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Discount pricing in word-of-mouth marketing: An optimal control approach

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Abstract

This paper addresses the discount pricing in word-of-mouth (WOM) marketing. First, a dynamic model capturing WOM spreading processes is suggested. Second, the problem of finding an optimal discount strategy boils down to an optimal control problem. Third, the existence of an optimal control for the control problem is proved, and an optimality system for finding an optimal control is presented. Thereby, the dynamic discount strategy associated with the optimal control is recommended. Some examples of the optimal control are given. Finally, the influence of different factors on the optimal expected net profit is examined.
Keywords: word-of-mouth marketing, discount strategy, dynamic model, optimal expected net profit, optimal control, optimality system

1. Introduction

Dynamic pricing, which is defined as the dynamic adjustment of market price, has been widely adopted by marketing researchers and practitioners [1–6]; this is because price is one of the most controllable variables. In the past, dynamic pricing strategies were widely applied to traditional marketing, with a significant cost associated with changing prices [7–9]. With the advent of the Internet marketing, the overhead has greatly diminished [10–13]. Dynamic discount pricing is a common form of dynamic pricing. In contrast to a merely low price, a discounted price typically implies the high quality and unusual bargain of the item and, hence, has strong attraction to rational customers. Consequently, the design and evaluation of dynamic discount pricing strategies have long been a research hotspot in the marketing field [14–20].

As compared to traditional advertizing marketing, the emerging word-of-mouth (WOM) marketing can bring about higher marketing profit at a much lower cost [21–25]; this is especially the case with the popularity of online social networks [26]. To evaluate the performance of different WOM marketing strategies, a number of dynamic

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models capturing WOM spreading processes have been suggested [27–36]. However, these dynamic models are all population-level, which cannot accommodate the WOM network. As a result, the accuracy of the estimated performance is questionable. To overcome this flaw, individual-level dynamic models, which have been widely applied to areas such as epidemic spreading [37–40], malware spreading [41–47] and cybersecurity [48–50], should be borrowed to more accurately characterize WOM spreading processes.

The optimal control theory provides an appropriate mathematical framework for the design of better dynamic strategies [51, 52]. For instance, the optimal control approach has been applied to the dynamic control of malware [53–55], where the virus spreading models accommodate the virus propagation network. To the best of our knowledge, to date the dynamic pricing in WOM marketing has not been dealt with in the framework of optimal control theory.

This paper addresses the dynamic discount pricing in WOM marketing by the optimal control approach. First, an individual-level differential dynamic model capturing WOM spreading processes is established. This model accurately characterizes the dynamic discount mechanism and perfectly accommodates the WOM network. On this basis, the expected profit of a WOM marketing champaign is evaluated, and the problem of finding an optimal discount strategy boils down to an optimal control problem. Next, the existence of an optimal control for the control problem is proved, and an optimality system for finding an optimal control is presented. The dynamic discount strategy associated with the optimal control is recommended, because it could bring out the highest possible marketing profit. Finally, the influence of different factors on the highest expected profit is examined.

The materials of this paper are organized as follows. Section 2 formulates the dynamic discount pricing problem. Section 3 models WOM spreading processes. The dynamic discount pricing problem is reduced to an optimal control problem in Section 4. Section 5 theoretically analyzes the optimal control problem. Some examples of the optimal control are given in Section 6. Section 7 examines the influence of different factors on the optimal expected net profit. This work is closed by Section 8.

2. A discount pricing problem

Suppose a marketer is planning a WOM marketing campaign, with the goal of gaining the highest possible marketing profit. This section is devoted to formulating the discount pricing problem to be solved in this paper. For this purpose, let us first introduce some terminologies and notations as follows.

Suppose the marketing campaign is conducted in the finite time horizon $[0, T]$ among a population of N persons labeled $1, 2, \cdots, N$. Let $G = (V, E)$ denote the WOM network related to the campaign, where each node stands for a person in the population, i.e., $V = \{1, 2, \cdots, N\}$, and $(i, j) \in E$ if and only if person j has direct positive influence on person i in terms of his/her purchase decision. Let $\mathbf{A} = \left(a_{ij}\right)_{N\times N}$ denote the adjacency matrix for the WOM network, i.e., $a_{ij} = 1$ or 0 according as $(i, j) \in E$ or not. The influence of node i in the WOM network is measured by the following normalized quantity:

$$d_i = \frac{\sum_{j=1}^{N} a_{ji}}{\max_{1 \le k \le N} \sum_{j=1}^{N} a_{kj}}. \tag{1}$$

Clearly, $0 \le d_i \le 1$.

