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Note

A note on the number of spanning trees of line digraphs

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ABSTRACT

Let G be a digraph and $\mathcal{L}G$ be its line digraph. Levine gave a formula that relates the number of rooted spanning trees of $\mathcal{L}G$ and that of G, with the restriction that G has no sources. In this note, we show that this restriction can be removed, thus his formula holds for all digraphs.

1. Terminology and notation

Since the discussion is based on the results of Levine [3], some notation from that paper are used in this note for consistency. The digraphs considered here may have loops and multiple edges. For terminology and notation not defined we refer the reader to Bondy and Murty [1].

Let G be a finite digraph. We use $V(G)$ and $E(G)$ to denote the vertex set and edge set of G respectively. For an edge e of G which directs from a vertex u to a vertex v, u is said to be the *tail* of e, denoted by $t(e) = u$, and v the *head* of e, denoted by $h(e) = v$. The *out-degree* and *in-degree* of vertex v are defined by $\text{outdeg}(v) = \left|\{e : t(e) = v\}\right|$ and $\text{indeg}(v) = \left|\{e : h(e) = v\}\right|$, respectively. G is said to be *k-out-regular* (*k-in-regular*) if $\text{outdeg}(v) = k$ ($\text{indeg}(v) = k$) for every vertex v of G. A *source* of G is a vertex with in-degree 0. Note that if there is a loop e at v ($t(e) = h(e) = v$), then v is not a source.

A *rooted spanning tree* of a digraph G is an oriented spanning tree such that every vertex except one (the *root*) has out-degree 1. The number of rooted spanning trees of G is denoted by $\kappa(G)$. Let $\{x_v\}_{v\in V(G)}$ and $\{x_e\}_{e\in E(G)}$ be the indeterminates on the vertex set and edge set of G, which can be regarded as weights on the vertices and edges respectively. In our discussion, we will assume that $x_v > 0$ for all $v \in V(G)$ and $x_e > 0$ for all $e \in E(G)$. Consider the following two polynomials

$$\kappa^{edge}(G, \mathbf{x}) = \sum_T \prod_{e\in T} x_e \quad \text{and} \quad \kappa^{vertex}(G, \mathbf{x}) = \sum_T \prod_{e\in T} x_{h(e)},$$

where the sums are taken over all the rooted spanning trees of G. It is trivial that $\kappa^{edge}(G, \mathbf{1}) = \kappa^{vertex}(G, \mathbf{1}) = \kappa(G)$, where $\mathbf{1}$ is the vector with 1 as all of its components.

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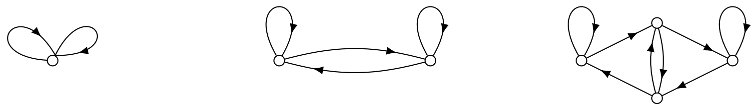

Fig. 1. Digraphs G, $\mathcal{L}G$, and $\mathcal{L}^2G$.

The *line digraph* of G is the digraph $\mathcal{L}G$ with vertex set $V(\mathcal{L}G) = E(G)$ and edge set $E(\mathcal{L}G) = \big\{(e, f) \in E(G) \times E(G) | h(e) = t(f)\big\}$ (see Fig. 1). The vertex weight of $\mathcal{L}G$ is the same as the edge weight of G, i.e., the vertex e in $\mathcal{L}G$ has weight x_e.

For an unweighted digraph G, the iterated line digraph $\mathcal{L}^tG = \mathcal{L}\big(\mathcal{L}^{t-1}G\big)$ is the t-th line digraph of G. The vertex set of $\mathcal{L}^tG$ is denoted by

$$E_t = \Big\{(e_1, e_2, \ldots, e_t) \in E^n \Big| t(e_{i+1}) = h(e_i), i = 1, \ldots, t-1\Big\},$$

in which whose elements are directed walks in G with t edges. This definition of E_t is slightly different from the definition in [3] where the elements of E_t are directed paths in G with t edges.

