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Lie symmetry analysis for a parabolic Monge-Ampère equation in the optimal investment theory

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Abstract

In this paper, Lie symmetry analysis is performed on the parabolic Monge-Ampère equation $u_s u_{yy} + ryu_y u_{yy} - \theta u_y^2 = 0$ arising from the optimal investment theory. Lie symmetry and optimal system of this equation are derived. In particular, based on optimal system, symmetry reductions and invariant solutions are obtained.

Keywords: The parabolic Monge-Ampère equation; Lie symmetry analysis; Symmetry reductions; Invariant solutions

1. Introduction

In recent years, there are many researches use Lie symmetry analysis to partial differential equations (PDEs) which arising from physics, chemistry , economics and other fields [1–7]. The investigation of exact analytical solutions of PDEs play an important role for a long time. Lie symmetry analysis is one of the most effective methods for finding the exact analytical solutions of differential equations, and many authors used this method to find the analytical solutions of PDEs [8–12]. Lie symmetry analysis method was originally developed in the 19th century by the Norwegian mathematician Sophus Lie and developed in differential equations since Bluman and Cole proposed similarity theory for differential equations in 1970s [3, 7].

Over the last forty years, there was a considerable development in PDEs which arise in mathematical finance [13–19]. It was worth pointing out that Bordag and Chmakova's pioneering paper studied the evaluation of an option hedge-cost under relaxation of the price-taking assumption by Lie symmetry analysis method.

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In particular, they found some analytical solutions to a nonlinear Black-Scholes equation which incorporates the feedback-effect of a large trader in case of market illiquidity and showed that the typical solution would have a payoff which approximates a strangle, then used these solutions to test numerical schemes for solving a nonlinear Black-Scholes equation [16].

In this paper, we consider the following parabolic Monge-Ampère equation in the optimal investment theory [17]:

$$u_s u_{yy} + ryu_y u_{yy} - \theta u_y^2 = 0, \tag{1.1}$$

where $r, \theta > 0$ are constants. Existence of solutions to initial value problem for this equation were showed in [18].

The rest of the paper is organized as follows. In Section 2, we recall the model Eq.(1.1) arise from optimal investment of mathematical finance theory. In Section 3, vector fields and optimal system are given by employing Lie symmetry analysis method. In Section 4, the similarity variables and analytical solutions of Eq.(1.1) are obtained by using optimal system. Finally, conclusions are presented at the end of the paper.

2. The model arise from optimal investment of mathematical finance theory

In this section, we recall the model equation presented in [17].

There are $n+1$ assets continuously traded. The 0-th asset is bond, and the last n are stocks. The price process of the i-th asset is denoted by $P_i(t)$ and they satisfy the following system:

$$\begin{cases} dP_0(t) = r(t)P_0(t)dt, \\ dP_i(t) = b_i(t)dt + P_i(t)\sum_{j=1}^{d} \sigma_{ij} dW_j(t), \quad 1 \le i \le n, P_i(0) = p_i, \quad 0 \le i \le n, \end{cases} \tag{2.1}$$

where $r(t), b_i(t)$ and $\sigma_{ij}(t)$ are the interest rate, the appreciation rate, and the volatility, respectively. $W(t) = (W_1(t), W_2(t), ..., W_n(t))$ is a d-dimensional standard Brownian motion. Denote $b(t) = (b_1(t), b_2(t), ..., b_n(t))$, $\sigma(t) = (\sigma_{ij}(t))_{n\times d}$.

Let an investor have an initial wealth $y \in \mathbb{R}$ and invest this amount in the market described above and the wealth process $Y(t)$ satisfies the stochastic differential equation (SDE) as follows:

$$\begin{cases} dY(t) = \{r(t)Y(t) + \langle b(t) - r(t)\vec{1}, \pi(t)\rangle\}dt + \langle \pi(t), \sigma(t)dW(t)\rangle, \quad t \in [0, T], \\ Y(0) = y, \end{cases}$$

$$(2.2)$$

where $(\pi(\cdot), N(\cdot)) \in \prod[0,T] \times N[0,T]$. For this particular investor has his own attitude to the risk versus the gain at the final time T, which is described by a strictly increasing and concave utility function $g : \mathbb{R} \to \mathbb{R}$. The investor would like to maximize the payoff functional $J(\pi(t)) = E[g(Y(t))]$ by choosing a suitable strategy $(\pi(\cdot), N(\cdot)) \in \prod[0,T] \times N[0,T]$. This is so-called self-financing optimal investment problems:

