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Multistability and instability analysis of recurrent neural networks with time-varying delays

Fanghai Zhang *, Zhigang Zeng

School of Automation, Huazhong University of Science and Technology, Wuhan 430074, China
Key Laboratory of Image Processing and Intelligent Control of Education Ministry of China, Wuhan 430074, China

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ABSTRACT

This paper provides new theoretical results on the multistability and instability analysis of recurrent neural networks with time-varying delays. It is shown that such n-neuronal recurrent neural networks have exactly $(4k+3)^{k_0}$ equilibria, $(2k+2)^{k_0}$ of which are locally exponentially stable and the others are unstable, where k_0 is a nonnegative integer such that $k_0 \leq n$. By using the combination method of two different divisions, recurrent neural networks can possess more dynamic properties. This method improves and extends the existing results in the literature. Finally, one numerical example is provided to show the superiority and effectiveness of the presented results.

1. Introduction

Recently, cellular neural networks (CNNs) have attracted much attention from both academic and industry communities due to their wide applications in image processing, pattern recognition, associative memory and their ability to tackle complex problems, for example Chen and Rong (2003), Chen, Zhang, and Lin (2016), Chua and Yang (1988), Cohen and Grossberg (1983), Kosko (1988), Liu, Li, Tong, and Chen (2016c), Liu and Michel (1993), Lu, Wang, and Chen (2011), Maundy and El-Masry (1990), Thiran, Crounse, Chua, and Halser (1995), Wang, Shen, Yin, and Zhang (2015), Wang, Sun, and Mazenc (2016), Wen, Zeng, Chen, and Huang (2017), Wen, Zeng, Huang, Yu, and Xiao (2015) and Yuan and Cao (2005). Recurrent neural networks (RNNs) are regarded as another kind of neural networks, which have more abundant dynamic properties. It is necessary to further research and study recurrent neural network.

The stability analysis of neural networks for steady-state solution (equilibria or invariant orbit) is the prerequisite and foundation in practical application, see, e.g., Cao (2001), Chen, Ge, Wu, and Gong (2015), Chen and Wang (2007), Di Marco, Forti, and Pancioni (2016), Huang, Fan, and Mitra (2017), Nie and Zheng (2015a), Wen, Zeng, Huang, and Zhang (2014), Zeng, Wang, and Liao (2003) and Zhang and Shen (2015). In an associative memory neural network, the dynamic evolution process from any initial state to its adjacent equilibrium points or adjacent periodic orbits can be considered as a process of associative memory, which requires multistability or multiperiodicity to provide theoretical analysis. In other words, in order to have the effect of associative memory in neural networks, memory model is designed for equilibrium points or periodic orbits. In addition, multistability or multiperiodicity is of great interest in both theory and practice (Cao, Feng, & Wang, 2008; Liu, Zeng, & Wang, 2016a; Nie, Zheng, & Cao, 2015; Shayer & Campbell, 2000; Zhang, Yi, & Yu, 2008; Zhang & Zeng, 2016).

In recent years, there are still many interesting topics of the multistability of neural networks and the topics have been widely discussed (Cheng, Lin, & Shih, 2006; Cheng, Lin, Shih, & Tseng, 2015; Di Marco, Forti, & Pancioni, 2017; Kaslik & Sivasundaram, 2011; Liu, Zeng, & Wang, 2016b; Nie, Cao, & Fei, 2013; Nie & Zheng, 2015b; Nie, Zheng, & Cao, 2016; Wang & Chen, 2012, 2014, 2015; Zeng & Wang, 2006; Zhang, Yi, Zhang, & Heng, 2009). It should be noted that most existing results are concerned with the neural networks with bounded activation function or bounded time delays. For instance, in Zeng and Wang (2006), by decomposition of state space $\Re^n$ into 3^n areas, some conditions were derived to ensure the existence of the multiperiodicity of CNNs, and to acquire 2^n stable periodic trajectories. Specially, 3^n equilibria in the Hopfield-type neural networks are obtained in Cheng et al. (2006). Besides, it was shown that convergence and multistability of DM-CNNs in the general case of nonsymmetric interconnections could be investigated in Di Marco et al. (2017).

In order to make storage capacity greater, in Bao and Zeng (2012), the neural networks with discontinuous activation functions were considered, and it was proved that the n-neuronal dynamical networks can obtain $(4k-1)^n$ locally exponentially stable equilibrium points. More generally, in Nie et al. (2013), the

* Correspondence to: School of Automation, Huazhong University of Science and Technology, Wuhan 430074, China.
E-mail addresses: fhzhanghust@163.com (F. Zhang), zgzeng@hust.edu.cn (Z. Zeng).

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n-neuronal competitive neural networks with exactly $(2r+1)^n$ equilibria were discussed, $(r+1)^n$ of which were exponentially stable, where the output function of network belonged to a class of piecewise linear functions with $2r(r \geq 1)$ corner points. Moreover, based on the geometrical properties of the activation functions in Gong, Liang, and Zhang (2016), the multistability of complex-valued neural networks with appropriate real-imaginary-type activation functions and distributed delays was addressed. With the development of multistability, the other relevant works could be found in Huang, Zhang, and Wang (2012, 2014), Nie and Zheng (2016) and Wang, Lu, and Chen (2010).

In this paper, by using the combination method of two different divisions, our aim is to further explore the multistability of recurrent neural networks with the piecewise linear activation function. Different from the previous division of state space, we have increased the dimensional division of state space. Some sufficient criteria are obtained to ensure that an n-neuronal recurrent neural network with $(k+1)$-stair activation function can have $(4k+3)^{k_0}$ equilibrium points and $(2k+2)^{k_0}$ of them are locally exponentially stable, where k_0 is a nonnegative integer such that $k_0 \leq n$. By contrast with most of the contributions available in the literature, the dimensional division of state space can lead to neural networks more abundant in dynamic behavior, and some conclusions extend conclusions produced by the division of state space.

Similar activation functions were also presented in Zeng, Huang, and Zheng (2010) and Zeng and Zheng (2012, 2013). The traditional division of state space rely heavily on the dimension of state space. The new way of division(i.e., coupled division) is presented, which reduces dependency on the dimension of state space. Note that by using the dimensional reconstruction and division of state space, the coupled division allows the division of space to be more diverse, and our conclusions extend the existing results of multistability. As a result, it has been well recognized that the different regions of parameter are given by means of coupling division and they can be chosen freely, which is helpful to improve the range of the regions of parameter.

The rest of the paper is organized as follows. Section 2 describes model and preliminaries which will be used later. In Section 3, sufficient conditions are derived for the existence, instability and local stability of the equilibrium points for the recurrent neural networks with time-varying delays. In Section 4, one example is provided to demonstrate the effectiveness of the obtained results. Some concluding remarks are drawn in Section 5.

2. Preliminaries

2.1. Notations

Let $\mathcal{C}([t_0-\tau, t_0], \mathcal{D})$ be the Banach space of functions mapping $[t_0-\tau, t_0]$ into $\mathcal{D} \subseteq \Re^n$ with norm defined $\|\phi\|_\infty = \max_{1\leq i\leq n}\{\sup_{r\in[t_0-\tau,t_0]}|\phi_i(r)|\}$, where $\phi(s) = (\phi_1(s), \phi_2(s), \ldots, \phi_n(s))^T \in \mathcal{C}([t_0-\tau, t_0], \mathcal{D})$. Denote $\|x\|_\infty = \max_{1\leq i\leq n}\{|x_i|\}$ as the vector norm of the vector $x = (x_1, x_2, \ldots, x_n)^T$; Card$(G)$ as the number of elements in the set G; $|A|$ as the absolute-value matrix of $A = [a_{ij}]$, i.e., $|A| = [|a_{ij}|]$.

