregation operators for EHFLTSs. Zhang and Wu [146] developed a family of aggregation operators for the HFLS-Is, including the hesitant fuzzy linguistic weighted averaging (HFLWA) operator, the hesitant fuzzy linguistic weighted geometric (HFLWG) operator, and their ordered weighted forms, generalized forms and induced forms. They also investigated the aggregation operators for the HFULSs. Wang and Xu [95] developed the OWA operator for EHFLTSs based on the total orders of EHFLTSs.

Wei and Liao [107] introduced the MHTWA operator, the MHTOWA operator, and the MHTWOWA operator to aggregate the H2TSs.

Zhang et al. [142] developed the WA and OWA operators for the LDAs. Chen et al. [8] proposed the PHFLWA operator and the PHFLOWA operator to aggregate the PHFLTSs. Wu and Xu [119] introduced the HFLWA and HFLOWA operators for PDHFLTSs.

After introducing the operations of LHFSs, Meng et al. [68] further proposed some aggregation operators for LHFSs, including the generalized linguistic hesitant fuzzy hybrid weighted averaging (GLHFHWA) operator, the generalized linguistic hesitant fuzzy hybrid geometric mean (GLHFHGM) operator, the generalized linguistic hesitant fuzzy hybrid Shapley weighted averaging (GLHFHSWA) operator, the generalized linguistic hesitant fuzzy hybrid Shapley geometric mean (GLHFHSGM) operator, etc. Later, Yu et al. [136] improved the operations of LHFSs and introduced some Heronian aggregation operators for the LHFSs. Meng et al. [69] introduced some aggregation operators for LHFSs based on additive measures and fuzzy measures. Zhu et al. [154] developed some power aggregation operators for LHFSs and studied their properties.

For the HFLS-IIs, Lin et al. [52] proposed the weighted average operator, the ordered weighted average operator and the hybrid average operator. Wu et al. [115] further introduced some generalized prioritized aggregation operators for HFLS-IIs and then proposed a MCDM method based on these operators. Liu [53] developed the hesitant fuzzy linguistic Bonferroni mean (HFLBM) operator and hesitant fuzzy linguistic weighted Bonferroni mean (HFLWBM) operator for HFLS-IIs, while Gu et al. [27] proposed the hesitant fuzzy linguistic geometric Bonferroni mean (HFLGBM) operator and hesitant fuzzy linguistic weighted geometric Bonferroni mean (HFLWGBM) operator for HFLS-IIs. Wang and Li [104] developed the HFL Einstein Bonferroni mean operator for HFLS-IIs. Wei et al. [112] proposed the HFLWA, HFLOWA, HFLHA operators for HFLS-IIs and the DHFLWA, DHFLOWA, DHFLHA, DHFLWG, DHFLOWG, DHFLHG operators for DHFLSs. To show the interrelationships among the given values, Yu et al. [135] developed two harmonic averaging operators for HFLS-IIs, which are the HFL Maclaurin symmetric mean and the HFL weighted Maclaurin symmetric mean operator.

Wang et al. [101] proposed some prioritized aggregation operators for the IVHFLSs and discussed their properties. Zhang et al. [143] developed two Choquet integral aggregation operators and two generalized Shapley Choquet integral aggregation operators for IVHFLSs. Li et al. [39] proposed some geometric aggregation operators for the HFULS-IIs. Afterward, Huo and Zhou [31] proposed the hesitant fuzzy uncertain linguistic correlated aggregation (HFULCA) operator and hesitant fuzzy uncertain linguistic correlated geometric (HFULCG) operator for the HFULS-IIs. Zheng [150] developed the hesitant fuzzy uncertain linguistic power weighted average (HFULPWA) operator. Later, Zhao et al. [149] proposed the hesitant fuzzy uncertain linguistic power weighted geometric (HFULPWG) operator. Jin and Liao [32] also investigated the HFULPWA and HFULPWG operators. Luo et al. [63] defined the induced correlated geometric operator for HFULS-IIs. Peng and Wang [77] proposed some power aggregation operators for HULZNs. Yang and Ju [128] introduced some geometric aggregation operators and some prioritized aggregation operators for the DHFLEs. Liu et al. [58] developed some aggregation operators for HIFLSs and HIFULSs.

Wei et al. [113] proposed the IVDHFLWA operator, the IVDHFLOWA operator and the IVDHFLHA operator for the IVDHFLSs. Wei et al. [114] also proposed some geometric operator for the IVDHFLSs. Qi et al. [78] developed some generalized power aggregation operators for IVDHFLSs.

Yang et al. [129] also investigated the aggregation operators for HILFSs (the same as HIFLSs). Yang et al. [132] proposed some generalized hybrid aggregation operators for the LHIFSs (the same as HIFLSs and HILFSs). Yang et al. [131] further introduced some Choquet aggregation operators for LHIFSs. Yang et al. [133] developed some generalized aggregation operators for the

HIVIFLSs. Ju et al. [33] developed some geometric aggregation operators for the DHTFLSs and used them for MCDM. Liu et al. [59] introduced some generalized aggregation operators for the IVHFULSs. Wei [111] also proposed some aggregation operators for IVHFULSs, including the IVHFULWA, IVHFULOWA, IVHFULHA, IVHFULWG, IVHFULOWG, IVHFULHG, IVHFULCG, induced IVHFULOWA, induced IVHFULOWG, induced IVHFULCA, induced IVHFULCG, IVHFUL prioritized average, IVHFUL prioritized geometric, IVHFUL power weighted average, and IVHFUL power weighted geometric operators. Lu and Wei [62] gave some aggregation operators for the DHFULTSs.

4.1.2 Hierarchical Hesitant Fuzzy Linguistic Model

Yavuz et al. [134] proposed a hierarchical hesitant fuzzy linguistic model to handle the MCDM problem with HFLEs. In this method, the HFLEs are transformed into the envelopes and then a vector of intervals of collective preferences for the alternatives is obtained and then ranked by the preference degree of intervals. From this point of view, this hierarchical hesitant fuzzy linguistic model is indeed very simple and easy to be understood, which is quite different from the traditional AHP model.

Recently, Tüysüz and Şimşek [91] also tried to use the HFL-AHP method to solve a performance evaluation problem. However, their method is also very simple and far away from the traditional AHP framework. After constructing a set of HFLPRs, they directly obtained the pessimistic and optimistic collective preference relations in the form of 2 tuple linguistic forms, without any aggregation process. Then, the interval utility of each alternative was obtained irrationally. The alternatives were ranked according to the so-called midpoints of the intervals, which also makes the method not convincing. Thus, this method cannot be taken as regular HFL-AHP method given that it does not conduct any work on consistency checking, consistency improving and prioritization processes.

García-Lapresta and PérezRomán [20] proposed an agglomerative hierarchical clustering process for the problem in which the agents evaluate the alternatives by linguistic terms or HFLTS and then applied this method to the field experiment to assess five fruits' suitability to combine with dark chocolate.

4.1.3 HFL-TOPSIS & HFL-VIKOR & HFL-MULTIMOORA

Beg and Rashid [2] introduced a hesitant fuzzy linguistic TOPSIS (HFL-TOPSIS) method for MCDM with HFLEs based on the distance measure between the envelopes of HFLEs. After giving a family of distance and similar measures, Liao et al. [49] proposed a satisfaction degree-based method to solve the hesitant fuzzy linguistic MCDM problems. This method is a little similar to the HFL-TOPSIS method proposed by Beg and Rashid [2], but the distance measures they used are quite different. Liao et al. [50, 141] further investigated the HFL-VIKOR method for the MCDM problems in which the criteria are conflicting with each other and no optimal solutions exist but only compromise solutions. Later, Liao and Xu [47] proposed some cosine distance measures of HFLTSs and then investigated the HFL-TOPSIS and HFL-VIKOR methods based on these cosine distance measures. After developing a weight determining method, Li et al. [40] investigated the distance-based HFL-TOPSIS method for individual research output evaluation. Based on the introduced distance measures, Tan et al. [88] improved the HFL-TOPSIS method. Based on the trapezoidal fuzzy envelope of HFLE, Liu and Rodríguez [55] used the HFL-TOPSIS method to solve the supplier selection problem in which all the HFLEs were transferred to the envelopes in the form of trapezoidal fuzzy sets. Farhadinia [18] proposed an entropy-based method to derive the weights of criteria and then developed a MCDM algorithm based on the HFL-VIKOR method.

Wei et al. [110] studied the TOPSIS-based method for the MCDM problems in which the assessments are given in EHFLTSs. Beg and Rashid [3] proposed a TOPSIS-based method for group MCDM problems with H2TLTSs. Pang et al. [75] developed a TOPSIS-based MCDM method with PLTSs. Meng and Chen [67] developed a method for multi-granularity decision making with hesitant fuzzy linguistic information, and the main idea of this method is similar to the HFL-TOPSIS method. Ghadikolaed et al. [21] investigated the extensions of VIKOR method within the context of EHFLTS based on the comparison laws and distance measures of EHFLTSs. After introducing some Hausdorff distance between HFLS-IIs, Wang et al. [102] proposed a VIKOR based method for MCDM with HFLS-IIs. Dong et al. [12] extended the VIKOR method to linguistic hesitant fuzzy sets. Yang et al. [130] introduced the distance measures between LHIFS and then extended the TOPSIS and VIKOR methods to accommodate linguistic hesitant intuitionistic fuzzy information. Gou et al. [23] calculated the distance between the reference point and each alternative whose evaluation values are represented by DHHFLTSs, and then introduced a double hierarchy hesitant fuzzy linguistic MULTIMOORA (DHHFL-MULTIMOORA) method.