For given initial endowment $y \in \mathbb{R}$, find a portfolio $\bar{\pi}(\cdot) \in \prod[0,T]$, such that

$$J(\bar{\pi}) = \sup_{\bar{\pi}(\cdot)\in\prod[0,t]} J(\pi(\cdot)). \tag{2.3}$$

Any $\pi(\cdot) \in \Pi[0,T]$ satisfies (2.3) is called an optimal portfolio and the corresponding wealth $Y(\cdot)$ is again called an optimal wealth process.

In order to use the dynamic programming, we need to consider the optimal investment problem on the time interval $[s,T]$ with $s \in [0,T]$,i.e,

$$\begin{cases} dY(t) = \{r(t)Y(t) + \langle b(t) - r(t)\vec{1}, \pi(t)\rangle\}dt + \langle \pi(t), \sigma(t)dW(t)\rangle, \quad t \in [s,T], \\ Y(s) = y. \end{cases} \tag{2.4}$$

Define $J(s,y;\pi(\cdot)) = Eg(Y(T;s,y,\pi(\cdot)))$, and

$$\begin{cases} V(s,y) = \sup_{\bar{\pi}(\cdot)\in\prod[s,t]} J(s,y;\pi(\cdot)), \quad (s,y) \in [0,T) \times \mathbb{R}, \\ V(T,y) = g(y), \ y \in \mathbb{R}, \end{cases} \tag{2.5}$$

where $Y(\cdot;s,y,\pi(\cdot))$ is the solution of (2.4), V(s,y) is called the value function of self-financing optimal investment problem.

Next, introducing the Hamilton for self-financing optimal investment problem

$$\begin{cases} \wp(s,y,p,G,\pi) \equiv p[r(s)y + \langle b(s) - r(s)\vec{1}, \pi\rangle] + \frac{1}{2}G|\sigma^T(s)\pi|^2, \\ \forall (s,y,p,G,\pi) \in [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}, \end{cases} \tag{2.6}$$

and

$$\begin{cases} H(s,y,p,G) = \sup_{\pi\in\mathbb{R}^n} \wp(s,y,p,G,\pi), \\ \forall (s,y,p,G) \in [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}. \end{cases} \tag{2.7}$$

Then we have the Hamilton-Jacobi-Bellman equation associated with self-financing optimal investment problems

$$u_s(s,y) + H(s,y,u_y(s,y),u_{yy}(s,y)) = 0, \tag{2.8}$$

for all $(s, y) \in [0, T) \times \mathbb{R}$, such that

$$(s, y, u_y(s, y), u_{yy}(s, y)) \in D(H), \tag{2.9}$$

where $D(H) = \{(s, y, p, G)|H(s, y, p, G) < \infty\}$.

Now we consider a simple case: $n = d = 1$, and all the functions r, b, σ are constants with $\sigma > 0, b - r > 0$. Then from (2.8) we get

$$u_s + ryu_y - \frac{(b - r)u_y^2}{2\sigma u_{yy}} = 0, \tag{2.10}$$

or equivalently

$$u_s u_{yy} + ryu_y u_{yy} - \theta u_y^2 = 0, \tag{2.11}$$

where $\theta = \frac{b-r}{\sigma}$.

3. Lie symmetry

In this section, we shall perform Lie symmetry analysis for Eq.(1.1). The method of determining Lie symmetry for a partial differential equation is standard which is described in [1–7].