For the given integer $k \geq 1$ and the given constant $0 < E_s \in \Re$, $s = 1, 2, \ldots, 2k+1$, there exist Z_j^{i-}, $Z_j^{i+} \in \Re, j = 1, 2, \ldots, 4k+3$, such that $\forall i \in \{1, 2, \ldots, n\}$

$$\begin{aligned}&Z_1^{i-} < Z_1^{i+} < -E_{2k+1} < Z_2^{i-} < Z_2^{i+}\\&< -E_{2k} < \cdots < -E_1 < Z_{2k+2}^{i-} < Z_{2k+2}^{i+}\\&< E_1 < Z_{2k+3}^{i-} < Z_{2k+3}^{i+} < E_2 < \cdots < E_{2k}\\&< Z_{4k+2}^{i-} < Z_{4k+2}^{i+} < E_{2k+1} < Z_{4k+3}^{i-} < Z_{4k+3}^{i+}.\end{aligned}$$

For example, when $k = 1$ and $E_s = 3s-2$, there exist Z_j^{i-}, $Z_j^{i+} \in \Re, j = 1, 2, \ldots, 7$, such that $\forall i \in \{1, 2, \ldots, n\}$

$$\begin{aligned}&Z_1^{i-} < Z_1^{i+} < -7 < Z_2^{i-} < Z_2^{i+} < -4 < Z_3^{i-}\\&< Z_3^{i+} < -1 < Z_4^{i-} < Z_4^{i+} < 1 < Z_5^{i-} < Z_5^{i+}\\&< 4 < Z_6^{i-} < Z_6^{i+} < 7 < Z_7^{i-} < Z_7^{i+}.\end{aligned}$$

Let

$$I_1 = \{i | i = 1, 2, 3, \ldots, n\}$$

$$I_2 = \{i | i = 1, 2, 3, \ldots, 4k+3\}$$

$$I_3 = \{i \in I_2 | i = 2s-1, s = 1, 2, 3, \ldots, 2k+2\}$$

$$I_4 = \{2k+2\}$$

$$I_5 = \{4k+4\}$$

$$\mathcal{D}_{i1} = \{[Z_j^{i-}, Z_j^{i+}] | \forall j \in I_3\}$$

$$\mathcal{D}_{i2} = \{[Z_{2k+2}^{i-}, Z_{2k+2}^{i+}]\}$$

$$\mathcal{D}_{i3} = \{[Z_j^{i-}, Z_j^{i+}] | \forall j \in I_2 - I_3 - I_4\}.$$

Then, $\mathcal{D}_{i1}$ is composed of $(2k+2)$ intervals; $\mathcal{D}_{i2}$ is composed of one interval; $\mathcal{D}_{i3}$ is composed of $2k$ intervals. Since $l_{j(i)}^i \in \mathcal{D}_{i1} \bigcup \mathcal{D}_{i2} \bigcup \mathcal{D}_{i3}$ is an one-dimensional interval, we obtain that for $\forall j(i) \in I_2 \bigcup I_5$

$$\prod_{i=1}^{n} l_{j(i)}^i = l_{j(1)}^1 \times l_{j(2)}^2 \times \cdots \times l_{j(n)}^n.$$

It is easy to see that any $\prod_{i=1}^{n} l_{j(i)}^i$ is a compact and convex.

2.2. Model

In this paper, we consider a general class of recurrent neural networks with time-varying delays as follows: $\forall i \in I_1$

$$\begin{aligned}\dot{x}_i(t) = &- x_i(t) + \sum_{j=1}^{n} a_{ij} f(x_j(t))\\&+ \sum_{j=1}^{n} b_{ij} f(x_j(t-\tau_j(t))) + u_i \qquad (1)\end{aligned}$$

where $x = (x_1, x_2, \ldots, x_n)^T \in \Re^n$ is the state vector; $A = [a_{ij}]$ and $B = [b_{ij}]$ are connection weight matrices that are not assumed to be symmetric; $u = (u_1, u_2, \ldots, u_n)^T \in \Re^n$ is an input vector; for $\forall t \geq t_0$, $\forall j \in \{1, 2, \ldots, n\}$, $\tau_j(t)$ with respect to τ satisfies $0 \leq \tau_j(t) \leq \tau = \max_{1\leq i\leq n}\{\sup\{\tau_i(t), t \geq t_0\}\}$. One activation function with $(k+1)$-stair is given by

$$f(x) = \sum_{s=1}^{N} \frac{(m_{s-1} - m_s)}{2} (|x + E_s| - |x - E_s|) \qquad (2)$$

where $N = 2k+1$, $E_s = 3s-2$, $m_0 = 1$, $m_s = 1 + (-1)^s$, $s = 1, 2, \ldots, 2k+1$. In particular, when $k = 0$ in (2), $f(x) = \frac{1}{2}(|x+1| - |x-1|)$ is the common saturated function.

In general, such neural network (1) not only represents the network with delays, but also indicates the network without delays. Denote RNN (1) with activation function (2) to RNN (1′).

2.3. Properties

The initial condition of RNN (1′) with activation function $f(x(t))$ is:

$$\phi(\vartheta) = (\phi_1(\vartheta), \phi_2(\vartheta), \ldots, \phi_n(\vartheta))^T \tag{3}$$

where $\phi(\vartheta) \in \mathcal{C}([t_0-\tau, t_0], \mathcal{D})$, $\mathcal{D} \subseteq \Re^n$. Let $x(t; x_0, \phi)$ stand for the solution of RNN (1′) with initial condition (3). Then $x(t; x_0, \phi)$ is continuous differential function and satisfies RNN (1′), and $x(t; x_0, \phi) = \phi(s)$, for all $s \in [t_0-\tau, t_0]$. Also let $x(t)$ be the solution of RNN (1′).

Definition 1. The equilibrium point x^* of RNN (1) is locally exponentially stable in region $\mathcal{D}$, if there exist constants $\alpha > 0, \beta > 0$ such that $\forall t \geq t_0$

$$\|x(t; t_0, \phi) - x^*\| \leq \beta\|\phi - x^*\| exp\{-\alpha(t-t_0)\}$$

where $x(t; x_0, \phi)$ is the solution of RNN (1) with any initial condition $\phi(\vartheta) \in \mathcal{C}([t_0-\tau, t_0], \mathcal{D})$, and $\mathcal{D}$ is a locally exponentially attractive set of the equilibrium point x^*. When $\mathcal{D} = \Re^n$, x^* is globally exponentially stable.

Definition 2. A set $\mathcal{D}$ is said to have the Brouwer fixed point x^* if and only if, for any continuous map $H : \mathcal{D} \to \mathcal{D}$, there exists at least $x^* \in \mathcal{D}$ such that $H(x^*) = x^*$, where $\mathcal{D}$ is a compact and convex subset of the Euclidean space $\Re^n$.

Let

$$G_i(x_i) = -x_i + (a_{ii} + b_{ii})f(x_i)$$

$$G_{1i} = \{x_i | G_i(x_i) = 0, x_i \in \Re\}$$

$$G_{2i} = \{x_i | G_i(x_i) + c_i = 0, |c_i| \leq \beta_i, x_i \in \Re\}.$$

If $c_i = 0$ in G_{2i}, it is easy to see that G_{2i} is simplified as G_{1i}. Without loss of generality, further assume that there are some ordered elements in G_{2i} such that

$$\begin{aligned} &Z^i_j \in G_{2i},\ j = 1, 2, \ldots, 4k+3 \\ &Z^i_1 < Z^i_2 <, \ldots, < Z^i_{j-1} < Z^i_j \\ &< Z^i_{j+1} <, \ldots, < Z^i_{4k+2} < Z^i_{4k+3} \end{aligned}$$

and

$$\begin{aligned} &Z^i_1 < -E_{2k+1} < Z^i_2 < -E_{2k} <, \ldots, < Z^i_{2k+1} \\ &< -E_1 < Z^i_{2k+2} < E_1 < Z^i_{2k+3} <, \ldots, < E_{2k} \\ &< Z^i_{4k+2} < E_{2k+1} < Z^i_{4k+3}. \end{aligned}$$

Define upper and lower functions, for $\forall\, i \in I_1$,

$$F_i(x) = -x_i + (a_{ii} + b_{ii})f(x_i) + \sum_{j=1, j\neq i}^{n} (a_{ij} + b_{ij})f(x_j) + u_i$$

$$F_{1i}(x_i) = -x_i + (a_{ii} + b_{ii})f(x_i) + \sum_{j=1, j\neq i}^{n} (a_{ij} + b_{ij})f(x_j) + u_i$$

$$F_{2i}(x_i) = -x_i + (a_{ii} + b_{ii})f(x_i) - Q_i$$

$$F_{3i}(x_i) = -x_i + (a_{ii} + b_{ii})f(x_i) + Q_i$$

$$Q_i = \sum_{j=1, j\neq i}^{n} (6k+1)(|a_{ij}| + |b_{ij}|) + |u_i|.$$

It is also clear that for any $x_i \in \Re$ and $i \in I_1$, $F_{2i}(x_i) \leq F_{1i}(x_i) \leq F_{3i}(x_i)$ holds. When Q_i satisfies some certain conditions, the following formulas are established for F_{2i} and F_{3i}

$$\begin{aligned} &F_{2i}(Z^{i-}_j) = 0,\ j = 1, 2, \ldots, 4k+3 \\ &Z^{i-}_1 < Z^{i-}_2 <, \ldots, < Z^{i-}_{j-1} < Z^{i-}_j \\ &< Z^{i-}_{j+1} <, \ldots, < Z^{i-}_{4k+2} < Z^{i-}_{4k+3} \end{aligned}$$

and

$$\begin{aligned} &F_{3i}(Z^{i+}_j) = 0,\ j = 1, 2, \ldots, 4k+3 \\ &Z^{i+}_1 < Z^{i+}_2 <, \ldots, < Z^{i+}_{j-1} < Z^{i+}_j \\ &< Z^{i+}_{j+1} <, \ldots, < Z^{i+}_{4k+2} < Z^{i+}_{4k+3}. \end{aligned}$$

