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The bankruptcy problem in financial networks

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HIGHLIGHTS

- Bargaining Theory was previously applied to derive rules governing default of a single debtor having multiple creditors.
- Bankruptcy resolution within payments and interbank loan networks is more complex, due to multiple debtors owing multiple creditors.
- Bargaining Theory is extended to default resolution within such networks.
- Popular resolution rules, derived from Bargaining Theory for single debtor situations, are not similarly justified in financial networks.

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ABSTRACT

The *bankruptcy problem* of resolving a single debt owed to multiple creditors is extended to financial networks, where there are multiple debtors and creditors.

1. Introduction

The law and economics of default resolution in bankruptcy typically presume a single debtor that owes multiple creditors. But *multiple* debtors owe multiple creditors within interbank payment networks like Fedwire and CHIPS, as well as in decentralized networks created by complex interbank loan obligations. Nash Bargaining theory has previously been used to justify two popular rules for resolving default of a single debtor. I describe how to extend these rules to financial networks, but find that Nash Bargaining theory does not justify applying those rules to resolve defaults in those networks.

2. Default resolution rules

Dagan and Volij (1993) consider the following *bankruptcy problem*: one agent owes non-negative amounts denoted $c_1, c_2, \dots, c_n$ to each of n creditors, but possesses an ability to pay only $E \leq \sum_{j=1}^{n} c_j$. The problem arises when the inequality is strict, in which case default is inevitable and must be resolved. A feasible *allocation* is a vector of default resolution payments $x_1, x_2, \dots, x_n$ such that $\forall j, x_j \leq c_j$ and $\sum_j x_j = E$.

Dagan and Volij (op.cit.) study two default resolution rules :

(a) The *Proportional rule* allocation $x_1, x_2, \dots, x_n$ requires the debtor to pay the *same* fraction $\lambda \leq 1$ in resolution to each creditor, i.e. $x_j = \lambda c_j$ where $\lambda \sum_j c_j = E$.

The Proportional rule has a very long history and is the basis for the extant resolution procedure in bankruptcy law (Kadens, 2010).

(b) The *Constrained Equality (CE) rule* allocates an equal *dollar* payment, denoted x, to each creditor, subject to the constraint that no one is paid more than originally *owed*.[1] That is, the CE rule finds an amount x and requires $x_j = \min(x, c_j)$ and $\sum_j [x_j = \min(x, c_j)] = E$.

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1 Levinthal's (1918) history of early bankruptcy law notes that ancient Jewish law required equal *dollar* (rather than percentage) payments to creditors, subject to the constraints that this would not compensate any creditors more than they were owed (op.cit, p. 234).

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Largely using their notation to facilitate comparisons, I generalize this framework to financial networks. In such networks, represent the payment c_{ij} *owed* by agent i to agent j by a nonnegative matrix:

$$\mathbf{C} = \begin{bmatrix} 0 & c_{12} & \dots & c_{1n} \\ c_{21} & 0 & \dots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \dots & 0 \end{bmatrix}. \tag{1}$$

Such networks include interbank lending, checks drawn on one bank that must be deposited in accounts at another bank, or payments owed as a result of mutual trading activities on an asset exchange. It is the nature of these networks that both c_{ij} and c_{ji} could be positive, and that there is no partition of indices into debtors and creditors.

A feasible network allocation is a matrix of resolution payments x_{ij} such that $\forall i, j \neq i : x_{ij} \leq c_{ij}$ and $\sum_{j\neq i} x_{ij} = E_i$, where E_i is agent i's ability to pay. Unlike the classical bankruptcy problem, here an agent's ability to pay is *endogenous* with the allocation, because any agent i's ability to pay includes payments it receives from other agents, i.e. $E_i \geq \sum_{j\neq i} x_{ij}$. In order to focus on the role of the interagent obligations matrix (1) rather than the exogenous component of the distribution of ability to pay, we assume strict equality, so that default resolution allocations must arise from redistributing obligations in (1), resulting in a matrix with each row total equal to its corresponding column total:

$$\mathbf{X} = \begin{bmatrix} 0 & x_{12} & \dots & x_{1n} \\ x_{21} & 0 & \dots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \dots & 0 \end{bmatrix}. \tag{2}$$