We shall focus our attention on a type of dynamic discount strategies, where at any time each node is assigned a discount that is linearly proportional to his/her influence in the WOM network. We refer to these discount strategies as *influence-based dynamic discount* (IBDD) strategies. Let $\theta(t) \in [0, 1]$ denote the *basic discount rate* (BDR) at time t. Then the discount assigned to node i at time t is $d_i\theta(t)$. We refer to the function $\theta(t), t \in [0, T]$ as a *BDR function*. Then an IBDD strategy is represented by the associated BDR function. In what follows, we assume that the admissible set of BDR functions is

$$\Theta = \left\{\theta \in L^2[0, T] : 0 \le \theta(t) \le 1, t \in [0, T]\right\}, \tag{2}$$

where the symbol $L^2[0, T]$ stands for the set of all Lebesgue square integrable functions on the interval $[0, T]$. For the relevant knowledge, see Ref. [56].

Our problem is formulated as follows.

Discount pricing problem: Find a BDR function from the admissible set of BDR functions so that the associated expected marketing profit is maximized.

For this purpose, the expected marketing profit must be estimated. And this is the main goal of the subsequent two sections.

3. The modeling of word-of-mouth spreading processes

For the purpose of estimating the expected marketing profit of a WOM marketing campaign, the marketing process needs to be modeled as a differential dynamical system. For the fundamental knowledge on differential dynamical systems, see Ref. [57].

For our purpose, it is assumed that at any time in the time horizon, each and every node in the WOM network is in one of three possible states: *dormant*, *potential* and *adopting*. A dormant node stands for a person who has no will of buying a product, a potential node stands for a person who has the will of buying a product, and an adopting node stands for a person who has bought a product. Let $X_i(t) = 0, 1$ and 2 denote the event that customer i is dormant, potential and adopting at time t, respectively. Let $D_i(t)$, $P_i(t)$ and $A_i(t)$ denote the probability of the event that customer i is dormant, potential and adopting at time t, respectively. That is,

$$D_i(t) = \{X_i(t) = 0\}, \quad P_i(t) = \{X_i(t) = 1\}, \quad A_i(t) = \{X_i(t) = 2\}, \quad 1 \le i \le N. \tag{3}$$

As $D_i(t) + P_i(t) + A_i(t) \equiv 1$ for $t \in [0, T]$, the vector

$$\mathbf{x}(t) = (P_1(t), \cdots, P_N(t), A_1(t), \cdots, A_N(t))^T$$

represents the expected state of the WOM network at time t. Next, let us impose a basic set of hypotheses as follows.

(H_1) (Poisson property) Each node buys no more than one item in an infinitesimal interval of time.

(H_2) (Unit price) The original price per item is one unit.

(H_3) (Controllability) The IBDD strategy, θ, is under full control of the marketer.

(H_4) (WOM) Due to the influence of neighboring adopting nodes, at time t a dormant node i has the will of buying an item and hence becomes potential at rate $\alpha \sum_{j=1}^{N} a_{ij}A_j(t)$, where $\alpha > 0$ is referred to as the *WOM force*. This hypothesis implies that a more influential node contributes more to the marketing than a less influential node.

(H_5) (Rigid demand) Due to the rigid demand, at time t a potential node i buys an item and hence becomes adopting at rate β_1, where $\beta_1 > 0$ is referred to as the *rigid demand*.

(H_6) (Lure) Due to the lure of discount, at time t a potential node i buys an item and hence becomes adopting at rate $\beta_2 d_i \theta(t)$, where $\beta_2 > 0$ is referred to as the *lure force*.

(H_7) (Viscosity) At any time an adopting node temporarily loses the will of buying another item and hence becomes dormant at rate $\gamma > 0$, where γ is referred to as the *viscosity*.

Fig. 1 shows hypotheses (H_4)-(H_7) schematically.

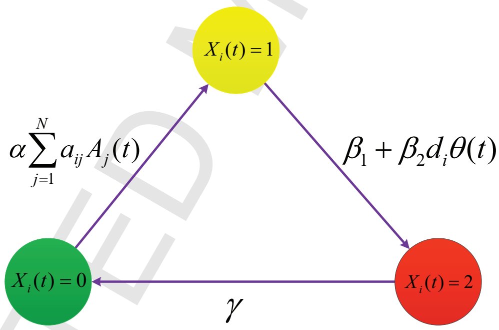

Figure 1. The diagram of state transitions of a customer under the individual-level DPA model.