Further inferences are presented

$$\begin{aligned} &Z^{i-}_1 < Z^{i+}_1 < -E_{2k+1} < Z^{i-}_2 < Z^{i+}_2 < -E_{2k} \\ &<, \ldots, < Z^{i-}_{2k+1} < Z^{i+}_{2k+1} < -E_1 < Z^{i-}_{2k+2} < Z^{i+}_{2k+2} \\ &< E_1 < Z^{i-}_{2k+3} < Z^{i+}_{2k+3} < E_2 <, \ldots, < E_{2k} \\ &< Z^{i-}_{4k+2} < Z^{i+}_{4k+2} < E_{2k+1} < Z^{i-}_{4k+3} < Z^{i+}_{4k+3}. \end{aligned}$$

Lemma 1. *For the given integer $k \geq 1$, if $\beta_i \leq Q_i$ and*

$$a_{ii} + b_{ii} < \frac{3}{4} - \frac{Q_i}{2} - |\frac{1}{4} - \frac{Q_i}{2}| \tag{4}$$

then $\mathrm{Card}(G_{2i}) = 1$.

Proof. From (4), $2(a_{ii} + b_{ii}) - 1 < \frac{3}{2} - Q_i - |\frac{1}{2} - Q_i| - 1 \leq 0$. Since $G_i(x_i) + c_i$ is strictly monotonously decreased for $x_i \in \Re$ as depicted in Fig. 1, together with $G_i(x_i) + c_i \to +\infty$, as $x_i \to -\infty$ and $G_i(x_i) + c_i \to -\infty$, as $x_i \to +\infty$, we can obtain $\mathrm{Card}(G_{2i}) = 1$.

Lemma 2. *If $I - |\tilde{C}| - |\tilde{D}|$ is a nonsingular M matrix, and*

$$\dot{x}_i(t) = -x_i(t) + \sum_{j=1}^{n} \tilde{c}_{ij}x_j(t) + \sum_{j=1}^{n} \tilde{d}_{ij}x_j(t - \tau_j(t)) \tag{*}$$

then (*) is locally exponentially stable, where $\forall i \in I_1$, $\tilde{C} = [\tilde{c}_{ij}]$ and $\tilde{D} = [\tilde{d}_{ij}]$ are connection weight matrices, $0 \leq \tau_j(t) \leq \tau$, $|\tilde{C}| = [|\tilde{c}_{ij}|]$ and $|\tilde{D}| = [|\tilde{d}_{ij}|]$.

Proof. See related corollary proof of Theorem 1 in Zeng et al. (2003).

When $a_{ii} + b_{ii} < \frac{3}{4} - \frac{Q_i}{2} - |\frac{1}{4} - \frac{Q_i}{2}|$, assume that Z^{i-}_{4k+4}, Z^{i}_{4k+4} and Z^{i+}_{4k+4} are respectively zero solution of the three equations $F_{2i} = 0$, $G_i(x_i) + c_i = 0$ and $F_{3i} = 0$, then $-1 < Z^{i-}_{4k+4} \leq Z^{i}_{4k+4} \leq Z^{i+}_{4k+4} < 1$. Denote

$$\mathcal{D}_{i4} = \{[Z^{i-}_{4k+4}, Z^{i+}_{4k+4}]\}$$

$$\begin{aligned} M_1 = \{i \in I_1 | a_{ii} + b_{ii} < \frac{3}{4} - \frac{Q_i}{2} - |\frac{1}{4} - \frac{Q_i}{2}| \\ \text{and} \quad a_{ii} \geq 0,\ b_{ii} \geq 0\} \end{aligned} \tag{5}$$

$$\begin{aligned} M_2 = \{i \in I_1 | Q_i < 3(a_{ii} + b_{ii} - 1)(1-k) \\ + \frac{1}{2}[3 - |(a_{ii} + b_{ii} - 1)(4-6k) + 3|], \\ \text{and} \quad Q_i < 1\,\} \end{aligned} \tag{6}$$

$$\begin{aligned} \Omega_1 = \{\prod_{i=1}^{n} l^i_{j(i)} | \, l^i_{j(i)} \in \mathcal{D}_{i1} \bigcup \mathcal{D}_{i4}, \\ j(i) \in I_3 \text{ when } i \in M_2, \\ \text{or } j(i) \in I_5 \text{ when } i \in M_1\} \end{aligned} \tag{7}$$

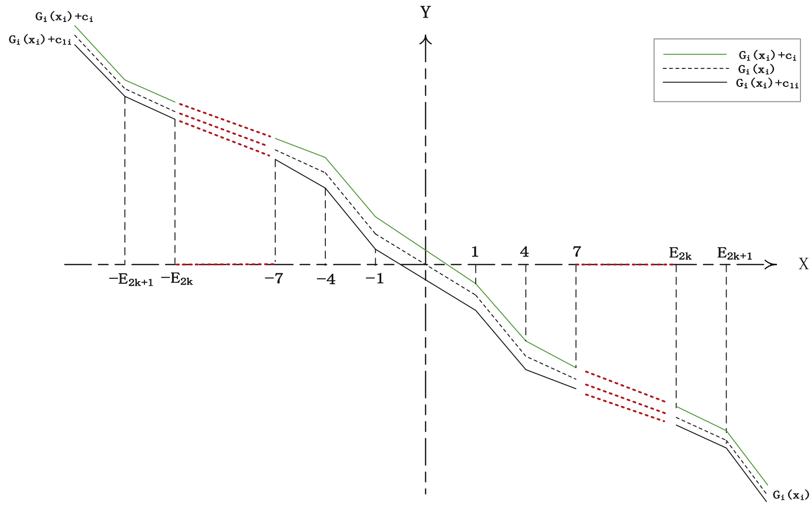

Fig. 1. Functions $G_i(x_i)$ and $G_i(x_i) + c_i$ in Lemma 1.

$$\Omega_2 = \{\prod_{i=1}^{n} l^i_{j(i)} |\, l^i_{j(i)} \in \mathcal{D}_{i1} \bigcup \mathcal{D}_{i2} \bigcup \mathcal{D}_{i3} \bigcup \mathcal{D}_{i4},$$
$$j(i) \in I_2 \text{ when } i \in M_2, \text{ or } j(i) \in I_5 \text{ when } i \in M_1\} \quad (8)$$

$$\Theta_1 = \bigcup_{i \in M_1} [Z^{i-}_{4k+4}, Z^{i+}_{4k+4}]$$

$$\Theta_2 = \bigcup_{j \in I_3} \bigcup_{i \in M_2} [Z^{i-}_{j}, Z^{i+}_{j}]$$

$$\Theta_3 = \bigcup_{j \in I_2 - I_3 - I_4} \bigcup_{i \in M_2} [Z^{i-}_{j}, Z^{i+}_{j}]$$

$$\Theta_4 = \bigcup_{i \in M_2} [Z^{i-}_{2k+2}, Z^{i+}_{2k+2}]$$

When $i \in M_1$, then $\mathcal{D}_{i1} = \mathcal{D}_{i2} = \mathcal{D}_{i3} = \emptyset$; when $i \in M_2$, then $\mathcal{D}_{i4} = \emptyset$. When $i \in M_1$, from the definition of M_1, $a_{ii} + b_{ii} < \frac{3}{4} - \frac{Q_i}{2} - |\frac{1}{4} - \frac{Q_i}{2}| \leq \frac{1}{2}$. When $i \in M_2$, from the definition of M_2, $a_{ii} + b_{ii} > 1 + Q_i \geq 1$. Hence, it is easy to verify $M_1 \bigcap M_2 = \emptyset$.

3. Main results

In this part, some new criteria are obtained and we finally present that an n-neuronal RNN can obtain $(2k+2)^{k_0}$ stable equilibrium points, and $(4k+3)^{k_0} - (2k+2)^{k_0}$ equilibria are unstable. The following three theorems are obtained: (i) Theorem 1 addresses whether there is equilibrium point; (ii) Theorem 2 presents new theoretical result about the exact number of equilibria; (iii) Theorem 3 provides analysis of stability and instability.

