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Reliability assessment of reinforced concrete structures based on random damage model

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ABSTRACT

A stochastic model is developed to describe the fracture behaviour of concrete fibres at micro-level, and a probabilistic relation between the fibre strain and the concrete damage variable is established. In this context, the concrete damage evolution can be quantified by two representative random variables. In this regard, the number of random variables employed in potential reliability assessment studies can be greatly reduced. The accuracy of the proposed method is verified by comparing the first- and second-order moments of the stochastic damage evolution with the corresponding closed form solutions. Further, the proposed method is applied to the non-linear analysis and reliability assessment of a five-story reinforced concrete frame, and the results show that it is quite efficient for stochastic response determination and reliability evaluation of complex structures.

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Reliability assessment; stochastic damage model; reinforced concrete structure; random field; non-linear analysis

1. Introduction

In the concrete structural reliability assessment, the material damage and its influence on the response are often of particular concern. Generally, the continuum stress–strain law with random coefficients is adopted in the simulation. However, the concrete material is heterogeneous, non-uniform and random at micro-level. In this context, the non-uniformity and randomness at micro-level may influence the constitutive law of the concrete in a stochastic manner. In addition, this randomness finally affects the structural behaviours and reliability assessment. In this regard, it is more reasonable to consider the stochastic damage of the concrete material at micro-level, and study its effects on the non-linear response and reliability evaluation of structures.

The concrete fracture or damage has been widely investigated and a number of models (Beck & Gomes, 2013; Curtin, 1998; Strauss, Zimmermann, Lehký, Novák, & Keršner, 2014) have been proposed. Among these models, the fiber bundle model (Li & Zhang, 2001; Peirce,1926) is the most commonly used, and also effective in practical applications at micro-level. The fiber bundle model research could be traced to the pioneering work of Peirce (1926). Later, Daniels (1945) studied the influence of microscopic randomness to the macro-strength of material, and proposed the primary form of stochastic damage model. In 1986, Krajcinovic and Fanella (1986) established the classic parallel bundle model for concrete damage mechanics analysis, and the degradation of concrete was denoted by the failure probability of the bundle.

It should be noted that the damage variable is deterministic in Krajcinovic's model. Therefore, Kandarpa, Kirkner, and Spencer (1996) improved Krajcinovic's model by introducing a random field to describe the stochastic damage evolution at micro-level. Li and Zhang (2001) developed a uniaxial stochastic damage model for concrete, and the fracture strains of the micro-elements were also defined as random field. Li and Yang (2009) further modified the classic fiber bundle model by introducing the remnant strain. The fiber bundle model uses a series of fibres to simulate the concrete material, and appears in an extremely simple form. Nonetheless, it can achieve a perfect balance between complexity and comprehensiveness. Furthermore, it can be quite effectively used in the simulation of stochastic fracture and damage of material.

At present, the calculation of the stochastic damage is mainly based on two types of methods. One is the moment-based method (Krajcinovic & Fanella, 1986; Kandarpa et al., 1996) and the other is the direct numerical simulation method (Iwan, 1967). The moment-based method, in general, obtains the first- and second-order moments of the stochastic damage while the direct numerical simulation is usually time-consuming in computation, especially when a large number of random variables are required. In this regard, reducing the random variables would be the key issue for the numerical simulation of the problem.

In this paper, a stochastic model is developed and used to establish a probabilistic relation between the fibre strain and the concrete damage variable. In this context, the stochastic damage evolution of the concrete can be quantified by two representative random variables. In this regard, the number of random variables employed in potential reliability assessment studies can be greatly reduced. In this context, the paper is structured as follows: the stochastic fracture/damage fiber bundle model is first reviewed in Section 2. And Section 3 describes a stochastic model to simulate

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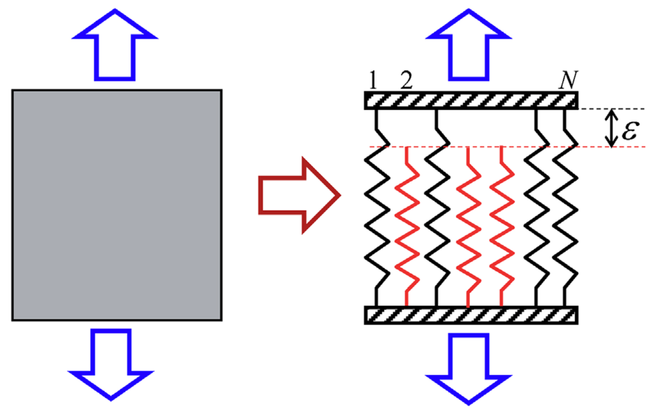


Figure 1. Fiber bundle model.

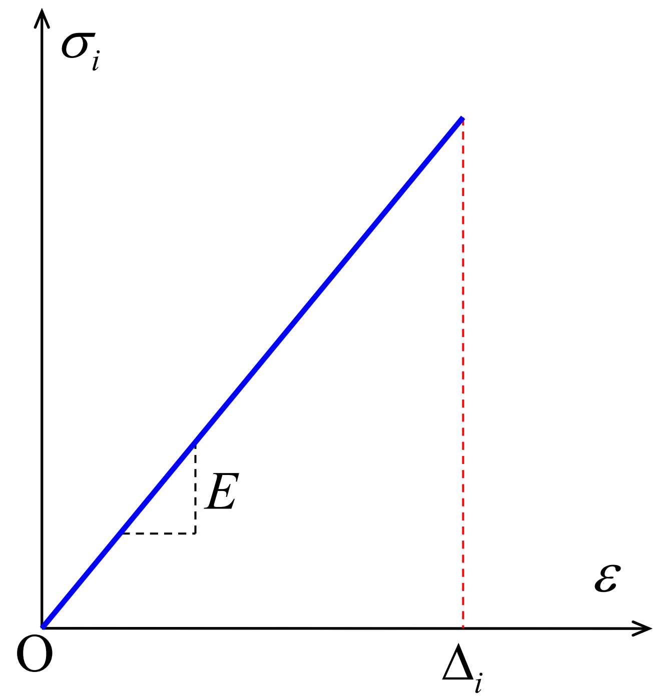


Figure 2. Elastic-rupture behaviour.

the stochastic fracture/damage system. The procedure of reliability assessment of reinforced concrete (RC) structure is presented in Section 4. Section 5 shows some numerical verifications of the model, and the non-linear reliability analysis of a RC frame structure. Conclusions are summarised in the last section.

2. Stochastic fracture/damage fiber bundle model

Concrete is the most widely used material in infrastructure engineering. Due to its composition, the response of concrete members and structures shows stochastic characteristics, especially to its damage evolution. In order to study the stochastic damage evolution of the concrete, the fiber bundle model was put forward from micro/meso point of view. In the bundle model, the concrete is simulated by a series of fibers linked in parallel with two rigid ending plates (Figure 1). The cracking at micro-level is denoted by the rupture of fibres. As shown in Figure 1, the un-ruptured fibres have the same strain denoted by ε, while the ruptured ones retract to the initial length. The elastic-rupture behaviour (Figure 2) was adopted for each fibre.

Based on the fiber bundle model (Kandarpa et al., 1996), the stress–strain law in continuum level can be expressed in the form of conventional damage constitutive relation as follows:

$$\sigma = (1 - D)E\varepsilon, \tag{1}$$

and the damage variable is expressed as:

$$D(\varepsilon) = \frac{1}{A_\Omega}\int_\Omega \mathrm{H}[\varepsilon - \Delta(\mathbf{x})]\mathrm{d}A, \tag{2}$$

where Ω denotes the occupied domain of the fiber bundle (Figure 1); $A_\Omega = \int_\Omega \mathrm{d}A$ represents the total measure of the occupied domain Ω; and $\mathrm{H}(\cdot)$ is the Heaviside function with $\mathrm{H}(x) = \begin{cases} 0 & x \le 0 \\ 1 & x > 0 \end{cases}$; $\Delta(\mathbf{x})$ is the limit strain of each fibre, which describes the stochastic fracture at micro-level and can be represented by a stationary random field; $\mathbf{x}$ denotes the coordinate within Ω.

