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Optimal reinsurance and investment strategies for insurer under interest rate and inflation risks

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HIGHLIGHTS

- Establish risk model with stochastic interest rate, inflation index, bonds and TIPS.
- Study the optimal reinsurance and investment problem under maximizing CRRA utility.
- Derive closed-forms of the optimal utility, reinsurance and investment strategies.
- Give a sensitivity analysis to clarify the behavior of the risk model.

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ABSTRACT

In this paper, we investigate an optimal reinsurance and investment problem for an insurer whose surplus process is approximated by a drifted Brownian motion. Proportional reinsurance is to hedge the risk of insurance. Interest rate risk and inflation risk are considered. We suppose that the instantaneous nominal interest rate follows an Ornstein–Uhlenbeck process, and the inflation index is given by a generalized Fisher equation. To make the market complete, zero-coupon bonds and Treasury Inflation Protected Securities (TIPS) are included in the market. The financial market consists of cash, zero-coupon bond, TIPS and stock. We employ the stochastic dynamic programming to derive the closed-forms of the optimal reinsurance and investment strategies as well as the optimal utility function under the constant relative risk aversion (CRRA) utility maximization. Sensitivity analysis is given to show the economic behavior of the optimal strategies and optimal utility.

1. Introduction

Optimal investment strategy for insurer has recently become an important subject. The insurer can participate in the financial market to avoid risk. More recently, many literatures have studied maximizing the utility of terminal value or minimizing the

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probability of ruin for the insurer. Browne (cf. Browne, 1995) initiated the study of explicit solution for a firm to maximize the exponential utility of terminal wealth and minimize the probability of ruin with its surplus process given by the Lundberg risk model. For different claim sizes of insurers, the optimal strategy was given by the Bellman equation in Hipp and Plum (2000) to minimize the ruin probability. Wang, Xia and Zhang (cf. Wang et al., 2007) efficiently applied martingale method to study the optimal portfolio selection for insurer under the mean–variance criterion as well as the expected constant absolute risk aversion (CARA) utility maximization. The readers are referred to, for example, Yang and Zhang (2005), Wang (2007), Liu and Yang (2004), Bai and Guo (2008) and references therein.

In addition to the risk of market, the insurer also takes into account the risk of insurance. The risk of insurance cannot be avoided by singly investing in the bond and other assets in the market. However, the business of reinsurance provides a way for the insurer to hedge this risk, and this way has also recently drawn much concern. The business of reinsurance comes up in different forms. Quota-share reinsurance and investment were originally considered by Promislow and Young (cf. Promislow and Young, 2005). Proportional reinsurance was accessible in Bäuerle (2005) in which the author minimized the expected quadratic distance of the terminal value over a positive constant and successfully solved the related mean–variance problem. Zeng and Li (cf. Zeng and Li, 2011) essentially got the mean–variance efficient frontier of the diffusion model with multiple risky assets in the case of proportional reinsurance. The stock price in the above models generally follows a geometric Brownian motion and the market price of the risk correlated with the stock is constant. But in the real market, the stock price may have other features, for example, stochastic volatility. Liang, Yuen and Guo (cf. Liang et al., 2011) characterized the instantaneous rate of the stock by Ornstein–Uhlenbeck process and derived the optimal reinsurance and investment strategies. The constant elasticity of variance (CEV) model was established in Gu et al. (2012) in which the insurer can buy excess-of-loss reinsurance. In Bäuerle and Blatter (2011), both the surplus of the insurer and the stock index in the market followed the Lévy process, and optimal investment and reinsurance policies were explicitly derived. Moreover, the optimal investment strategy was beautifully solved by Badaoui and Fernández (cf. Badaoui and Fernández, 2013) when the instantaneous rate and the volatility were related with a common stochastic factor.

Based on the investment and reinsurance strategy, the insurer can successfully avoid its risk. However, the time of investment may be long for the insurer, so it is natural to take the risk of interest rate into account. So far, few literature is available for insurer under stochastic interest rate. Elliott and Siu (cf. Elliott and Siu, 2011) used the so called game theoretic approach to find the best allocations in the market when the interest rate was given by a regime-switching model. In fact, most of the work of investment under stochastic interest focus on portfolio selection. In the case of stochastic interest rate, zero coupon bonds, delivering a fixed return of \$1 at maturity, are issued in the market to hedge the risk of interest rate. With the help of zero coupon bonds, we can establish a complete market. Bajeux-Besnainou and Portait (cf. Bajeux-Besnainou and Portait, 1998) first solved the portfolio selection problem when the instantaneous interest rate was stochastic. They introduced the pricing kernel and derived the mean–variance efficient frontier under the generalized Vasicek model. Bajeux-Besnainou, Jordan and Portait (cf. Bajeux-Besnainou et al., 2003) considered a case when the interest rate followed an Ornstein–Uhlenbeck process and got the optimal investment strategies to maximize CRRA and hyperbolic absolute risk aversion (HARA) utility for investors by martingale methods. Mean–variance problem with extended Cox–Ingersoll–Ross (CIR) stochastic interest rate model was studied by Ferland and Waiter (cf. Ferland and Waiter, 2010). Besides, Boulier, Huang and Taillard (cf. Boulier et al., 2001), Josa-Fombellida and Rincón-Zapatero (cf. Josa-Fombellida and Rincón-Zapatero, 2010) solved the optimal investment problem under stochastic interest rate in defined contribution (DC) and defined benefit (DB) pension plans, respectively.