3.1. Existence of equilibria in regions of Ω_2

Theorem 1. *For the given integer $k \geq 1$, if $M_1 \bigcup M_2 = I_1$ and $\text{Card}(M_2) = k_0$, then for any region $\Lambda \in \Omega_2$, there exists at least one equilibrium point of RNN (1′) located in Λ.*

Proof. Consider a compact region $\Lambda = \prod_{i=1}^{n} l^i_{j(i)} \in \Omega_2$. Obviously, there are $(4k+3)^{k_0}$ elements Λ in Ω_2, and each element Λ is a subregion in $\Re^n$. For any given $(\zeta_1, \zeta_2, \ldots, \zeta_n) \in \Lambda$, x_i satisfies the following

$$\bar{F}_{1i}(x_i) = 0 \quad (9)$$
$$F_{2i}(x_i) = 0$$
$$F_{3i}(x_i) = 0$$

where $\bar{F}_{1i}(x_i) = -x_i + (a_{ii} + b_{ii})f(x_i) + J_i$, and $J_i = \sum_{j=1, j\neq i}^{n}(a_{ij} + b_{ij})f(\zeta_j) + \mu_i$, $\forall i \in M_1 \bigcup M_2$. Note that each $\bar{F}_{1i}$ is a vertical shift of $-x_i + (a_{ii} + b_{ii})f(x_i)$ and lies between F_{2i} and F_{3i}, where $-Q_i \leq J_i \leq Q_i$. Thus, for $i \in M_1$, there exists one solution of (9), which lies in region $[Z^{i-}_{4k+4}, Z^{i+}_{4k+4}]$. For $i \in M_2$, we can always find $4k + 3$ solutions of (9), which lie in regions $[Z^{i-}_{1}, Z^{i+}_{1}], [Z^{i-}_{2}, Z^{i+}_{2}], \ldots, [Z^{i-}_{4k+2}, Z^{i+}_{4k+2}]$ and $[Z^{i-}_{4k+3}, Z^{i+}_{4k+3}]$, respectively. Next, consider a mapping

$$H : \Lambda \to \Lambda$$

defined by

$$H(\zeta_1, \zeta_2, \ldots, \zeta_n) = (\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n)^T.$$

where $\hat{x}_i$ can be defined by $(\bar{F}_{1i}|_{l^i_{j(i)}})^{-1}(0)$. If $i \in M_1$, then $j(i) \in I_5$; if $i \in M_2$, then $j(i) \in I_2$. That is, $\hat{x}_i$ is the solution of (9) lying in $[Z^{i-}_{\sigma}, Z^{i+}_{\sigma}]$, $\sigma \in I_2 \bigcup I_5$. The defined mapping H is continuous, since the function f is continuous. It follows from the Brouwer fixed point definition that there exists one fixed point $x^* = (x^*_1, x^*_2, \ldots, x^*_n)$ of H in Λ, which is also a zero of the function $F(x) = (F_1(x), F_2(x), \ldots, F_n(x))^T$, where $F_i(x)$ is defined as above. Consequently, there exist at least $(4k+3)^{k_0}$ equilibria for RNN (1′) and each of the $(4k+3)^{k_0}$ different subregions Λ contains at least one of the equilibria.

3.2. Exact number of equilibria in regions of Ω_2

Theorem 2. *For the given integer $k \geq 1$, if $M_1 \bigcup M_2 = I_1$ and $\text{Card}(M_2) = k_0$, then there exist exactly $(4k+3)^{k_0}$ equilibria in RNN (1′), and each region $\Lambda \in \Omega_2$ contains exactly one of these $(4k+3)^{k_0}$ equilibria.*

Proof. According to Theorem 1, there exists at least an equilibrium point in $\Lambda = \prod_{i=1}^{n} l^i_{j(i)} \in \Omega_2$. Without loss of generality, there exists an equilibrium point $x^* = (x^*_1, x^*_2, \ldots, x^*_n)^T \in \Lambda$ in RNN (1′) such that

$$-x^*_i - \sum_{j=1, j\neq i}^{n} (a_{ij} + b_{ij})f(x^*_j) + u_i = 0. \quad (10)$$

(10) is equivalent to

$$-Ix+(A+B)G(x)+u=0 \tag{11}$$

and (11) has at least one solution in Λ, where $G(x) = (f(x_1), f(x_2), \dots, f(x_n))^T$. When Λ is given, then for all $s \in I_1$, each $l^{(s)}_{j(s)}$ is also determined. It follows from $f(x_i)$ that linear function with respect to one variable in each $l^{(s)}_{j(s)}$. (11) can be converted to the following form

$$\hat{A}x+\hat{b}=0 \tag{12}$$

and (12) has at least one solution in Λ, where $\hat{A} \in \Re^{n\times n}$, $\hat{b} \in \Re^{n\times 1}$. We will prove that the matrix $\hat{A}$ is invertible for a variety of $G(x)$ situations, so that a uniqueness equilibrium point can be derived in Eq. (12). Notice that if $\hat{A} = (\hat{a}^T_1, \hat{a}^T_2, \dots, \hat{a}^T_n)^T$ is invertible, then the matrix $(\hat{a}^T_{\sigma(1)}, \hat{a}^T_{\sigma(2)}, \dots, \hat{a}^T_{\sigma(n)})^T$ is also invertible, where $\{\sigma(1), \sigma(2), \dots, \sigma(n)\} = I_1$, $\hat{a}_s$, $\hat{a}_{\sigma(s)} \in \Re^{1\times n}$ and $s \in I_1$.

When $i \in M_1$,

$$\begin{aligned}|a_{ii}+b_{ii}-1| &= -a_{ii}-b_{ii}+1\\ &> 1-\frac{3}{4}+\frac{Q_i}{2}+|\frac{1}{4}-\frac{Q_i}{2}|\\ &\ge Q_i\\ &\ge 2\sum_{j=1,j\neq i}^{n}(|a_{ij}|+|b_{ij}|)\\ &\ge \sum_{j=1,j\neq i}^{n}(|a_{ij}|+|b_{ij}|).\end{aligned}$$

If taking $G(X) = (x_1, x_2, \dots, x_n)^T$, $\hat{A} = A+B-I$, then $\hat{A}$ herein is also the strictly diagonally dominant matrix. Hence, the matrix $\hat{A}$ is the invertible matrix.

When $i \in M_2$,

$$\begin{aligned}a_{ii}+b_{ii}-1 &\ge 3(a_{ii}+b_{ii}-1)(1-k)\\ &\quad+\frac{1}{2}[3-|(a_{ij}+b_{ij}-1)(4-6k)+3|]\\ &> Q_i\\ &\ge 2\sum_{j=1,j\neq i}^{n}(|a_{ij}|+|b_{ij}|)\\ &\ge \sum_{j=1,j\neq i}^{n}(|a_{ij}|+|b_{ij}|);\\ 2a_{ii}+2b_{ii}-1 &> Q_i\\ &\ge 2\sum_{j=1,j\neq i}^{n}(|a_{ij}|+|b_{ij}|)\\ &\ge \sum_{j=1,j\neq i}^{n}(|a_{ij}|+|b_{ij}|);\\ |-1| = 1 &> Q_i\\ &\ge 2\sum_{j=1,j\neq i}^{n}(|a_{ij}|+|b_{ij}|)\\ &\ge \sum_{j=1,j\neq i}^{n}(|a_{ij}|+|b_{ij}|).\end{aligned}$$

If taking $G(X) = (x_1, x_2, \dots, x_n)^T$, $\hat{A} = A+B-I$, then $\hat{A}$ is the strictly diagonally dominant matrix. If taking $G(X) = (2x_1+\bar{c}_1, 2x_2+\bar{c}_2, \dots, 2x_n+\bar{c}_n)^T$, $\hat{A} = 2(A+B)-I$, then $\hat{A}$ is the strictly diagonally dominant matrix, where $\bar{c}_i \in \{\pm(6j-5)|j = 2, 3, \dots, k+1\}$. If taking $G(X) = (\tilde{c}_1, \tilde{c}_2, \dots, \tilde{c}_n)^T$, we can have $\hat{A} = -I$, where $\tilde{c}_i \in \{\pm(6j-5)|j = 1, 2, \dots, k+1\}$. Obviously, $\hat{A}$ herein is the strictly diagonally dominant matrix. Hence, in three cases, the matrix $\hat{A}$ is reversible.