Based on the above hypotheses, the time evolution of the expected state of the WOM network obeys the following differential dynamical system:

$$\begin{cases} \dfrac{dP_i(t)}{dt} = \alpha\left[1 - P_i(t) - A_i(t)\right] \displaystyle\sum_{j=1}^{N} a_{ij}A_j(t) - (\beta_1 + \beta_2 d_i \theta(t))\, P_i(t), & t \in [0, T], 1 \le i \le N; \\ \dfrac{dA_i(t)}{dt} = (\beta_1 + \beta_2 d_i \theta(t))\, P_i(t) - \gamma A_i(t), & t \in [0, T], 1 \le i \le N. \end{cases} \tag{4}$$

We refer to the model as the *controlled DPA model*, where $\theta \in \Theta$ is the BDR function representing an IBDD strategy.

The DPA model can be written in matrix notation as

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{F}(\mathbf{x}(t), \theta(t)), \quad t \in [0, T]. \tag{5}$$

4. The optimal control problem associated with the dynamic pricing problem

The performance of a BDR function can be measured by the associated expected marketing profit; the higher the expected marketing profit, the better the BDR function. According to the DPA model and in line with hypotheses (H_1)-(H_3), the expected gross profit associated a BDR function θ is

$$P(\theta) = \int_0^T \sum_{i=1}^{N} [\beta_1 + \beta_2 d_i \theta(t)]\, P_i(t)\, [1 - d_i \theta(t)]\, dt.$$

On the other hand, the expected cost associated with the BDR function θ is

$$C(\theta) = \int_0^T \sum_{i=1}^{N} d_i \theta(t) P_i(t) dt. \quad (6)$$

Consequently, the expected net profit associated with the BDR function θ is

$$J(\theta) = P(\theta) - C(\theta) = \int_0^T L(\mathbf{x}(t), \theta(t)) dt, \quad (7)$$

where

$$L(\mathbf{x}(t), \theta(t)) = -\left[\beta_2 \sum_{i=1}^{N} d_i^2 P_i(t)\right] \theta^2(t) + \left[(\beta_2 - \beta_1 - 1) \sum_{i=1}^{N} d_i P_i(t)\right] \theta(t) + \beta_1 \sum_{i=1}^{N} P_i(t). \quad (8)$$

Therefore, the dynamic pricing problem comes down to the following optimal control problem:

$$\text{(P) Maximize}_{\theta \in \Theta} J(\theta) = \int_0^T L(\mathbf{x}(t), \theta(t)) dt,$$
$$\text{subject to } \frac{d\mathbf{x}(t)}{dt} = \mathbf{F}(\mathbf{x}(t), \theta(t)), \quad 0 \le t \le T,$$
$$\mathbf{x}(0) = \mathbf{x}_0.$$

5. A theoretical analysis of the optimal control problem

This section is dedicated to developing a numerical algorithm for solving the optimal control problem (P). For fundamental knowledge on optimal control theory, see Refs. [51, 52].

5.1. The existence of an optimal control

First, let us show that the problem (P) admits an optimal control. The following lemma comes from Ref. [52].

Lemma 1. *Problem (P) has an optimal control if the following five conditions hold simultaneously.*

(C_1) *Θ is closed and covex.*

(C_2) *There is $\theta \in \Theta$ such that the constraint dynamical system is solvable.*

(C_3) *$\mathbf{F}(\mathbf{x}, \theta)$ is bounded by a linear function in $\mathbf{x}$.*

(C_4) *$L(\mathbf{x}, \theta)$ is concave on Θ.*

(C_5) $L(\mathbf{x}, \theta) \leq c_1\theta^\rho + c_2$ *for some* $\rho > 1$, $c_1 < 0$ *and* c_2.

Next, let us show that the five conditions in Lemma 1 hold true.

Lemma 2. *The admissible set* Θ *is closed and convex.*

Proof: Let θ be a limit point of Θ. Then there is a sequence of points, $\{\theta_n\}_{n=1}^{\infty}$, in Θ that approaches θ. As $L^2[0,T]$ is complete, we have $\theta \in L^2[0,T]$. Hence, the closedness of Θ follows from $0 \leq \theta = \lim_{n\to\infty} \theta_n \leq 1$.

Let $\theta_1, \theta_2 \in \Theta$, $0 < \eta < 1$. As $L^2[0,T]$ is a real vector space, we have $(1-\eta)\theta_1 + \eta\theta_2 \in L^2[0,T]$. Hence, the convexity of Θ follows from $0 \leq (1-\eta)\theta_1 + \eta\theta_2 \leq 1$.

Lemma 3. *The constraint dynamical system with* $\theta = 1$ *is solvable.*

Proof: In view of the continuous differentiability of the function $\mathbf{F}(\mathbf{x}, 1)$, the claim follows from the Continuation Theorem for Differential Systems [57].

