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Numerical Solution of Interval Linear Programming

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Abstract

This paper presents an algorithm for solving interval linear programming(ILP) problems. Interval inequality constraints and equality constraints are discussed separately. The aim of the paper is to show that (ILP) problems can be decomposed into two general linear programming(LP) by the monotonicity of (LP) problems , and we can gain the interval objective values. Finally, the proposed method have virtually the same results with paper [1].

Keywords: interval linear programming, interval numbers, optimal objective interval values

1 Introduction

In traditional mathematical programming, the coefficients of the problems are always treated as deterministic values. However, uncertainty always exits in practical engineering problems[1-3]. In order to deal with the uncertain optimization problems, fuzzy[4] and stochastic[5-6] approaches are frequently used to describe the imprecise characteristics. In these two approaches,the membership function and probability distribution play important roles. However, it is sometimes difficult to specify an appropriate membership function or accurate probability distribution in an uncertain environment[7]. Therefore, interval optimization problems may provide an alternative choice for solving uncertainty optimization problems.

The solution methods to interval linear programming(ILP) problems were explored by some scholars [1][7-9]. An interval linear programming is defined as follows:

$$\begin{aligned} &\min z = C^I X \\ &s.t.\ A^I X \le b^I, \\ &\quad x_j \ge l_j, j = 1, ..., n. \end{aligned} \tag{1}$$

Where $C^I \in \{R^{\pm}\}^{1\times n}, b^I \in \{R^{\pm}\}^{n\times 1}$, $A^I \in \{R^{\pm}\}^{m\times n}$, $R^{\pm}$ denotes a set of interval numbers, and $X = (x_1, \cdots, x_n)^T$ is an n-dimensional design vector.

2 A general model

In this paper, a more general model of interval linear programming (ILP) than (1) is defined as follows:

$$\begin{aligned} &\min z = \sum_{j=1}^{n} c_j^I x_j \\ &s.t. \quad \sum_{j=1}^{n} a_{ij}^I x_j \le b_i^I, i = 1, ..., l, \\ &\qquad \sum_{j=1}^{n} a_{ij}^I x_j = b_i^I, i = l+1, ..., m, \\ &\qquad x_j \ge l_j, j = 1, ..., n. \end{aligned} \tag{2}$$

where $c_j^I \in [c_j^L, c_j^U]$, $a_{ij}^I \in [a_{ij}^L, a_{ij}^U]$, $b_i^I \in [b_i^L, b_i^U]$, $0 < l_j^I \in [l_j^L, l_j^U]$ are interval numbers.

Let $S = \{(c_j^I, a_{ij}^I, b_i^I, l_j^I) | c_j^I \in [c_j^L, c_j^U], a_{ij}^I \in [a_{ij}^L, a_{ij}^U], b_i^I \in [b_i^L, b_i^U], l_j^I \in [l_j^L, l_j^U], i = 1, ..., m, j = 1, ..., n\}$.

About programming (2), there are some discussions as follows. This paper's innovation lies in theorem 3.3, by which we can decompose programming (2) into two deterministic linear programming to gain the optimal objective interval values of the interval linear programming (2).

3 Results and Discussion

3.1 Equality constraints

Theorem 3.1. *In programming (2), denote* $A = \{X | \sum_{j=1}^{n} a_{ij}^I x_j = b_i^I, \forall a_{ij}^I \in [a_{ij}^L, a_{ij}^U], x_j \ge l_j^I, b_i^I \in [b_i^L, b_i^U], l_j^I \in [l_j^L, l_j^U], i = l+1, ..., m, j = 1, ..., n\}$, $B = \{X | \sum_{j=1}^{n} a_{ij}^L x_j \le b_i^U, \sum_{j=1}^{n} a_{ij}^U x_j \ge b_i^L, x_j \ge l_j^I, l_j^I \in [l_j^L, l_j^U], i = l+1, ..., m, j = 1, ..., n\}$, *then* $A = B$.

Proof. Firstly, proofing $A \subseteq B$. $\forall X^* \in A$, $\exists a_{ij}^I \in [a_{ij}^L, a_{ij}^U], b_i^I \in [b_i^L, b_i^U], l_j^I \in [l_j^L, l_j^U]$, s.t. $\sum_{j=1}^n a_{ij}^I x_j^* = b_i^I, x_j^* \geq l_j^I (\geq 0)$. Then $\sum_{j=1}^n a_{ij}^L x_j^* \leq \sum_{j=1}^n a_{ij}^I x_j^* = b_i^I \leq b_i^U$ and $\sum_{j=1}^n a_{ij}^U x_j^* \geq \sum_{j=1}^n a_{ij}^I x_j^* = b_i^I \geq b_i^L$. Clearly, $X^* \in B$. $A \subseteq B$ holds.