Notice that when $\hat{A}$ is an invertible matrix, m_1 principal minor determinant of $\hat{A}$ (it is expressed as $\hat{A}(\sigma(1), \sigma(2), \dots, \sigma(m_1))$, $\sigma(i) \in I_1$, $\sigma(1) < \sigma(2) <, \dots, < \sigma(m_1)$) is also an invertible matrix, where m_1 is a integer such that $m_1 \le n$.

When $i \in M_1 \bigcup M_2$, without loss of generality, assume that there exist positive integers m_1 and m_2 such that $G(X) = (x_1, x_2, \dots, x_{m_1}, 2x_{m_1+1}+\check{c}_{m_1+1}, 2x_{m_1+2}+\check{c}_{m_1+2}, \dots, 2x_{m_1+m_2}+\check{c}_{m_1+m_2}, \check{c}_{m_1+m_2+1}, \check{c}_{m_1+m_2+2}, \dots, \check{c}_n)^T$, where $\check{c}_{m_1+1}, \check{c}_{m_1+2}, \dots, \check{c}_n \in \{\pm(6j-5)|j = 1, 2, \dots, k+1\}$. Notice that when $m_1 = 0$, we can have $\hat{A} = -I$; when $m_1 = n$, we also get $\hat{A} = A+B-I$. We can obtain $\hat{A} = -I + (\hat{a}^T_1, \hat{a}^T_2, \dots, \hat{a}^T_{m_1}, 2\hat{a}^T_{m_1+1}, 2\hat{a}^T_{m_1+2}, \dots, 2\hat{a}^T_{m_2}, 0^T, 0^T, \dots, 0^T)^T$, where $0^T \in \Re^{n\times 1}$. Due to $det(\hat{A}) = (-1)^{n-m_2-m_1} det\hat{A}(1, 2, \dots, m_1+m_2) \neq 0$, thus the matrix $\hat{A}$ is reversible. This completes the proof.

Remark 1. In Theorems 1 and 2, it is proved the coexistence of multiple equilibrium points and the accurate number of equilibria. For any compact region $\Lambda \in \Omega_2$ in Theorems 1 and 2, the area $\bigcup_{\Lambda\in\Omega_2}\Lambda$ containing multiple equilibria can be continuation to almost the whole phase space $\Re^n$, in which the accurate number of equilibria remains the same. In comparison with Zeng et al. (2010) and Zeng and Zheng (2013), by linearly partitioning state space, selecting the division of region is more precise and more flexible. In addition, Theorem 2 also avoids using contraction mapping theory to derive the accurate number of equilibria in regions of Ω_2.

3.3. Stability and instability analysis of equilibria

Theorem 3. *For the given integer $k \ge 1$, if $M_1 \bigcup M_2 = I_1$ and $\mathrm{Card}(M_2) = k_0$, then there exist exactly $(2k+2)^{k_0}$ exponentially stable equilibria for RNN (1′), and $(4k+3)^{k_0} - (2k+2)^{k_0}$ equilibria are unstable, which are respectively located in the regions of $\Omega_2 \setminus \Omega_1$.*

Proof. Case (I): each region $\Lambda \in \Omega_1$ contains exactly one of these $(2k+2)^{k_0}$ exponentially stable equilibria.

According to Theorem 2, for RNN (1′), there exists a unique equilibrium point $x^* = (x^*_1, x^*_2, \dots, x^*_n)^T \in \Omega_1$. Let $z_i(t) = x_i(t) - x^*_i$. So when $x(t)$, $x(t-\tau(t))$, and $x^* \in \Omega_1$,

$$\begin{aligned}\frac{dz_i(t)}{dt} &= -z_i(t)+\sum_{j=1}^{n}a_{ij}f'(y_j)z_j(t)+\sum_{j=1}^{n}b_{ij}f'(\bar{y}_j)z_j(t-\tau_j(t))\\ &= -z_i(t)+\sum_{j\in M_1\bigcup M_2}a_{ij}f'(y_j)z_j(t)\\ &\quad+\sum_{j\in M_1\bigcup M_2}b_{ij}f'(\bar{y}_j)z_j(t-\tau_j(t))\end{aligned} \tag{13}$$

$$f'(y_j) = \begin{cases}1, & y_j \in \Theta_1\\ 0, & y_j \in \Theta_2\end{cases} \quad f'(\bar{y}_j) = \begin{cases}1, & \bar{y}_j \in \Theta_1\\ 0, & \bar{y}_j \in \Theta_2\end{cases}$$

where y_j is some number between x^*_j and $x_j(t)$, and $\bar{y}_j$ is some number between x^*_j and $x_j(t-\tau_j(t))$. It can be converted into

$$\frac{dz_i(t)}{dt} = -z_i(t)+\sum_{j\in M_1}a_{ij}z_i(t)+\sum_{j\in M_1}b_{ij}z_j(t-\tau_j(t)) \tag{14}$$

Herein, for $M_1 \subset I_1$, there are 2^n kinds of options in I_1. Without loss of generality, we only consider $M_1 = \{1, 2, \dots, m_1\}$ and $M_2 = \{m_1+1, m_1+2, \dots, n\}$. Herein, the mathematical form of (14) is represented by

$$\dot{z}(t) = -Iz(t)+\tilde{A}z(t)+\tilde{B}z(t-\tau(t)) \tag{15}$$

where $\tilde{A} = [C_{nm_1}, 0_{n(n-m_1)}]$, $\tilde{B} = [\bar{C}_{nm_1}, 0_{n(n-m_1)}]$

$$C_{nm_1} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m_1} \\ a_{21} & a_{22} & \cdots & a_{2m_1} \\ a_{31} & a_{32} & \cdots & a_{3m_1} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm_1} \end{bmatrix}$$

$$\bar{C}_{nm_1} = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1m_1} \\ b_{21} & b_{22} & \cdots & b_{2m_1} \\ b_{31} & b_{32} & \cdots & b_{3m_1} \\ \vdots & \vdots & \vdots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nm_1} \end{bmatrix}$$

$$0_{n(n-m_1)} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

We will assert that $D := I - |\tilde{A}| - |\tilde{B}| = [d_{ij}]$ is a nonsingular M matrix.

(i) $d_{ii} > 0$ and $d_{ij} \le 0 (i \ne j)$ for $\forall i, j \in I_1$ need to be validated.

When $i \in M_1$,

$$\begin{aligned} 1 - |a_{ii}| - |b_{ii}| &> 1 - \frac{3}{4} + \frac{Q_i}{2} + |\frac{1}{4} - \frac{Q_i}{2}| \\ &\ge 0. \end{aligned}$$

Thus

$$d_{ii} = \begin{cases} 1 - |a_{ii}| - |b_{ii}|, & i \in M_1 \\ 1, & i \in M_2 \end{cases}$$

$$d_{ij} = \begin{cases} -|a_{ij}| - |b_{ij}|, & i \in M_1 \setminus \{j\} \\ 0, & i \in M_2 \setminus \{j\} \end{cases}$$

Hence, $d_{ii} > 0$ and $d_{ij} \le 0 (i \ne j)$.

(ii) That $-D$ is a Hurwitz matrix which need to be validated.

We will claim that $Re\lambda_i(|\tilde{A}| + |\tilde{B}| - I) < 0$, where $\lambda_i(|\tilde{A}| + |\tilde{B}| - I)$ is the eigenvalue of matrix $|\tilde{A}| + |\tilde{B}| - I$. Let

$$r_i = \sum_{j=1, j \ne i}^{n} (|a_{ij}| + |b_{ij}|) = \sum_{j \in M_1 \setminus \{i\}} (|a_{ij}| + |b_{ij}|)$$

According to Gerschgorin's theorem,

$$\lambda_i(|\tilde{A}| + |\tilde{B}| - I) \in \bigcup_{i \in M_1} B(|a_{ii}| + |b_{ii}| - 1, r_i) \bigcup_{i \in M_2} B(-1, r_i)$$

for all $i \in M_1 \bigcup M_2$, where $B(|a_{ii}| + |b_{ii}| - 1, r_i) := \{\zeta \in \mathbb{C} |\ |\zeta - |a_{ii}| - |b_{ii}| + 1| < r_i\}$ and $B(-1, r_i) := \{\zeta \in \mathbb{C} |\ |\zeta + 1| < r_i\}$.