Lemma 4. $\mathbf{F}(\mathbf{x}, \theta)$ *is bounded by a linear function in* $\mathbf{x}$.

Proof: The claim follows the observation that for $t \geq 0$, $i = 1, 2, \cdots, N$,

$$-(\beta_1 + \beta_2 d_i) P_i \leq \alpha(1 - P_i - A_i)\sum_{j=1}^{N} a_{ij}A_j - (\beta_1 + \beta_2 d_i\theta) P_i \leq \alpha \sum_{j=1}^{N} a_{ij}A_j - \beta_1 P_i,$$
$$\beta_1 P_i - \gamma A_i \leq (\beta_1 + \beta_2 d_i\theta) P_i - \gamma A_i \leq (\beta_1 + \beta_2 d_i) P_i - \gamma A_i.$$

Lemma 5. $L(\mathbf{x}, \theta)$ *is concave on* Θ.

Proof: The claim follows from $\frac{\partial^2 L(\mathbf{x},\theta)}{\partial\theta^2} = -2\beta_2 \sum_{i=1}^{N} d_i^2 P_i \leq 0$.

Lemma 6. $L(\mathbf{x}, \theta) \leq -\theta^2 + 1 + \beta_1 \sum_{i=1}^{N}(d_i - 1)P_i$.

Proof: $L(\mathbf{x}, \theta) \leq \beta_1 \sum_{i=1}^{N} d_i P_i - \beta_1 \sum_{i=1}^{N} P_i \leq -\theta^2 + 1 + \beta_1 \sum_{i=1}^{N}(d_i - 1)P_i$.

From Lemmas 1-6, we have the following main result.

Theorem 1. *The problem (P) has an optimal control.*

This theorem implies that an optimal IBDD strategy does exist.

5.2. The optimality system

It is well known that the optimality system for an optimal control problem offers a numerical method for solving the problem. To present the optimality system for the problem (P), consider the Hamiltonian

$$\begin{aligned} H(\mathbf{x}, \theta, \lambda, \mu) = &-\beta_2\theta^2 \sum_{i=1}^{N} d_i^2 P_i + (\beta_2 - \beta_1 - 1)\,\theta \sum_{i=1}^{N} d_i P_i + \beta_1 \sum_{i=1}^{N} P_i \\ &+ \sum_{i=1}^{N} \lambda_i \left[\alpha(1 - P_i - A_i)\sum_{j=1}^{N} a_{ij}A_j - (\beta_1 + \beta_2 d_i\theta) P_i\right] + \sum_{i=1}^{N} \mu_i \left[(\beta_1 + \beta_2 d_i\theta) P_i - \gamma A_i\right], \end{aligned} \tag{9}$$

where $\lambda = (\lambda_1, \cdots, \lambda_N)^T$ and $\mu = (\mu_1, \cdots, \mu_N)^T$ are the adjoints.

A necessary condition for the optimal control of the problem (P) is presented below.

Theorem 2. *Suppose θ is an optimal control for the problem (P),* **x** *the solution to the constraint dynamical system with θ. Then there exist λ and μ with $\lambda(T) = \mu(T) = \mathbf{0}$ such that for $0 \le t \le T$, $1 \le i \le N$, we have*

$$\begin{cases} \dfrac{d\lambda_i(t)}{dt} = \alpha\lambda_i(t)\displaystyle\sum_{j=1}^{N} a_{ij}A_j(t) + (\beta_1 + \beta_2 d_i\theta(t))(\lambda_i(t) - \mu_i(t)) + \beta_2 d_i^2\theta^2(t) - (\beta_2 - \beta_1 - 1)d_i\theta(t) - \beta_1, \\ \dfrac{d\mu_i(t)}{dt} = \alpha\lambda_i(t)\displaystyle\sum_{j=1}^{N} a_{ij}A_j(t) - \alpha\sum_{j=1}^{N} a_{ji}\lambda_j(t)\left(1 - P_j(t) - A_j(t)\right) + \gamma\mu_i(t), \\ \theta(t) = \max\left\{\min\left\{\dfrac{(\beta_2 - \beta_1 - 1)\sum_{i=1}^{N} d_iP_i(t) + \beta_2\sum_{i=1}^{N} d_iP_i(t)\left[\mu_i(t) - \lambda_i(t)\right]}{2\beta_2\sum_{i=1}^{N} d_i^2P_i(t)}, 1\right\}, 0\right\}. \end{cases} \tag{10}$$