Also, the inflation risk is an important factor in the long run of investment. To hedge the inflation risk, in the case of optimal asset allocation with inflation, Treasury Inflation Protected Securities (TIPS) are needed. There are many TIPS in practice, in which people often use inflation-indexed zero coupon bond in the market. The model of inflation often includes nominal interest rate, real interest rate and the inflation index. The inflation index is also a factor to characterize the connection between the nominal market and the real market. The most famous equation between them is given by the famous Fisher equation. Jarrow and Yildirim (cf. Jarrow and Yildirim, 2003) made a breakthrough in establishing the Jarrow–Yildirim (JY) model to characterize the inflation index, the forward nominal interest rate and forward real interest rate. Brennan and Xia (cf. Brennan and Xia, 2002) modeled the inflation index in a different framework and obtained the optimal investment strategies under inflation. Besides, Zhang, Korn and Ewald (cf. Zhang et al., 2007) extended the Fisher equation under the risk-neutral measure and used the martingale method to derive the optimal allocations. Later, Han and Hung (cf. Han and Hung, 2012) first introduced the risks of inflation and interest rate in a DC pension fund model.

Unfortunately, as far as we are concerned, no literature of insurer cares about the above two important risks of market at the same time. But when we concern the optimal reinsurance and investment strategies for a long time, the both risks of interest rate and inflation should be included. More precisely, in this paper, we will concentrate on studying the optimal reinsurance and investment problem for an insurer under risks of interest rate and inflation. The objective of the insurer is to maximize the expected CRRA utility of the terminal real wealth, where we assume that the nominal interest rate follows an Ornstein–Uhlenbeck process, the connections among real interest rate, nominal interest rate and the inflation index are given by the famous Fisher equation. To make the market complete and hedge the risk of market, zero-coupon bonds, TIPS and stocks are also included in the financial market. Moreover, we also assume that the proportional reinsurance is allowed. By using the stochastic dynamic programming method, we first derive the Hamilton–Jacobi–Bellman (HJB) equations for the problem, and then solve it by employing a variable change technique, finally get the closed-forms of the optimal reinsurance and investment strategies in the dynamic optimization problem. However, since the existence of insurance, we will not get a self-financing wealth process and this makes the problem very difficult. To handle this situation, auxiliary process will be introduced to make the market also self-financing, and the auxiliary process will help to solve the optimal reinsurance and investment problem for insurers.

The paper is organized as follows. The model of proportional reinsurance with stochastic nominal interest rate and inflation index is presented in Section 2, and the dynamics of zero coupon bonds and TIPS are also given. Section 3 introduces an auxiliary problem and derives the optimal reinsurance and investment strategies by stochastic dynamic programming. Section 4 provides a sensitivity analysis to clarify the behavior of our model. Section 5 is a conclusion.

2. The risk model

In this section, we obtain a financial market for an insurer with risks of inflation and interest rate. $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbf{P})$ is a filtered complete probability space, where $\mathcal{F}_t$ is the collection of information in the market until time t. $[0, T]$ is a fixed time horizon. All the processes involved below are presumed to be adapted to $\{\mathcal{F}_t, t \in [0, T]\}$.

2.1. Market

We study an insurer whose surplus process is modeled by the classical Lundberg model

$$dX(t) = c dt - d\left\{\sum_{i=1}^{N_t} Y_i, \right\}$$

where c is the premium rate of the insurer, Y_i is the ith claim, the number of claims up to time t is denoted by the homogeneous Poisson process N_t with intensity $\lambda > 0$, and $N = \{N_t\}$ is independent of $\{Y_i\}$. All the claims $Y_i,\ i = 1, 2, 3, \ldots$ are assumed to be independent and identically distributed (i.i.d.) with $\mathbf{E}[Y_i] = \mu_1$ and $\mathbf{E}[Y_i^2] = \mu_2$. To prevent the insurer from bankruptcy immediately, $c > \lambda\mu_1$ is required. For simplicity, we set $c = \lambda\mu_1(1+\eta)$ according to the expected value principle (cf. Kaas et al., 2009), where $\eta > 0$ is the safety loading. In addition, reinsurance is allowed and we consider the proportional reinsurance here. Denote the reinsurance proportion by $a(t)$, which means that $100(1 - a(t))\%(a(t) \geq 0)$ of the insurance risk is divided to a reinsurer at t. When the ith claim Y_i occurs, the insurer pays only $a(t)Y_i$ while the reinsurer pays the rest. However, based on the expected value principle, the insurer has to pay a premium at the rate of $(1+\theta)\lambda\mu_1(1-a(t))(\theta > 0)$ to the reinsurer due to the reinsurance business. In general, $\theta > \eta$, otherwise, arbitrage will exist. The insurer can hedge its insurance risk by the reinsurance strategy $a(t)$. If $a(t)$ is small, the insurer takes a little risk of insurance by himself and divides most of the risk to the reinsurer. $a(t) > 1$ implies taking new reinsurance business from the insurance market. In this case, the surplus process $X(t)$ takes the following form:

$$dX(t) = \lambda\mu_1[a(t)(1+\theta) - (\theta - \eta)]dt - a(t)d\left\{\sum_{i=1}^{N_t} Y_i\right\}. \tag{2.1}$$