When $i \in M_1$,

$$\begin{aligned} |a_{ii}| + |b_{ii}| - 1 &= a_{ii} + b_{ii} - 1 \\ &< -\frac{1}{4} - \frac{Q_i}{2} - |\frac{1}{4} - \frac{Q_i}{2}| \\ &\le -Q_i \\ &\le -\sum_{j \in M_1 \setminus \{i\}} (|a_{ij}| + |b_{ij}|); \end{aligned}$$

when $i \in M_2$,

$$-1 < -Q_i \le -\sum_{j \in M_1 \setminus \{i\}} (|a_{ij}| + |b_{ij}|).$$

Hence, for each $i \in M_1 \bigcup M_2$, there exists i_0 such that

$$Re\lambda_i(|\tilde{A}| + |\tilde{B}| - I) < |a_{i_0 j}| + |b_{i_0 j}| - 1 + r_{i_0} < 0$$

or

$$Re\lambda_i(|\tilde{A}| + |\tilde{B}| - I) < -1 + r_{i_0} < 0.$$

It follows that $Re\lambda_i(|\tilde{A}| + |\tilde{B}| - I) < 0$. Thus, for $i \in M_1 \bigcup M_2$, all complex eigenvalues of $|\tilde{A}| + |\tilde{B}| - I$ have negative real part, and $-D$ is a Hurwitz matrix.

From Lemma 2, equilibria in Λ is locally exponentially stable. Therefore, there are $(2k+2)^{k_0}$ exponentially stable equilibria for RNN (1′).

Case (II), Each equilibria in region $\Lambda \in \Omega_2 \setminus \Omega_1$ is unable. That is, there are $(4k+3)^{k_0} - (2k+2)^{k_0}$ equilibria that are unstable.

If $M_2 = \emptyset$, it is easy to see that zero equilibria is unstable. We assume that $M_2 \ne \emptyset$ in the following proof. Take $\forall \Lambda = \prod_{i=1}^{n} l_{j(i)}^{i} \in \Omega_2 \setminus \Omega_1$, then there exists $\sigma \in M_2$ such that $j(\sigma) \in I_2 \setminus I_3$. Otherwise, it will be in contradiction with $\Lambda \in \Omega_1$. Herein, $\sigma \in M_2$, it follows that

$$\begin{aligned} a_{\sigma\sigma} + b_{\sigma\sigma} - 1 - Q_\sigma >& a_{\sigma\sigma} + b_{\sigma\sigma} - 1 - 3(a_{\sigma\sigma} + b_{\sigma\sigma} - 1)(1-k) \\ &- \frac{1}{2}\{3 - |(a_{\sigma\sigma} + b_{\sigma\sigma} - 1)(4 - 6k) + 3|\} \\ =& (a_{\sigma\sigma} + b_{\sigma\sigma} - 1)(3k - 2) - \frac{3}{2} \\ &+ |(a_{\sigma\sigma} + b_{\sigma\sigma} - 1)(2 - 3k) + \frac{3}{2}| \\ \ge& 0. \end{aligned}$$

Without loss of generality, let $x^* \in \Lambda$ be only one isolated equilibrium point, and initial condition ϕ of RNN (1) locates $\prod_{i=1}^{n} l_{j(i)}^{i} \bigcap B(x^*, r)$, where $B(x^*)$ is a neighborhood of x^*. Denote

$$l_{j(\sigma)}^{\sigma} \in \{[Z_j^{\sigma-}, Z_j^{\sigma+}] | j \in I_2 \setminus I_3\}$$

$$\Theta = \left\{ \begin{array}{c} [E_{2s}, E_{2s+1}],\ [-E_1, E_1], \\ [-E_{2s+1}, -E_{2s}], \\ s = 1, 2, \ldots, k \end{array} \right\}$$

then there exists $\upsilon \in \Theta$ such that $l_{j(\sigma)}^{\sigma} \subset \upsilon$. Hence, it is obvious that

$$0 < \mathcal{U}(l_{j(\sigma)}^{\sigma}) \le \begin{cases} 2, & l_{j(\sigma)}^{\sigma} \subset [-E_1, E_1] \\ 3, & l_{j(\sigma)}^{\sigma} \subset \upsilon \in \Theta \setminus \{[-E_1, E_1]\} \end{cases}$$

where $\mathcal{U}$ represents Lebesgue measure. Assume that equilibrium point $x^* = (x_1^*, x_2^*, \ldots, x_\sigma^*, \ldots, x_n^*)^T \in \Lambda$ is stable. For $x(t), x(t - \tau(t))$, and $x^* \in \Lambda \bigcap B(x^*, r)$, $t \ge t_0 - \tau$, let $z_i(t) = x_i(t) - x^*$, then

$$\frac{dz_i(t)}{dt} = -z_i(t) + \sum_{j=1}^{n} a_{ij} f'(y_j) z_j(t) + \sum_{j=1}^{n} b_{ij} f'(\bar{y}_j) z_j(t - \tau_{ij}(t))$$

$$f'(y_j) = \begin{cases} 1, & y_j \in \Theta_1 \bigcup \Theta_4 \\ 0, & y_j \in \Theta_2 \\ 2, & y_j \in \Theta_3 \end{cases}$$

$$f'(\bar{y}_j) = \begin{cases} 1, & \bar{y}_j \in \Theta_1 \bigcup \Theta_4 \\ 0, & \bar{y}_j \in \Theta_2 \\ 2, & \bar{y}_j \in \Theta_3 \end{cases}$$

where y_j is some number between x_j^* and $x_j(t)$, and $\bar{y}_j$ is some number between x_j^* and $x_j(t - \tau_j(t))$.

Let $\varepsilon(t) = \max_{i \in I_1}(\max_{t_0 - \tau \le s \le t} |x_i(s; t_0, \phi) - x_i^*| / \nu_i)$, $\nu_i \in \Re$ and $\varepsilon(t) \le \mathcal{U}(l_{j(\sigma)}^{\sigma})$. Without loss of generality, taking $z_\sigma(\tilde{t}) = \varepsilon(t)$, when $l_{j(\sigma)}^{\sigma} \subset [-E_1, E_1]$

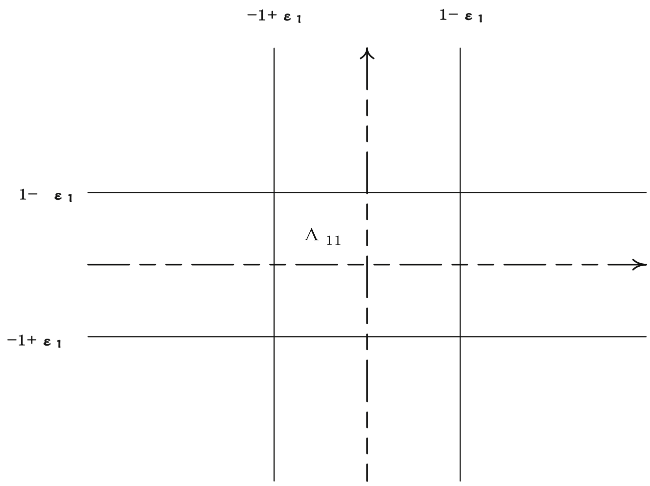

Fig. 2. Ω_2^- is composed of 1 part, when $M_1 = \{1, 2\}, M_2 = \emptyset$ and $k_0 = 0$.

$$
\begin{aligned}
\frac{dz_\sigma(t)}{dt}|_{t=\tilde{t}} &= -z_\sigma(\tilde{t}) + \sum_{j=1}^{n} a_{\sigma j} f'(y_j) z_j(\tilde{t}) \\
&\quad + \sum_{j=1}^{n} b_{\sigma j} f'(\bar{y}_j) z_j(\tilde{t} - \tau_j(t)) \\
&\geq \{a_{\sigma j} + b_{\sigma j} - 1 - 2 \sum_{j=1, j\neq\sigma}^{n} (|a_{\sigma j}| + |b_{\sigma j}|)\} \varepsilon(t) \\
&\geq \{a_{\sigma j} + b_{\sigma j} - 1 - Q_\sigma\} \varepsilon(t) \\
&> 0;
\end{aligned}
$$

when $l_{j(\sigma)}^{\sigma} \subset [-E_{2s+1}, -E_{2s}]$ or $[E_{2s}, E_{2s+1}]$, for $\forall s = 1, 2, \ldots, k$

$$
\begin{aligned}
\frac{dz_\sigma(t)}{dt}|_{t=\tilde{t}} &\geq \{2a_{\sigma j} + 2b_{\sigma j} - 1 - 2 \sum_{j=1, j\neq\sigma}^{n} (|a_{\sigma j}| + |b_{\sigma j}|)\} \times \varepsilon(t) \\
&\geq \{a_{\sigma j} + b_{\sigma j} - 1 - Q_\sigma\} \varepsilon(t) \\
&> 0.
\end{aligned}
$$

According to definition of $\varepsilon(t)$, $\varepsilon(t)$ is strictly monotone increasing and unbounded function, which is in contradiction with $\varepsilon(t) \leq \mathcal{U}(l_{j(\sigma)}^{\sigma})$. So this assumption is not satisfied. That is, equilibria in Λ is unstable. Therefore, there are $(4k+3)^{k_0} - (2k+2)^{k_0}$ unstable equilibria. The proof is completed.