2. Main results

There is a well-known result which relates the number of spanning trees of a regular (undirected) graph and its line graph (see [2, pp. 218]): If a graph G is k-regular, then

$$\kappa(\mathcal{L}G) = 2^{m-n+1}k^{m-n-1} \cdot \kappa(G).$$

A directed version of this result follows from the following theorem.

Theorem 1 (*Zhang, Zhang and Huang [4]*)**.** *Let G be a digraph with order n. If G is both k-out-regular and k-in-regular, then*

$$\kappa(\mathcal{L}^tG) = k^{(k^t-1)n} \cdot \kappa(G).$$

By setting $t = 1$ in Theorem 1, for digraphs which are both k-out-regular and k-in-regular there holds

$$\kappa(\mathcal{L}G) = k^{m-n} \cdot \kappa(G).$$

In 2011, Levine [3] gave a formula that relates $\kappa^{vertex}(\mathcal{L}G, \mathbf{x})$ and $\kappa^{edge}(G, \mathbf{x})$.

Theorem 2 (*Levine [3]*)**.** *Let $G = (V, E)$ be a finite digraph with no sources. Then*

$$\kappa^{vertex}(\mathcal{L}G, \mathbf{x}) = \kappa^{edge}(G, \mathbf{x}) \cdot \prod_{v\in V}\Bigg(\sum_{t(e)=v} x_e\Bigg)^{\text{indeg}(v)-1}.$$

Here we find that the restriction G has no sources can be removed.

Theorem 3. *Let $G = (V, E)$ be a finite digraph. Then*

$$\kappa^{vertex}(\mathcal{L}G, \mathbf{x}) = \kappa^{edge}(G, \mathbf{x}) \cdot \prod_{v\in V}\Bigg(\sum_{t(e)=v} x_e\Bigg)^{\text{indeg}(v)-1}.$$

Proof. If G has no sources, the result follows from Theorem 2. Now we assume that G has a nonempty set of sources $S = \{s_1, s_2, \ldots, s_t\}$. By adding a weighted loop l_i at each s_i for $1 \le i \le s$, there obtains another digraph G^* (see Fig. 2). It is trivial that

$$\kappa^{edge}(G^*, \mathbf{x}) = \kappa^{edge}(G, \mathbf{x}).$$

The out-neighbours of l_i in $\mathcal{L}G^*$ are exactly the out-going edges of v_i in G, and l_i has no in-neighbours in $\mathcal{L}G^*$, for all $1 \le i \le t$. There holds

$$\kappa^{vertex}(\mathcal{L}G^*, \mathbf{x}) = \kappa^{vertex}\big(\mathcal{L}(G^* - l_i), \mathbf{x}\big) \cdot \sum_{\substack{e\in E(G)\\ t(e)=s_i}} x_e, \quad 1 \le i \le t.$$

Moreover, note that $\mathcal{L}G = \mathcal{L}G^* - \{l_1, l_2, \ldots, l_t\}$, and $\{l_1, l_2, \ldots, l_t\}$ forms an independent set of $\mathcal{L}G^*$. There follows

$$\kappa^{vertex}(\mathcal{L}G^*, \mathbf{x}) = \kappa^{vertex}(\mathcal{L}G, \mathbf{x}) \cdot \prod_{v\in S}\Bigg(\sum_{\substack{e\in E(G)\\ t(e)=v}} x_e\Bigg). \quad (1)$$