4.1.4 Possibility Degree Matrix-Based Method

Based on the different possibility degree measures given in Sect. 2.4, we can calculate the possibility degree of the alternative a_i over the alternative a_k as:

$$\pi(a_i, a_k) = \sum_{j=1}^{n} p(h_S^{ij}, h_S^{kj})\omega_j \tag{4.1}$$

where $p(h_S^{ij}, h_S^{kj})$ is the possibility degree between the alternatives a_i and a_k on the criterion c_j and ω_j is the weight of the criterion c_j. It is proven that $\pi(a_i, a_k) + \pi(a_k, a_i) = 1$ [19]. Thus, the matrix $(\pi(a_i, a_k))_{m\times m}$ is a fuzzy preference relation. Then, we can use some priority determining methods to derive the ranking of alternatives.

Besides the above method, Gou et al. [25] introduced a hesitant fuzzy linguistic alternative queuing method, which is quite similar to the possibility degree matrix-based method, but the weights of criteria were derived based on the entropy measures of HFLTSs.

4.1.5 HFL-LINMAP Method

The linear programming technique for multidimensional analysis of preference (LINMAP) is a widely used MCDM method which constructs a linear programming based on the defined consistency and inconsistency indices to derive the ideal solution and criterion weights. Based on the introduced hesitant fuzzy linguistic weighted distance (HFLWD) operator, Xu et al. [121] proposed the HFL-LINMAP method based on the traditional LINMAP method. The basic idea of the HFL-LINMAP method involves the following steps: (1) build the HFL decision matrix and determine the HFL positive ideal solution (HFLPIS); (2) get the preferences over the alternatives from decision makers; (3) calculate the introduced measures (such as the distance measure, the correlation measure, etc.) between the HFLPIS and each alternative to obtain the inconsistency and consistency indices, and then construct a programming model based on these indices to derive the weights of criteria; (4) rank the alternatives with respect to the values of measures. The objective of the linear programming is to minimize the total hesitant fuzzy linguistic inconsistency index.

Liu et al. [61] extended the LINMAP method to solve the group MCDM problems with LHFSs. Liao et al. [43] investigated the LINMAP method with PLTSs.

4.1.6 Satisfactory-Based Interactive Method

Motivated by Liao and Xu [46]'s work, Da and Xu [11] proposed a satisfactory-based interactive method for the HFL-MCDM problems. In this method, the weights of criteria are completely unknown or partly unknown. We first define a formula of satisfactory degree of each alternative, and then build some optimization models to maximize the satisfactory degree of each alternative and then derive the weights of criteria. The ranking of alternatives can be obtained according to the aggregated values of the alternatives.

Zhou et al. [151] introduced an evidential reasoning-based method for MCDM with LHFSs. Sun et al. [87] proposed a projection-based multi-attributive border approximation area comparison (MABAC) method with HFLTSs.

4.2 Outranking Methods

Some scholars have investigated the outranking methods for the HFL-MCDM problems. The outranking methods are built based on the binary comparisons which further lead to the concordance and discordance indices.

4.2.1 HFL-PROMETHEE

Based on Eq. 4.1, the positive outranking flow and the negative outranking flow of the alternative a_i can be calculated as [19]:

$$\begin{cases} \psi^{+}(a_i) = \dfrac{1}{m-1}\sum_{k=1}^{m} \pi(a_i, a_k) \\ \psi^{-}(a_i) = \dfrac{1}{m-1}\sum_{k=1}^{m} \pi(a_k, a_i) \end{cases} \tag{4.2}$$

Then the alternatives can be ranked by the net outranking flow of the alternatives. Feng et al. [19] developed the HFL-PROMETHEE method. Later, for the group MCDM problem whose judgments are represented by HIFLTSs, Faizi et al. [17] proposed an outranking method to rank the alternatives based on the net outranking flow index of each alternative, which is combined by the support function, risk function and credibility function of HIFLTS.

4.2.2 HFL-TODIM

Motivated by the traditional TODIM method, Wei et al. [108] introduced the HFL-TODIM method. The main idea of this method is to compute the dominance degrees between the alternatives a_i and a_k by the prospect value function as shown below:

$$\Theta_j(a_i, a_k) = \begin{cases} \sqrt{\dfrac{w_{jr} d(h_{ij}, h_{kj})}{\sum_{j=1}^{n} w_{jr}}}, & \varsigma_3(h_{ij}) > \varsigma_3(h_{kj}) \\ 0, & \varsigma_3(h_{ij}) = \varsigma_3(h_{kj}) \\ -\dfrac{1}{\theta}\sqrt{\left(\sum_{j=1}^{n} w_{jr}\right)\dfrac{d(h_{ij}, h_{kj})}{w_{jr}}}, & \varsigma_3(h_{ij}) < \varsigma_3(h_{kj}) \end{cases} \tag{4.3}$$

where $d(h_{ij}, h_{kj})$ is a distance measure, denoting the gain of the alternative a_i over a_k with respect to the criterion c_j if $\varsigma_3(h_{ij}) > \varsigma_3(h_{kj})$ and the loss if $\varsigma_3(h_{ij}) < \varsigma_3(h_{kj})$. θ denotes the attenuation parameter of the losses. w_{jr} is the relative weight of the criterion c_j to the reference criterion c_r.

Aggregating all $\Theta_j(a_i, a_k)$ with respect to all criteria derives the dominance degree $\Theta(a_i, a_k)$ of the alternative a_i over a_k, and the overall dominance degree of each alternative can be calculated and the ranking of the alternatives can be derived.

After proposing the likelihood function of MHFLTEs, Wang et al. [96] extended the TODIM method to the context that each judgment value is represented in MHFLE.

4.2.3 HFL-ELECTRE

By combining the classical ELECTRE (*Elimination et Choix Traduisant la Réalité*, means elimination and choice that translates reality) I method with the HFLTSs, Wang et al. [100] introduced the HFL-ELECTRE method and then applied it to choose the best anti-air information warfare system. Later, Fahmi et al. [16] also investigated the HFL-ELECTRE method and they divided it into the HFLTS operation phase and the ELECTRE I outranking phase. The main idea of the HFL-ELECTRE method is illustrated as follows:

Let $S = \{s_0, s_1, \ldots, s_g\}$ be a LTS, and h_S^1 and h_S^2 be two HFLEs. The degree to which h_S^1 outranks h_S^2 is defined as [100]:

$$r\left(h_S^1, h_S^2\right) = \frac{1}{L_1 L_2} \sum_{s_i \in h_S^1, s_j \in h_S^2} P\left(s_i, s_j\right) \tag{4.4}$$

where the binary relation P for s_i and s_j is defined as:

$$P\left(s_i, s_j\right) = \begin{cases} 1, & \text{if } s_i > s_j \\ 0, & \text{if } s_i > s_j \end{cases} \tag{4.5}$$

(1) If $r(h_S^1, h_S^2) = 1$, then h_S^1 strongly dominates h_S^2, denoted as $h_S^1 >_s h_S^1$; (2) If $0 \le r(h_S^1, h_S^2) < 1$, then h_S^1 weakly dominates h_S^2, denoted as $h_S^1 >_w h_S^1$; (3) If $h_S^{1+} = h_S^{2+}$ and $h_S^{1-} = h_S^{2-}$, i.e., $h_S^1 = h_S^2$ according to Scheme 2.1, then h_S^1 is indifferent to h_S^2, denoted as $h_S^1 \sim h_S^1$; (4) If none of the above relations satisfied, then h_S^1 and h_S^2 are incomparable, denoted as $h_S^1 \perp h_S^1$.

The concordance index between a_i and a_k is

$$c_{ik} = \sum_{j \in C_s(a_i, a_k)} w_j + \sum_{j \in C_w(a_i, a_k)} w_j r(a_{ij}, a_{kj}) \tag{4.6}$$

where $C_s(a_i, a_k) = \{j | a_{ij} >_s a_{kj}\}$, $C_w(a_i, a_k) = \{j | a_{ij} >_w a_{kj}\}$.

The discordance index between a_i and a_k is

$$d_{ik} = \frac{\max\limits_{j \in D_s(a_i, a_k) \cup D_s(a_i, a_k)} w_j d(a_{kj}, a_{ij})}{\max\limits_{j \in J} w_j d(a_{kj}, a_{ij})} \tag{4.7}$$

where $D_s(a_i, a_k) = \{j | a_{ij} <_s a_{kj}\}$, $C_w(a_i, a_k) = \{j | a_{ij} <_w a_{kj}\}$.

The net concordance index is defined as:

$$c_i = \sum_{k=1, k \ne i}^{m} c_{ik} + \sum_{k=1, k \ne i}^{m} c_{ki} \tag{4.8}$$

The net discordance index is defined as:

$$d_i = \sum_{k=1, k \ne i}^{m} d_{ik} + \sum_{k=1, k \ne i}^{m} d_{ki} \tag{4.9}$$

The net concordance index denotes the degree to which a_i dominates all the other alternatives, while the net discordance index implies the intensity that a_i is inferior to all the other alternatives. Thus, c_i should be as great as possible, while d_i should be as small as possible.

Khishtandar et al. [34] slightly modified the final aggregation process of the above HFL-ELECTRE method and then used it to assess bioenergy production technologies. On the basis of the introduced linguistic scale function and the distance measures, Wang et al. [97] defined some new versions of formulas of the strong dominance, weak dominance, indifference and incomparable relation, and then a different version of the HFL-ELECTRE outranking method was developed. Rashid et al. [79] investigated the ELECTRE-based MCDM method in which the evaluation values are given in HIFLSs.