We will use the following numerical example for illustration throughout:

$$\mathbf{C} = \begin{bmatrix} 0 & 0 & 10 & 0 \\ 30 & 0 & 20 & 20 \\ 10 & 30 & 0 & 10 \\ 10 & 0 & 20 & 0 \end{bmatrix}. \tag{3}$$

Inspecting (3), we see that agent #2 owes 70 in total but is owed only 30 in total. Hence it will have to default on some of these payments, triggering defaults by agents #3 and #4.

Elimam et al. (1996) and Eisenberg and Noe (2001) focused on a network default resolution rule in which defaulting agents are required to proportionally renege on all their respective creditors. This is the natural generalization of the classical proportional rule (a). We state this formally below:

($\mathrm{a}^{\mathrm{Net}}$) The *Network Proportional rule* is a nonnegative matrix (2) requiring each agent i to pay a fraction $\lambda_i \leq 1$ in resolution of what it owes to each of the other agents, i.e.

$$x_{ij} = \lambda_i c_{ij} \text{ where } \sum_{j\neq i} x_{ij} = \lambda_i \sum_{j\neq i} c_{ij} = \sum_{j\neq i} x_{ji} = \sum_{j\neq i} \lambda_j c_{ji}. \tag{4}$$

The existence of a vector of fractions $(\lambda_1, \lambda_2, \dots, \lambda_n)$, that in conjunction with the obligations matrix $\mathbf{C}$ determines the Network Proportional rule allocation, was established by Eisenberg and Noe (op.cit.), who showed that one can be computed by solving a linear programming problem. A simplified exposition is given in Demange (2015), who solves the following linear program:

$$\begin{aligned} &\boldsymbol{\lambda}^* = \arg\max_{\lambda_1,\dots,\lambda_n} \sum_i \lambda_i \sum_{j\neq i} c_{ij} \\ &s.t.\ \lambda_i \sum_{j\neq i} c_{ij} - \sum_{j\neq i} \lambda_j c_{ji} \leq 0;\ \ i = 1, \dots, n. \end{aligned} \tag{5}$$

The constraints require that the solution fractions $\boldsymbol{\lambda}^*$ result in an aggregate of resolution payments *from* each agent i that does not exceed the aggregate of payments *to* it. The objective function is the aggregate of payments made throughout the network.

In our illustrative example (3), the numerical solution of (5) is $\boldsymbol{\lambda}^* = (1.0, 14.8\%, 34.4\%, 21.3\%)$. The associated *default resolution payments* x_{ij} are:

$$\mathbf{X}^P = \begin{bmatrix} 0 & 0 & 10 & 0 \\ 4.426 & 0 & 2.951 & 2.951 \\ 3.443 & 10.328 & 0 & 3.443 \\ 2.131 & 0 & 4.262 & 0 \end{bmatrix}. \tag{6}$$

The natural network extension of the CE rule (b) is the following *Network CE rule:*

($\mathrm{b}^{\mathrm{Net}}$) The *Network CE rule* is a nonnegative matrix (2) requiring each agent i to pay the same dollar amount x_i to each of its creditors, subject to the constraint that none of its creditors is paid more than originally *owed*. That is, the rule is a vector $x_1, x_2, \dots, x_n$ and requires that

$$\begin{aligned} &x_{ij} = \min(x_i, c_{ij}) \text{ and} \\ &\sum_{j\neq i} [x_{ij} = \min(x_i, c_{ij})] = E_i = \sum_{j\neq i} [x_{ji} = \min(x_j, c_{ji})]. \end{aligned} \tag{7}$$