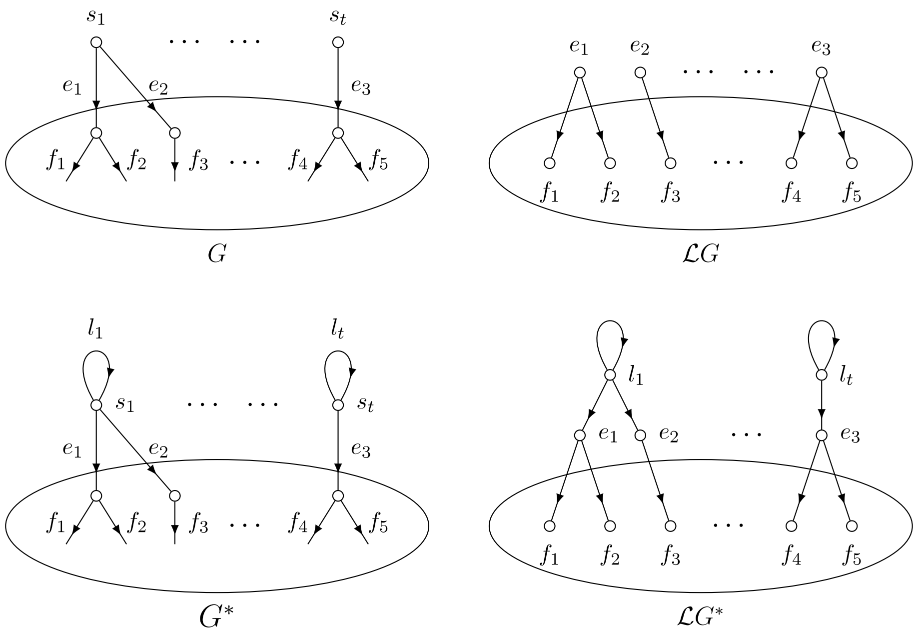

Fig. 2. Digraphs G, $\mathcal{L}G$, G^* and $\mathcal{L}G^*$.

Let $\text{indeg}^*(v)$ be the indegree of v in G^*. Note that $\text{indeg}^*(v) = 1$, $\text{indeg}(v) = 0$, if $v \in S$; and $\text{indeg}^*(v) = \text{indeg}(v)$, if $v \in V\backslash S$. Since G^* has no sources, it follows from Theorem 2 that

$$\begin{aligned}
\kappa^{vertex}(\mathcal{L}G^*, \mathbf{x}) &= \kappa^{edge}(G^*, \mathbf{x}) \cdot \prod_{v\in V}\Bigg(\sum_{\substack{e\in E(G^*)\\ t(e)=v}} x_e\Bigg)^{\text{indeg}^*(v)-1}\\
&= \kappa^{edge}(G, \mathbf{x}) \cdot \prod_{v\in V\backslash S}\Bigg(\sum_{\substack{e\in E(G^*)\\ t(e)=v}} x_e\Bigg)^{\text{indeg}^*(v)-1}\\
&= \kappa^{edge}(G, \mathbf{x}) \cdot \prod_{v\in V\backslash S}\Bigg(\sum_{\substack{e\in E(G)\\ t(e)=v}} x_e\Bigg)^{\text{indeg}(v)-1}\\
&= \kappa^{edge}(G, \mathbf{x}) \cdot \prod_{v\in V}\Bigg(\sum_{\substack{e\in E(G)\\ t(e)=v}} x_e\Bigg)^{\text{indeg}(v)-1} \cdot \prod_{v\in S}\Bigg(\sum_{\substack{e\in E(G)\\ t(e)=v}} x_e\Bigg).
\end{aligned} \tag{2}$$

Since every source $v \in S$ has outgoing edges in G, there holds $\prod_{v\in S}\left(\sum_{\substack{e\in E(G)\\ t(e)=v}} x_e\right) \neq 0$. The result follows from Eqs. (1) and (2). □

Corollary 1. *Let G be a digraph with order n and size m. If $\sum_{t(e)=v} x_e = f(\mathbf{x})$ for every vertex v of G, then*

$$\kappa^{vertex}(\mathcal{L}G, \mathbf{x}) = f(\mathbf{x})^{m-n} \cdot \kappa^{edge}(G, \mathbf{x}).$$

As noted in [3], setting all $x_e = 1$ in Theorem 3 yields a product formula.