Following the same process as in Grandll (1991), Liang and Huang (2011) and Liang and Sun (2011), the above process can be approximated by the following drifted process:

$$dX(t) = \lambda\mu_1(\eta - \theta)dt + \lambda\mu_1\theta a(t)dt + \sqrt{\lambda\mu_2}a(t)dW_0(t), \tag{2.2}$$

where $W_0(t)$ is a standard Brownian motion on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbf{P})$.

In the market, risks of interest rate and inflation are considered. To simplify the model, we assume that the instantaneous nominal rate $r_n(t)$ and the inflation index $I(t)$ are stochastic processes while the instantaneous real rate $r_r(t)$ is a deterministic function of t. The stochastic nominal rate $r_n(t)$ is mean-reverting and driven by the following Ornstein–Uhlenbeck equation:

$$dr_n(t) = a(b - r_n(t))dt - \sigma_{r_n} dW_{r_n}(t),$$

where a, b, σ_{r_n} are positive constants and $W_{r_n}(t)$ is a standard Brownian motion, and it is independent of $W_0(t)$.

The original Fisher equation only describes the relationships among the real interest rate, nominal interest rate and the inflation risk in the discrete time case. We formulate the continuous time model of inflation index, based on the extended Fisher equation given by Zhang (cf. Zhang et al., 2007), as follows:

$$\begin{cases} r_n(t) - r_r(t) = \lim\limits_{\Delta t\to 0} \dfrac{1}{\Delta t}\widetilde{\mathbf{E}}[i(t, t+\Delta t)|\mathcal{F}_t], \\ i(t, t+\Delta t) = \dfrac{I(t+\Delta t) - I(t)}{I(t)}, \end{cases} \tag{2.3}$$

where $\widetilde{\mathbf{E}}$ is the expectation with respect to the risk neutral measure $\widetilde{P}$. $i(t, t+\Delta t)$ denotes the inflation rate within time horizon $[t, t+\Delta t]$, and the stochastic inflation index $I(t)$ is given by the following stochastic differential equation:

$$\frac{dI(t)}{I(t)} = (r_n(t) - r_r(t))dt + \sigma_{I_1} d\widetilde{W}_{r_n}(t) + \sigma_{I_2} d\widetilde{W}_I(t), \tag{2.4}$$

where $\widetilde{W}_{r_n}(t)$ and $\widetilde{W}_I(t)$ are two standard Brownian motions with respect to the risk-neutral measure $\widetilde{P}$.

Assume that the market price of risk of $W_I(t)$ is λ_I, then by the Girsanov theorem we know that the stochastic inflation index $I(t)$ under the original measure P can be defined by

$$\begin{aligned} \frac{dI(t)}{I(t)} &= (r_n(t) - r_r(t))dt + \sigma_{I_1}[\lambda_{r_n}dt + dW_{r_n}(t)] \\ &\quad + \sigma_{I_2}[\lambda_I dt + dW_I(t)]. \end{aligned} \tag{2.5}$$

The risk-free asset price $S_0(t)$ evolves according to

$$dS_0(t) = S_0(t)r_n(t)dt, \quad S_0(0) = 1. \tag{2.6}$$

To make the market complete, zero-coupon bonds are issued in the market to hedge the risk of nominal interest rate. A zero-coupon bond $B_n(t, T)$ is a contract at time t with final payment of \$1 at maturity T, and we assume that $B_n(t, T)$ satisfies the following partial differential equation:

$$\begin{cases} \dfrac{\partial B_n(t,T)}{\partial t} + [a(b - r_n) + \lambda_{r_n}\sigma_{r_n}]\dfrac{\partial B_n(t,T)}{\partial r_n} \\ \quad + \dfrac{1}{2}\sigma_{r_n}^2 \dfrac{\partial^2 B_n(t,T)}{\partial r_n^2} = r_n B(t,T), \\ B(T,T) = 1, \end{cases} \tag{2.7}$$

where λ_{r_n} is the market price of risk on $W_{r_n}(t)$. Then $B_n(t, T)$ has the following closed-form:

$$B_n(t,T) = \exp[r_n(t)C(t,T) - A(t,T)], \tag{2.8}$$

where $C(t,T) = \frac{e^{-a(T-t)} - 1}{a}$, $A(t,T) = -\int_t^T [(ab + \lambda_{r_n}\sigma_{r_n})C(s,T) + \frac{1}{2}\sigma_{r_n}^2 C(s,T)^2]ds$.