Proof: According to the Pontryagin Maximum Principle [51], there exist λ and μ such that

$$\frac{d\lambda_i(t)}{dt} = -\frac{\partial H(\mathbf{x}(t), \theta(t), \lambda(t), \mu(t))}{\partial P_i}, \quad \frac{d\mu_i(t)}{dt} = -\frac{\partial H(\mathbf{x}(t), \theta(t), \lambda(t), \mu(t))}{\partial A_i}, \quad t \in [0, T], 1 \le i \le N.$$

Thus, the first $2N$ equations in the claim follow by direct calculations. As the terminal cost is unspecified and the final state is free, the transversality conditions $\lambda(T) = \mu(T) = \mathbf{0}$ hold. By using the optimality condition, we have

$$\theta(t) = \arg\max_{\theta^*\in\Theta} H(\mathbf{x}(t), \theta^*(t), \lambda(t), \mu(t)), \quad t \in [0, T].$$

So, either

$$\frac{\partial H(\mathbf{x}(t), \theta(t), \lambda(t), \mu(t))}{\partial\theta} = -2\beta_2\theta(t)\sum_{i=1}^{N} d_i^2P_i(t) + (\beta_2 - \beta_1 - 1)\sum_{i=1}^{N} d_iP_i(t) - \beta_2\sum_{i=1}^{N} d_iP_i(t)\left[\lambda_i(t) - \mu_i(t)\right] = 0,$$

or $\theta(t) = 0$, or $\theta(t) = 1$.

Eqs. (3) and Eqs. (9) together consist the optimality system of the optimal control problem (P).

6. Examples of the optimal control

Applying the well-known forward-backward Euler scheme to the optimality system of the problem (P) presented in the previous section, we can obtain an optimal control, which represents an optimal IBDD strategy, as well as the resulting expected net profit. This section is intended to give three examples of the optimal control obtained by solving the optimality system. For comparison purpose, for an admissible control θ to the problem (P), let $EP(t)$ denote the resulting expected net profit in the time horizon $[0, t]$,

$$EP(t) = \int_0^t \sum_{i=1}^{N} L(\mathbf{x}(s), \theta(s))ds, \quad 0 \le t \le T. \tag{11}$$

6.1. Scale-free network

Networks that approximately follow a power-law degree distribution are referred to as being *scale-free*. Empirical studies show that many real-world networks are scale-free [58]. Therefore, scale-free networks are regarded as a suitable model of real-world networks. Pajek is one of the most famous pieces of software for performing social network analysis [59]. By using Pajek, setting the number of nodes as $N = 100$, and setting the number of edges as $E = 165$, we get a synthetic scale-free network, denoted G_{SF}. See Fig. 2.

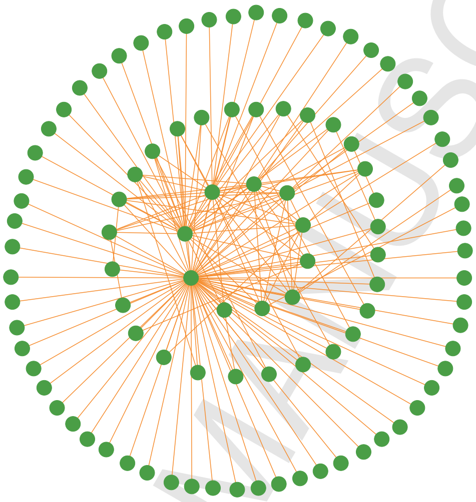

Figure 2. A synthetic scale-free network G_{SF}.

Example 1. *Consider an instance of the optimal control problem (P), where* $G = G_{SF}$, $T = 20$, $\alpha = 0.3$, $\beta_1 = 0.2$, $\beta_2 = 0.5$, *and* $\gamma = 0.1$. *The initial conditions are* $P_i(0) = A_i(0) = 0.1, 0 \leq i \leq N$. *By solving the associated optimality system, we get an optimal control, which is shown in Fig. 3(a). For comparion purpose, Fig. 3(b) plots* $EP(t)$, $t \in [0, T]$, *for the optimal control and three static controls. It is seen that the optimal control is superior to the static controls in terms of the expected net profit.*

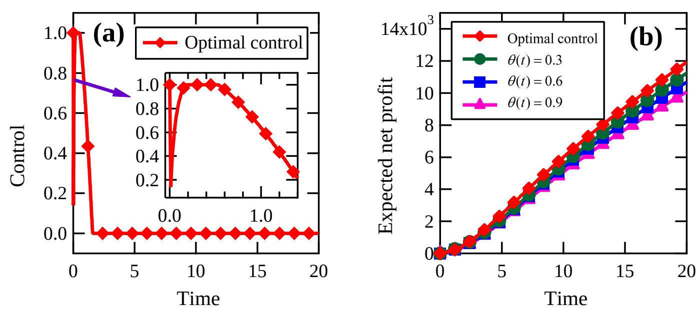

Figure 3. (a) The optimal control in Example 1, and (b) a comparison between the optimal control and three static controls in terms of the expected net profit.