Corollary 2. *Let G be a digraph. Then*

$$\kappa(\mathcal{L}G) = \kappa(G) \cdot \prod_{v\in V} \text{outdeg}(v)^{\text{indeg}(v)-1}.$$

When G is k-out-regular, there holds $m = \sum_v \text{indeg}(v) = \sum_v \text{outdeg}(v) = kn$.

Corollary 3. *If a digraph G is k-out-regular, then*

$$\kappa(\mathcal{L}G) = k^{m-n} \cdot \kappa(G).$$

The following corollary is a generalization of Theorem 1. Note that if G is an unweighted k-out-regular digraph with n vertices and m edges, then $\mathcal{L}G$ is also k-out-regular. It follows that $|V(\mathcal{L}G)| = m = kn$ and $|E(\mathcal{L}G)| = k|V(\mathcal{L}G)| = km = k^2n$. Thus $|V(\mathcal{L}^tG)| = k^tn$ and $|E(\mathcal{L}^tG)| = k^{t+1}n$. There follows from Corollary 3 that

$$\begin{aligned}\kappa(\mathcal{L}^tG) &= k^{(k^tn-k^{t-1}n)}\cdot\kappa(\mathcal{L}^{t-1}G)\\ &= k^{(k^tn-k^{t-1}n)}\cdot k^{(k^{t-1}n-k^{t-2}n)}\cdot\kappa(\mathcal{L}^{t-2}G)\\ &\vdots\\ &= k^{(k^tn-n)}\cdot\kappa(\mathcal{L}G).\end{aligned}$$

Corollary 4. *If a digraph G is k-out-regular, then*

$$\kappa(\mathcal{L}^tG) = k^{(k^t-1)n}\cdot\kappa(G).$$

For iterated line digraphs $\mathcal{L}^tG$, let $p(t,v) = \big|\{(e_1,\ldots,e_t) : h(e_i) = t(e_{i+1}),\ 1\le i\le t-1 \text{ and } h(e_t) = v\}\big|$ be the number of directed walks of t edges in G ending at vertex v. (The definition of $p(t,v)$ in [3] is defined to be the number of directed paths of t edges in G ending at vertex v.) Levine [3] proved the following theorem.

Theorem 4 (*Levine [3]*)**.** *Let $G = (V,E)$ be a finite digraph with no sources. Then*

$$\kappa(\mathcal{L}^tG) = \kappa(G)\cdot\prod_{v\in V}\operatorname{outdeg}(v)^{p(t,v)-1}.$$

By using Theorem 3 instead of Theorem 2 in the proof of this theorem, the restriction that G has no sources can also be removed.

Theorem 5. *Let $G = (V,E)$ be a finite digraph. Then*

$$\kappa(\mathcal{L}^tG) = \kappa(G)\cdot\prod_{v\in V}\operatorname{outdeg}(v)^{p(t,v)-1}.$$

Proof. Let e be an edge of digraph G, it is easy to see that

$$\begin{aligned}&\operatorname{outdeg}(e) = \operatorname{outdeg}\big(h(e)\big)\\ &\operatorname{indeg}(e) = \operatorname{indeg}\big(t(e)\big).\end{aligned}$$

For $i\ge 2$, let $(e_1,e_2,\ldots,e_i)$ be an edge of $\mathcal{L}^iG$. It follows from the definition that

$$\begin{aligned}h\big((e_1,e_2,\ldots,e_i)\big) &= (e_2,\ldots,e_i)\\ t\big((e_1,e_2,\ldots,e_i)\big) &= (e_1,\ldots,e_{i-1}).\end{aligned}$$

Thus, for any $i\ge 0$, let $(e_1,e_2,\ldots,e_t)$ be an edge of $\mathcal{L}^iG$, there holds

$$\begin{aligned}&\operatorname{outdeg}\big((e_1,e_2,\ldots,e_i)\big) = \operatorname{outdeg}\big(h(e_i)\big)\\ &\operatorname{indeg}\big((e_1,e_2,\ldots,e_i)\big) = \operatorname{indeg}\big(h(e_1)\big).\end{aligned}$$