It can be seen that Equation (2) describes the upscaling from the microscopic random field to the damage evolution in continuum level. According to Kandarpa et al.(1996), Li and Zhang (2001) and Li and Yang (2009), the mean value and standard deviation (STD) of damage variables can be expressed as:

$$\begin{cases} \mu_D(\varepsilon) = F_\Delta(\varepsilon) \\ V_D = \sqrt{\frac{1}{A_\Omega^2}\int_\Omega\int_\Omega F_{\Delta_1\Delta_2}(\varepsilon,\varepsilon;|\mathbf{x}_2 - \mathbf{x}_1|)\mathrm{d}A\mathrm{d}A - \mu_D^2(\varepsilon)} \end{cases}, \tag{3}$$

where $F_\Delta(\cdot)$ and $F_{\Delta_1\Delta_2}(\Delta_1, \Delta_2;|\mathbf{x}_2 - \mathbf{x}_1|)$ are the 1-D and 2-D cumulative distribution functions (CDF) of the random field, respectively.

The higher order moment solutions of stochastic damage may also be obtained following the similar procedure mentioned above. However, the higher order integrations required in the expressions may prohibit their applications in structural reliability analysis. Further, the probability density function of damage evolution, which represents the complete stochastic information, may be very difficult (or even impossible) to derive analytically. Thus, the stochastic damage model expressed in the present form, simply with the mean value and STD of damage variable, could not be easily used in the analysis of structural reliability. On the other hand, the stochastic damage evolution (2) can also be estimated by the direct Monte Carlo simulation (MCS) with respect to random field $\Delta(\mathbf{x})$. However, the MCS often needs many terms, e.g. 100~200, to achieve the required accuracy.

As is shown above, neither the moment-based method nor the direct MCS offers a satisfactory solution of the stochastic damage evolution. In this regard, a random functional model is proposed, and a probabilistic relation between the stochastic fibre strain and damage evolution is established, with only two random variables needed to represent the concrete damage evolution.

3. Stochastic damage model based on functional method

3.1. Mathematical framework

The basis of the random field and stochastic damage evolution is clarified and some strict definition about sub-σ-algebra is firstly presented. Consider a random field $\{\Delta(\mathbf{x})\}_{\mathbf{x}\in\Omega}$ defined on a probability space $(\Theta, \mathcal{F}, P)$ and assume values in $\mathbf{R}$, and Ω is the parameter space. All samples of the random field are considered as follows $\{\Delta_1(\mathrm{x}), \Delta_2(\mathrm{x}), \cdots, \Delta_n(\mathrm{x}), \cdots\}_{\mathrm{x}\in\Omega}$. Moreover, the n-th sample $\Delta_n(\mathbf{x})$ still defines a probability space for random variable $\Delta \in (\Delta_n(\mathbf{x}), \mathcal{F}_n, P_n)_{\mathbf{x}\in\Omega}$, where $\mathcal{F}_n \subseteq \mathcal{F}$ is sub-σ-algebra. In addition, the probability P_n could be defined by the conditional probability:

$$P_n(\cdot) = P(\cdot|\mathcal{F}_n) \tag{4}$$

while the conditional probability density function (CPDF) and the conditional cumulative distribution function (CCDF) are given as follows:

$$\begin{cases} f_n(\Delta) = f(\Delta|\mathcal{F}_n) \\ F_n(\Delta) = F(\Delta|\mathcal{F}_n) \end{cases} \tag{5}$$

Substituting $\Delta_n(\mathbf{x})$ into the stochastic damage evolution (2) and based on the properties of conditional expectation (Øksendal, 2005), we get:

$$\begin{aligned} D_n(\varepsilon) &= \frac{1}{A_\Omega}\int_\Omega H\left[\varepsilon - \Delta_n(\mathbf{x})\right]\mathrm{d}A = \mu\left[H(\varepsilon - \Delta)|\mathcal{F}_n\right] \\ &= P(\Delta \le \varepsilon|\mathcal{F}_n) = F_\Delta(\varepsilon|\mathcal{F}_n) \end{aligned} \tag{6}$$

where $\mu(\cdot|\mathcal{F}_n)$ is the conditional expectation operator defined by $\mathcal{F}_n$. In most cases, conditional CDF $F_\Delta(\varepsilon|\mathcal{F}_n)$ in Equation (6) could be expressed by a series of parameters related to the random field:

$$F_\Delta(\varepsilon|\mathcal{F}_n) = F_\Delta(\varepsilon;\boldsymbol{\omega}_n) \tag{7}$$

where $\boldsymbol{\omega}_n = \left[\omega_{n1}, \omega_{n2}, \ldots, \omega_{nr}, \ldots\right]^T$ is the parameter vector representing the random fields. Substituting Equation (7) into Equation (6) and recalling the definition of strong convergence, it can be derived that:

$$D(\varepsilon) = F_\Delta(\varepsilon;\boldsymbol{\omega}) \tag{8}$$

where the random parameter vector $\boldsymbol{\omega} = \left[\omega_1, \omega_2, \ldots, \omega_r, \ldots\right]^T$ represents the characteristic value of the random field, usually denoted by the moments of different orders. Herein, the components of $\boldsymbol{\omega}$ are the moments and defined as the integration of random field $\Delta(\mathbf{x})$:

$$\omega_r = \mathcal{I}_r[\Delta(\mathbf{x})] \tag{9}$$

where $\mathcal{I}_r[\cdot]$ is the r-th integration operator and ω_r is the r-th moment.

Using Equation (8), the relation between the damage evolution and probability function of fibre strain is established. From Equations (8) and (9), it can be noticed that the damage evolution can be expressed by the CDF of the random field parameters $\Delta(\mathbf{x})$. For some special distribution, such as the commonly used Gaussian distribution, only two random variables are needed to describe the cumulative distribution function. Therefore, the randomness of damage evolution could be quantified by a few random parameters. Thus, the stochastic damage constitutive law of the concrete material could be represented by the above-mentioned random parameters. Actually, the functional-based method proposed here could be treated as a kind of stochastic homogenisation (Vanmarcke, 2010) over the microscopic cell.

3.2. Gaussian random field

Taking the most commonly used Gaussian distribution random field as an example, the non-Gaussian cases can be transformed into Gaussian distribution. For the Gaussian random field, the marginal distribution function is $\Phi_\Delta\left(x;\mu_0,\sigma_0^2\right)$, where μ_0, σ_0^2 are the mean value and variance of the random field samples, which need to be determined first, and can be determined from concrete material experiments. The exponential autocorrelation function was adopted:

$$R(|\mathbf{x}_2 - \mathbf{x}_1|) = \exp\left[-\left(\frac{|\mathbf{x}_2 - \mathbf{x}_1|}{L_0}\right)^m\right] \tag{10}$$

where L_0 is the correlation length of the material and the positive number m is the correlation order; and $(\mathbf{x}_1, \mathbf{x}_2)$ are spatial coordinates.