First of all, let us consider a one-parameter group of infinitesimal transformation:

$$\begin{aligned} \tilde{s} &= s + \epsilon\xi(s, y, u) + O(\epsilon^2), \\ \tilde{y} &= y + \epsilon\tau(s, y, u) + O(\epsilon^2), \\ \tilde{u} &= u + \epsilon\phi(s, y, u) + O(\epsilon^2), \end{aligned} \tag{3.1}$$

where $\epsilon \ll 1$ is a group parameter. The vector field associated with the above group of transformations can be written as

$$V = \tau(s, y, u)\frac{\partial}{\partial s} + \xi(s, y, u)\frac{\partial}{\partial y} + \phi(s, y, u)\frac{\partial}{\partial u}. \tag{3.2}$$

Thus, the second prolongation $Pr^{(2)}V$ is

$$Pr^{(2)}V = V + \phi^s\frac{\partial}{\partial u_s} + \phi^y\frac{\partial}{\partial u_y} + \phi^{yy}\frac{\partial}{\partial u_{yy}}, \tag{3.3}$$

where

$$
\begin{aligned}
\phi^s &= D_s(\phi) - u_s D_s(\tau) - u_y D_s(\xi),\\
\phi^y &= D_y(\phi) - u_s D_y(\tau) - u_y D_y(\xi),\\
\phi^{yy} &= D_y(\phi^y) - u_{sy} D_y(\tau) - u_{yy} D_y(\xi),
\end{aligned}
$$

and D_s and D_y denote the total derivative operator with respect to s and y.

Applying the second prolongation $Pr^{(2)}V$ to Eq.(1.1), we find that the coefficient functions $\tau(s, y, u), \xi(s, y, u)$ and $\phi(s, y, u)$ must satisfy the following invariant condition:

$$
Pr^{(2)}V(\Delta)|_{\Delta=0} = \xi r u_y u_{yy} + \phi^s u_{yy} + \phi^y r y u_{yy} - 2\theta\phi^y u_y + \phi^{yy} u_s + \phi^{yy} r y u_y = 0, \quad (3.4)
$$

where $\Delta = u_s u_{yy} + r y u_y u_{yy} - \theta u_y^2 = 0$. Then we obtain an over determined system of equations as follows:

$$
\begin{aligned}
\xi_u &= \xi_{yy} = 0,\\
\xi_s &= -r(y\xi_y - \xi),\\
\tau_s &= \tau_u = \tau_y,\\
\phi_s &= \phi_y = \phi_{uu}.
\end{aligned} \quad (3.5)
$$

Solving above Eqs.(3.5), one can get

$$
\begin{aligned}
\xi &= C_1 y + C_2,\\
\tau &= C_3,\\
\phi &= C_4 u + C_5,
\end{aligned}
$$

where C_1, C_2, C_3, C_4 and C_5 are arbitrary constants. Hence the Lie algebra of infinitesimal symmetries of Eq.(1.1) is spanned by the following vector fields:

$$
\begin{aligned}
V_1 &= y\frac{\partial}{\partial y},\\
V_2 &= e^{rs}\frac{\partial}{\partial y},\\
V_3 &= \frac{\partial}{\partial s},\\
V_4 &= u\frac{\partial}{\partial u},\\
V_5 &= \frac{\partial}{\partial u}.
\end{aligned}
$$

Then, all of the infinitesimal generators of Eq.(1.1) can be expressed as

$$V = C_1V_1 + C_2V_2 + C_3V_3 + C_4V_4 + C_5V_5. \tag{3.6}$$

The commutation relations of Lie algebra determined by V_1, V_2, V_3, V_4, V_5, are shown in Table 1. It is obvious that $\{V_1, V_2, V_3, V_4, V_5\}$ is commute under the Lie bracket.

Table 1: The commutation table of Lie algebra

$[V_i, V_j]$	V_1	V_2	V_3	V_4	V_5
V_1	0	$-V_2$	0	0	0
V_2	V_2	0	$-rV_2$	0	0
V_3	0	rV_2	0	0	0
V_4	0	0	0	0	$-V_5$
V_5	0	0	0	V_5	0