In addition, $B_n(t, T)$ also satisfies the following backward stochastic differential equation (BSDE):

$$\begin{cases} \dfrac{dB_n(t,T)}{B_n(t,T)} = r_n(t)dt + \sigma_{B_1}(T-t)[\lambda_{r_n}dt + dW_{r_n}(t)], \\ B_n(T,T) = 1, \end{cases} \tag{2.9}$$

where $\sigma_{B_1}(t) = \frac{1-e^{-at}}{a}\sigma_{r_n}$.

We can invest into the asset $B(t, T)$ by selling $B(t - dt, T)$ at $t - dt$ and purchasing $B(t, T)$ at time t. Hence, the maturity of zero coupon bond we invest in changes over time. However, as is stated in Boulier, Huang and Taillard (cf. Boulier et al., 2001), there may not exist zero-coupon bonds with any maturity $t > 0$ in the market, so we need to introduce a rolling bond with a constant maturity K_1. The rolling bond $B_{K_1}(t)$ is of the form

$$\frac{dB_{K_1}(t)}{B_{K_1}(t)} = r_n(t)dt + \sigma_{B_1}(K_1)[\lambda_{r_n}dt + dW_{r_n}(t)]. \tag{2.10}$$

The relationship between B_{K_1} and $B_n(t, T)$ is given by

$$\begin{aligned} \frac{dB_n(t,T)}{B_n(t,T)} &= \left(1 - \frac{\sigma_{B_1}(T-t)}{\sigma_{B_1}(K_1)}\right)\frac{dS_0(t)}{S_0(t)} \\ &\quad + \frac{\sigma_{B_1}(T-t)}{\sigma_{B_1}(K_1)}\frac{dB_{K_1}(t)}{B_{K_1}(t)}. \end{aligned} \tag{2.11}$$

We see that the stochastic inflation index model in this paper is a particular case of the JY model in Jarrow and Yildirim (2003) when the nominal interest rate is an Ornstein–Uhlenbeck process and the real interest rate is non-random. To reduce the risk of inflation, TIPS is available in the market. We consider a particular TIPS named indexed zero coupon bond $P(t,T)$ which delivers real money of \$1, i.e. $I(t)$ at maturity T. Using the general pricing theory of derivatives, we know that the bond $P(t,T)$ satisfies

$$\begin{cases} \dfrac{\partial P}{\partial t}+\dfrac{P}{\partial r_n}[a(b-r_n)+\lambda_{r_n}\sigma_{r_n}]+P_I I(r_n-r_r) \\ \quad +\dfrac{1}{2}P_{r_n r_n}\sigma_{r_n}^2+\dfrac{1}{2}P_{II}I^2(\sigma_{I_1}^2+\sigma_{I_2}^2)-P_{Ir_n}I\sigma_{r_n}\sigma_{I_1}=r_nP, \\ P(T,T)=I(T). \end{cases} \tag{2.12}$$

The closed-form of $P(t,T)$ is given by

$$P(t,T)=I(t)\exp\left[-\int_t^T r_r(s)\mathrm{d}s\right]. \tag{2.13}$$

Moreover, the $P(t,T)$ satisfies the following BSDE:

$$\begin{cases} \dfrac{\mathrm{d}P(t,T)}{P(t,T)}=r_n(t)\mathrm{d}t+\sigma_{I_1}[\lambda_{r_n}\mathrm{d}t+\mathrm{d}W_{r_n}(t)] \\ \qquad +\sigma_{I_2}[\lambda_I\mathrm{d}t+\mathrm{d}W_I(t)], \\ P(T,T)=1. \end{cases} \tag{2.14}$$

We also consider a rolling indexed bond $P_{K_2}(t)$ with constant maturity K_2 satisfying

$$\begin{aligned} \frac{\mathrm{d}P_{K_2}(t)}{P_{K_2}(t)} &= r_n(t)\mathrm{d}t+\sigma_{I_1}[\lambda_{r_n}\mathrm{d}t+\mathrm{d}W_{r_n}(t)] \\ &\quad +\sigma_{I_2}[\lambda_I\mathrm{d}t+\mathrm{d}W_I(t)]. \end{aligned} \tag{2.15}$$

The relationship between $P_{K_2}(t)$ and $P(t,T)$ is

$$\frac{\mathrm{d}P(t,T)}{P(t,T)}=\frac{\mathrm{d}P_{K_2}(t)}{P_{K_2}(t)}. \tag{2.16}$$

It is easy to see that the differential of TIPS $P(t,T)$ is not correlated with its maturity T.

Furthermore, there is a stock in the market, and we assume that the price of the stock follows the following stochastic differential equation:

$$\begin{aligned} \frac{\mathrm{d}S_1(t)}{S_1(t)} &= r_n(t)\mathrm{d}t+\sigma_{S_1}(\lambda_{r_n}\mathrm{d}t+\mathrm{d}W_{r_n}(t)) \\ &\quad +\sigma_{S_2}(\lambda_I\mathrm{d}t+\mathrm{d}W_I(t))+\sigma_{S_3}(\lambda_S\mathrm{d}t+\mathrm{d}W_S(t)), \end{aligned} \tag{2.17}$$

where λ_S is the market price of risk of a standard Brownian motion $W_S(t)$ on $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\in[0,T]},\mathbf{P})$, and the standard Brownian motions $\{W_0(t)\}$, $\{W_{r_n}(t)\}$, $\{W_I(t)\}$ and $\{W_S(t)\}$ are independent.