Combining Equations (6) and (8), the damage evolution defined by one of the random field samples can be expressed as:

$$D(\varepsilon) = \frac{1}{A_\Omega}\int_\Omega \mathrm{H}[\varepsilon - \Delta(\mathbf{x})]\mathrm{d}A = \Phi_\Delta\left(\varepsilon;\omega_1,\omega_2\right) = \Phi\left(\frac{\varepsilon - \omega_1}{\omega_2}\right) \tag{11}$$

where $\Phi(\Box)$ denotes the CDF of standard normal distribution; ω_1 and ω_2 are the random parameters related to one random field sample, which can be obtained by the first- and second-order local averaging of the random field (Vanmarcke, 2010) as follows:

$$\begin{cases} \omega_1 = \lim\limits_{N\to\infty} \dfrac{\sum_{i=1}^{N}\Delta(\mathbf{x}_i)A}{\sum_{i=1}^{N}A} = \frac{1}{A_\Omega}\int_\Omega \Delta(\mathbf{x})\mathrm{d}A \\ \omega_2 = \lim\limits_{N\to\infty} \dfrac{\sum_{i=1}^{N}\left[\Delta(\mathbf{x}_i) - \omega_1\right]^2 A}{\sum_{i=1}^{N}A} = \frac{1}{A_\Omega}\int_\Omega \Delta^2(\mathbf{x})\mathrm{d}A - \omega_1^2 \end{cases} \tag{12}$$

From Equation (11), it can be seen that the damage evolution is only governed by the random variables (ω_1, ω_2). Thus, the stochastic damage evolution could be solved as long as the correlated density function $p_{\omega_1\omega_2}(x, y)$ is known. And then samples of damage evolution are generated and introduced into stress–stain law used in the simulation of structural analysis.

The deviation of the first and second moments of variables (ω_1, ω_2) can be found in Appendix 1. And based on the deduction in Appendix 1, it can be found that the random variables ω_1, ω_2 are independent to each other. Therefore, the joint probability density function of ω_1, ω_2 could be expressed as follows:

$$p_{\omega_1\omega_2}(x, y) = p_{\omega_1}(x)p_{\omega_2}(y) \tag{13}$$

It is well known that the integration of Gaussian field is still Gaussian distribution; thus, the marginal distribution of ω_1 is also Gaussian distributed:

$$p_{\omega_1}(x) = \frac{1}{\sqrt{2\pi}\sigma_{\omega_1}}\exp\left[-\frac{(x - \mu_{\omega_1})^2}{2\sigma_{\omega_1}^2}\right] \tag{14}$$

where $\mu_{\omega_1} = \mu_0, \sigma_{\omega_1}^2 = \varphi_1\sigma_0^2$ (see Appendix 1) are the mean value and the variance of variable ω_1, respectively.

By observing Equation (12), it is evident that ω_2 is the summation of squares of normal distribution. Thus, the marginal distribution of ω_2 could be approximated by Gamma distribution as follows:

$$p_{\omega_2}(y) = \begin{cases} 0 & y < 0 \\ \frac{x^{a-1}e^{-x/b}}{b^a\Gamma(a)} & y \geq 0 \end{cases} \tag{15}$$

where the parameters (see Appendix 1):

$$\begin{cases} a = \frac{\mu_{\omega_2}^2}{\sigma_{\omega_2}^2} = \frac{(1-\varphi_1)^2}{2(\varphi_2-\varphi_1^2)} \\ b = \frac{\sigma_{\omega_2}^2}{\mu_{\omega_2}} = \frac{2(\varphi_2-\varphi_1^2)\sigma_0^2}{(1-\varphi_1)} \end{cases} \tag{16}$$

According to the Appendix 1 followed, the values of φ_1, φ_2 can be calculated by:

$$\begin{cases} \varphi_1 = \frac{1}{A_\Omega^2}\int_\Omega\int_\Omega R(|\mathbf{x}_2-\mathbf{x}_1|)\mathrm{d}A\mathrm{d}A \\ \varphi_2 = \frac{1}{A_\Omega^2}\int_\Omega\int_\Omega \left[R(|\mathbf{x}_2-\mathbf{x}_1|)\right]^2\mathrm{d}A\mathrm{d}A \end{cases} \tag{17}$$

Combining the stochastic damage evolution (11) with the joint probability density function (13), the statistical information of damage evolution could be determined by numerical integration. However, many random quantities could not be directly represented by Gaussian distribution. For example, the material strength of solid could not be accurately modelled by Gaussian distribution because their values always remain positive. Thus, it is necessary to take non-Gaussian random field into consideration.

3.3. Non-Gaussian random field

Non-Gaussian random field is rather difficult for theoretical development because many good theoretical tools have been developed merely in case of Gaussian distribution. Nevertheless, the non-Gaussian random field can be transferred to Gaussian random field according to the classic structural reliability theory (Ditlevsen & Madsen, 2005). In the present work, the following transform between Gaussian and non-Gaussian distribution (Nataf transform) is considered:

$$F_{\tilde{\Delta}}(y) = \Phi_\Delta(x;\omega_1,\omega_2) \tag{18}$$

where $F_{\tilde{\Delta}}(\cdot)$ is the cumulative distribution function of a non-Gaussian random field $\tilde{\Delta}$. The stochastic damage evolution (2) governed by non-Gaussian random field can be expressed as:

$$\begin{aligned} D(\varepsilon) &= \frac{1}{A_\Omega}\int_\Omega \mathrm{H}\left[\varepsilon - \tilde{\Delta}(\mathbf{x})\right]\mathrm{d}A \\ &= \frac{1}{A_\Omega}\int_\Omega \mathrm{H}\left\{\Phi_\Delta^{-1}\left[F_{\tilde{\Delta}}(\varepsilon);\omega_1,\omega_2\right] - \Delta(\mathbf{x})\right\}\mathrm{d}A \end{aligned} \tag{19}$$

Comparing Equations (11) and (19) yields the damage evolution function as follows:

$$D(\varepsilon) = \Phi_\Delta\left\{\Phi_\Delta^{-1}\left[F_{\tilde{\Delta}}(\varepsilon);\omega_1,\omega_2\right];\omega_1,\omega_2\right\} \overset{\text{def}}{=} F_\Phi\left(\varepsilon;\omega_1,\omega_2\right) \tag{20}$$

It should be noted that the non-linear transformation (18) also changes the correlation structure of the random field. To address this issue, Der Kiureghian and Liu (1986) proposed a set of semi-empirical formulas for the ratio between the original Gaussian distribution and the transformed non-Gaussian distribution:

$$\rho = \frac{R_0\left(\left|\mathbf{x}_2-\mathbf{x}_1\right|\right)}{R\left(\left|\mathbf{x}_2-\mathbf{x}_1\right|\right)}, \tag{21}$$

where $R_0\left(\left|\mathbf{x}_2-\mathbf{x}_1\right|\right)$ and $R\left(\left|\mathbf{x}_2-\mathbf{x}_1\right|\right)$ denote the autocorrelation functions of the original normal distribution and generated non-linear distribution, respectively.

3.4. Statistical properties of damage evolution

Using the aforementioned stochastic damage evolution (20), the joint PDF Equation (13) and the ratio Equation (21), the complete statistical properties of damage evolution can be calculated in a consistent way. The n-th order moment of damage evolution is:

$$m_D(\varepsilon;n) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left[F_\Phi\left(\varepsilon;x,y\right)\right]^n p_{\omega_1\omega_2}(x,y)\mathrm{d}x\mathrm{d}y \tag{22}$$

and the n-th order central moment is:

$$m_D'(\varepsilon;n) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left[F_\Phi\left(\varepsilon;x,y\right) - m_D(\varepsilon;1)\right]^n p_{\omega_1\omega_2}(x,y)\mathrm{d}x\mathrm{d}y \tag{23}$$

The corresponding CDF of damage can be expressed as:

$$\begin{aligned} lF_D(d;\varepsilon) &= P\{D(\varepsilon)\leq d\} = P\left\{F_\Phi\left(\varepsilon;x,y\right)\leq d\right\} \\ &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\mathrm{H}\left[d - F_\Phi\left(\varepsilon;x,y\right)\right]p_{\omega_1\omega_2}(x,y)\mathrm{d}x\mathrm{d}y \end{aligned} \tag{24}$$

Differentiating Equation (24) yields the PDF of damage evolution as follows:

$$f_D(d;\varepsilon) = \frac{\partial F_D(d;\varepsilon)}{\partial d} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\delta\left[d - F_\Phi\left(\varepsilon;x,y\right)\right]p_{\omega_1\omega_2}(x,y)\mathrm{d}x\mathrm{d}y \tag{25}$$

where $\delta(\cdot)$ denotes the Kronecker delta function.