It follows from Corollary 2 that

$$\begin{aligned}\kappa(\mathcal{L}^{i+1}G) &= \kappa(\mathcal{L}^iG)\cdot\prod_{(e_1,e_2,\ldots,e_i)\in E_i}\operatorname{outdeg}\big((e_1,e_2,\ldots,e_i)\big)^{\operatorname{indeg}((e_1,e_2,\ldots,e_i))-1}\\ &= \kappa(\mathcal{L}^iG)\cdot\prod_{(e_1,e_2,\ldots,e_i)\in E_i}\operatorname{outdeg}\big(h(e_i)\big)^{\operatorname{indeg}(t(e_1))-1}\\ &= \kappa(\mathcal{L}^iG)\cdot\prod_{v\in V}\prod_{\substack{(e_1,e_2,\ldots,e_i)\in E_i\\ h(e_i)=v}}\operatorname{outdeg}(v)^{\operatorname{indeg}(t(e_1))-1}.\end{aligned}$$

Since $\sum_{\substack{(e_1,e_2,\ldots,e_i)\in E_i\\ h(e_i)=v}} 1 = p(i,v)$ and $\sum_{\substack{(e_1,e_2,\ldots,e_i)\in E_i\\ h(e_i)=v}} \operatorname{indeg}(t(e_1)) = p(i+1,v)$, there follows

$$\kappa(\mathcal{L}^{i+1}G) = \kappa(\mathcal{L}^iG)\cdot\prod_{v\in V}\operatorname{outdeg}(v)^{p(i+1,v)-p(i,v)}.$$

Thus

$$\kappa(\mathcal{L}^t G) = \kappa(G) \cdot \prod_{0\le i\le t-1} \prod_{v\in V} \mathrm{outdeg}(v)^{p(i+1,v)-p(i,v)}$$
$$= \kappa(G) \cdot \prod_{v\in V} \mathrm{outdeg}(v)^{p(t,v)-1}. \quad \square$$

In the rest of this section, we will discuss spanning trees with a fixed root. The following theorem relates the numbers of spanning trees with a fixed root of $\mathcal{L}G$ and G. Let G be a digraph with a given vertex v_*, we define

$$\kappa^{edge}(G, v_*, \mathbf{x}) = \sum_{\mathrm{root}(T)=v_*} \prod_{e\in T} x_e, \qquad \kappa^{vertex}(G, v_*, \mathbf{x}) = \sum_{\mathrm{root}(T)=v_*} \prod_{e\in T} x_{h(e)}.$$

Theorem 6 (*Levine [3]*). *Let $G = (V, E)$ be a finite digraph, and let $e_* = (w_*, v_*)$ be an edge of G. If $\mathrm{indeg}(v) \ge 1$ for all vertices $v \in V$, and $\mathrm{indeg}(v_*) \ge 2$, then*

$$\frac{\kappa^{vertex}(\mathcal{L}G, e_*, \mathbf{x})}{x_{e_*}\kappa^{edge}(G, w_*, \mathbf{x})} = \left(\sum_{t(e)=v_*} x_e\right)^{\mathrm{indeg}(v_*)-2} \prod_{v\ne v_*}\left(\sum_{t(e)=v} x_e\right)^{\mathrm{indeg}(v)-1}.$$

This theorem also can be generalized to all kinds of digraphs.

Lemma 1. *Let $G = (V, E)$ be a finite digraph, and let $e_* = (w_*, v_*)$ be an edge of G. If $\mathrm{indeg}(v_*) \ge 2$, then*

$$\frac{\kappa^{vertex}(\mathcal{L}G, e_*, \mathbf{x})}{x_{e_*}\kappa^{edge}(G, w_*, \mathbf{x})} = \left(\sum_{t(e)=v_*} x_e\right)^{\mathrm{indeg}(v_*)-2} \prod_{v\ne v_*}\left(\sum_{t(e)=v} x_e\right)^{\mathrm{indeg}(v)-1}.$$

Proof. The proof is similar to the proof of Theorem 3. If G has no sources, the result follows from Theorem 6. Now we assume that G has a nonempty set of sources $S = \{s_1, s_2, \ldots, s_t\}$. By adding a weighted loop l_i to each source s_i, we obtain another digraph G^*. (See Fig. 2.)