To get symmetry groups, we should solve the following ordinary differential equations with initial problems:

$$\begin{cases} \frac{d\tilde{s}}{d\epsilon} = \xi(\tilde{s}, \tilde{y}, \tilde{u}), \\ \tilde{s}|_{\epsilon=0} = s, \end{cases} \quad \begin{cases} \frac{d\tilde{y}}{d\epsilon} = \tau(\tilde{s}, \tilde{y}, \tilde{u}), \\ \tilde{y}|_{\epsilon=0} = y, \end{cases} \quad \begin{cases} \frac{d\tilde{u}}{d\epsilon} = \phi(\tilde{s}, \tilde{y}, \tilde{u}), \\ \tilde{u}|_{\epsilon=0} = u, \end{cases}$$

then we obtain one-parameter symmetry groups $g_i : (s, y, u) \to (\tilde{s}, \tilde{y}, \tilde{u})$ of above corresponding the infinitesimal generators $V_i (i = 1, 2, 3, 4, 5)$ are given as follows:

$$\begin{aligned}
g_1 &: (s, y, u) \to (s, ye^{\epsilon}, u), \\
g_2 &: (s, y, u) \to (s, y + \epsilon e^{rs}, u), \\
g_3 &: (s, y, u) \to (s + \epsilon, y, u), \\
g_4 &: (s, y, u) \to (s, y, ue^{\epsilon}), \\
g_5 &: (s, y, u) \to (s, y, u + \epsilon).
\end{aligned}$$

Thus the following theorem holds:

Theorem 3.1. *If $u = f(y, s)$ is a known solution of Eq.(1.1), then by using the above groups $g_i(i = 1, 2, 3, 4, 5)$, the corresponding new solutions $u_i(i = 1, 2, 3, 4, 5)$*

can be obtained respectively as follows:

$$u_1 = f(ye^{-\epsilon}, s),$$
$$u_2 = f(y - \epsilon e^{rs}, s),$$
$$u_3 = f(y, s - \epsilon),$$
$$u_4 = e^{-\epsilon} f(y, s),$$
$$u_5 = f(y, s) - \epsilon.$$

Using the Table 1 and the following Lie series

$$Ad(\exp(\epsilon V_i))V_j = V_j - \epsilon[V_i, V_j] + \frac{\epsilon^2}{2!}[V_i, [V_i, V_j]] - \frac{\epsilon^3}{3!}[V_i, [V_i, [V_i, V_j]]] + \cdots,$$

we obtain the adjoint representation in Table 2.

Table 2: The adjoint representation

$Ad(\exp(\epsilon V_i))V_j$	V_1	V_2	V_3	V_4	V_5
V_1	V_1	$e^{\epsilon}V_2$	V_3	V_4	V_5
V_2	$V_1 - \epsilon V_2$	V_2	$V_3 + \epsilon r V_2$	V_4	V_5
V_3	V_1	$(1 - r + re^{\epsilon})V_2$	V_3	V_4	V_5
V_4	V_1	V_2	V_3	V_4	$e^{\epsilon}V_5$
V_5	V_1	V_2	V_3	$V_4 - \epsilon V_5$	V_5

Based on the adjoint representation, we have the following theorem:

Theorem 3.2. *The optimal system of one-dimensional subalgebras of the Lie algebra spanned by V_1, V_2, V_3, V_4, V_5 of Eq.(1.1) given by*

$$V_1 \pm V_3, V_1 \pm V_4, V_1 \pm V_3 \pm V_4, V_2 \pm V_4, V_3, V_3 \pm V_4, V_5.$$

4. Symmetry reductions and invariant solutions

In this section, making use of optimal system in Theorem 3.2, we will get derive several types of symmetry reductions and invariant solutions.

Case 1. For the infinitesimal generator $V_1 + V_3 = y\frac{\partial}{\partial y} + \frac{\partial}{\partial s}$, the similarity variables are $\xi = s - \ln y$, $f(\xi) = u$, and the group-invariant solution is $u = f(\xi)$.