4.2.4 HFL-QUALIFLEX

Qualitative flexible multiple criteria method (QUALIFLEX) is another useful outranking method, which is also based on the concordance index and the discordance index. However, different from the ELECTRE method that calculates the concordance index and the discordance index via the combined weights of different types of criteria, the QUALIFLEX identifies the concordance index and the discordance index via the differences between the objective ranking and the supposed pre-order between the alternatives. Liu et al. [56] extended the QUALIFLEX into the context of H2TLTS context. Based on the introduced likelihood preference measure given as Eq. (2.18), Zhou et al. [89] introduced the HFL-QUALIFLEX method.

According to Scheme 2.8, if the pre-order between two alternatives a_i and a_k with respective to the criterion c_j is equal, then we should have $L(h_{ij} \ge h_{kj}) = L(h_{ij} \le h_{kj}) = 0.5$. Thus, for any permutation P_f of the alternatives, we can use the index $I_j^f(a_i, a_k) = L(h_{ij} \ge h_{kj}) - 0.5$ to measure the concordance or discordance between the pre-order of alternatives and the objective ranking derived by some methods. (1) If $I_j^f(a_i, a_k) > 0$, then the concordance exists; (2) If $I_j^f(a_i, a_k) = 0$, then the ex aequo exists; (3) If $I_j^f(a_i, a_k) < 0$, then the discordance exists.

Based on the above analysis, the weighted concordance/discordance index between the pre-order of any permutation P^f and the objective ranking is

$$I^f(a_i, a_k) = \sum_{j=1}^{n} \left(L(h_{ij} \geq h_{kj}) - 0.5\right) w_j \tag{4.10}$$

Thus, the overall concordance/discordance index for the permutation P^f is

$$I^f = \sum_{j=1}^{n} \sum_{a_i, a_k \in A} \left(L(h_{ij} \geq h_{kj}) - 0.5\right) w_j \tag{4.11}$$

The larger the value of I^f is, the more reliable of the final ranking result should be. Thus, the best ranking is obtained by comparing the value of I^f for each permutation, i.e., $I^* = \max_{f=1}^{m!}\{I^f\}$.

Remark 4.1 Besides the above four outranking methods with HFL information, recently, Liao et al. [44] investigated another outranking method, namely the HFL-ORESTE method. After introducing a knowledge-based paradigm for comparing HFLTSs, Sellak et al. [86] proposed a novel outranking method.

Remark 4.2 Some scholars investigated the multi-expert multi-criteria decision-making (MEMCDM) model with HFLEs. Such type of problems also can be taken as the group MCDM (GMCDM) problems. Montes et al. [70] studied this problem with the help of the 2-tuple linguistic representation model. Different decision matrices were constructed in HFLEs and then aggregated to a collective matrix by *min_upper* operator and *max_lower* operator as well as the arithmetic weighted extended mean. Liu et al. [61] investigated the GMCDM method with LHFSs based on the extended LINMAP method. Liao et al. [43] studied the GMCDM problem based on the PL-LINMAP method. Wu and Xu [119] studied the GMCDM with PDHFLTSs by considering the consensus among different decision makers.

5 Hesitant Fuzzy Linguistic Preference Relation Theory

When evaluating a set of alternatives $X = \{x_1, x_2, \ldots, x_n\}$, sometimes, the experts prefer to provide pairwise comparison judgments of the alternatives and then construct a preference relation. The hesitant fuzzy linguistic preference relation (HFLPR) has been investigated by many scholars since it was proposed. This section mainly reviews the HFLPR in terms of definition, consistency and consensus for GDM.

5.1 Definition of HFLPR

Let $S = \{s_t | t = -\tau, \ldots, 0, \cdots, \tau\}$ be a LTS. Rodríguez et al. [84] firstly introduced the concept of hesitant fuzzy linguistic preference relation (HFLPR) in which each element is originated from the comparative linguistic expression. After transforming the comparative linguistic expressions to HFLEs, the HFLPR they proposed was then represented by the linguistic intervals based on the envelope of each HFLE and some aggregation operators for linguistic intervals were used for aggregation.

Note that the HFLPR proposed by Rodríguez et al. [84] is not symmetric. Afterward, Liu et al. [54] further improved the definition of HFLPR based on the traditional linguistic fuzzy preference relation. The elements p_{ij} $(i, j = 1, 2, \ldots, n)$ in the HFLPR $P = \left(p_{ij}\right)_{n\times n}$ they defined involve the following forms:

1. a single term $s_l \in S$;
2. the expression "at least s_l," $s_l \in S$;
3. the expression "at most s_l," $s_l \in S$;
4. the expression "between s_k and s_l," $s_k < s_l$, $s_k, s_l \in S$.

All the elements in an HFLPR should satisfy the reciprocal property, i.e.,

1. $p_{ii} = s_0$;
2. If $p_{ij} = s_l$, then $p_{ji} = s_{-l}$;
3. If $p_{ij} =$ "at least s_l," then $p_{ji} =$ "at most s_{-l}";
4. If $p_{ij} =$ "at most s_l," then $p_{ji} =$ "at least s_{-l}";
5. If $p_{ij} =$ "between s_k and s_l," then $p_{ji} =$ "between s_{-l} and s_{-k}."

Zhu and Xu [153] defined the HFLPR on S in mathematical form.

Definition 5.1 [153] A HFLPR H is presented by a matrix $H = (h_{ij})_{n\times n} \subset X \times X$, where $h_{ij} = \{s_{\phi_l} | s_{\phi_l} \in S; l = 1, 2, \ldots, L_{ij}\}$ (L_{ij} is the number of linguistic terms in b_{ij}) is a HFLE, indicating the hesitant degrees to which x_i is preferred to x_j. For all $i, j = 1, 2, \ldots, n$, $h_{ij} (i < j)$ should satisfy the following conditions:

$$h_{ij}^{\phi_l} \oplus h_{ji}^{\phi_l} = s_0, h_{ii} = \{s_0\}, L_{ij} = L_{ji}, h_{ij}^{\phi_l} < h_{ij}^{\phi_{(l+1)}}, h_{ji}^{\phi_{(l+1)}} < h_{ji}^{\phi_l} \tag{5.1}$$

where $h_{ij}^{\phi_l}$ and $h_{ji}^{\phi_l}$ are the lth elements in h_{ij} and h_{ji}, respectively.

Remark 5.1 The HFLPR defined by Rodríguez et al. [84] is not symmetric with respect to the diagonal, while the HFLPR introduced by Zhu and Xu [153] is symmetric regarding to the diagonal. Also note that in Definition 5.1, the linguistic terms in h_{ij} $(i < j)$ are arranged in ascending order while the linguistic terms in h_{ji} $(i < j)$ are arranged in descending order.

If all the elements in the HFLPR are extended to the same length, then this HFLPR is called the normalized HFLPR (NHFLPR) [153].

Observe that Wang and Xu [94] proposed the extended HFLPR in which all the elements are represented by EHFLEs. The addition operation shown as (1) in Definition 2.8 is used for the EHFLPR, while the addition operation shown as (2) in Definition 2.8 is utilized for the HFLPR.

5.2 Consistency of HFLPR

Consistency related to the transitivity of the opinions over the objects is a basic property of any preference relations. Some scholars have studied the consistency of the NHFLPR.

5.2.1 Additive Consistency

After transforming the HFLTS to the linguistic 2-tuple form, Liu et al. [54] investigated its additive consistency based the additive consistency of the traditional linguistic fuzzy preference relation and then proposed a method to improve the inconsistent ones. Zhu and Xu [153] gave a simple definition of additive consistency of a HFLPR as follows:

Definition 5.2 [153] Given a HFLPR $H = (h_{ij})_{n\times n}$ and its corresponding NHFLPR$\bar{H} = (\bar{h}_{ij})_{n\times n}$, if

$$\bar{h}_{ij}^{\phi(l)} = \bar{h}_{ik}^{\phi(l)} \oplus \bar{h}_{kj}^{\phi(l)} \; (i,j,k = 1,2,\ldots,n) \tag{5.2}$$

then $H = (h_{ij})_{n\times n}$ is an additive consistent HFLPR.

Wang and Xu [94] also defined an additive consistency condition for the EHFLPR as:

$$h_{ij} \cong h_{ik} \oplus h_{kj}, \quad (i,j,k = 1,2,\ldots,n) \tag{5.3}$$

where "≅" denotes that the means of the two HFLEs are equal. They further pointed out that $\dot{H} = (\dot{h}_{ij})_{n\times n}$ is additive if

$$\dot{h}_{ij} = \begin{cases} \frac{1}{n} \oplus_{k=1}^{n} (h_{ik} \oplus h_{kj}), & i,j = 1,2,\ldots,n, i \neq j \\ s_0, & \text{otherwise} \end{cases} \tag{5.4}$$

Wang and Xu [94] then proposed a method to derive the reduced linguistic preference relation with the highest additive consistency from the EHFLPR. The weak consistency of an EHFLPR was also introduced.

Based on the 2 tuple linguistic model and the possibility distribution defined as Eq. (3.5), Wu and Xu [118] gave the additive consistency condition of the HFLPR $H = (h_{ij})_{n\times n}$ as

$$\Delta E(h_{ij}) = \Delta E(h_{ik}) \oplus \Delta E(h_{kj}), \quad (i,j,k = 1,2,\ldots,n) \tag{5.5}$$

where Δ is a function that returns the associated linguistic 2-tuples of $E(h_{ij})$ and $E(h_{ij})$ is the mean of h_{ij}.