Thus, in the above market, the wealth $X(t)$ of the insurer must satisfy the following SDE:

$$\begin{aligned} \mathrm{d}X(t) &= \lambda\mu_1(\eta-\theta)\mathrm{d}t+\lambda\mu_1\theta a(t)\mathrm{d}t+\sqrt{\lambda\mu_2}a(t)\mathrm{d}W_0(t) \\ &\quad +\theta_0(t)\frac{\mathrm{d}S_0(t)}{S_0(t)}+\theta_B(t)\frac{\mathrm{d}B_{K_1}(t)}{B_{K_1}(t)}+\theta_P(t)\frac{\mathrm{d}P_{K_2}(t)}{P_{K_2}(t)} \\ &\quad +\theta_S(t)\frac{\mathrm{d}S_1(t)}{S_1(t)}, \end{aligned} \tag{2.18}$$

where $\theta_0(t),\theta_B(t),\theta_P(t),\theta_S(t)$ are the money invested in the cash, zero coupon bond, TIPS and the stock, respectively. The wealth of our model is $X(t)=\theta_0(t)+\theta_B(t)+\theta_P(t)+\theta_S(t)$, and $\bar{u}(t)\triangleq(a(t),\theta_B(t),\theta_P(t),\theta_S(t))^T$ is called a strategy. $\bar{u}(t)$ is a combination of the reinsurance strategy and the investment strategy. We say $\bar{u}(t)$ is an admissible strategy if $\bar{u}(t)$ is adapted to the filtration $\mathcal{F}=\{\mathcal{F}_t\}_{t\in[0,T]}$ and the reinsurance strategy $a(t)$ in $\bar{u}(t)$ is not less than zero. Moreover, the wealth process $X(t)$ corresponding to $\bar{u}(t)$ should satisfy $X(t)\geq 0$. Substituting (2.6), (2.10), (2.15) and (2.17) into the last equation, we can rewrite $X(t)$ in the following compact form:

$$\mathrm{d}X(t)=\lambda\mu_1(\eta-\theta)\mathrm{d}t+\bar{u}(t)^T\sigma[\Lambda\mathrm{d}t+\mathrm{d}W(t)], \tag{2.19}$$

where

$$\Lambda\triangleq\begin{pmatrix}\dfrac{\lambda\mu_1\theta}{\sqrt{\lambda\mu_2}}\\ \lambda_{r_n}\\ \lambda_I\\ \lambda_S\end{pmatrix},\qquad \sigma\triangleq\begin{pmatrix}\sqrt{\lambda\mu_2}&0&0&0\\ 0&\sigma_{B_1}(K_1)&0&0\\ 0&\sigma_{I_1}&\sigma_{I_2}&0\\ 0&\sigma_{S_1}&\sigma_{S_2}&\sigma_{S_3}\end{pmatrix},$$

$$\mathrm{d}W(t)\triangleq\begin{pmatrix}\mathrm{d}W_0(t)\\ \mathrm{d}W_{r_n}(t)\\ \mathrm{d}W_I(t)\\ \mathrm{d}W_S(t)\end{pmatrix}.$$

2.2. The optimization problem

In this paper, we intend to maximize the expected utility of the terminal wealth by continuously arranging the allocations in the assets and the reinsurance proportion within time horizon $[0,T]$. Because inflation risk exists in the market, we need to maximize the expected utility of the real value of the terminal wealth $X(T)$. So the optimization problem can be written as

$$\begin{cases}\max\left\{\mathbf{E}\left[U\left(\dfrac{X(T)}{I(T)}\right)\right]\right\}\\ \text{subject to: } X(0)=x,\ \bar{u}(t)\text{ admissible.}\end{cases} \tag{2.20}$$

The CRRA utility function is

$$U(x)=\frac{x^{1-\gamma}}{1-\gamma},\quad \gamma>0,\ \gamma\neq 1 \tag{2.21}$$

and γ is the relative risk aversion.

3. Solution of the optimization problem

The optimization problem (2.20) is not a classical self-financing problem. In the insurance market, the insurer has a continuous income of the premium. So maybe we cannot solve the problem via the classical methods. Besides, the problem involves reinsurance strategy and investment strategies, so it is not a single investment problem. The existence of reinsurance can affect the solution of the optimal strategy. However, similar to that of the single-agent consumption and investment problem in Karatzas and Shreve (1998), we have the following lemma on the $X(t)$ defined by (2.19).