Substituting the group-invariant solution into Eq.(1.1), we obtain the following reduction equation:

$$(1 + r)f'f'' - \theta f'^2 = 0. \tag{4.1}$$

Solving above reduction equation, we obtain solution of Eq.(1.1) as follows:

$$u = c_1 + c_2 e^{\frac{\theta(s-\ln y)}{1+r}}. \tag{4.2}$$

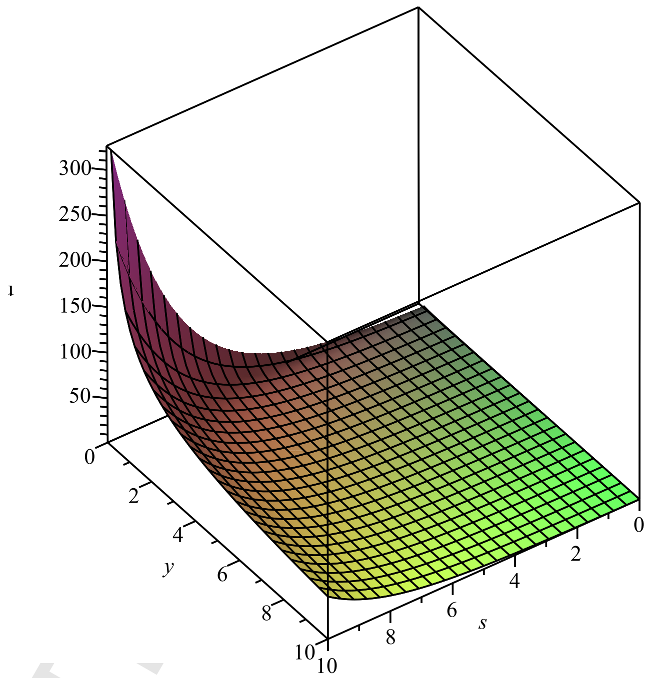

Figure 1: Solution (4.2) with $c_1 = c_2 = r = \theta = 1$

Case 2. For the infinitesimal generator $V_1 + V_4 = y\frac{\partial}{\partial y} + u\frac{\partial}{\partial u}$, the similarity variables are $\xi = s, f(\xi) = \frac{u}{y}$, and the group-invariant solution is $u = yf(\xi)$. Substituting the group-invariant solution into Eq.(1.1), we obtain the following reduction equation:

$$-\theta f(s) = 0. \tag{4.3}$$

Therefore, Eq.(1.1) has a solution $u = 0$. Obviously, the solution is not meaningful.

Case 3. For the infinitesimal generator $V_1 + V_3 + V_4 = y\frac{\partial}{\partial y} + \frac{\partial}{\partial s} + u\frac{\partial}{\partial u}$, the similarity variables are $\xi = s - \ln y, f(\xi) = \frac{u}{y}$, and the group-invariant solution is $u = yf(\xi)$. Substituting the group-invariant solution into Eq.(1.1), we obtain the following reduction equation:

$$(r-\theta-1)f'^2 + (2\theta - r)ff' + (1-r)f'f'' + rff'' - \theta f^2 = 0. \tag{4.4}$$

Case 4. For the infinitesimal generator $V_3 = \frac{\partial}{\partial s}$, the similarity variables are $\xi = y, f(\xi) = u$, and the group-invariant solution is $u = f(\xi)$. Substituting the group-invariant solution into Eq.(1.1), we obtain the following reduction equation:

$$r\xi f'f'' - \theta f'^2 = 0. \tag{4.5}$$

Solving above reduction equation, we obtain solution of Eq.(1.1) as follows:

$$u = c_1 + c_2 y^{\frac{\theta+r}{r}}. \tag{4.6}$$

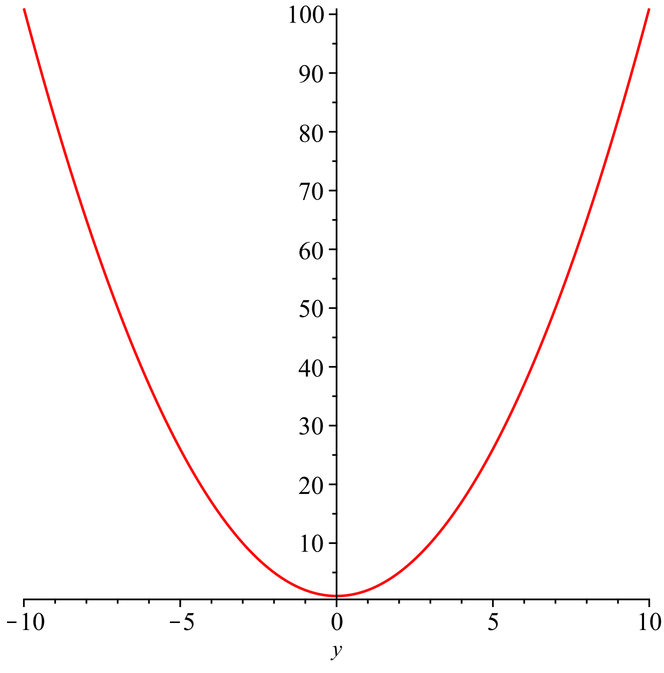

Figure 2: Solution (4.6) with $c_1 = c_2 = r = \theta = 1$

Case 5. For the infinitesimal generator $V_2+V_4 = e^{rs}\frac{\partial}{\partial y}+u\frac{\partial}{\partial u}$, the similarity variables are $\xi = s, f(\xi) = ue^{-ye^{-rs}}$, and the group-invariant solution is $u = e^{ye^{-rs}}f(\xi)$.

Substituting the group-invariant solution into Eq.(1.1), we obtain the following reduction equation:

$$ff' - \theta f^2 = 0. \tag{4.7}$$

Solving above reduction equation, we have $f = ce^{\theta s}$. Therefore, Eq.(1.1) has solution as follows:

$$u = ce^{\theta s + ye^{-rs}}. \tag{4.8}$$

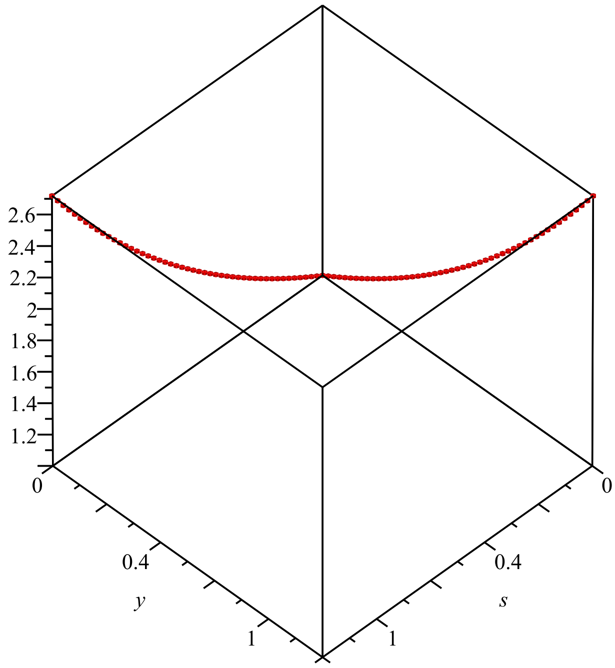

Figure 3: Solution (4.8) with $c = r = \theta = 1$

Case 6. For the infinitesimal generator $V_3 + V_4 = \frac{\partial}{\partial s} + u\frac{\partial}{\partial u}$, the similarity variables are $\xi = y, f(\xi) = ue^{-s}$, and the group-invariant solution is $u = e^s f(\xi)$. Substituting the group-invariant solution into Eq.(1.1), we obtain the following reduction equation:

$$ff'' + r\xi f' f'' - \theta f'^2 = 0. \tag{4.9}$$

5. Conclusions

In this paper, we study Lie symmetry analysis for a parabolic Monge-Ampère equation in the optimal investment theory . As a result, infinitesimal generator, commutation table of Lie algebra and optimal system of this equation are derived. With the help of optimal systems, symmetry reductions and invariant solutions are obtained. It is shown that the the value function of self-financing optimal investment problem is a smooth and monotonically function of initial wealth y. For n-dimensional case, let $r = \sqrt{y_1^2 + \cdots + y_n^2}$, Eq.(2.8) becomes Eq.(2.11), so discuss the case $n = d = 1$ that involve the general case. The solutions which we obtain can be used to self-financing optimal investment problem and to check on the accuracy and reliability of numerical algorithm of Hamilton-Jacobi-Bellman equation (2.8).

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