4. Structural reliability assessment procedure

The reliability of a structure P_r is usually defined as:

$$P_r = \Pr(R > P), P_f = 1 - P_r \tag{26}$$

where R is structural resistance, P is load and P_f is the probability of failure. Assuming R and P are statistical independent, the reliability in Equation (26) becomes:

$$P_r = 1 - \int_0^{+\infty}\left[1 - F_P(x)\right]f_R(x)\mathrm{d}x, P_f = \int_0^{+\infty}\left[1 - F_P(x)\right]f_R(x)\mathrm{d}x \tag{27}$$

where $f_R(x)$ is the probability density function (PDF) of structure resistance R and $F_P(x)$ is the cumulative distribution function (CDF) of P.

The structural resistance R is a function of a series of variables expressed as follows:

$$R = R(\text{random variables;deterministic variables}) \quad (28)$$

Based on the method mentioned in this paper, ω_1 and ω_2 were adopted as random variables to describe the stochastic damage of concrete material. Thus, Equation (28) could be rewritten as:

$$R = R(\omega_1, \omega_2;\text{deterministic variables}) \stackrel{\text{def}}{=} R(\omega_1, \omega_2) \quad (29)$$

Then, the CDF of R is calculated by:

$$\begin{aligned} lF_R(x) &= \Pr\{R \le x\} = \Pr\left\{R(\omega_1, \omega_2) \le x\right\} \\ &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} H[x - R(s,t)]p_{\omega_1\omega_2}(s,t)\mathrm{d}s\mathrm{d}t \end{aligned} \quad (30)$$

The corresponding PDF of R is:

$$f_R(x) = \frac{\mathrm{d}}{\mathrm{d}x}F_R(x) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \delta[x - R(s,t)]p_{\omega_1\omega_2}(s,t)\mathrm{d}s\mathrm{d}t \quad (31)$$

which could be solved by numerical integration in 2-D. By substituting Equation (31) into Equation (27) and giving the CDF of load $F_p(x)$, the reliability of a structure P_r could be easily solved.

The numerical quadrature for double integration in Equations (30) and (31) needs the value of structure resistance at each quadrature point, which can be denoted by $R_I = R(s_I, t_I)$, where $I = 1, 2, \ldots \ldots$ is the index of quadrature points. Each (s_I, t_I) defines one sample of stress–strain law curves, and indicates one non-linear structural analysis, together with other deterministic prescribed parameters. In this regard, $R_I = R(s_I, t_I)$ can be obtained through deterministic structural simulation.

5. Numerical verification and application

5.1. Damage evolution comparison with analytical result

Consider a one-dimensional lognormal random field, with parameter domain $x \in [0, 1]$. The mean and variance of the random field are taken as: $\mu_0 = 4.5$ and $\sigma_0 = 0.5$. The corresponding stochastic damage evolution is shown by Equation (20). The first-order exponential autocorrelation function (Equation 10) is chosen ($m = 1$) and the correlation length is $L_0 = 0.1$.

Substituting the values of μ_0 and σ_0 into Equation (21) yields $\rho = 1.003$. So, $R(|\mathbf{x}_2 - \mathbf{x}_1|) \approx R_0(|\mathbf{x}_2 - \mathbf{x}_1|)$ is adopted without much loss of accuracy. The joint PDF $p_{\omega_1\omega_2}(x, y)$ can be calculated with Equations (13) to (17). The results are shown in Figure 3. Using Equations (22) and (23), the moments of damage evolution were calculated and shown in Figures 4 to 7. It can be seen that the results of stochastic functional method proposed here agree well with the corresponding analytical solutions. Moreover, high-order statistics could also be calculated, without much additional computational effort. The skewness (*3rd* central moment) and the kurtosis (*4th* central moment) curves are illustrated in Figures 6 and 7, respectively.

The probability density function was also calculated, as presented in Figures 8 and 9. It can be seen that the damage variable grows with the fibre strain and its PDF exhibits distinct patterns for different strain values. As mentioned by Chen and Li (2005), the subtle information provided by the probability density reveals the transmission of randomness throughout the loading process. From the information indicated here, it can be seen that the probability information will change during the loading process.

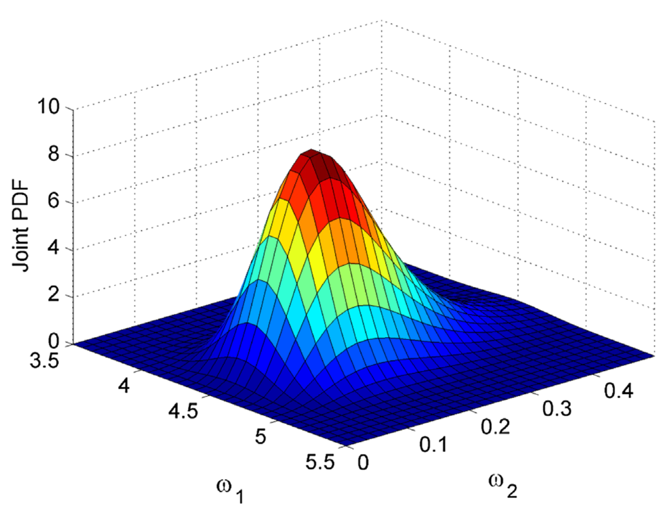


Figure 3. Joint PDF of (ω_1, ω_2).

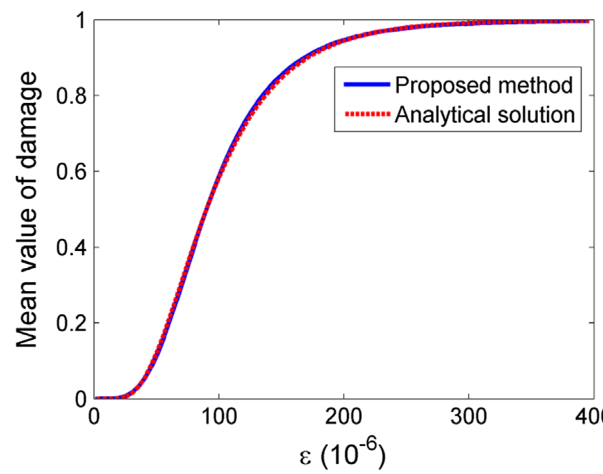


Figure 4. Mean value of damage evolution.

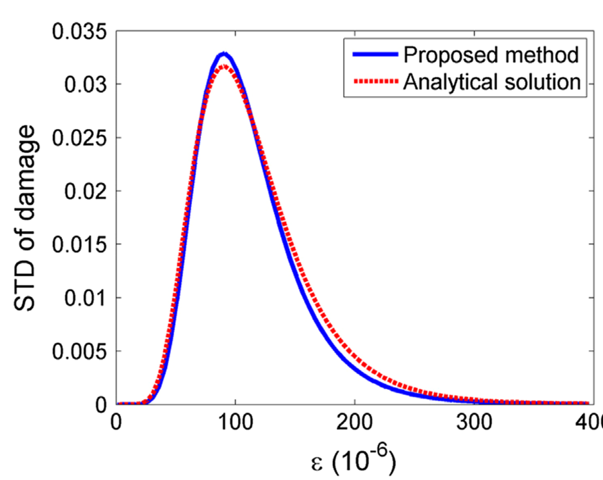


Figure 5. Standard derivation of damage evolution.