Secondly proofing $B \subseteq A$. $\forall X^* \in B$, assuming the function $f(a_{i1}^I, a_{i2}^I, a_{i3}^I, ..., a_{in}^I) = \sum_{j=1}^n a_{ij}^I x_j^* (i = l+1, ..., m)$ are continuous on $[a_{i1}^L, a_{i1}^U] \times [a_{i2}^L, a_{i2}^U] \times ... \times [a_{in}^L, a_{in}^U]$, because $\forall X^* \in B$, we have $\sum_{j=1}^n a_{ij}^L x_j^* \leq b_i^U$ and $\sum_{j=1}^n a_{ij}^U x_j^* \geq b_i^L$. And then we give a $a_{ij}^I \in [a_{ij}^L, a_{ij}^U]$ in order to $b_i^L \leq \sum_{j=1}^n a_{ij}^I x_j^* \leq b_i^U$. Since the intermediate value theorem, $\exists b_i^I \in [b_i^L, b_i^U]$, s.t. $\sum_{j=1}^n a_{ij}^I x_j^* = b_i^I$, and so $X^* \in A$. Therefore, $B \subseteq A$ holds. From above, we can attain $A = B$. □

From theorem 3.1, we know that $\sum_{j=1}^n a_{ij}^I x_j = b_i^I$ is equivalent to $\sum_{j=1}^n a_{ij}^L x_j \leq b_i^U$ and $\sum_{j=1}^n a_{ij}^U x_j \geq b_i^L$. So the minimum optimal solution of interval programming (2) must satisfy both $\sum_{j=1}^n a_{ij}^L x_j \leq b_i^U$ and $\sum_{j=1}^n a_{ij}^U x_j \geq b_i^L$.

Lemma 3.2 (see[1]). *The maximum optimal solution of interval programming (2) must satisfy one of* $\sum_{j=1}^n a_{ij}^L x_j = b_i^U$ *and* $\sum_{j=1}^n a_{ij}^U x_j = b_i^L$.

3.2 The monotonicity of programming (3.1)

Assuming $c_j, a_{ij}, b_i, l_j (i = 1, \cdots, j)$ are real numbers of $[c_j^L, c_j^U]$, $[a_{ij}^L, a_{ij}^U]$, $[b_i^L, b_i^U]$, $[l_j^L, l_j^U]$ respectively, programming (2) will be a general linear programming(LP) problems:

$$\begin{aligned} \min z = & \sum_{j=1}^n c_j x_j \\ s.t. \quad & \sum_{j=1}^n a_{ij} x_j \leq b_i, i = 1, ..., l, \\ & x_j \geq l_j, j = 1, ..., n. \end{aligned} \tag{3}$$

where has no equation constraints.

Suppose that S' is the feasible region of programming (2), $X_{S'}^*$ is the optimal solution of programming (3), and $G = \min_{X \in S'} z(X) = z(X_{S'}^*)$.

Clearly, if $c_j, a_{ij}, b_i, l_j (i = 1, \cdots, l, j = 1, \cdots, n)$ are deterministic values of $[c_j^L, c_j^U]$, $[a_{ij}^L, a_{ij}^U]$, $[b_i^L, b_i^U]$, $[l_j^L, l_j^U]$ respectively, $G = G(c_j, a_{ij}, b_i, l_j)$ is the function of c_j, a_{ij}, b_i, l_j. Now, we will explain the relation between $G = G(c_j, a_{ij}, b_i, l_j)$ and c_j, a_{ij}, b_i, l_j, which is an important characteristic of programming (3).

Theorem 3.3. *G is an increasing function of c_j, a_{ij}, l_j and decreasing function of b_i, namely,* $\frac{\partial G}{\partial c_j} \geq 0$, $\frac{\partial G}{\partial a_{ij}} \geq 0, \frac{\partial G}{\partial l_j} \geq 0$, $\frac{\partial G}{\partial b_i} \leq 0$.

Proof. For c_j, giving an increasing variable $\Delta c_j > 0$, $c_j + \Delta c_j > 0$, then we can get $0 \leq \sum_{i=1}^{n} c_j x_j \leq \sum_{i=1}^{n} (c_j + \Delta c_j) x_j$, namely, $G(c_j, a_{ij}, b_i, l_j) \leq G(c_j + \Delta c_j, a_{ij}, b_i, l_j)$, therefore $\frac{\partial G}{\partial c_j} \geq 0$.