Using (3), it is easy to verify that the vector $(x_1 = 10, x_2 = 5/3, x_3 = 5, x_4 = 10/3)$ enables feasible allocation of the following Network CE rule resolution payments:

$$\mathbf{X}^{\mathbf{CE}} = \begin{bmatrix} 0 & 0 & 10 & 0 \\ 5/3 & 0 & 5/3 & 5/3 \\ 5 & 5 & 0 & 5 \\ 10/3 & 0 & 10/3 & 0 \end{bmatrix}. \tag{8}$$

Using (7), denote $\sum_{j\neq i} x_{ij} \equiv l_i(\overrightarrow{x})$, where $l_i(\overrightarrow{x})$ denotes the total resolution payments *made* (i.e. liabilities) by agent i, and $\sum_{j\neq i} x_{ji} = a_i(\overrightarrow{x})$, the total resolution payments *received* (i.e. assets) by agent i. We prove the following constructive existence proposition below:

Proposition 1. *Under the Network* CE *rule* (7), *the feasible allocation set is nonempty. Moreover, the maximal* $\overrightarrow{x}^* = (x_1^*, x_2^*, \dots, x_n^*)$ *defining its required payments* $x_{ij}^* = min(c_{ij}, x_i^*)$ *can be found by solving*

$$\begin{aligned} &\max_{X_{ij}} \sum_i \sum_{j\neq i} x_{ij} \equiv \max_{x_1,\dots,x_n} \sum_i \sum_{j\neq i} min(x_i, c_{ij})\, s.t. \\ &l_i(\overrightarrow{x}) \equiv \sum_{j\neq i} min(x_i, c_{ij}) = a_i(\overrightarrow{x}) \\ &\qquad \equiv \sum_{j\neq i} min(x_j, c_{ji}),\ \ i = 1, \dots, n. \end{aligned} \tag{9}$$

Proof. Define the vector-valued maps $\mathbf{l}(\overrightarrow{x}) \equiv (l_1(\overrightarrow{x}), \dots, l_n(\overrightarrow{x}))$ and $\mathbf{a}(\overrightarrow{x}) \equiv (a_1(\overrightarrow{x}), \dots, a_n(\overrightarrow{x}))$ on the subset S of vectors $\overrightarrow{x}$ in which $x_i \in [0, \max_{j\neq i} c_{ij}], i = 1, \dots, n$. This subset is a complete lattice with the usual ordering $\leq$ on n-vectors. $\mathbf{l}(\overrightarrow{x})$ is monotone increasing on S, and hence has an inverse. Because $\mathbf{a}(\overrightarrow{x})$ is monotone nondecreasing on S, the map $\mathbf{f} : S \to S; f(\overrightarrow{x}) = \mathbf{l}^{-1}(\mathbf{a}(\overrightarrow{x}))$ is monotone on the complete lattice S. A fixed point of $\mathbf{f}$ satisfies the constraints in (9). By the Knaster–Tarski Fixed Point Theorem,[2] the map has a set of fixed points which is also a complete lattice (and hence nonempty), and hence has a maximal element. So it can be found by solving (9). □

The matrix (8) was found by substituting (3) into problem (9) and numerically solving it.

[2] See https://en.wikipedia.org/wiki/Knaster%E2%80%93Tarski_theorem.

3. Bargaining theory characterizations

Dagan and Volij assume linear utility of payments made to each creditor of a single debtor, and that the creditors' threat points are all 0. They accordingly define the Nash Bargaining solution to be the feasible allocation that maximizes the product of payments to the creditors. In their Proposition 1 (op.cit., p. 292), they prove that the Nash Bargaining solution is the CE rule defined above in Section 1(b).

It is insightful to generalize and formally prove this result when utilities are strictly concave, as I now do.