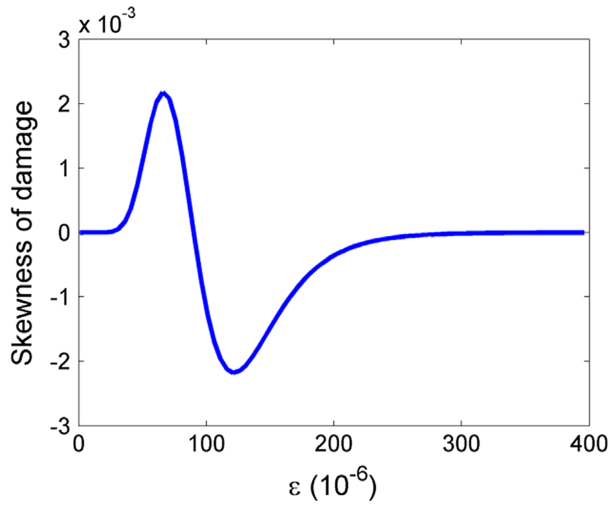


Figure 6. Skewness ($m'_D(3)$) of damage evolution.

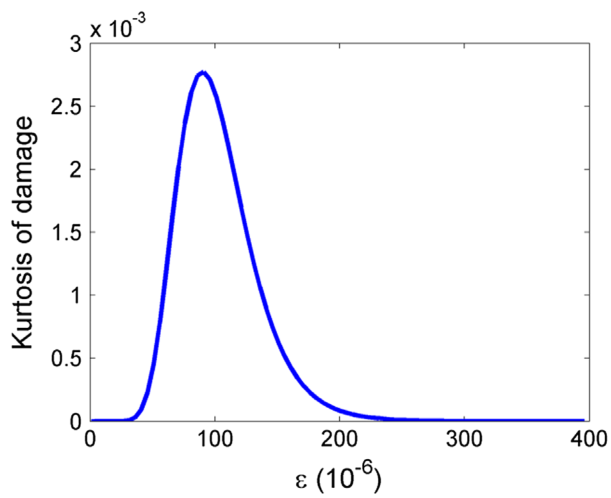


Figure 7. Kurtosis ($m'_D(4)$) of damage evolution.

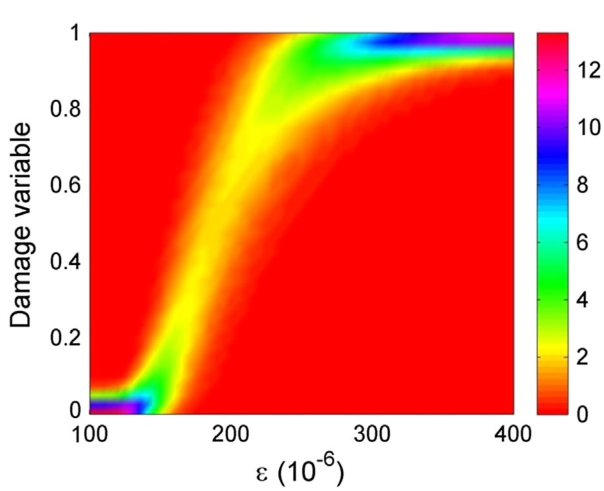


Figure 8. Probability density contour of damage evolution.

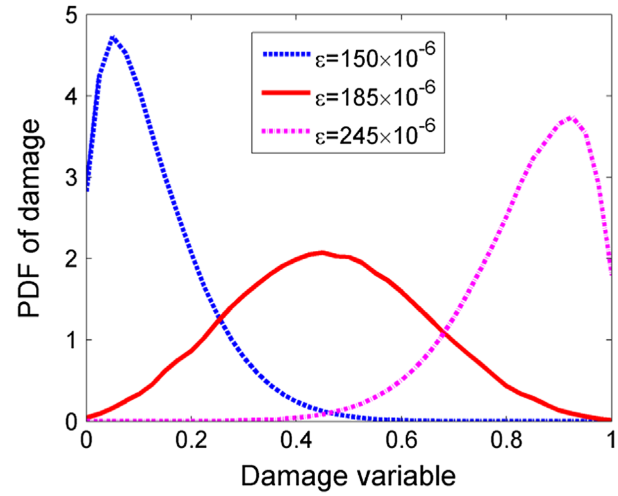


Figure 9. Probability density curves of damage at certain strain levels.

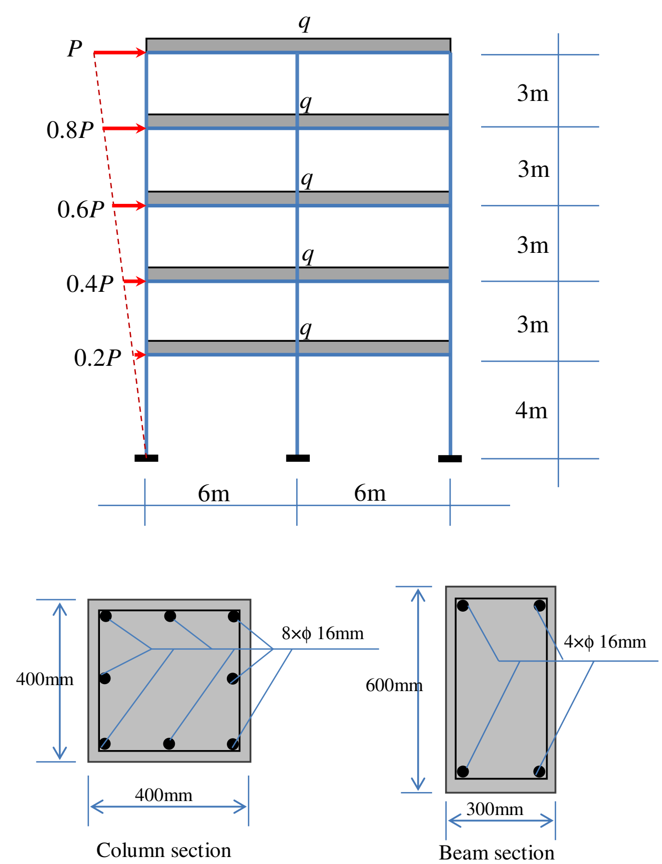


Figure 10. RC frame structure.

5.2. Non-linear reliability analysis of RC structure

The reliability of a five-story reinforced concrete (RC) frame structure was investigated as an example. Figure 10 shows the configuration of the RC frame. In this paper, the structure was simulated by the finite element package ABAQUS. The beams