Remark 2. It is proved not only that some equilibrium points are locally stable in Theorem 3, but also that the other equilibria are unstable. When $m = 2k+1$, $(-1)^{s+1}\mu_s = \frac{m_{s-1}-m_s}{2}$, $z_s = E_s$, $k_0 = n$, $f(x) = \sum_{s=1}^{m} (-1)^{s+1} \mu_s (|x + z_s| - |x - z_s|)$. In Zeng and Zheng (2013), it was shown that an n-neuronal neural network can obtain $(2m+1)^n$ equilibria in saturation regions and these equilibria are locally exponentially attractive. Therefore, Theorem 3 improves and extends the existing result of multistability.

Remark 3. Note that the sets M_1 and M_2 involve a lot of conditions on parameters. By the coupled division, the n-neuronal neural network improves the range of the regions of parameter, and the coupled division allows the division of space to be more diverse. Although the result of the design scheme is more generally in mathematical form, there are some difficulties in the construction sets and some conditions on parameters should be verified strictly in some applications. In what follows, according to Theorem 3, several simple corollaries are obtained. These corollaries are derived by selecting special parameters and these are also easy to verify.

Corollary 1. *If $M_2 = I_1$, then there are $(4k+3)^n$ equilibria and $(2k+2)^n$ of them are locally exponentially stable in RNN (1′).*

Corollary 2. *If $M_2 = \emptyset$, then there is only one equilibrium point and it is also locally exponentially stable in RNN (1′).*

Remark 4. By contrast with Cheng et al. (2015), the activation function in the paper is obviously nonsmooth function, and there are more equilibria in RNN (1′). The different regions of parameter are given by means of coupling division, and we allow regions of parameter to have more options. As a result, the dynamic behavior of multiple equilibria is more abundant. In particular, the multiple attractors also enhance the ability of associative memory of the networks.

4. Numerical example

In this section, we will give one example to illustrate the effectiveness of our results.

Choosing the continuation of intervals and areas, we can structure sets as follows

$$
\begin{aligned}
\mathcal{D}_{i1}^- &= \{[-\frac{1}{\epsilon}, -7-\epsilon],\ [-4+\epsilon, -1-\epsilon], \\
&\quad [7+\epsilon, \frac{1}{\epsilon}],\ [1+\epsilon, 4-\epsilon]\} \\
\mathcal{D}_{i2}^- &= \{[-1+\epsilon, 1-\epsilon]\}
\end{aligned}
$$

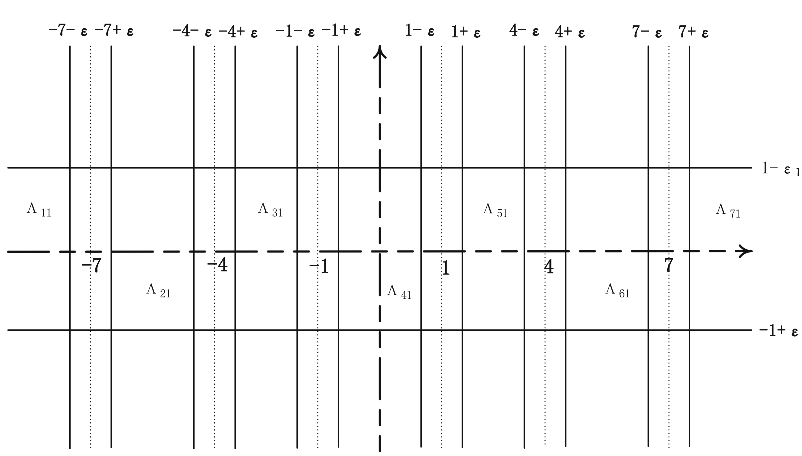

Fig. 3. Ω_2^- is composed of 7 parts, when $M_1 = \{2\}, M_2 = \{1\}$ and $k_0 = 1$.

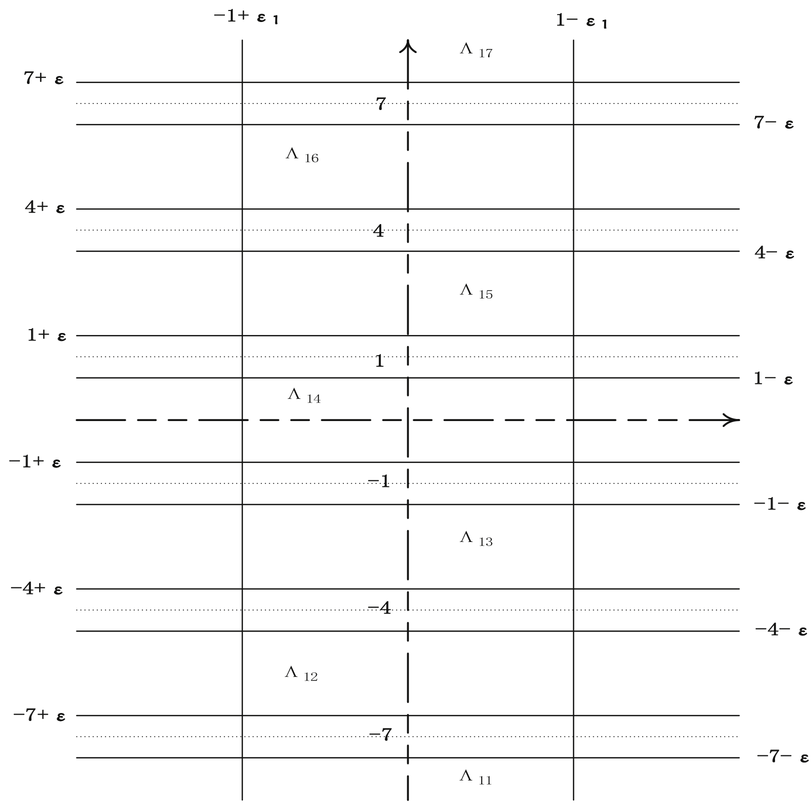

Fig. 4. Ω_2^- is composed of 7 parts, when $M_1 = \{1\}$, $M_2 = \{2\}$ and $k_0 = 1$.

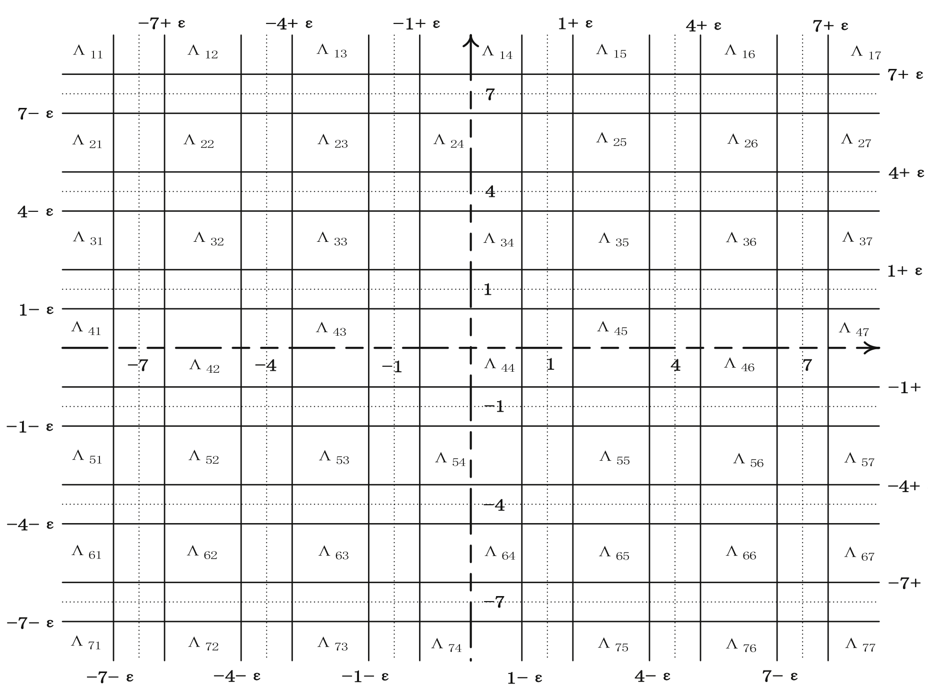

Fig. 5. Ω_2^- is composed of 7^2 parts, when $M_1 = \emptyset$, $M_2 = \{1, 2\}$ and $k_0 = 2$.