Corollary to Dagan and Volij's Proposition 1. *When there is a single defaulting agent, and creditors $j = 1, 2, \ldots, n$ have concave utilities $U_j(x_j)$ with identical threat points $U_j(0) = 0$, the Nash Bargaining solution is the* CE *rule.*

Proof. The Lagrangian for the Nash Bargaining problem is

$$L = \prod_j U_j(x_j) - \sum_j \delta_j(x_j - c_j) - \gamma\left(\sum_j x_j - E\right).$$

We assume $U_j(x_j > 0) > 0$ and that $U_j' > 0$ and $U_j'' \leq 0$. Then one can replace the objective function $\prod_j U_j(x_j) := \log \sum_j U_j(x_j)$ in the Lagrangian. The first order conditions are:

$$\frac{U_j'(x_j^*)}{U_j(x_j^*)} = \delta_j^* + \gamma^*, \ \forall j. \tag{10}$$

Because we assume concave utility, $U_j'' \leq 0$ so $\frac{d}{dx_j}\left[\frac{U_j'(x_j^*)}{U_j(x_j^*)}\right] = \frac{U_j''U_j - U_j'^2}{U^2} < 0$. Hence the left hand side of each equation j in the first order condition is monotone (decreasing) and must have an inverse, dubbed I_j. So each $x_j^* = I_j(\delta_j^* + \gamma^*)$. If $x_j^* < c_j$, the complementary slackness condition implies that $\delta_j^* = 0$ in (10), in which case the equations yield $x_j^* = I_j(\gamma^*) \equiv x, \ \forall j : x_j^* < c_j$. In other words, all creditors who are not made whole receive equal dollar payments$x$. So $x_j^* = min(x, c_j)$ in accord with the single defaulter CE rule of Section 1(b). □

We will now see that *Dagan and Volij's Proposition 1 does not extend to financial networks*. To see why, note that the Network Nash Bargaining solution solves the following problem:

$$\begin{aligned} &\max_{x_{ij}} \prod_i U_i\left(\sum_{j\neq i} x_{ji}\right) \ s.t. \\ &x_{ij} \leq c_{ij} \ \forall i, \ j \neq i \\ &\sum_{j\neq i} x_{ij} = \sum_{j\neq i} x_{ji}, \ i = 1, \ldots, n. \end{aligned} \tag{11}$$

The objective function in (11) is the product of each agent's utility of total payments received in resolution, subject to the usual constraints that in resolution nobody is paid more than originally owed and that the total amount paid by each agent equals what is received in resolution.

The Lagrangian is:

$$\begin{aligned} L = &\prod_i U_i\left(\sum_{j\neq i} x_{ji}\right) - \sum_i \sum_{j\neq i} \delta_{ij}(x_{ij} - c_{ij}) \\ &- \sum_i \gamma_i\left(\sum_{j\neq i} x_{ij} - \sum_{j\neq i} x_{ji}\right). \end{aligned}$$

As in the above proof of our corollary, we can substitute the log of the product of utilities and write the first order conditions with respect to each x_{ij} as:

$$\frac{U_j'\left(\sum_{j\neq i} x_{ji}^*\right)}{U_j\left(\sum_{j\neq i} x_{ji}^*\right)} = \delta_{ij}^* + \gamma_i^* - \gamma_j^*, \ \forall i, j \neq i. \tag{12}$$

Also as in the proof of our corollary, for any j we can invert the left hand side to derive

$$\sum_{j\neq i} x_{ji}^* = I_j(\delta_{ij}^* + \gamma_i^* - \gamma_j^*) \forall i, \ j \neq i.$$

Now if agent i pays two agents j and j' less than they are respectively owed, complementary slackness implies that $\delta_{ij}^* = \delta_{ij'}^* = 0$. But due to the presence of γ_j^* and $\gamma_{j'}^*$ in system (12), this does *not* force $x_{ij} = x_{ij'} \equiv x_i$ as it does in the first order condition (10) that leads to our corollary of Dagan and Volij's Proposition 1. That is, an agent i does *not* have to pay the same amount to two other agents who each receive less than owed. *In networks, the Nash Bargaining solution is not the CE rule.*