Based on the distance measure between $\dot{H}$ and H, Zhu and Xu [153] introduced the consistency index of the HFLPRs H as

$$\mathrm{CI}(H) = d(\dot{H}, H) = \sqrt{\frac{2}{n(n-1)} \sum_{i<j}^{n} (d(\bar{h}_{ij}^{1}, \bar{h}_{ij}^{2}))^2} \tag{5.6}$$

where $\bar{h}_{ij}$ and $\dot{h}_{ij}$ are defined as Eqs. (5.2) and (5.4), respectively. Wu and Xu [118] introduced another consistency index, but their definitions are quite similar to Eq. (5.6). Based on the consistency index, some inconsistency improving algorithms can be proposed.

5.2.2 Multiplicative Consistency

The additive consistency property sometimes is unreasonable when we directly add two linguistic terms together [147]. Thus, the multiplicative consistency of HFLPR was proposed.

Definition 5.3 [147] Given a HFLPR $H = (h_{ij})_{n\times n}$ and its corresponding NHFLPR $\bar{H} = (\bar{h}_{ij})_{n\times n}$, if

$$I\left(\bar{h}_{ik}^{\phi_l}\right) I\left(\bar{h}_{kj}^{\phi_l}\right) I\left(\bar{h}_{ji}^{\phi_l}\right) = I\left(\bar{h}_{ij}^{\phi_l}\right) I\left(\bar{h}_{jk}^{\phi_l}\right) I\left(\bar{h}_{ki}^{\phi_l}\right) \quad (i,j,k = 1,2,\ldots,n) \tag{5.7}$$

where I $\left(\bar{h}_{ij}^{\phi_l}\right)$ is the subscript of the lth element in $\bar{h}_{ij}$, then $H = (h_{ij})_{n\times n}$ is an additive consistent HFLPR.

Zhang and Wu [147] defined the distance between two HFLPRs, based on which, a consistency index of the HFLPR was proposed:

$$\mathrm{CI}(H) = 1 - d(\bar{H}, \tilde{H}) \tag{5.8}$$

Given a threshold value $\bar{CI}$, the acceptable multiplicative consistent HFLPR satisfies $\mathrm{CI}(H) \geq \bar{CI}$. Zhang and Wu [147] further proposed an approach to repair the inconsistent HFLPR. The ranking of alternatives was derived by directly aggregating the HFLEs in their method.

5.3 Consensus and GDM with HFLPRs

Based on the distance formula shown as Eq. (2.22), Dong et al. [13] defined the consensus level of a DM d_i.

Definition 5.4 [13] let A_i represent the individual opinion of d_i and $\bar{A}$ represent the group opinion, then the consensus level of d_i is defined as:

$$\mathrm{CL}_i = 1 - \frac{d(A_i, \bar{A})}{l_{A_i \cup \bar{A}}} = \frac{l_{A_i \cap \bar{A}}}{l_{A_i \cup \bar{A}}} \tag{5.9}$$

where CL_i means the proportion of the same terms between A_i and $\bar{A}$.

With the above consensus level, Dong et al. [13] then proposed a two-stage algorithm to get the consensus reaching suggestions for the HFL-GDM problems, which can minimize the number of iterations. Zhang et al. [139] developed a minimum adjustment distance consensus rule for consensus reaching process of GDM with HFLPRs.

Generally, there are two categories of consensus measures: One is based on the distances between experts, and the other is based on the distance between each individual preference to the collective preference [41]. For the first type, Wu and Xu [118] introduced the similarity measure between any two HFLTSs based on the distance between the means of these two HFLTSs; then some similarity matrices can be constructed to identify the similarity degrees between any pair of experts; by aggregating the similarity matrices, the consensus degree of the group of experts can be calculated. Wu and Xu [118] further provided a feedback mechanism to improve the consensus degree of a group. Montserrat-Adell et al. [73] defined a collective degree of consensus and an individual degree of consensus respectively and then made a comparison over the existing consensus measures for GDM with HFLPRs.

In addition, Wu and Xu [117] used the HFLWA operator to fuse the group preference values and then defined the consensus degree as the total similarity degree of all individual judgment matrices to the collective judgment matrix. They then proposed an interactive consensus reaching model and compared it with that developed in Ref. [118]. It was found that this model is less restrictive and doing less rounds of iterations than that in Ref. [118].

Based on the concept of LDA, Zhang et al. [142] investigated the consistency and consensus of the distribution linguistic preference relations.

Compatibility is another topic in group decision making with preference relations. Gou et al. [22] defined some compatibility measures for HFLPRs and then developed a method to solve the HFL-GDM problems.

Besides the consistency, consensus and compatibility, how to derive the priorities of alternatives from the HFLPR is also an essential research issue. In this regard, Wang and Gong [103] established the ranking of objects based on chance-restricted programming.

As an extension of the HFLPR, the consistency and consensus of the probabilistic linguistic preference relations (PLPR) have been investigated by many scholars [144, 145].

6 Applications of the HFLTSs

As a flexible tool in representing people's qualitative and subjective cognitions, the HFLTS has gained great success in practical applications. Table 2 lists the recent application areas of HFLTS.

From Table 2, we can find that the HFLTS has been implemented to human resource management, investment management, supply chain management, business management, recommender system, healthcare management, telecommunication management, electrical power systems management, emergency management, resources management and so on. It seems that any evaluation problems with qualitative assessment information can be solved by the hesitant fuzzy linguistic decision-making approaches. With the promotion and development of the research on hesitant linguistic theory, more and more practical problems can be solved by this comprehensively qualitative tool.

7 Challenges and Future Research Directions Over HFLTSs

After the comprehensive review on the advances of hesitant fuzzy linguistic theory, we can summerize the challenges in this field and some potential research directions in the coming future, which are highlighted as follows:

1. As we can see, many scholars have introduced different kinds of extensions of HFLTSs. Even though some of them are appropriate and reasonable, many extension forms are far complex and do not have good applicable potentials. We need to unify some of the extensions together.
2. It is observed that for different extensions of HFLTSs and even for the definition of HFLTS itself, the given LTSs are quite different. Thus, it is challenging for us to find a unified LTS to act as a standard for the expert's evaluation. Meanwhile, as different people have different cognitions over the given problem, it is also common that the experts may use different numbers of linguistic terms, also named as multi-granular linguistic scales, to evaluate the decision-making problems. An overview about multi-granularity linguistic decision making can be found in [126].
3. As we can see from Table 1 as well as Sect. 4, there are many scholars crowed in the direction of hesitant fuzzy linguistic MCDM. However, more than half of these scholars were focused on the aggregation methods of HFLTSs. In addition, it is the truth that all of these aggregation operators are based on the distinct

Table 2 Applications of the HFLTSs

Applications	Papers	Year
Investment management	[2]	2013
	[101, 115, 146],	2014
	[33, 58, 128]	
	[67, 120, 132],	2015
	[19, 80]	
	[3, 98, 102, 152]	2016
	[63, 75, 118, 121]	
	[135]	2017
Human resource management	[109, 146]	2014
	[93, 136, 150]	2015
	[8, 69, 88]	2016
	[61, 119, 154]	
	[79]	2017
Enterprise management	[129],	2014
	[70]	2015
	[62, 104, 130]	2016
	[112, 149]	
	[131]	2017
Logistics management	[55]	2014
	[97, 110]	2015
	[16, 69, 96, 114]	2016
	[6, 117]	
	[91]	2017
Healthcare management	[51, 107]	2015
	[26, 43, 138]	2016
	[56, 141]	
	[1, 24, 25]	2017
ERP system and software selection	[39, 52]	2014
	[47, 50]	2015
	[77, 89, 111]	2016
Energy management	[53, 134],	2015
	[95]	2016
	[34]	2017
Education management	[84]	2013
	[40]	2015
	[32]	2016
Emergency management	[27]	2015
	[59, 78, 143]	2016
Movie recommender system	[49]	2014
	[18]	2016
Building construction management	[31]	2015
	[17]	2017
Telecommunications service selection	[108]	2015
	[21]	2017
New technology management	[68, 100]	2014
	[10]	2016
	[71]	2017
Environment management	[23]	2017
Field experiment management	[20]	2016
Person-job fit problem	[140]	2016
Urban management	[11]	2016

traditional aggregation operators under quantitative circumstances but hardly take into account the hesitant fuzzy linguistic characteristics. In other words, most of the existing aggregation operators are based on the calculation on subscripts or labels of the HFLEs, but do not consider the semantics of the hesitant fuzzy linguistic information deeply. In addition, the existing aggregation operators do not consider the rationality of aggregation. For example, it is not convincing to fuse the heterogeneous information together. From this point of view, we need to develop more reasonable aggregation operators for HFLEs.