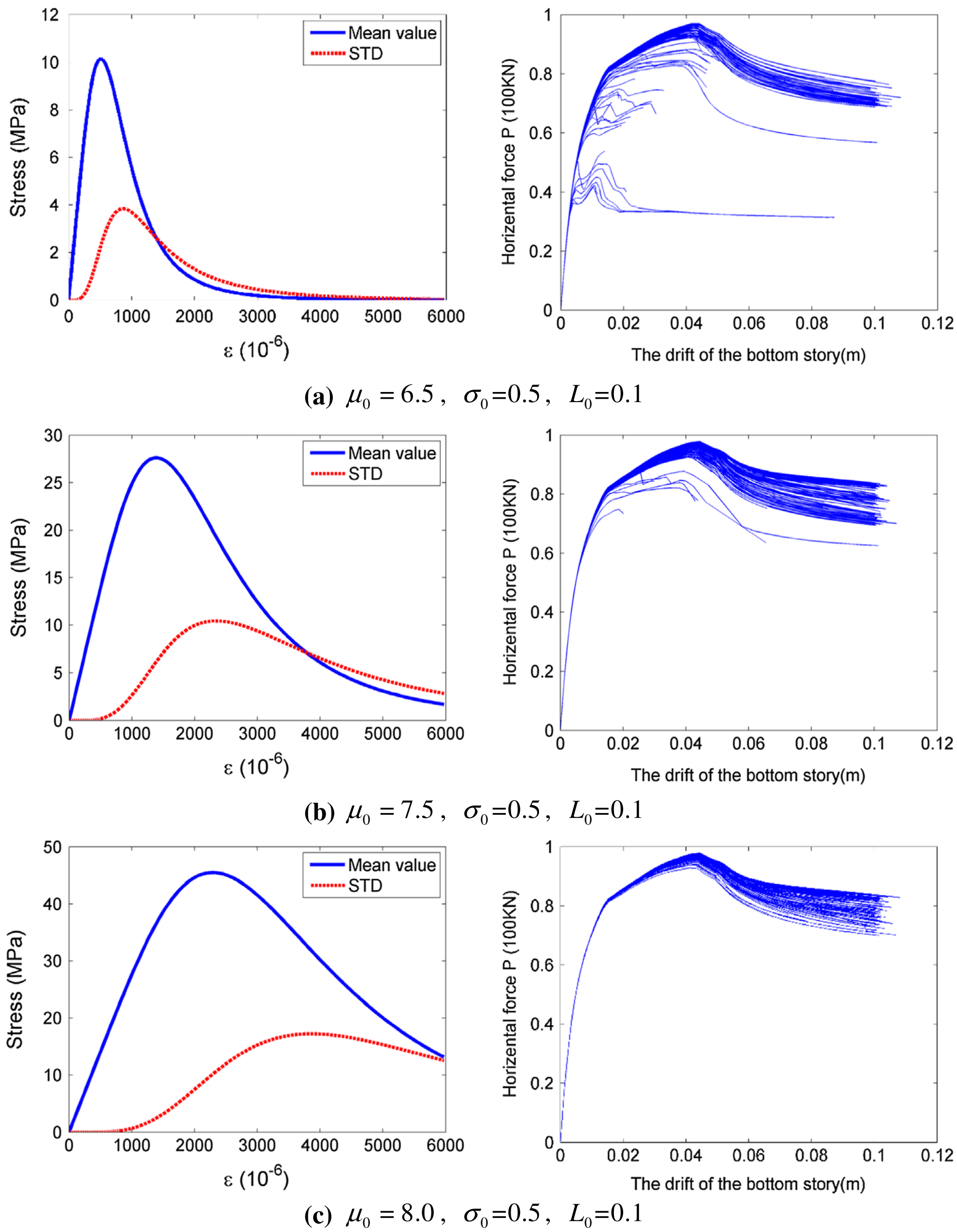


Figure 11. Simulated stress–strain curves and load–displacement curves with different values of μ_0.

and columns were modelled as beam elements with steel rebar. And the fibre-based stochastic damage model of concrete was implemented in ABAQUS by UMAT and Python.

The non-linear behaviours of concrete were considered by the stochastic damage model with distributive parameters μ_0 and σ_0. The exponential autocorrelation function is chosen to describe the spatial correlation with correlation length $L_0 = 0.1$. And Young's modulus of concrete is $E_c = 28$GPa. The steel reinforcements were considered as elasto-plastic materials, with Young's modulus $E_s = 210$ GPa and yield stress $f_y = 360$ MPa. The structure is subjected to a uniform vertical force of intensity $q = 6$KN/m on each story, and thereafter a triangle distribution horizontal load with maximum value P is applied from one side of the frame.

The horizontal load P is expressed by the following equation:

$$P = \eta G \tag{32}$$

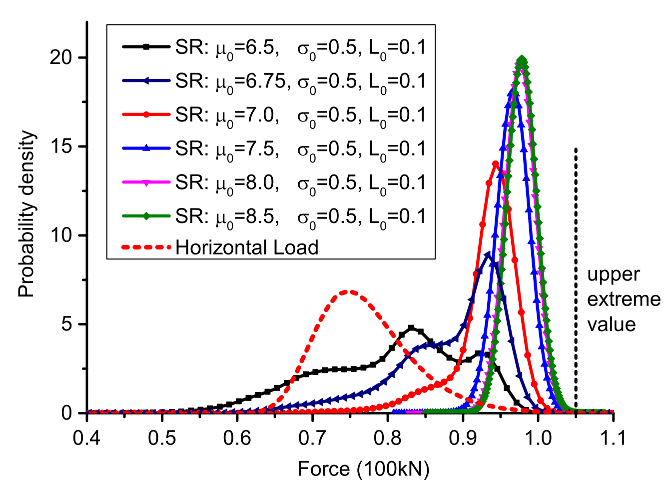
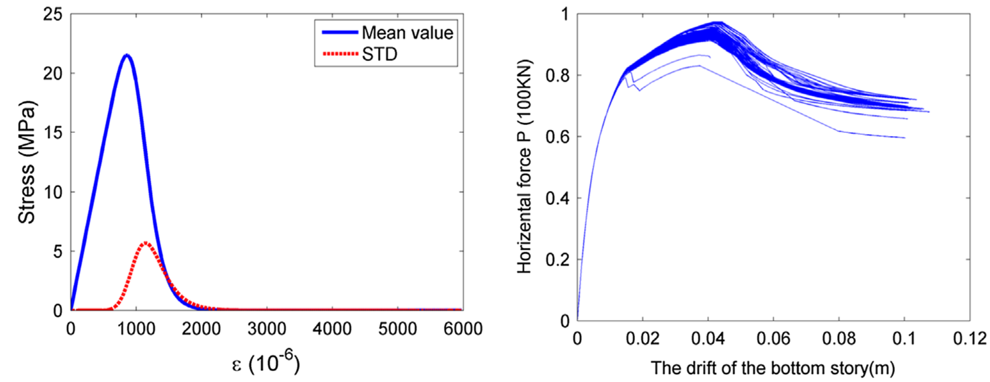
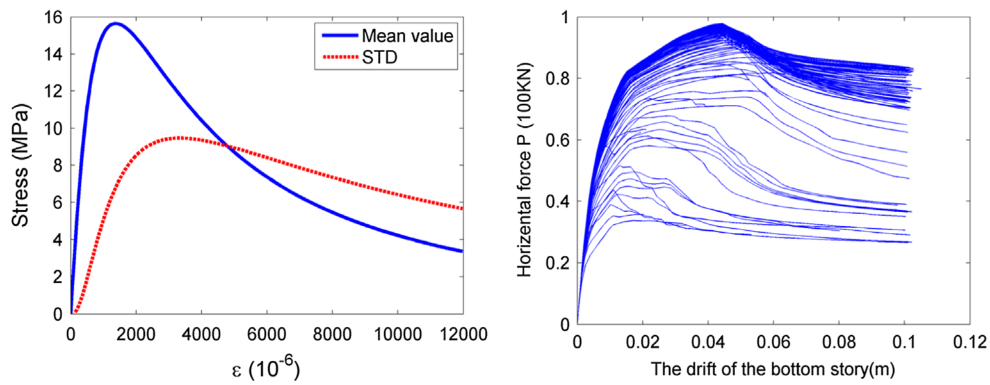


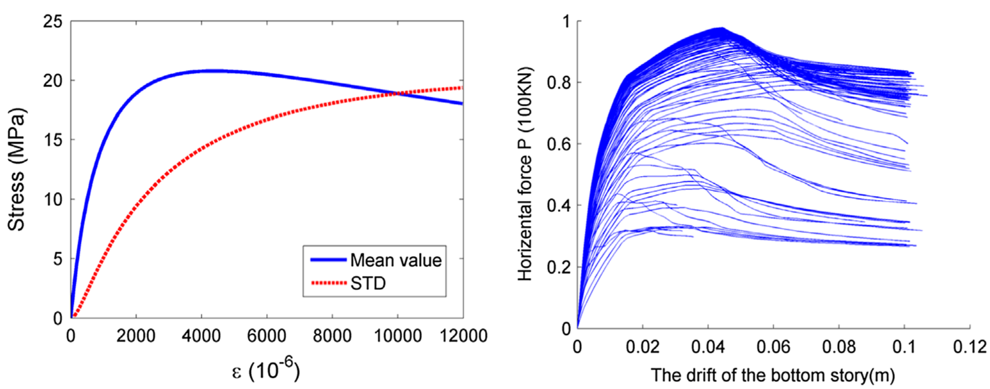
Figure 12. PDFs of horizontal load vs. structural resistance with different values of μ_0.