Lemma 3.1. *Let* $H(t)=\exp\{\int_0^t(r_n(s)+\frac{1}{2}\|\Lambda\|^2)\mathrm{d}s+\int_0^t\Lambda^T\mathrm{d}W(s)\}$. *Then* $H(t)$ *satisfies the following SDE:*

$$\frac{\mathrm{d}H(t)}{H(t)}=[r_n(t)+\Lambda^T\Lambda]\mathrm{d}t+\Lambda^T\mathrm{d}W(t),\quad H(0)=1.$$

Moreover, the $X(t)$ *must be the following form:*

$$X(t)=\mathbf{E}\left[-\int_t^T\frac{\lambda\mu_1(\eta-\theta)H(t)}{H(s)}\mathrm{d}s+\frac{X(T)H(t)}{H(T)}\middle|\mathcal{F}_t\right],\quad t\in[0,T].$$

Proof. Applying Itô formula to the process $\{\frac{X(t)}{H(t)}\}$, we have

$$\mathrm{d}\left(\frac{X(t)}{H(t)}\right) = \lambda\mu_1(\eta-\theta)H^{-1}(t)\mathrm{d}t + [H^{-1}(t)\bar{u}^T(t) - X(t)H^{-1}\Lambda^T]\mathrm{d}W(t). \tag{3.1}$$

And so

$$\mathrm{d}\left[\frac{X(t)}{H(t)} - \int_0^t \frac{\lambda\mu_1(\eta-\theta)}{H(s)}\mathrm{d}s\right] = [H^{-1}(t)\bar{u}^T(t) - X(t)H^{-1}\Lambda^T]\mathrm{d}W(t). \tag{3.2}$$

Then $\{\frac{X(t)}{H(t)} - \int_0^t \frac{\lambda\mu_1(\eta-\theta)}{H(s)}\mathrm{d}s, 0 \le t \le T\}$ is a martingale. This lemma follows. □

Similar to the general investment problem, $H(t)$ may act as the pricing kernel of the financial market. Because of the existence of reinsurance risk, $H(t)$ is in fact the combination of risk of reinsurance and the financial market. In addition, in the self-financing case, we simply have $X(t) = \mathbf{E}[\frac{X(T)H(t)}{H(T)}|\mathcal{F}_t]$, which means that $X(t)$ is a martingale under the risk neutral measure. However, we can see that the wealth of the insurer is a supermartingale under a certain measure in the proof of Lemma 3.1. The term $\lambda\mu_1(\eta-\theta)\mathrm{d}t$ in (2.19) acts as a continuously outcome of the wealth. When the insurer chooses the reinsurance and investment strategies, the effect of the outcome should also be considered. We denote $F(t) = \mathbf{E}[\int_t^T \frac{\lambda\mu_1(\eta-\theta)H(t)}{H(s)}\mathrm{d}s|\mathcal{F}_t]$, and it can be seen as the discounted expected value of the continuous outcome $\lambda\mu_1(\eta-\theta)\mathrm{d}t$ of the wealth $X(t)$, and we have the following lemma to calculate $F(t)$.

Lemma 3.2. *The discounted value $F(t)$ can be written as $F(t) = \lambda\mu_1(\eta-\theta)\int_t^T B_n(t,s)\mathrm{d}s$, and $F(t)$ satisfies the following BSDE:*

$$\begin{cases} \mathrm{d}F(t) = -\lambda\mu_1(\eta-\theta)\mathrm{d}t + F(t)[r_n(t) + \lambda_{r_n}\sigma_F(t,T)]\mathrm{d}t \\ \quad +F(t)\sigma_F(t,T)\mathrm{d}W_{r_n}(t), \\ F(T) = 0, \end{cases} \tag{3.3}$$

where $\sigma_F(t,T) = \int_t^T \frac{\lambda\mu_1(\eta-\theta)\sigma_{B_1}(s-t)B_n(t,s)}{F(t)}\mathrm{d}s$.

Proof. Since

$$F(t) = \lambda\mu_1(\eta-\theta)\int_t^T \mathbf{E}\left[\frac{H(t)}{H(s)}\bigg|\mathcal{F}_t\right]\mathrm{d}s, \tag{3.4}$$

it suffices to calculate $\mathbf{E}[\frac{H(t)}{H(s)}|\mathcal{F}_t]$, $s \geqslant t$. By the independence of $W_0(t)$, $W_{r_n}(t)$, $W_I(t)$ and $W_S(t)$, it easily follows

$$\begin{cases} \mathbf{E}\left[\frac{H(t)}{H(s)}\bigg|\mathcal{F}_t\right] = \mathbf{E}\Big[-\int_t^s \left(r_n(u) + \frac{1}{2}\|\Lambda\|^2\right)\mathrm{d}u \\ \quad -\int_t^s \Lambda^T \mathrm{d}W(u)|\mathcal{F}_t\Big] \\ = \mathbf{E}\left[-\int_t^s \left(r_n(u) + \frac{1}{2}\lambda_{r_n}^2\right)\mathrm{d}u - \int_t^s \lambda_{r_n}\mathrm{d}W_{r_n}(u)|\mathcal{F}_t\right] \\ = \tilde{\mathbf{E}}\left[-\int_t^s (r_n(u))|\mathcal{F}_t\right] \\ = B_n(t,s). \end{cases} \tag{3.5}$$