6.2. *Small-world network*

Networks that have a small diameter and a high clustering coefficient are referred to as being *small-world*. Empirical studies show that many real-world networks are small-world [58]. Therefore, small-world networks are adopted as an appropriate model of real-world networks. By using Pajek, setting the number of nodes as $N = 100$, and setting the edge-rewiring probability as $p = 0.1$, we get a synthetic small-world network, denoted G_{SW}. See Fig. 4.

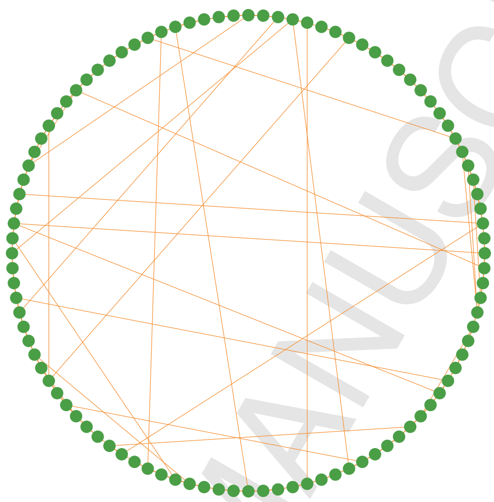

Figure 4. A synthetic small-world network G_{SW}.

Example 2. *Consider an instance of the optimal control problem (P), where* $G = G_{SW}$, $T = 20$, $\alpha = 0.5$, $\beta_1 = 0.6$, $\beta_2 = 0.5$, *and* $\gamma = 0.2$. *The initial conditions are* $P_i(0) = A_i(0) = 0.1, 0 \le i \le N$. *By solving the associated optimality system, we get an optimal control, which is depicted in Fig. 5(a). For comparion purpose, Fig. 5(b) plots* $EP(t)$, $t \in [0, T]$, *for the optimal control and three static controls. It is seen that the optimal control outperforms the static controls in terms of the expected net profit.*

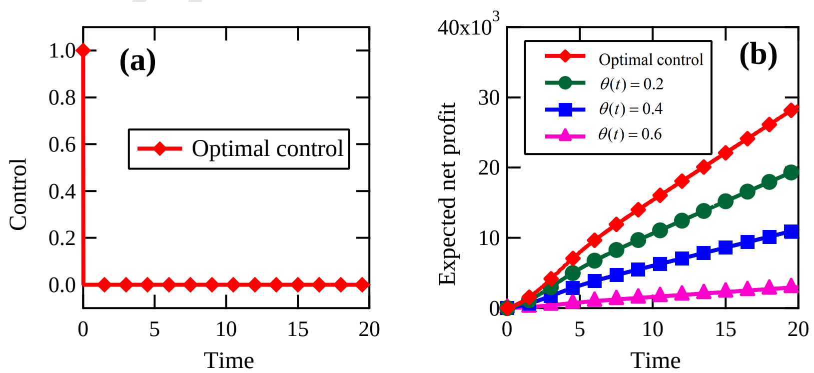

Figure 5. (a) The optimal control in Example 2, and (b) a comparison between the optimal control and three static controls in terms of the expected net profit.

6.3. Email network

Fig. 6 exhibits a realistic email network with $N = 100$ nodes, denoted G_{EM} [60].

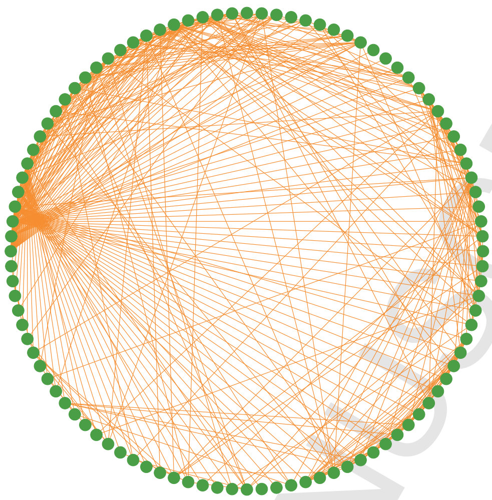

Figure 6. A realistic email network G_{EM}.