To illustrate this using (3), following Dagan and Volij we assume linear utilities and numerically solve (11) to find:

$$\mathbf{X}^{NB} = \begin{bmatrix} 0 & 0 & 10 & 0 \\ 0 & 0 & 10 & 20 \\ 0 & 30 & 0 & 10 \\ 10 & 0 & 20 & 0 \end{bmatrix} \tag{13}$$

which clearly is not the allocation (8) that does satisfy the Network CE rule. We state this as our Proposition 1 below:

Proposition 2. *The Network Nash Bargaining solution solving (11) is not the Network* CE *rule allocation (7), even when utilities are linear as Dagan and Volij assumed.*

Dagan and Volij also used bargaining theory to rationalize the Proportional rule (see our Section 1(a)) when there is only a single debtor. They defined a *c-asymmetric Nash Bargaining* solution in which each creditor j's (linear) utility is raised to the power c_j, i.e. $x_j^{c_j}$ before these terms are all multiplied together to form the defining objective function. Dagan and Volij's Proposition 2 (op.cit, p. 293) shows that the resulting solution is a single debtor Proportional rule allocation defined in our Section 1(b).

I now show that this result does not generalize to nonlinear utilities, even when there is only a single debtor as they assumed.

Corollary to Dagan and Volij's Proposition 2. *When there is a single defaulting agent and n creditors with strictly concave utilities, the c-asymmetric Nash Bargaining solution is* not *necessarily a Proportional rule.*

Proof. To see this, take the log of the weighted objective function to find the Lagrangian for that problem:

$$L = \sum_j c_j \log(U_j(x_j)) - \sum_j \delta_j(x_j - c_j) - \gamma(\sum_j x_j - E).$$

With first order conditions:

$$\frac{U_j'(x_j^*)}{U_j(x_j^*)} = \frac{\delta_j^* + \gamma^*}{c_j}, \ \forall j.$$

If $x_j^* < c_j$, $\delta_j^* = 0$ and upon inverting the first order conditions one obtains $x_j^* = I_j\left(\frac{\gamma^*}{c_j}\right)$. Because I_j is monotone decreasing, x_j^* is a monotone *increasing* function of c_j, denoted $x_j^* = f_j(c_j)$. For some other agent j', if $x^*_{j'} < c_{j'}$, $f_{j'}$ is a *possibly different* increasing function of $c_{j'}$. This is because it is derived from its own creditor's marginal log utility, which may not be the same function of a different creditor. But the Proportional rule requires that these be the *same* increasing *linear* function $x^*_j = \lambda^* c_j$, for j, j'. This will not generally be the case. □

In light of the negative result in our Proposition 2, there is no point in formally proving that the Network Proportional rule will also not be rationalizable as a c-asymmetric Network Nash Bargaining solution. But that network solution is easy to calculate. Using linear utilities as Dagan and Volij did, and our example obligations matrix (3), a c-asymmetric Network Nash Bargaining solution allocation is:

$$\mathbf{X}^{\textbf{c-asymmetric}} = \begin{bmatrix} 0 & 0 & 10 & 0 \\ 0 & 0 & 13.75 & 16.25 \\ 3.75 & 30 & 0 & 10 \\ 6.25 & 0 & 20 & 0 \end{bmatrix}. \tag{14}$$

Not surprisingly, the c-asymmetric solution (14) is inconsistent with the Network Proportional rule, e.g. agent #2 pays 0% of the 30 it owed to Agent #1, while paying around 69% of the 20 it owed agent #3, and around 81% of the 20 it owed agent #4. *In summary, Dagan and Volij's (op.cit.) Proposition 2 neither extends to nonlinear utilities nor to financial networks.*

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