So $F(t) = \lambda\mu_1(\eta-\theta)\int_t^T B_n(t,s)\mathrm{d}s$. Differentiating it directly, we obtain the second equation. □

3.1. An auxiliary problem

In this paper, we consider an auxiliary process $Y(t)$ defined by $Y(t) = X(t) + F(t)$ with initial value $F(0) = f$. By (2.19) and (3.3), we have

$$\begin{aligned} \mathrm{d}Y(t) &= \mathrm{d}X(t) + \mathrm{d}F(t) \\ &= r_n(t)Y(t)\mathrm{d}t + \begin{pmatrix} a(t) \\ \theta_B(t) + \dfrac{F(t)\sigma_F(t,T)}{\sigma_{B_1}(K_1)} \\ \theta_P(t) \\ \theta_S(t) \end{pmatrix}^T \\ &\quad \times \sigma[\Lambda\mathrm{d}t + \mathrm{d}W(t)] \\ &= r_n(t)Y(t)\mathrm{d}t + u(t)^T\sigma[\Lambda\mathrm{d}t + \mathrm{d}W(t)], \end{aligned} \tag{3.6}$$

where $u(t) = \bar{u}(t) + (0, \frac{F(t)\sigma_F(t,T)}{\sigma_{B_1}(K_1)}, 0, 0)^T$. Since $F(T) = 0$ and what we care about is the terminal value at time T, we can transform the original problem (2.20) into the following auxiliary self-financing problem:

$$\begin{cases} \max \mathbf{E}\left[U\left(\frac{Y(T)}{I(T)}\right)\right] \\ \text{subject to: } Y(0) = x + f \text{ and } u(t) \text{ is admissible.} \end{cases} \tag{3.7}$$

It is easy to see that, to get the self-financial problem, the insurer should buy more zero-coupon bond to hedge the risk of market due to the outcome of the wealth. In addition, $Y(0) \geq 0$ should be satisfied in our model, otherwise, bankruptcy may take place within $[0, T]$.

3.2. Solution to the auxiliary problem

As the problem introduced above is a self-financing problem, it is solvable. There are mainly two methods to solve it, one is the stochastic dynamic programming method, and the other one is the martingale method. In this paper, we will solve it by the former. Define

$$V(t, r_n, I, y) \triangleq \max_{u(t)}\left\{\mathbf{E}\left[U\left(\frac{Y(T)}{I(T)}\right)\bigg| r_n(t) = r_n \text{ and } I(t) = I, Y(t) = y\right]\right\}.$$

We have the following.

Theorem 3.3. *The associated HJB equation of the auxiliary problem (3.7) is*

$$\begin{cases} \sup\Big\{V_t + V_y[r_n y + u^{*T}(t)\sigma\Lambda] + V_{r_n}a(b - r_n) \\ \quad +V_I I(r_n - r_r + \sigma_{I_1}\lambda_{r_n} + \sigma_{I_2}\lambda_I) \\ \quad +\frac{1}{2}V_{yy}u^{*T}(t)\sigma\sigma^T u^*(t) + \frac{1}{2}V_{r_n r_n}\sigma_r^T\sigma_r \\ \quad +\frac{1}{2}V_{II}I^2\sigma_I^T\sigma_I + V_{yr_n}u^{*T}(t)\sigma\sigma_r \\ \quad +V_{yI}I\sigma\sigma_I + V_{Ir_n}I\sigma_r^T\sigma_I\Big\} = 0, \end{cases} \tag{3.8}$$

where $\sigma_r = (0, -\sigma_{r_n}, 0, 0)^T$, $\sigma_I = (0, \sigma_{I_1}, \sigma_{I_2}, 0)$.

Proof. The proof is very standard, see Merton (1969), Fleming and Soner (1993), Vigna and Haberman (2001), He and Liang (2009) and references therein, which we omit it here. □

We can get the optimal feedback function $u^*(t,y)$:

$$u^*(t,y) = -\frac{V_y\Sigma^{-1}\sigma\Lambda}{V_{yy}} - \frac{V_{yI}I\Sigma^{-1}\sigma\sigma_I}{V_{yy}} - \frac{V_{yr_n}\Sigma^{-1}\sigma\sigma_r}{V_{yy}}, \tag{3.9}$$

where $\Sigma \triangleq \sigma\sigma^T$. Substituting $u^*(t,y)$ into the HJB equation, we can get the closed-form of $V(t, r_n, I, y)$ and thus the optimal strategy $u^*(t) = u^*(t, Y^*(t))$, where $\{Y^*(t)\}$ is the unique solution of the SDE (3.6) with replacing the coefficient $u(t)$ there by $u^*(t, Y^*(t))$. So we have the following.