By a similar analysis, it is easy to see that: $\mathrm{indeg}^*(v) = 1$, $\mathrm{indeg}(v) = 0$, if $v \in S$; and $\mathrm{indeg}^*(v) = \mathrm{indeg}(v)$, if $v \in V\backslash S$. Moreover,

$$\kappa^{edge}(G^*, w_*, \mathbf{x}) = \kappa^{edge}(G, w_*, \mathbf{x});$$
$$\kappa^{vertex}(\mathcal{L}G^*, w_*, \mathbf{x}) = \kappa^{vertex}(\mathcal{L}G, w_*, \mathbf{x}) \cdot \prod_{v\in S}\left(\sum_{t(e)=v} x_e\right).$$

Thus

$$\frac{\kappa^{vertex}(\mathcal{L}G^*, e_*, \mathbf{x})}{x_{e_*}\kappa^{edge}(G^*, w_*, \mathbf{x})} = \frac{\kappa^{vertex}(\mathcal{L}G, e_*, \mathbf{x})}{x_{e_*}\kappa^{edge}(G, w_*, \mathbf{x})} \cdot \prod_{v\in S}\left(\sum_{t(e)=v} x_e\right).$$

Since every source must have outgoing edges, we have $\prod_{v\in S}\left(\sum_{t(e)=v} x_e\right) \ne 0$. On the other hand, $\mathrm{indeg}^*(v) \ge 1$ for all vertices $v \in V$ and $\mathrm{indeg}^*(v_*) = \mathrm{indeg}(v_*) \ge 2$. It follows from Theorem 6 that

$$\frac{\kappa^{vertex}(\mathcal{L}G, e_*, \mathbf{x})}{x_{e_*}\kappa^{edge}(G, w_*, \mathbf{x})} = \prod_{v\in S}\left(\sum_{t(e)=v} x_e\right)^{-1} \cdot \frac{\kappa^{vertex}(\mathcal{L}G^*, e_*, \mathbf{x})}{x_{e_*}\kappa^{edge}(G^*, w_*, \mathbf{x})}$$
$$= \prod_{v\in S}\left(\sum_{t(e)=v} x_e\right)^{-1} \cdot \left(\sum_{t(e)=v_*} x_e\right)^{\mathrm{indeg}^*(v_*)-2} \prod_{v\ne v_*}\left(\sum_{t(e)=v} x_e\right)^{\mathrm{indeg}^*(v)-1}.$$
$$= \left(\sum_{t(e)=v_*} x_e\right)^{\mathrm{indeg}(v_*)-2} \prod_{v\ne v_*}\left(\sum_{t(e)=v} x_e\right)^{\mathrm{indeg}(v)-1}. \quad \square$$

Theorem 7. *Let $G = (V, E)$ be a finite directed graph, and let $e_* = (w_*, v_*)$ be an edge of G. If $\sum_{t(e)=v_*} x_e = 0$, then*

$$\frac{\kappa^{vertex}(\mathcal{L}G, e_*, \mathbf{x})}{x_{e_*}\kappa^{edge}(G, w_*, \mathbf{x})} = \prod_{v\ne v_*}\left(\sum_{t(e)=v} x_e\right)^{\mathrm{indeg}(v)-1};$$