(a) $\mu_0 = 7.0$, $\sigma_0 = 0.2$, $L_0 = 0.1$



(a) $\mu_0 = 7.0$, $\sigma_0 = 1.0$, $L_0 = 0.1$



(c) $\mu_0 = 7.0$, $\sigma_0 = 1.5$, $L_0 = 0.1$

Figure 13. Simulated stress–strain curves and load–displacement curves with different values of σ_0.

where $G = 100$ kN and the horizontal force factor η was assumed to be governed by the following extreme value distribution (Davenport, 1964).

$$F_\eta(x) = \exp\left[-A\exp\left(-\frac{x^2}{2B^2}\right)\right], x \geq 0 \quad (33)$$

while the corresponding PDF is:

$$f_\eta(x) = \frac{Ax}{B^2}\exp\left[-A\exp\left(-\frac{x^2}{2B^2}\right) - \frac{x^2}{2B^2}\right], x \geq 0 \quad (34)$$

where parameters $A = 1000$ and $B = 0.2$ were adopted in this study.

A case study with different distributive parameters μ_0, σ_0 was conducted to investigate the influences of different non-linear material properties on the structural behaviours. Figures 11 and 12 display the influence of variable μ_0, while Figures 13 and 14 depict the influence of variable σ_0.

Figures 11 and 13 show the stress–strain curves and load–displacement curves with different values of μ_0 and σ_0. It is evident that most of the simulated pushover load–displacement curves exhibit typical patterns as RC structure. At the initial loading stage, the structural response is nearly linear elastic. After the cracking of concrete, the non-linearities become more apparent. As the deformation increased, the reinforcement yields and the whole load–displacement curve go up to the peak value. After the peak value, the structure enters into the softening period due to the crush of concrete, and the responses are more divergent,

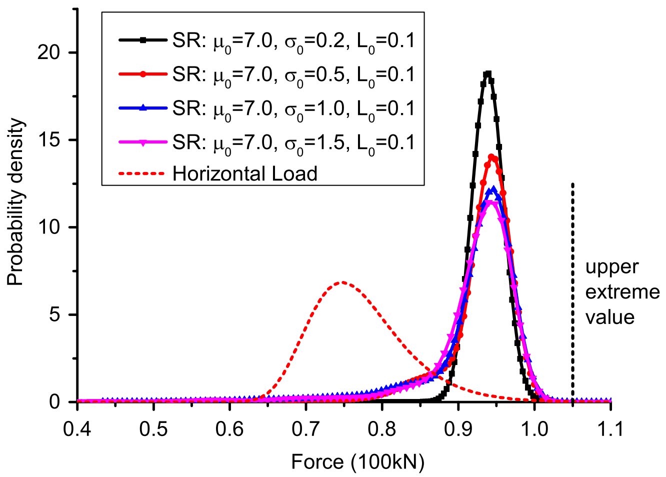


Figure 14. PDFs of horizontal load vs. structural resistance with different values of σ_0.

especially with higher values of σ_0. This could be explained because the randomness induced by the damage evolution in the material is suppressed by the structure at the elastic stage and will be more apparent as the damage of the concrete developed.

Additional interesting results could also be observed in Figure 11 (a)~(c). The randomness of load–displacement responses is suppressed by increasing concrete strength. The reason may be that the structural failure is governed by the reinforcement, as the concrete strength becomes sufficiently high. Taking the peak value of each load–displacement curve as the structural resistance R_l and substituting it into the numerical integration of Equation (31), the probability density function (PDF) of structural resistance, i.e. $f_R(x)$, could be obtained.

The simulated PDFs of structural resistance against the horizontal load P are plotted in Figures 12 and 14. Figure 12 depicts the PDFs calculated for different values of μ_0, which governs the absolute strength of concrete. It can be seen that the upper extreme values of the structural resistance for all cases are almost identical, which means the upper bound is governed by the reinforcement rather than concrete. Another interesting result is that as the concrete strength decreases, the mean value of structural resistance decreases, while its random deviation increases. Further, it can be seen that the PDF of structure resistance exhibits substantial random deviation for the rather low strength concrete. From Figure 14, it can be noticed that as the material random derivation increases, the same phenomenon occurs as well.

The structural reliability, as well as the failure probability, for different cases is listed in Tables 1 and 2. From Table 1, it can be seen that the higher the strength of concrete, the higher the reliability of the structure. Table 2 gives the reliability and failure probability with same strength mean value and different random deviations. It can be seen that as the material random deviation increases, the reliability of structure decreases.

Other interesting aspects can also be observed from the simulation results: the response branching. From Figures 11 and 13, it can be seen that under some conditions (Figures 11(a) and 13(b)–(c)), the branching is quite obvious, while in Figures 11(b)–(c) and 13(a), the branching is not quite obvious, and the simulated pushover curves are distributed in a relatively regular

Table 1. Simulated structural reliabilities for different values of μ_0.

Case	μ_0	σ_0	Structural reliability	Probability of failure
1	6.5	0.5	0.6325	0.3675
2	6.75	0.5	0.8673	0.1327
3	7.0	0.5	0.9627	0.0373
4	7.5	0.5	0.9902	0.0098
5	8.0	0.5	0.9925	0.0075
6	8.5	0.5	0.9929	0.0071

Table 2. Simulated structural reliabilities for different values of σ_0.

Case	μ_0	σ_0	Structural reliability	Probability of failure
7	7.0	0.2	0.9816	0.0184
8	7.0	0.5	0.9627	0.0373
9	7.0	1.0	0.9348	0.0652
10	7.0	1.5	0.9320	0.0680

way. This is also reflected in the characterised PDFs of structural resistance, which are similar to Gaussian distribution. The above cases occur at a relatively higher mean value and lower variation. For the cases shown in Figures 11(a) and 13(b)–(c), the pushover curves show quite obvious branching. The corresponding PDF of structural resistance is multi-peaks in Figure 12 and skewness in Figure 14. The multi-peaks and branching in the stochastic behaviours of a structure usually indicate the change of failure modes. In this context, it may be envisioned that the failure modes of the structure change considering the random damage evolutions of concrete. Nevertheless, the failure mode changes due to the random material performance. This still an open field and deserves further explorations and efforts.

6. Conclusions

Based on the theoretical analysis and the numerical example, several conclusions can be obtained as follows:

(1) The fiber bundle model can simulate the random fracture of concrete fibres, and a stochastic model is developed to describe the behaviour of concrete fibre at the micro-level.
(2) A probabilistic relation between the damage variable and the fibre strain is established. The damage evolution can be quantified by two random variables.
(3) The mean and standard deviation of damage evolution agree quite well with the corresponding analytical solutions. The probability density function of the damage can also be obtained.
(4) The proposed method is quite efficient for stochastic response determination and reliability evaluation of complex structures.