$$\mathcal{D}_{i3}^- = \{[4+\epsilon, 7-\epsilon],\ [-7+\epsilon, -4-\epsilon]\}$$

$$\mathcal{D}_{i4}^- = \{[-1+\epsilon_1, 1-\epsilon_1]\}$$

then

$$\Omega_1^- = \{\prod_{i=1}^{2} l_{j(i)}^i | l_{j(i)}^i \in \mathcal{D}_{i1}^- \bigcup \mathcal{D}_{i4}^-,$$

$$j(i) = 1, 3, 5 \text{ or } 7 \text{ when } i = 2$$

$$j(i) = 8 \text{ when } i = 1\}$$

$$\Omega_2^- = \{\prod_{i=1}^{2} l_{j(i)}^i | l_{j(i)}^i \in \mathcal{D}_{i1}^- \bigcup \mathcal{D}_{i2}^- \bigcup \mathcal{D}_{i3}^- \bigcup \mathcal{D}_{i4}^-,$$

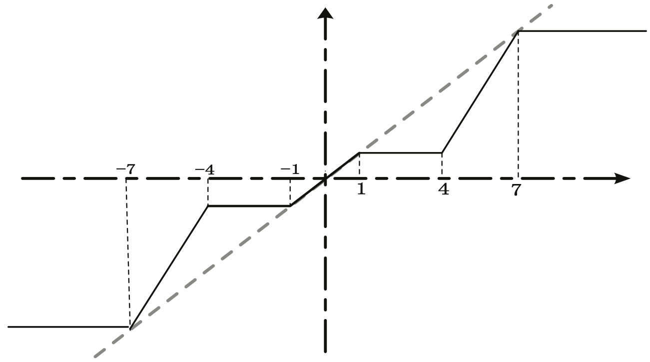

Fig. 6. When $k = 1$, $f(x)$ is plotted in the real line.

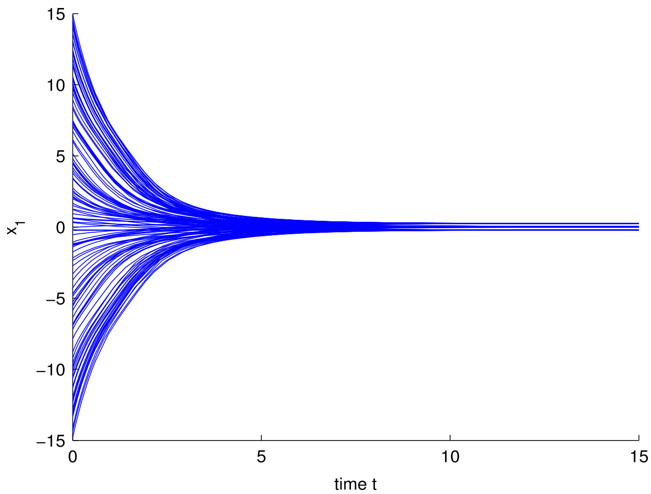

Fig. 7. State trajectories of $x_1(t)$, when $M_1 = \{1\}$, $M_2 = \{2\}$ and $k_0 = 1$.

$$j(i) = 1, 2, 3, 4, 5, 6, 7 \text{ when } i = 2$$
$$j(i) = 8 \text{ when } i = 1\}$$

where $0 < \epsilon, \epsilon_1 \ll \frac{1}{7}$, ϵ and ϵ_1 are small enough.

When $k = 1$ in the definition of $f(x)$, it is obvious that Ω_2^- is composed of 7^{k_0} parts, meanwhile, all parts of Ω_2^- are plotted in plane, shown in Figs. 2–5.

Example 1. Consider the following 2-neuronal recurrent neural network with activation function (2)

$$\begin{cases} \dot{x}_1 = -x_1(t) + 0.3f(x_1(t)) + 0.1f(x_1(t-0.1)) \\ \qquad -0.02f(x_2(t-0.2)) + 0.04f(x_2(t)) + 0.01 \\ \dot{x}_2 = -x_2(t) + f(x_2(t)) + 0.8f(x_2(t-0.2)) \\ \qquad -0.02f(x_1(t-0.1)) + 0.07f(x_1(t)) - 0.14 \end{cases} \quad (16)$$

where $k = 1$, $N = 3$, $s \in \{1, 2, 3\}$, $E_1 = 1$, $E_2 = 4$ and $E_3 = 7$.

As shown in Fig. 6, the activation function $f(x)$ is defined as follows:

$$f(x) = \begin{cases} 7, & r \in [7, +\infty) \\ 2x-7, & r \in [4, 7), \\ 1 & r \in [1, 4), \\ r, & r \in (-1, 1), \\ -1, & r \in (-4, 1], \\ 2x+7, & r \in (-7, -4], \\ -7, & r \in (-\infty, 3-4k]. \end{cases}$$

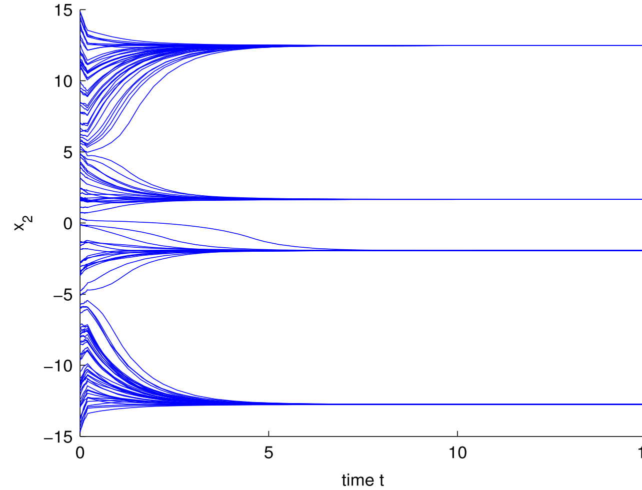

Fig. 8. State trajectories of $x_2(t)$, when $M_1 = \{1\}$, $M_2 = \{2\}$ and $k_0 = 1$.

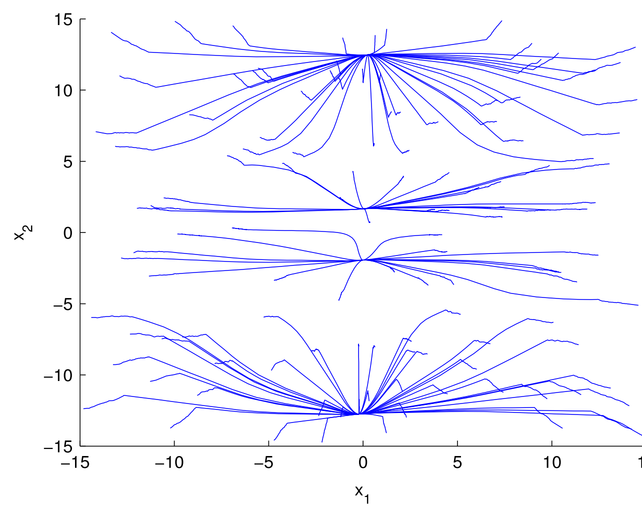

Fig. 9. Phase plots of $x_1(t)$ and $x_2(t)$, when $M_1 = \{1\}$, $M_2 = \{2\}$ and $k_0 = 1$.

Obviously, it is nonmonotonic. According to Theorems 2 and 3, when $M_1 = \{1\}$, $M_2 = \{2\}$ and $k_0 = 1$, neural network (16) has 7 equilibrium points in Ω_2^- and 4 of them are locally exponentially stable in Ω_1^-. As plotted in Figs. 7–9, the trajectories of (16) with 120 random initial values are presented.

5. Conclusion

In this paper, we have discussed the multistability and instability issue of delayed recurrent neural networks. By the division of state space and the dimensional space reconstruction, some sufficient criteria have been established to ensure the existence of $(2k+2)^{k_0}$ locally exponentially stable equilibria, and $(4k+3)^{k_0} - (2k+2)^{k_0}$ equilibria are unstable, where k_0 is a nonnegative integer such that $k_0 \leq n$. These new criteria improve and extend the existing results of multistability in the literature. By means of coupling division, it also reveals that the different regions of parameter are influenced by division, and these regions of parameter are allowed to have more options, in which the dynamic behaviors are more abundant. Finally, a numerical simulation is conducted to illustrate the derived theoretical results.

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