For a_{ij}, giving an increasing variable $\Delta a_{ij} > 0$, $a_{ij} + \Delta a_{ij} > 0$, let $S_1 = \{X | \sum_{j=1}^{n} a_{ij} x_j \leq b_i, i = 1, ..., l\}$, $S_2 = \{X | \sum_{i=1}^{n} (a_{ij} + \Delta a_{ij}) x_j \leq b_i, i = 1, ..., l\}$. If $X \in S_2$, $\sum_{i=1}^{n} (a_{ij} + \Delta a_{ij}) x_j \leq b_i, i = 1, ..., l$. However, $\sum_{j=1}^{n} a_{ij} x_j \leq \sum_{j=1}^{n} (a_{ij} + \Delta a_{ij}) x_j$ $\leq b_i, i = 1, ..., l$, therefore, $X \in S_1$. And then $S_2 \subseteq S_1$, let $S_1 = S_2 \cup T$, $S_2 \cap T = \varphi$, so $G|_{X \in S_1}$=min$\{G|_{X \in S_2}, G|_{X \in T}\} \leq G|_{X \in S_2}$, namely, $\frac{\partial G}{\partial a_{ij}} \geq 0$.

Similarly, we can have $\frac{\partial G}{\partial l_j} \geq 0$, $\frac{\partial G}{\partial b_i} \leq 0$. □

We only discuss inequality constraints $(i = 1, \cdots, l, j = 1, \cdots, n)$ in programming (3), for equality constraints $\sum_{j=1}^{n} a_{ij}^I x_j = b_i^I$ equivalent to $\sum_{j=1}^{n} a_{ij}^L x_j \leq b_i^U$ and $\sum_{j=1}^{n} a_{ij}^U x_j \geq b_i^L$ or one of $\sum_{j=1}^{n} a_{ij}^L x_j = b_i^U$ and $\sum_{j=1}^{n} a_{ij}^U x_j = b_i^L$, which are all deterministic inequality constraints, and won't affect programming (2)'s monotonicity .

3.3 The lower and upper bound of programming (2)

As in [1], in order to compute the interval numbers of objective function , we need to find the lower and upper bound of the objective function value of programming (2).

To calculate the bounds of programming (2) by theorem (3.1), lemma(3.2) and theorem(3.3), programming (2) can be translated into the following two programming problems (4) and (5).

$$\begin{aligned} z^L = & \min_{(c_j^I, a_{ij}^I, b_i^I, l_j^I)\in S} \min_X z = \sum_{j=1}^n c_j^I x_j \\ s.t. \quad & \sum_{j=1}^n a_{ij}^I x_j \le b_i^I, i = 1, ..., l, \\ & \sum_{j=1}^n a_{ij}^L x_j \le b_i^U, i = l+1, ..., m, \\ & \sum_{j=1}^n a_{ij}^U x_j \ge b_i^L, i = l+1, ..., m, \\ & x_j \ge l_j, j = 1, ..., n. \end{aligned} \tag{4}$$

$$\begin{aligned} z^U = & \max_{(c_j^I, a_{ij}^I, b_i^I, l_j^I)\in S} \min_X z = \sum_{j=1}^n c_j^I x_j \\ s.t. \quad & \sum_{j=1}^n a_{ij}^I x_j \le b_i^I, i = 1, ..., l, \\ & \sum_{j=1}^n a_{ij}^L x_j = b_i^U, i = l+1, ..., m, \\ (or \quad & \sum_{j=1}^n a_{ij}^U x_j = b_i^L, i = l+1, ..., m,) \\ & x_j \ge l_j, j = 1, ..., n. \end{aligned} \tag{5}$$

In programming (4) and (5), programming (2)'s equality constrains have been translated into corresponding inequality constraints.

For the objective function in (4) , z^L can be written as $z^L = \min_{(c_j, a_{ij}, b_i, l_j)\in S} \min_{X\in S'} z = \sum_{j=1}^n c_j^I x_j$, according to theorem 3.3 , G is an increasing function of c_j , so $z^L = \min_{(c_j, a_{ij}, b_i, l_j)\in S} G(c_j, a_{ij}, b_i, l_j) = \min_X \; z = \sum_{j=1}^n c_j^L x_j$, which is as same as the result of [1].

For the inequality constraints in (4) $\sum_{j=1}^n a_{ij}^I x_j \le b_i^I, i = 1, ..., l$, and $x_j \ge l_j^I, j = 1, ..., n$, according to theorem 3.3, G is the increasing function of a_{ij}, l_j and is the decreasing function of b_i, so that in order to calculate the lower bound, $\sum_{j=1}^n a_{ij}^I x_j \le b_i^I, i = 1, ..., l$, $x_j \ge l_j^I, j = 1, ..., n$ can be written as $\sum_{j=1}^n a_{ij}^L x_j \le b_i^U, i = 1, ..., l$ and $x_j \ge l_j^L, j = 1, ..., n$.