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Appendix 1. Derivation of first and second moments for ω_1 and ω_2

The marginal distribution of ω_1 could be estimated by its mean value and variance. The mean value of ω_1 is given by:

$$\mu_{\omega_1} = \mu\left[\frac{1}{A_\Omega}\int_\Omega \Delta(\mathbf{x})\mathrm{d}A\right] = \frac{1}{A_\Omega}\int_\Omega \mu[\Delta(\mathbf{x})]\mathrm{d}A = \mu_0 \tag{A1}$$

while the second-order moment is:

$$\begin{aligned} m_{\omega_1}^2 &= \mu\left[\frac{1}{A_\Omega}\int_\Omega \Delta(\mathbf{x})\mathrm{d}A\right]^2 = \frac{1}{A_\Omega^2}\int_\Omega\int_\Omega \mu\left[\Delta(\mathbf{x}_1)\Delta(\mathbf{x}_2)\right]\mathrm{d}A\mathrm{d}A \\ &= \sigma_0^2\frac{1}{A_\Omega^2}\int_\Omega\int_\Omega R(|\mathbf{x}_2-\mathbf{x}_1|)\mathrm{d}A\mathrm{d}A + \mu_0^2 \end{aligned} \tag{A2}$$

Then, the variance becomes:

$$\sigma_{\omega_1}^2 = m_{\omega_1}^2 - \mu_0^2 = \sigma_0^2\left[\frac{1}{A_\Omega^2}\int_\Omega\int_\Omega R(|\mathbf{x}_2-\mathbf{x}_1|)\mathrm{d}A\mathrm{d}A\right] \tag{A3}$$

Defining the coefficient by:

$$\varphi_1 = \frac{1}{A_\Omega^2}\int_\Omega\int_\Omega R(|\mathbf{x}_2-\mathbf{x}_1|)\mathrm{d}A\mathrm{d}A \tag{A4}$$

where $R(|\mathbf{x}_2-\mathbf{x}_1|) \le 1$, gives:

$$\varphi_1 \le 1 \Rightarrow \sigma_{\omega_1}^2 \le \sigma_0^2 \tag{A5}$$

Similarly, the marginal distribution for ω_2 could also be expressed by its corresponding mean value and variance; thus, the mean of ω_2 is given by:

$$\mu_{\omega_2} = \frac{1}{A_\Omega}\int_\Omega \mu\left[\Delta^2(\mathbf{x})\right]\mathrm{d}A - \mu\left(\omega_1^2\right) \tag{A6}$$

The first term on the right-hand side (RHS) in Equation (A6) is determined as:

$$\frac{1}{A_\Omega}\int_\Omega \mu\left[\Delta^2(\mathbf{x})\right]\mathrm{d}A = \frac{1}{A_\Omega}\int_\Omega \left(\mu_0^2+\sigma_0^2\right)\mathrm{d}A = \mu_0^2+\sigma_0^2 \tag{A7}$$

while the second term is:

$$\mu\left(\omega_1^2\right) = \mu_{\omega_1}^2 + \sigma_{\omega_1}^2 = \mu_0^2 + \varphi_1\sigma_0^2 \tag{A8}$$

Substituting Equations (A6) and (A7) into Equation (A4) yields:

$$\mu_{\omega_2} = (1-\varphi_1)\sigma_0^2 \tag{A9}$$

and we consider the variance of ω_2 as follows:

$$\sigma_{\omega_2}^2 = \mu\left[(\omega_2)^2\right] - \left[\mu(\omega_2)\right]^2 \tag{A10}$$

The second term on the RHS of Equation (A10) has been solved by Equation (A9). The first right term in Equation (A10) is derived by:

$$\begin{aligned} \mu\left[(\omega_2)^2\right] &= \mu\left\{\left[\frac{1}{A_\Omega}\int_\Omega \Delta^2(\mathbf{x})\mathrm{d}A - \left(\frac{1}{A_\Omega}\int_\Omega \Delta(\mathbf{x})\mathrm{d}A\right)^2\right]^2\right\} \\ &= \frac{1}{A_\Omega^2}\int_\Omega\int_\Omega \mu\left[\Delta^2(\mathbf{x}_1)\Delta^2(\mathbf{x}_2)\right]\mathrm{d}A\mathrm{d}A \\ &\quad - \frac{2}{A_\Omega^3}\int_\Omega\int_\Omega\int_\Omega \mu\left[\Delta(\mathbf{x}_1)\Delta(\mathbf{x}_2)\Delta^2(\mathbf{x}_4)\right]\mathrm{d}A\mathrm{d}A\mathrm{d}A \\ &\quad + \frac{1}{A_\Omega^4}\int_\Omega\int_\Omega\int_\Omega\int_\Omega \mu\left[\Delta(\mathbf{x}_1)\Delta(\mathbf{x}_2)\Delta(\mathbf{x}_3)\Delta(\mathbf{x}_4)\right]\mathrm{d}A\mathrm{d}A\mathrm{d}A\mathrm{d}A \end{aligned} \tag{A11}$$

Recalling the properties of the fourth-order central moments of multivariate normal distribution, Equation (A11) further converts to:

$$\mu\left[(\omega_2)^2\right] = (1+2\varphi_2-2\varphi_1-\varphi_1^2)\sigma_0^4 \tag{A12}$$

where:

$$\varphi_2 = \frac{1}{A_\Omega^2}\int_\Omega\int_\Omega \left[R(|\mathbf{x}_2-\mathbf{x}_1|)\right]^2\mathrm{d}A\mathrm{d}A \tag{A13}$$

Substituting Equations (A9) and (A12) into Equation (A11) yields the variance of ω_2 as follows:

$$\sigma_{\omega_2}^2 = (1+2\varphi_2-2\varphi_1-\varphi_1^2)\sigma_0^4 - (1-\varphi_1)^2\sigma_0^4 = 2(\varphi_2-\varphi_1^2)\sigma_0^4 \tag{A14}$$

The correlation between ω_1 and ω_2 could be estimated by the covariance as follows:

$$\mathrm{cov}(\omega_1,\omega_2) = \mu(\omega_1\omega_2) - \mu(\omega_1)\mu(\omega_2) \tag{A15}$$

The first term on the RHS of Equation (A15) is formed as follows:

$$\begin{aligned}\mu(\omega_1\omega_2) &= \mu\left\{\frac{1}{A_\Omega}\int_\Omega \Delta(\mathbf{x})\mathrm{d}A\left[\frac{1}{A_\Omega}\int_\Omega \Delta^2(\mathbf{x})\mathrm{d}A - \left(\frac{1}{A_\Omega}\int_\Omega \Delta(\mathbf{x})\mathrm{d}A\right)^2\right]\right\}\\ &= \frac{1}{A_\Omega^2}\int_\Omega\int_\Omega \mu\left[\Delta(\mathbf{x}_1)\Delta^2(\mathbf{x}_2)\right]\mathrm{d}A\mathrm{d}A\\ &\quad - \frac{1}{A_\Omega^3}\int_\Omega\int_\Omega\int_\Omega \mu\left[\Delta(\mathbf{x}_1)\Delta(\mathbf{x}_2)\Delta(\mathbf{x}_3)\right]\mathrm{d}A\mathrm{d}A\mathrm{d}A\end{aligned} \tag{A16}$$

Recalling the properties of the third-order central moments of multivariate normal distribution, Equation (A16) converts to:

$$\mu(\omega_1\omega_2) = \mu\sigma_0^2(1-\varphi_1) \tag{A17}$$

Substituting Equations (A1), (A11) and (A17) into (A15) yields:

$$\mathrm{cov}(\omega_1, \omega_2) \equiv 0 \tag{A18}$$

which indicates that the correlation coefficient between ω_1 and ω_2 equals to zero. Therefore, it is reasonable to adopt that ω_1 and ω_2 are independent.