Example 3. *Consider an instance of the optimal control problem (P), where $G = G_{EM}$, $T = 20$, $\alpha = 0.3$, $\beta_1 = 0.2$, $\beta_2 = 0.5$, and $\gamma = 0.1$. The initial conditions are $P_i(0) = A_i(0) = 0.1, 0 \le i \le N$. By solving the associated optimality system, we get an optimal control, which is depicted in Fig. 7(a). For comparion purpose, Fig. 7(b) plots $EP(t)$, $t \in [0, T]$, for the optimal control and three static controls. It is seen that the optimal control outmatches the static controls in terms of the expected net profit.*

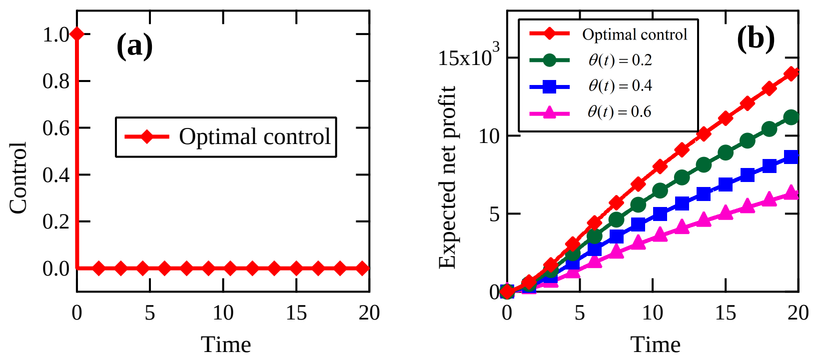

Figure 7. (a) The optimal control in Example 3, and (b) a comparison between the optimal control and three static controls in terms of the expected net profit.

It is seen from these examples that, for any instance of the optimal control problem (P), the optimal control obtained by solving the associated optimality system excels any other submissible control in terms of the expected net profit. Therefore, the IBDD strategy corresponding the optimal control is strongly recommended.

7. The influence of different factors on the optimal expected net profit

The goal of this section is to examine the influence of different factors on the optimal expected net profit, i.e., the expected net profit associated with the optimal control. For brevity, let J^* denote the optimal expected net profit.

7.1. The influence of the four model parameters

First, let us inspect the influence of the four model parameters (WOM force α, rigid demand β_1, lure force β_2, and viscosity γ) on the optimal expected net profit through experiments.

Example 4. *Consider the optimal control problem (P), where* $T = 20$, $G \in \{G_{SF}, G_{SM}, G_{EM}\}$. *The initial conditions are* $P_i(0) = A_i(0) = 0.1, 0 \leq i \leq N$. *Fig. 8(a)-(d) displays the influence of* α, β_1, β_2 *and* γ *on* J^*, *respectively.*

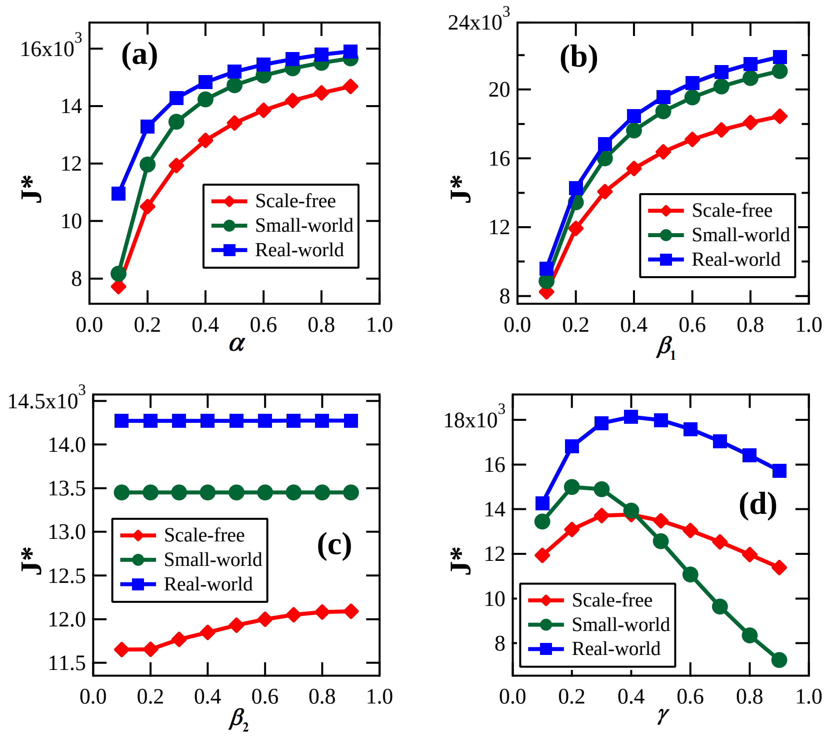

Figure 8. An illustration of the influence of α, β_1, β_2 and γ on J^*.

The following conclusions are drawn from this example.