4. For the MCDM methods, besides the aggregation operators, some scholars have also proposed different approaches to handle the HFL-MCDM problems. However, as we have discussed in this paper, those proposed methods are just a simple attempt and we still need to find more reasonable methods. For example, we need to build a more comprehensive framework of hierarchical hesitant fuzzy linguistic model. The evidential reasoning-based method should be further investigated. In addition, as interactive process is common and closer to real decision-making process, more attention should be paid on the interactive hesitant fuzzy linguistic methods. Even though the scholars have extended many classical outranking methods to hesitant fuzzy linguistic environment, there are still some other outranking methods, such as the MELCHIOR [36], PRAGMA [65], MAPPACC [65] and TACTIC [92]. We can further investigate these methods within the context of HFLEs.
5. Preference relation is a very important tool in aiding decision-making process. However, as we can see from Sect. 5, there is not too much work on the hesitant fuzzy preference relation theory. We are sure that this direction would be hot along the next several years. More definitions on different types of consistency of the HFLPR should be proposed. The inconsistency improving process, the incomplete reducing process, the group consensus reaching process are all very good topics in this research direction.
6. Clustering is a very important process in handling some complicated decision-making problems with

huge amount of alternatives, experts or even criteria (see the recent large-scale decision-making studies [60, 74, 116]). Fuzzy clustering has gained great success over the past several decades. However, as we can see in Table 1, there is only one paper focused on the clustering algorithm with HFL information. Thus, it is for sure that in the future, there would be more and more scholars engaging in this research direction. Given that more and more different types of measures of HFLTSs would be proposed, it is not very tough to investigate the clustering algorithms with hesitant fuzzy linguistic information.

7. It is also a very interesting topic to investigate the hesitant linguistic expressions, including linguistic modifiers and qualifiers [15].

8 Concluding Remarks

In this paper, we have made an overview on the state of the art of the researches on hesitant fuzzy linguistic theory during the period from 2012 to 2017 based on the selected 134 papers from the well-known database, Web of Sciences. The motivation, definitions and operations have been clearly summarized. As the comparison methods between HFLEs are essential for many decision-making methods, eight different comparison schemes have been reviewed in-depth. We have summerized the measures of HFLTSs. We have described all the distinct extensions on HFLTSs. We have conducted a survey on MCDM with HFLTSs in terms of aggregation operators and MCDM methods. We have made an overview on decision making with HFLPRs. The applications, research challenges and future directions have also been given.

Acknowledgements The work was supported by the National Natural Science Foundation of China (Nos. 71771156, 71501135, 71571123), the China Postdoctoral Science Foundation (2016T90863, 2016M602698), the Scientific Research Foundation for Excellent Young Scholars at Sichuan University (No. 2016SCU04A23), the Scientific Research Foundation for Scholars at Sichuan University (No. YJ201535), the 2016 Key Project of the Key Research Institute of Humanities and Social Sciences in Sichuan Province (CJZ16-01, CJCB2016-02, Xq16B04), the International Visiting Program for Excellent Young Scholars of SCU and the Grant from the FEDER funds (No. TIN2016-75850-R).

References

1. Bai, C.Z., Zhang, R., Qian, L.X., Wu, Y.N.: Comparisons of probabilistic linguistic term sets for multi-criteria decision making. Knowl. Based Syst. **119**, 284–291 (2017)
2. Beg, I., Rashid, T.: Topsis for hesitant fuzzy linguistic term sets. Int. J. Intell. Syst. **28**(12), 1162–1171 (2013)
3. Beg, I., Rashid, T.: Hesitant 2-tuple linguistic information in multiple attributes group decision making. J. Intell. Fuzzy Syst. **30**(1), 109–116 (2016)
4. Bustince, H., Barrenechea, E., Pagola, M., Fernandez, J., Xu, Z.S., Bedregal, B., Montero, J., Hagras, H., Herrera, F., De Baets, B.: A historical account of types of fuzzy sets and their relationships. IEEE Trans. Fuzzy Syst. **24**(1), 179–194 (2016)
5. Cabrerizo, F.J., Herrera-Viedma, E., Pedrycz, W.: A method based on PSO and granular computing of linguistic information to solve group decision making problems defined in heterogeneous contexts. Eur. J. Oper. Res. **230**(3), 624–633 (2013)
6. Chang, K.H.: A more general reliability allocation method using the hesitant fuzzy linguistic term set and minimal variance owga weights. Appl. Soft Comput. **56**, 589–596 (2017)
7. Chen, S.M., Hong, J.A.: Multicriteria linguistic decision making based on hesitant fuzzy linguistic term sets and the aggregation of fuzzy sets. Inf. Sci. **286**, 63–74 (2014)
8. Chen, Z.S., Chin, K.S., Li, Y.L., Yang, Y.: Proportional hesitant fuzzy linguistic term set for multiple criteria group decision making. Inf. Sci. **357**, 61–87 (2016)
9. Chen, Z.S., Chin, K.S., Mu, N.Y., Xiong, S.H., Chang, J.P., Yang, Y.: Generating hflts possibility distribution with an embedded assessing attitude. Inf. Sci. **394**, 141–166 (2017)
10. Cui, Y.: An approach to evaluating the performances of the photoelectric devices with hesitant fuzzy linguistic information. Int. J. Knowl. Based Intell. Eng. Syst. **20**(4), 245–249 (2016)
11. Da, T., Xu, Y.J.: Evaluation on connectivity of urban waterfront redevelopment under hesitant fuzzy linguistic environment. Ocean Coast. Manag. **132**, 101–110 (2016)
12. Dong, J.Y., Yuan, F.F., Wan, S.P.: Extended vikor method for multiple criteria decision-making with linguistic hesitant fuzzy information. Comput. Ind. Eng. **112**, 305–319 (2017)
13. Dong, Y.C., Chen, X., Herrera, F.: Minimizing adjusted simple terms in the consensus reaching process with hesitant linguistic assessments in group decision making. Inf. Sci. **297**, 95–117 (2015)
14. Dong, Y.C., Li, C.C., Herrera, F.: Connecting the linguistic hierarchy and the numerical scale for the 2-tuple linguistic model and its use to deal with hesitant unbalanced linguistic information. Inf. Sci. **367**, 259–278 (2016)
15. Durand, M., Truck, I.: A new proposal to deal with hesitant linguistic expressions on preference assessments. Inf. Fusion **41**, 176–181 (2018)
16. Fahmi, A., Kahraman, C., Bilen, Ü.: Electre I method using hesitant linguistic term sets: an application to supplier selection. Int. J. Comput. Intell. Syst. **9**(1), 153–167 (2016)
17. Faizi, S., Rashid, T., Zafar, S.: An outranking method for multi-criteria group decision making using hesitant intuitionistic fuzzy linguistic term sets. J. Intell. Fuzzy Syst. **32**(3), 2153–2164 (2017)
18. Farhadinia, B.: Multiple criteria decision-making methods with completely unknown weights in hesitant fuzzy linguistic term setting. Knowl. Based Syst. **93**, 135–144 (2016)
19. Feng, X.Q., Tan, Q.Y., Wei, C.P.: Hesitant fuzzy linguistic multi-criteria decision making based on possibility theory. Int. J. Mach. Learn. Cybern. (2017). http://dx.doi.org/10.1007/s13042-017-0659-7
20. García-Lapresta, J.L., Pérez-Román, D.: Consensus-based clustering under hesitant qualitative assessments. Fuzzy Sets Syst. **292**, 261–273 (2016)
21. Ghadikolaei, A.S., Madhoushi, M., Divsalar, M.: Extension of the VIKOR method for group decision making with extended hesitant fuzzy linguistic information. Neural Comput. Appl. (2017). http://dx.doi.org/10.1007/s00521-017-2944-5
22. Gou, X., Xu, Z., Liao, H.: Group decision making with compatibility measures of hesitant fuzzy linguistic preference