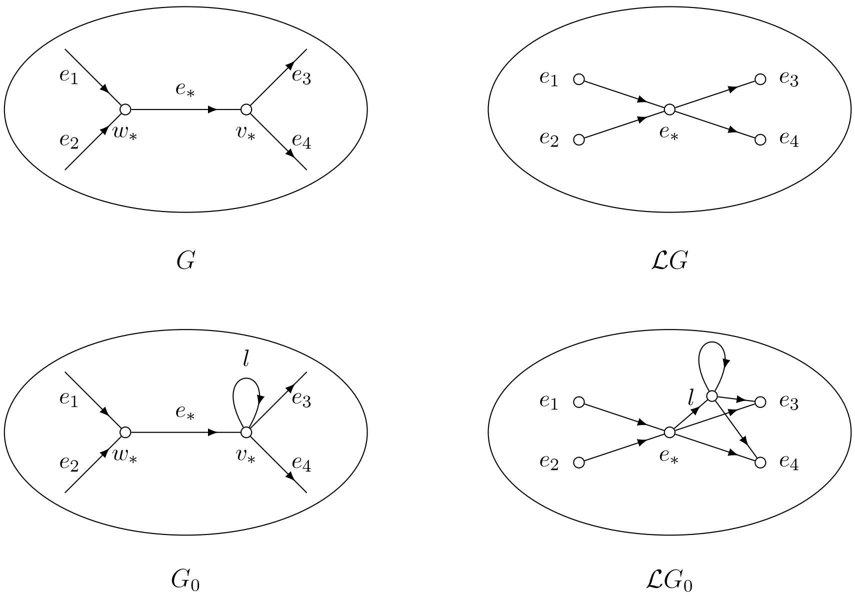

Fig. 3. Digraphs G, $\mathcal{L}G$, G_0 and $\mathcal{L}G_0$.

else if $\sum_{t(e)=v_*} x_e \neq 0$, *then*

$$\frac{\kappa^{vertex}(\mathcal{L}G, e_*, \mathbf{x})}{x_{e_*}\kappa^{edge}(G, w_*, \mathbf{x})} = \left(\sum_{t(e)=v_*} x_e\right)^{\mathrm{indeg}(v_*)-2} \prod_{v\neq v_*}\left(\sum_{t(e)=v} x_e\right)^{\mathrm{indeg}(v)-1}.$$

Proof. Since the result follows from Lemma 1 when $\mathrm{indeg}(v_*) \geq 2$, we may assume that $\mathrm{indeg}(v_*) = 1$. When $\sum_{t(e)=v_*} x_e = 0$, v_* has no outgoing edges and only one incoming edge. It is easy to see that

$$\kappa^{vertex}(\mathcal{L}G, \mathbf{x}) = \kappa^{vertex}(\mathcal{L}G, e_*, \mathbf{x})$$
$$\kappa^{edge}(G, \mathbf{x}) = x_{e_*}\kappa^{edge}(G, w_*, \mathbf{x}).$$

It follows from Theorem 3 that

$$\begin{aligned}\frac{\kappa^{vertex}(\mathcal{L}G, e_*, \mathbf{x})}{x_{e_*}\kappa^{edge}(G, w_*, \mathbf{x})} &= \frac{\kappa^{vertex}(\mathcal{L}G, \mathbf{x})}{\kappa^{edge}(G, \mathbf{x})} \\ &= \prod_{v\in V}\left(\sum_{t(e)=v} x_e\right)^{\mathrm{indeg}(v)-1} \\ &= \prod_{v\neq v_*}\left(\sum_{t(e)=v} x_e\right)^{\mathrm{indeg}(v)-1}. \end{aligned} \tag{3}$$

Now we assume that $\sum_{t(e)=v_*} x_e \neq 0$, and construct a new digraph G_0 by adding a weighted loop l at v_* in G. (See Fig. 3.) It is easy to see that $\mathrm{indeg}(v_*) = 1$ and $\mathrm{indeg}_0(v_*) = 2$. Moreover,

$$\kappa^{edge}(G_0, w_*, \mathbf{x}) = \kappa^{edge}(G, w_*, \mathbf{x});$$
$$\kappa^{vertex}(\mathcal{L}G_0, e_*, \mathbf{x}) = \kappa^{vertex}(\mathcal{L}G, e_*, \mathbf{x}) \cdot \sum_{t(e)=v_*} x_e.$$