(a) J^* goes up with α or β_1. This implies that an increase in WOM force or rigid demand could enhance the optimal expected net profit. In practice, WOM force can be enhanced through improvement of customers' experience. The rigid demand is typically uncontrollable.

(b) The influence of β_2 on J^* is negligible.

(c) With the rise of γ, J^* goes up first, then goes down. This suggests that a moderate viscosity could boost the maximum expected net profit. In marketing practice, measures such as pushing new promotion information help increase the viscosity to the turning point.

7.2. The influence of the WOM network

Second, let us examine the influence of the WOM network on the optimal expected net profit through experiments.

Example 5. *Consider the problem (P) with* $T = 20$, $\alpha = 0.3$, $\beta_1 = 0.2$, $\beta_2 = 0.5$, *and* $\gamma = 0.1$. *The WOM network* $G \in \{G_i : 1 \le i \le 7\}$, *where* G_i *is a scale-free network with* $N = 100$ *nodes and with a power-law exponent of* $r = 2.7 + 0.1 \times i$. *The initial conditions are* $P_i(0) = A_i(0) = 0.1, 0 \le i \le N$. *Figure 9(a) exhibits the influence of the power-law exponent on* J^*.

It is well known that the heterogeneity of a scale-free network is proportional to its power-law exponent. It is concluded from the example that J^* inclines with the heterogeneity of a scale-free WOM network. This result informs us that a highly heterogeneous WOM network helps enhance the optimal expected net profit.

Example 6. *Consider the problem (P) with* $T = 20$, $\alpha = 0.3$, $\beta_1 = 0.2$, $\beta_2 = 0.5$, *and* $\gamma = 0.1$. *The WOM network* $G \in \{G_i : 1 \le i \le 5\}$, *where* G_i *is a small-world network with* $N = 100$ *nodes and with an edge-rewiring probability of* $p = 0.1 \times i$. *The initial conditions are* $P_i(0) = A_i(0) = 0.1, 0 \le i \le N$. *Figure 9(b) demonstrates the influence of the edge-rewiring probability on* J^*.

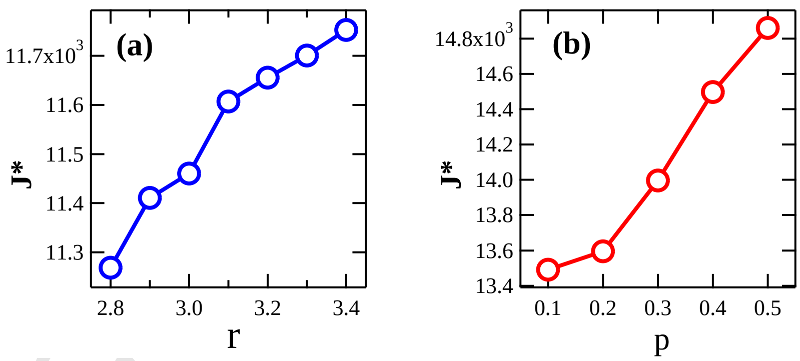

Figure 9. An illustration of how (a) the power-law exponent of a scale-free network, and (b) the edge-rewiring probability of a small-world network affects J^*.

It is concluded from the example that J^* rises with the randomness of a small-world WOM network. This result suggests that a highly random WOM network has positive influence on the optimal expected net profit.

8. Concluding remarks

This paper has addressed the discount pricing in word-of-mouth marketing. A discount pricing problem has been formulated, which has been modeled as an optimal control problem. A theoretical study shows that the optimal control problem is numerically solvable. Some examples of the optimal control have been given. The influence of different factors on the optimal expected net profit has been investigated. Thereby, some promotional measures have been recommended.

There are many related problems that are yet to be solved. Dynamic discount strategies that are similar ours but with different node influence indexes [61–66] should be investigated. This work builds on the rigorous premise that every customer purchases only a single item each time. In real-world scenarios, some customers may well buy multiple items one time. Therefore, a more general dynamic discount strategy has to be developed to adapt to these situations. In essence, discount pricing is a game between the marketer and the potential customers. Hence, this problem can be studied in the framework of game theory [67]. Last, but not least, the methodology used in this paper has many potential applications. In the context of rumor spreading, this technique can promote the circulation of the truth against the rumor [68–71]. As far as computer worm is concerned, this method helps the distributed dissemination of the patch [72].

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Highlights

- The discount pricing is modeled as to an optimal control problem.
- The existence of an optimal control for the control problem is proved.
- An optimality system for finding an optimal control is presented.
- The dynamic discount strategy associated with the optimal control is recommended.
- The influence of different factors on the optimal expected net profit is examined.