relations. Soft Comput. (2017). http://dx.doi.org/10.1007/s00500-017-2871-5
23. Gou, X.J., Liao, H.C., Xu, Z.S., Herrera, F.: Double hierarchy hesitant fuzzy linguistic term set and multimoora method: a case of study to evaluate the implementation status of haze controlling measures. Inf. Fusion **38**, 22–34 (2017)
24. Gou, X.J., Xu, Z.S.: Novel basic operational laws for linguistic terms, hesitant fuzzy linguistic term sets and probabilistic linguistic term sets. Inf. Sci. **372**, 407–427 (2016)
25. Gou, X.J., Xu, Z.S., Liao, H.C.: Hesitant fuzzy linguistic entropy and cross-entropy measures and alternative queuing method for multiple criteria decision making. Inf. Sci. **388**, 225–246 (2017)
26. Gou, X.J., Xu, Z.S., Liao, H.C.: Multiple criteria decision making based on bonferroni means with hesitant fuzzy linguistic information. Soft. Comput. **21**(21), 6515–6529 (2017)
27. Gu, G.Y., Wei, F.J., Zhou, S.H.: Risk assessment method for mass unexpected incident in city with hesitant fuzzy linguistic information. J. Intell. Fuzzy Syst. **29**(5), 2299–2304 (2015)
28. Herrera, F., Martínez, L.: A 2-tuple fuzzy linguistic representation model for computing with words. IEEE Trans. Fuzzy Syst. **8**(6), 746–752 (2000)
29. Hesamian, G., Shams, M.: Measuring similarity and ordering based on hesitant fuzzy linguistic term sets. J. Intell. Fuzzy Syst. **28**(2), 983–990 (2015)
30. Huang, H.C., Yang, X.J.: Pairwise comparison and distance measure of hesitant fuzzy linguistic term sets. Math. Probl. Eng. **2014** (2014). http://dx.doi.org/10.1155/2014/954040
31. Huo, Z.G., Zhou, Z.G.: Approaches to multiple attribute decision making with hesitant fuzzy uncertain linguistic information. J. Intell. Fuzzy Syst. **28**(3), 991–998 (2015)
32. Jin, M., Liao, M.J.: Approaches to multiple attribute decision making based on the hesitant fuzzy uncertain linguistic power aggregation operators and their applications to service quality evaluation in higher education. J. Comput. Theor. Nanosci. **13**(10), 7171–7175 (2016)
33. Ju, Y.B., Yang, S.H., Liu, X.Y.: A novel method for multiattribute decision making with dual hesitant fuzzy triangular linguistic information. J. Appli. Math. **2014** (2014). http://dx.doi.org/10.1155/2014/909823
34. Khishtandar, S., Zandieh, M., Dorri, B.: A multi criteria decision making framework for sustainability assessment of bioenergy production technologies with hesitant fuzzy linguistic term sets: the case of Iran. Renew. Sustain. Energy Rev. **77**, 1130–1145 (2017)
35. Kitchenham, B.: Procedures for performing systematic reviews. Keele UK Keele Univ. **33**(2004), 1–26 (2004)
36. Leclercq, J.: Propositions dextension de la notion de dominance en présence de relations dordre sur les pseudo-critères: MELCHIOR. Revue Belge de Recherche Opérationnelle, de Statistique et dInformatique **24**(1), 32–46 (1984)
37. Lee, L.W., Chen, S.M.: Fuzzy decision making based on likelihood-based comparison relations of hesitant fuzzy linguistic term sets and hesitant fuzzy linguistic operators. Inf. Sci. **294**, 513–529 (2015)
38. Li, C.C., Dong, Y., Herrera, F., Herrera-Viedma, E., Martínez, L.: Personalized individual semantics in computing with words for supporting linguistic group decision making. An application on consensus reaching. Inf. Fusion **33**, 29–40 (2017)
39. Li, Q.X., Zhao, X.F., Wei, G.W.: Model for software quality evaluation with hesitant fuzzy uncertain linguistic information. J. Intell. Fuzzy Syst. **26**(6), 2639–2647 (2014)
40. Li, Z.M., Xu, J.P., Lev, B., Gang, J.: Multi-criteria group individual research output evaluation based on context-free grammar judgments with assessing attitude. Omega **57**, 282–293 (2015)
41. Liao, H., Li, Z., Zeng, X.J., Liu, W.: A comparison of distinct consensus measures for group decision making with intuitionistic fuzzy preference relations. Int. J. Comput. Intell. Syst. **10**(1), 456–469 (2017)
42. Liao, H., Xu, Z., Zeng, X.J.: Novel correlation coefficients between hesitant fuzzy sets and their application in decision making. Knowl. Based Syst. **82**, 115–127 (2015)
43. Liao, H.C., Jiang, L.S., Xu, Z.S., Xu, J.P., Herrera, F.: A linear programming method for multiple criteria decision making with probabilistic linguistic information. Inf. Sci. **415–416**, 341–355 (2017)
44. Liao, H.C., Wu, X.L., Liang, X.D., Xu, J.P., Herrera, F.: A new hesitant fuzzy linguistic ORESTE method for hybrid multi-criteria decision making. IEEE Trans. Fuzzy Syst. (2017). Technique Report
45. Liao, H.C., Wu, X.L., Liang, X.D., Yang, J.B., Xu, D.L.: Continuous interval-valued linguistic ORESTE method for multi-criteria group decision making in mobile design. Knowl. Based Syst. (2017). Technique Report
46. Liao, H.C., Xu, Z.S.: Satisfaction degree based interactive decision making under hesitant fuzzy environment with incomplete weights. Int. J. Uncertain. Fuzziness Knowl. Based Syst. **22**(4), 553–572 (2014)
47. Liao, H.C., Xu, Z.S.: Approaches to manage hesitant fuzzy linguistic information based on the cosine distance and similarity measures for hfltss and their application in qualitative decision making. Expert Syst. Appl. **42**(12), 5328–5336 (2015)
48. Liao, H.C., Xu, Z.S., Xia, M.M.: Multiplicative consistency of hesitant fuzzy preference relation and its application in group decision making. Int. J. Inf. Technol. Decis. Mak. **13**(1), 47–76 (2014)
49. Liao, H.C., Xu, Z.S., Zeng, X.J.: Distance and similarity measures for hesitant fuzzy linguistic term sets and their application in multi-criteria decision making. Inf. Sci. **271**, 125–142 (2014)
50. Liao, H.C., Xu, Z.S., Zeng, X.J.: Hesitant fuzzy linguistic vikor method and its application in qualitative multiple criteria decision making. IEEE Trans. Fuzzy Syst. **23**(5), 1343–1355 (2015)
51. Liao, H.C., Xu, Z.S., Zeng, X.J., Merigó, J.M.: Qualitative decision making with correlation coefficients of hesitant fuzzy linguistic term sets. Knowl. Based Syst. **76**, 127–138 (2015)
52. Lin, R., Zhao, X.F., Wei, G.W.: Models for selecting an erp system with hesitant fuzzy linguistic information. J. Intell. Fuzzy Syst. **26**(5), 2155–2165 (2014)
53. Liu, D.N.: Model for evaluating the electrical power system safety with hesitant fuzzy linguistic information. J. Intell. Fuzzy Syst. **29**(2), 725–730 (2015)
54. Liu, H.B., Cai, J.F., Jiang, L.: On improving the additive consistency of the fuzzy preference relations based on comparative linguistic expressions. Int. J. Intell. Syst. **29**(6), 544–559 (2014)
55. Liu, H.B., Rodríguez, R.M.: A fuzzy envelope for hesitant fuzzy linguistic term set and its application to multicriteria decision making. Inf. Sci. **258**, 220–238 (2014)
56. Liu, H.C., You, J.X., Li, P., Su, Q.: Failure mode and effect analysis under uncertainty: an integrated multiple criteria decision making approach. IEEE Trans. Reliab. **65**(3), 1380–1392 (2016)
57. Liu, W.S., Liao, H.C.: A bibliometric analysis of fuzzy decision research during 1970–2015. Int. J. Fuzzy Syst. **19**(1), 1–14 (2017)
58. Liu, X.Y., Ju, Y.B., Yang, S.H.: Hesitant intuitionistic fuzzy linguistic aggregation operators and their applications to multiple attribute decision making. J. Intell. Fuzzy Syst. **27**(3), 1187–1201 (2014)
59. Liu, X.Y., Ju, Y.B., Yang, S.H.: Some generalized interval-valued hesitant uncertain linguistic aggregation operators and