Similarly, we write z^U of (5) as $z^U = \max_{(c_j, a_{ij}, b_i, l_j)\in S} G(c_j, a_{ij}, b_i, l_j) = \min_X \; z = \sum_{j=1}^n c_j^U x_j$, and inequality constraints $\sum_{j=1}^n a_{ij}^I x_j \le b_i^I, i = 1, ..., l$, and $x_j \ge l_j^I, j = 1, ..., n$ can be written as $\sum_{j=1}^n a_{ij}^U x_j \le b_i^L, i = 1, ..., l$ and $x_j \ge l_j^U, j = 1, ..., n$.

As a consequence, the smallest and largest objective value for z can be determined by mathematical programming problems

$$
\begin{aligned}
z^L &= \min_X \sum_{j=1}^n c_j^L x_j \\
s.t. \quad & \sum_{j=1}^n a_{ij}^L x_j \le b_i^U, i = 1, ..., l \\
& \sum_{j=1}^n a_{ij}^L x_j \le b_i^U \ , i = l+1, ..., m \\
& \sum_{j=1}^n a_{ij}^U x_j \ge b_i^L \ , i = l+1, ..., m \\
& x_j \ge l_j, j = 1, ..., n
\end{aligned} \tag{6}
$$

and

$$
\begin{aligned}
z^U &= \min_X \sum_{j=1}^n c_j^U x_j \\
s.t. \quad & \sum_{j=1}^n a_{ij}^U x_j \le b_i^L, i = 1, ..., l \\
& \sum_{j=1}^n a_{ij}^U x_j = b_i^L \ , i = l+1, ..., m \\
(or \quad & \sum_{j=1}^n a_{ij}^L x_j = b_i^U \ , i = l+1, ..., m) \\
& x_j \ge l_j, j = 1, ..., n
\end{aligned} \tag{7}
$$

Both programming (6) and (7) are traditional linear programming, and we can easily obtain the global optimum solution and the associated objective values z^L, z^U, the lower and upper bound of the objective values of the interval linear programming (2). We can find programming (6) and (7) is the same as [1]'s models. So the result is coincident. We will illustrate this fact further by an example in the next section.

3.4 Example

Example (see[1])

$$
\begin{aligned}
& \min z = [-1, 2]x_1 + x_2 \\
& s.t. \quad -x_1 + [1, 2]x_2 \ge [-2, -1], \\
& \quad [2, 3] \ x_1 + x_2 = [3, 4] \\
& \quad x_2 \le 3 \\
& \quad x_1 \ , x_2 \ge 0.
\end{aligned}
$$

According to analysis above, we can obtain

$$\begin{aligned} & z^L = \min z = -x_1 + x_2 \\ & s.t. \quad x_1 \text{ -}2x_2 \le 2 \\ & \qquad 2\ x_1 + x_2 \le 4 \\ & \qquad 3x_1 \ + x_2 \ge 3 \\ & \qquad x_2 \ \le 3 \\ & \qquad x_1\ , x_2 \ge 0. \end{aligned}$$

and

$$\begin{aligned} & z^U = \min z = 2x_1 + x_2 \\ & s.t. \quad x_1 \text{ -}x_2 \le 1, \\ & \qquad 2x_1 + x_2 = 4 \\ & \Big(\quad or \quad 3x_1 \ + x_2 = 3 \Big) \\ & \qquad x_2 \ \le 3 \\ & \qquad x_1\ , x_2 \ge 0. \end{aligned}$$

we can obtain the programming of z^L, z^U. By using the function "linprog" in Matlab 7.0, we can calculate z^L,$z^U$of two examples, whose results are listed in Table 1.

	Example 1
This Paper's result	$z^L = -2, x_1 = 2, x_2 = 0, z^U = 4, x_1 = 1.2377, x_2 = 1.5246$
[1]'s result	$z^L = -2, x_1 = 2, x_2 = 0, z^U = 4, x_1 = 5/3, x_2 = 2/3$

Table 1: numerical examples' results

In Table 1, this paper's result (lower bond z^L, and upper bound z^U), $[z^L, z^U] = [-2, 2]$, is the same as the [1], which can illustrate our method well.

4 Conclusion

In order to find optimal objective interval values $[z^L, z^U]$, Guo[1] discussed (ILP)'s objective function, inequality constraints and equality constraints separately. However, in this paper we needn't discuss these with hard work but equality constraints, and for monotonicity of function G ,we can easily gain the same result.

The method deserves to be considered widely while solving other interval programming problems, such as interval nonlinear programming.

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