Thus

$$\frac{\kappa^{vertex}(\mathcal{L}G_0, e_*, \mathbf{x})}{x_{e_*}\kappa^{edge}(G_0, w_*, \mathbf{x})} = \frac{\kappa^{vertex}(\mathcal{L}G, e_*, \mathbf{x})}{x_{e_*}\kappa^{edge}(G, w_*, \mathbf{x})} \cdot \sum_{t(e)=v_*} x_e.$$

On the other hand. Since $\mathrm{indeg}_0(v_*) = 2$, it follows from Lemma 1 that

$$\frac{\kappa^{vertex}(\mathcal{L}G, e_*, \mathbf{x})}{x_{e_*}\kappa^{edge}(G, w_*, \mathbf{x})} = \left(\sum_{t(e)=v_*} x_e\right)^{-1} \cdot \frac{\kappa^{vertex}(\mathcal{L}G_0, e_*, \mathbf{x})}{x_{e_*}\kappa^{edge}(G_0, w_*, \mathbf{x})}$$

$$= \left(\sum_{t(e)=v_*} x_e\right)^{-1} \cdot \left(\sum_{t(e)=v_*} x_e\right)^{\mathrm{indeg}_0(v_*)-2} \prod_{v\neq v_*}\left(\sum_{t(e)=v} x_e\right)^{\mathrm{indeg}_0(v)-1}$$

$$= \left(\sum_{t(e)=v_*} x_e\right)^{\mathrm{indeg}(v_*)-2} \prod_{v\neq v_*}\left(\sum_{t(e)=v} x_e\right)^{\mathrm{indeg}(v)-1}. \qquad \square$$

3. Remarks

Let $G = (V(G), E(G))$ be a digraph and $\mathcal{L}G$ be its line digraph. The vertex weights and edge weights of G are denoted by $\{x_v\}_{v\in V(G)}$ and $\{x_e\}_{e\in E(G)}$.

Although the definition of the vertex weights of $\mathcal{L}G$ is straight and reasonable, there are several different ways to define the edge weights of $\mathcal{L}G$. Let e, f be two consecutive edges in G with $h(e) = t(f) = v$, then ef is an edge in $\mathcal{L}G$. One way to define the weight of ef is to set it equal to x_v. Under this definition all the edges in $\mathcal{L}G$ generated by the incoming edges and outgoing edges of vertex v in G have the same edge weights x_v. Another way to define the weight of ef is to set it equal to $x_{h(f)}$. Under this definition, the edge weight of the rooted spanning trees of $\mathcal{L}G$ is denoted by $\kappa^{edge*}(\mathcal{L}G, \mathbf{x})$. There is a similar formula of Theorem 3 that relates $\kappa^{edge*}(\mathcal{L}G, \mathbf{x})$ and $\kappa^{vertex}(G, \mathbf{x})$.

Theorem 8. *Let $G = (V, E)$ be a finite directed graph. Then*

$$\kappa^{edge*}(\mathcal{L}G, \mathbf{x}) = \kappa^{vertex}(G, \mathbf{x}) \prod_{v\in V}\left(\sum_{t(e)=v} x_{h(e)}\right)^{\mathrm{indeg}(v)-1}.$$

Proof. For a vertex weighted digraph $G = (V, E)$ with vertex weights $\{x_v\}_{v\in V}$, let $G^* \cong G$ be the edge weighted digraph with edge weights $\{x_e = x_{h(e)}\}_{e\in E}$. It is easy to see that $\kappa^{vertex}(G) = \kappa^{edge}(G^*)$ and $\kappa^{vertex}(\mathcal{L}G^*) = \kappa^{edge}((\mathcal{L}G^*)^*)$. Moreover, $\kappa^{edge}((\mathcal{L}G^*)^*) = \kappa^{edge*}(\mathcal{L}G)$. The result follows from Theorem 3. $\square$

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