their applications to multiple attribute group decision making. Soft. Comput. **20**(2), 495–510 (2016)
60. Liu, Y., Fan, Z.P., Zhang, X.: A method for large group decision-making based on evaluation information provided by participators from multiple groups. Inf. Fusion **29**, 132–141 (2016)
61. Liu, Y.Z., Fan, Z.P., Gao, G.X.: An extended linmap method for magdm under linguistic hesitant fuzzy environment. J. Intell. Fuzzy Syst. **30**(5), 2689–2703 (2016)
62. Lu, M., Wei, G.W.: Models for multiple attribute decision making with dual hesitant fuzzy uncertain linguistic information. Int. J. Knowl. Based Intell. Eng. Syst. **20**(4), 217–227 (2016)
63. Luo, D.C., Yang, M.H., Gao, L.F.: Research on risk assessment of financial markets venture capital project with hesitant fuzzy uncertain linguistic information. J. Comput. Theor. Nanosci. **13**(10), 7115–7119 (2016)
64. Massanet, S., Riera, J.V., Torrens, J., Herrera-Viedma, E.: A new linguistic computational model based on discrete fuzzy numbers for computing with words. Inf. Sci. **258**, 277–290 (2014)
65. Matarazzo, B.: Multicriterion analysis of preferences by means of pairwise actions and criterion comparisons (MAPPACC). Appl. Math. Comput. **18**(2), 119–141 (1986)
66. Mendel, J.M., Zadeh, L.A., Trillas, E., Yager, R., Lawry, J., Hagras, H., Guadarrama, S.: What computing with words means to me [discussion forum]. IEEE Comput. Intell. Mag. **5**(1), 20–26 (2010)
67. Meng, F.Y., Chen, X.H.: A hesitant fuzzy linguistic multi-granularity decision making model based on distance measures. J. Intell. Fuzzy Syst. **28**(4), 1519–1531 (2015)
68. Meng, F.Y., Chen, X.H., Zhang, Q.: Multi-attribute decision analysis under a linguistic hesitant fuzzy environment. Inf. Sci. **267**, 287–305 (2014)
69. Meng, F.Y., Wang, C., Chen, X.H.: Linguistic interval hesitant fuzzy sets and their application in decision making. Cognit. Comput. **8**(1), 52–68 (2016)
70. Montes, R., Sánchez, A.M., Villar, P., Herrera, F.: A web tool to support decision making in the housing market using hesitant fuzzy linguistic term sets. Appl. Soft Comput. **35**, 949–957 (2015)
71. Montes, R., Sanchez, A.M., Villar, P., Herrera, F.: Teranga go!: carpooling collaborative consumption community with multi-criteria hesitant fuzzy linguistic term set opinions to build confidence and trust. Appl. Soft Comput. (2017). https://doi.org/10.1016/j.asoc.2017.05.039
72. Montserrat-Adell, J., Agell, N., Sánchez, M., Prats, F., Ruiz, F.J.: Modeling group assessments by means of hesitant fuzzy linguistic term sets. J. Appl. Logic **23**, 40–50 (2017)
73. Montserrat-Adell, J., Agell, N., Sánchez, M., Ruiz, F.J.: Consensus, dissension and precision in group decision making by means of an algebraic extension of hesitant fuzzy linguistic term sets. Inf. Fusion **42**, 1–11 (2018)
74. Palomares, I., Martinez, L., Herrera, F.: A consensus model to detect and manage noncooperative behaviors in large-scale group decision making. IEEE Trans. Fuzzy Syst. **22**(3), 516–530 (2014)
75. Pang, Q., Wang, H., Xu, Z.S.: Probabilistic linguistic term sets in multi-attribute group decision making. Inf. Sci. **369**, 128–143 (2016)
76. Pedrycz, W., Song, M.: A granulation of linguistic information in ahp decision-making problems. Inf. Fusion **17**, 93–101 (2014)
77. Peng, H.G., Wang, J.Q.: Hesitant uncertain linguistic z-numbers and their application in multi-criteria group decision-making problems. Int. J. Fuzzy Syst. **19**(5), 1300–1316 (2017)
78. Qi, X.W., Liang, C.Y., Zhang, J.L.: Multiple attribute group decision making based on generalized power aggregation operators under interval-valued dual hesitant fuzzy linguistic environment. Int. J. Mach. Learn. Cybernet. **7**(6), 1147–1193 (2016)
79. Rashid, T., Faizi, S., Xu, Z.S., Zafar, S.: Electre-based outranking method for multi-criteria decision making using hesitant intuitionistic fuzzy linguistic term sets. Int. J. Fuzzy Syst. (2017). http://dx.doi.org/10.1007/s40815-017-0297-y
80. Riera, J.V., Massanet, S., Herrera-Viedma, E., Torrens, J.: Some interesting properties of the fuzzy linguistic model based on discrete fuzzy numbers to manage hesitant fuzzy linguistic information. Appl. Soft Comput. **36**, 383–391 (2015)
81. Rodríguez, R.M., Bedregal, B., Bustince, H., Dong, Y.C., Farhadinia, B., Kahraman, C., Martínez, L., Torra, V., Xu, Y.J., Xu, Z.S., et al.: A position and perspective analysis of hesitant fuzzy sets on information fusion in decision making. Towards high quality progress. Inf. Fusion **29**, 89–97 (2016)
82. Rodríguez, R.M., Labella, A., Martínez, L.: An overview on fuzzy modelling of complex linguistic preferences in decision making. Int. J. Comput. Intell. Syst. **9**(sup1), 81–94 (2016)
83. Rodriguez, R.M., Martinez, L., Herrera, F.: Hesitant fuzzy linguistic term sets for decision making. IEEE Trans. Fuzzy Syst. **20**(1), 109–119 (2012)
84. RodríGuez, R.M., MartıNez, L., Herrera, F.: A group decision making model dealing with comparative linguistic expressions based on hesitant fuzzy linguistic term sets. Inf. Sci. **241**, 28–42 (2013)
85. Rodríguez, R.M., Martínez, L., Torra, V., Xu, Z.S., Herrera, F.: Hesitant fuzzy sets: state of the art and future directions. Int. J. Intell. Syst. **29**(6), 495–524 (2014)
86. Sellak, H., Ouhbi, B., Frikh, B.: A knowledge-based outranking approach for multi-criteria decision-making with hesitant fuzzy linguistic term sets. Appl. Soft Comput. (2017). https://doi.org/10.1016/j.asoc.2017.06.031
87. Sun, R., Hu, J., Zhou, J., Chen, X.: A hesitant fuzzy linguistic projection-based MABAC method for patients' prioritization. Int. J. Fuzzy Syst. (2017). http://dx.doi.org/10.1007/s40815-017-0345-7
88. Tan, Q.Y., Wei, C.P., Liu, Q., Feng, X.Q.: The hesitant fuzzy linguistic TOPSIS method based on novel information measures. Asia-Pacific J. Oper. Res. **33**(05), 1650035 (2016)
89. Tian, Z.P., Wang, J., Wang, J.Q., Zhang, H.Y.: A likelihood-based qualitative flexible approach with hesitant fuzzy linguistic information. Cognit. Comput. **8**(4), 670–683 (2016)
90. Torra, V.: Hesitant fuzzy sets. Int. J. Intell. Syst. **25**(6), 529–539 (2010)
91. Tüysüz, F., Şimşek, B.: A hesitant fuzzy linguistic term sets-based AHP approach for analyzing the performance evaluation factors: an application to cargo sector. Complex Intell. Syst. **3**, 167–175 (2017)
92. Vansnick, J.C.: On the problem of weights in multiple criteria decision making (the noncompensatory approach). Eur. J. Oper. Res. **24**(2), 288–294 (1986)
93. Wang, H.: Extended hesitant fuzzy linguistic term sets and their aggregation in group decision making. Int. J. Comput. Intell. Syst. **8**(1), 14–33 (2015)
94. Wang, H., Xu, Z.S.: Some consistency measures of extended hesitant fuzzy linguistic preference relations. Inf. Sci. **297**, 316–331 (2015)
95. Wang, H., Xu, Z.S.: Total orders of extended hesitant fuzzy linguistic term sets: definitions, generations and applications. Knowl. Based Syst. **107**, 142–154 (2016)
96. Wang, J., Wang, J.Q., Zhang, H.Y.: A likelihood-based todim approach based on multi-hesitant fuzzy linguistic information for evaluation in logistics outsourcing. Comput. Ind. Eng. **99**, 287–299 (2016)

97. Wang, J., Wang, J.Q., Zhang, H.Y., Chen, X.H.: Multi-criteria decision-making based on hesitant fuzzy linguistic term sets: an outranking approach. Knowl. Based Syst. **86**, 224–236 (2015)
98. Wang, J., Wang, J.Q., Zhang, H.Y., Chen, X.H.: Multi-criteria group decision-making approach based on 2-tuple linguistic aggregation operators with multi-hesitant fuzzy linguistic information. Int. J. Fuzzy Syst. **18**(1), 81–97 (2016)
99. Wang, J., Wang, J.Q., Zhang, H.Y., Chen, X.H.: Distance-based multi-criteria group decision-making approaches with multi-hesitant fuzzy linguistic information. Int. J. Inf. Technol. Decis. Mak. **16**(4), 1069–1099 (2017)
100. Wang, J.Q., Wang, J., Chen, Q.H., Zhang, H.Y., Chen, X.H.: An outranking approach for multi-criteria decision-making with hesitant fuzzy linguistic term sets. Inf. Sci. **280**, 338–351 (2014)
101. Wang, J.Q., Wu, J.T., Wang, J., Zhang, H.Y., Chen, X.H.: Interval-valued hesitant fuzzy linguistic sets and their applications in multi-criteria decision-making problems. Inf. Sci. **288**, 55–72 (2014)
102. Wang, J.Q., Wu, J.T., Wang, J., Zhang, H.Y., Chen, X.H.: Multi-criteria decision-making methods based on the hausdorff distance of hesitant fuzzy linguistic numbers. Soft. Comput. **20**(4), 1621–1633 (2016)
103. Wang, L., Gong, Z.: Priority of a hesitant fuzzy linguistic preference relation with a normal distribution in meteorological disaster risk assessment. Int. J. Environ. Res. Publ. Health **14**(10), 1203 (2017)
104. Wang, Y.H., Li, L.: Models for multiple attribute decision making with hesitant fuzzy linguistic information and their application to enterprise risk evaluation. J. Intell. Fuzzy Syst. **30**(3), 1531–1536 (2016)
105. Webster, J., Watson, R.T.: Analyzing the past to prepare for the future: writing a literature review. MIS Quart. **26**(2), xiii–xxiii (2002)
106. Wei, C., Rodríguez, R.M., Martínez, L.: Uncertainty measures of extended hesitant fuzzy linguistic term sets. IEEE Trans. Fuzzy Syst. (2017). https://doi.org/10.1109/TFUZZ.2017.2724023
107. Wei, C.P., Liao, H.C.: A multigranularity linguistic group decision-making method based on hesitant 2-tuple sets. Int. J. Intell. Syst. **31**(6), 612–634 (2016)
108. Wei, C.P., Ren, Z.L., Rodríguez, R.M.: A hesitant fuzzy linguistic todim method based on a score function. Int. J. Comput. Intell. Syst. **8**(4), 701–712 (2015)
109. Wei, C.P., Zhao, N., Tang, X.J.: Operators and comparisons of hesitant fuzzy linguistic term sets. IEEE Trans. Fuzzy Syst. **22**(3), 575–585 (2014)
110. Wei, C.P., Zhao, N., Tang, X.J.: A novel linguistic group decision-making model based on extended hesitant fuzzy linguistic term sets. Int. J. Uncertain. Fuzziness Knowl. Based Syst. **23**(03), 379–398 (2015)
111. Wei, G.W.: Interval valued hesitant fuzzy uncertain linguistic aggregation operators in multiple attribute decision making. Int. J. Mach. Learn. Cybernet. **7**(6), 1093–1114 (2016)
112. Wei, G.W., Alsaadi, F.E., Hayat, T., Alsaedi, A.: Hesitant fuzzy linguistic arithmetic aggregation operators in multiple attribute decision making. Iran. J. Fuzzy Syst. **13**(4), 1–16 (2016)
113. Wei, G.W., Lin, R., Wang, H.J., Ran, L.G.: Interval-valued dual hesitant fuzzy linguistic arithmetic aggregation operators in multiple attribute decision making. Int. Core J. Eng. **1**(6), 12–222 (2015)
114. Wei, G.W., Xu, X.R., Deng, D.X.: Interval-valued dual hesitant fuzzy linguistic geometric aggregation operators in multiple attribute decision making. Int. J. Knowl. Based Intell. Eng. Syst. **20**(4), 189–196 (2016)
115. Wu, J.T., Wang, J.Q., Wang, J., Zhang, H.Y., Chen, X.H.: Hesitant fuzzy linguistic multicriteria decision-making method based on generalized prioritized aggregation operator. Sci. World J. **2014** (2014). http://dx.doi.org/10.1155/2014/645341
116. Wu, Z., Xu, J.: A consensus model for large-scale group decision making with hesitant fuzzy information and changeable clusters. Inf. Fusion **41**, 217–231 (2018)
117. Wu, Z.B., Xu, J.P.: An interactive consensus reaching model for decision making under hesitation linguistic environment. J. Intell. Fuzzy Syst. **31**(3), 1635–1644 (2016)
118. Wu, Z.B., Xu, J.P.: Managing consistency and consensus in group decision making with hesitant fuzzy linguistic preference relations. Omega **65**, 28–40 (2016)
119. Wu, Z.B., Xu, J.P.: Possibility distribution-based approach for magdm with hesitant fuzzy linguistic information. IEEE Trans. Cybern. **46**(3), 694–705 (2016)
120. Xu, Y.J., Xu, A.W., Merigó, J.M., Wang, H.M.: Hesitant fuzzy linguistic ordered weighted distance operators for group decision making. J. Appl. Math. Comput. **49**(1–2), 285–308 (2015)
121. Xu, Y.J., Xu, A.W., Wang, H.M.: Hesitant fuzzy linguistic linear programming technique for multidimensional analysis of preference for multi-attribute group decision making. Int. J. Mach. Learn. Cybernet. **7**(5), 845–855 (2016)
122. Xu, Z., Wang, H.: On the syntax and semantics of virtual linguistic terms for information fusion in decision making. Inf. Fusion **34**, 43–48 (2017)
123. Xu, Z.S.: EOWA and EOWG operators for aggregating linguistic labels based on linguistic preference relations. Int. J. Uncertain. Fuzziness Knowl. Based Syst. **12**(06), 791–810 (2004)
124. Xu, Z.S.: Deviation measures of linguistic preference relations in group decision making. Omega **33**(3), 249–254 (2005)
125. Xu, Z.S.: Intuitionistic fuzzy aggregation operators. IEEE Trans. Fuzzy Syst. **15**(6), 1179–1187 (2007)
126. Xu, Z.S., Wang, H.: Managing multi-granularity linguistic information in qualitative group decision making: an overview. Granul. Comput. **1**(1), 21–35 (2016)
127. Yager, R.R.: Concepts, theory, and techniques a new methodology for ordinal multiobjective decisions based on fuzzy sets. Decis. Sci. **12**(4), 589–600 (1981)
128. Yang, S.H., Ju, Y.B.: Dual hesitant fuzzy linguistic aggregation operators and their applications to multi-attribute decision making. J. Intell. Fuzzy Syst. **27**(4), 1935–1947 (2014)
129. Yang, S.H., Sun, Z., Ju, Y.B., Qiao, C.Y.: A novel multiple attribute satisfaction evaluation approach with hesitant intuitionistic linguistic fuzzy information. Math. Probl. Eng. **2014** (2014). http://dx.doi.org/10.1155/2014/692782
130. Yang, W., Pang, Y.F., Shi, J.R., Wang, C.J.: Linguistic hesitant intuitionistic fuzzy decision-making method based on VIKOR. Neural Comput. Appl. (2016). http://dx.doi.org/10.1007/s00521-016-2526-y
131. Yang, W., Pang, Y.F., Shi, J.R., Yue, H.Y.: Linguistic hesitant intuitionistic fuzzy linear assignment method based on choquet integral. J. Intell. Fuzzy Syst. **32**(1), 767–780 (2017)
132. Yang, W., Shi, J.R., Pang, Y.F.: Generalized linguistic hesitant intuitionistic fuzzy hybrid aggregation operators. Math. Probl. Eng. **2015** (2015). http://dx.doi.org/10.1155/2015/983628
133. Yang, W., Shi, J.R., Zheng, X.Y., Pang, Y.F.: Hesitant interval-valued intuitionistic fuzzy linguistic sets and their applications. J. Intell. Fuzzy Syst. **31**(6), 2779–2788 (2016)
134. Yavuz, M., Oztaysi, B., Onar, S.C., Kahraman, C.: Multi-criteria evaluation of alternative-fuel vehicles via a hierarchical hesitant fuzzy linguistic model. Expert Syst. Appl. **42**(5), 2835–2848 (2015)
135. Yu, S.M., Zhang, H.Y., Wang, J.Q.: Hesitant fuzzy linguistic Maclaurin symmetric mean operators and their applications to multi-criteria decision-making problem. Int. J. Intell. Syst. (2017). http://dx.doi.org/10.1002/int.21907

136. Yu, S.M., Zhou, H., Chen, X.H., Wang, J.Q.: A multi-criteria decision-making method based on Heronian mean operators under a linguistic hesitant fuzzy environment. Asia-Pacific J. Oper. Res. **32**(05), 1550035 (2015)
137. Zadeh, L.A.: The concept of a linguistic variable and its application to approximate reasoning—I. Inf. Sci. **8**(3), 199–249 (1975)
138. Zhai, Y.L., Xu, Z.S., Liao, H.C.: Probabilistic linguistic vector-term set and its application in group decision making with multi-granular linguistic information. Appl. Soft Comput. **49**, 801–816 (2016)
139. Zhang, B.W., Liang, H.M., Zhang, G.Q.: Reaching a consensus with minimum adjustment in magdm with hesitant fuzzy linguistic term sets. Inf. Fusion **42**, 12–23 (2018)
140. Zhang, C., Li, D.Y., Liang, J.Y.: Hesitant fuzzy linguistic rough set over two universes model and its applications. Int. J. Mach. Learn. Cybern. (2016). http://dx.doi.org/10.1007/s13042-016-0541-z
141. Zhang, F.Y., Luo, L., Liao, H.C., Zhu, T., Shi, Y.K., Shen, W.W.: Inpatient admission assessment in west china hospital based on hesitant fuzzy linguistic VIKOR method. J. Intell. Fuzzy Syst. **30**(6), 3143–3154 (2016)
142. Zhang, G.Q., Dong, Y.C., Xu, Y.F.: Consistency and consensus measures for linguistic preference relations based on distribution assessments. Inf. Fusion **17**, 46–55 (2014)
143. Zhang, W.K., Ju, Y.B., Liu, X.Y.: Multiple criteria decision analysis based on Shapley fuzzy measures and interval-valued hesitant fuzzy linguistic numbers. Comput. Ind. Eng. **105**, 28–38 (2017)
144. Zhang, Y.X., Xu, Z.S., Liao, H.C.: A consensus process for group decision making with probabilistic linguistic preference relations. Inf. Sci. **414**, 260–275 (2017)
145. Zhang, Y.X., Xu, Z.S., Wang, H., Liao, H.C.: Consistency-based risk assessment with probabilistic linguistic preference relation. Appl. Soft Comput. **49**, 817–833 (2016)
146. Zhang, Z.M., Wu, C.: Hesitant fuzzy linguistic aggregation operators and their applications to multiple attribute group decision making. J. Intell. Fuzzy Syst. **26**(5), 2185–2202 (2014)
147. Zhang, Z.M., Wu, C.: On the use of multiplicative consistency in hesitant fuzzy linguistic preference relations. Knowl. Based Syst. **72**, 13–27 (2014)
148. Zhao, N., Xu, Z.S., Ren, Z.L.: Some approaches to constructing distance measures for hesitant fuzzy linguistic term sets with applications in decision making. Int. J. Inf. Technol. Decis. Mak. (2017). https://doi.org/10.1142/S0219622017500316
149. Zhao, X.Q., Yang, L.H., Wang, L.J.: Models for evaluating the resource integration capability of textile enterprise with hesitant fuzzy uncertain linguistic information. J. Intell. Fuzzy Syst. **31**(3), 2001–2008 (2016)
150. Zheng, X.M.: Methods for multiple attribute decision making with hesitant fuzzy uncertain linguistic information and their application for evaluating the college english teachers' professional development competence. J. Intell. Fuzzy Syst. **28**(3), 1243–1250 (2015)
151. Zhou, H., Wang, J.Q., Zhang, H.Y., Chen, X.H.: Linguistic hesitant fuzzy multi-criteria decision-making method based on evidential reasoning. Int. J. Syst. Sci. **47**(2), 314–327 (2016)
152. Zhou, W., Xu, Z.S.: Generalized asymmetric linguistic term set and its application to qualitative decision making involving risk appetites. Eur. J. Oper. Res. **254**(2), 610–621 (2016)
153. Zhu, B., Xu, Z.S.: Consistency measures for hesitant fuzzy linguistic preference relations. IEEE Trans. Fuzzy Syst. **22**(1), 35–45 (2014)
154. Zhu, C.X., Zhu, L., Zhang, X.Z.: Linguistic hesitant fuzzy power aggregation operators and their applications in multiple attribute decision-making. Inf. Sci. **367**, 809–826 (2016)

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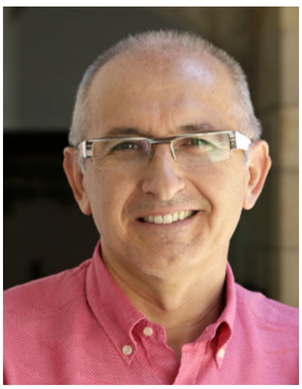

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