Proposition 3.4. *The optimal reinsurance–investment strategy $u^*(t)$ is*

$$u^*(t) = \frac{Y^*(t)}{\gamma}\Sigma^{-1}\sigma\Lambda + \left(1-\frac{1}{\gamma}\right)Y^*(t)\Sigma^{-1}\sigma\sigma_I$$
$$= \frac{X^*(t)+F(t)}{\gamma}\begin{pmatrix} \frac{\mu_1\theta}{\mu_2} \\ \frac{\lambda_{r_n}}{\sigma_{B_1}(K_1)} - \frac{\lambda_I\sigma_{P_1}}{\sigma_{B_1(K_1)}\sigma_{P_2}} + \frac{\lambda_S(\sigma_{P_1}\sigma_{S_2}-\sigma_{S_1}\sigma_{P_2})}{\sigma_{B_1}(K_1)\sigma_{P_2}\sigma_{S_3}} \\ \frac{\lambda_I}{\sigma_{P_2}} - \frac{\lambda_S\sigma_{S_2}}{\sigma_{P_2}\sigma_{S_3}} \\ \frac{\lambda_S}{\sigma_{S_3}} \end{pmatrix}$$
$$+ \left(1-\frac{1}{\gamma}\right)(X^*(t)+F(t))\begin{pmatrix}0\\0\\1\\0\end{pmatrix}. \tag{3.10}$$

The closed-form of $V(t, r_n, I, y)$ is

$$V(t, r_n, I, y) = \frac{1}{1-\gamma}\left(\frac{y}{I}\right)^{1-\gamma}h(t) \tag{3.11}$$

and

$$h(t) = \exp\left\{\int_t^T(\gamma-1)\left[-r_r(s) + \sigma_{I_1}\lambda_r + \sigma_{I_2}\lambda_I - \frac{1}{2\gamma}\Lambda^T\Lambda - \left(1-\frac{1}{\gamma}\right)\Lambda^T\sigma_I - \frac{1}{2\gamma}\sigma_I^T\sigma_I\right]\mathrm{d}s\right\}.$$

Proof. See Appendix.

The first term in $u^*(t)$ is the general form of the optimal strategy in self-financing framework. The second term is to invest only in the TIPS and it will be used to hedge the risk of inflation. Since, in our model, the nominal interest rate and the inflation are closely correlated and what we concern about is the real value and the real interest rate is deterministic, we only need to hedge the risk of inflation, and the risk of interest rate can be ignored. Besides, the optimal utility $V(t, r_n, I, y)$ is also not correlated with the nominal interest rate.

3.3. Solution to the original problem

Once we obtain the solution of the auxiliary problem, we can easily derive the solution of the original problem (2.20). The optimal reinsurance–investment strategies of the original problem are

$$\overline{u}^*(t) = u^*(t) - \left(0, \frac{F(t)\sigma_F(t,T)}{\sigma_{B_1}(K_1)}, 0, 0\right)^T, \tag{3.12}$$

i.e., since there is a continuously outcome in our model, we have to borrow $\frac{F(t)\sigma_F(t,T)}{\sigma_{B_1}(K_1)}$ zero coupon bond to get the optimal utility.

3.4. The optimal strategies

We observe that $Y^*(t)$ exists in the optimal strategies $\overline{u}^*(t)$. Indeed, in the market, $Y^*(t)$ is not observable. Noting that $F(t) = \lambda\mu_1(\eta-\theta)\int_t^T B_n(t,s)\mathrm{d}s$ in $Y^*(t) = X^*(t) + F(t)$, we can approximate $F(t)$ by zero coupon bonds with different maturities and thus obtain $Y^*(t)$. However, we may not have so many zero coupon bonds in the market, the method does not work. So we transform $Y^*(t)$ in terms of the assets and indexes in the market. By observing the values of the assets and indexes in the market, we can easily get $Y^*(t)$ and thus the optimal strategies. Substituting (3.10) into (3.6), we have

$$\mathrm{d}Y^*(t) = Y^*(t)\left\{\left[r_n(t) + \frac{1}{\gamma}\Lambda^T\Lambda + \left(1-\frac{1}{\gamma}\right)\sigma_I^T\Lambda\right]\mathrm{d}t + \left[\frac{1}{\gamma}\Lambda^T + \left(1-\frac{1}{\gamma}\right)\sigma_I^T\right]\mathrm{d}W(t)\right\}. \tag{3.13}$$

The formula indicates that $Y^*(t)$ follows a geometric Brownian motion. Because in the market we do not have any assets to represent the insurance risk, in order to represent $Y^*(t)$, we first need to introduce a fictitious asset $Z(t)$ defined by

$$\frac{\mathrm{d}Z(t)}{Z(t)} = \frac{\lambda\mu_1\theta}{\gamma\sqrt{\lambda\mu_2}}\mathrm{d}W_0(t). \tag{3.14}$$

By conjugation, we rewrite $Y^*(t)$ as

$$Y^*(t) = (x+f)e^{mt}\left(\frac{S_0(t)}{S_0(0)}\right)^{\alpha_1}\left(\frac{B_{K_1}(t)}{B_{K_1}(0)}\right)^{\alpha_2}\left(\frac{P_{K_2}(t)}{P_{K_2}(0)}\right)^{\alpha_3} \times \left(\frac{S_1(t)}{S_1(0)}\right)^{\alpha_4}\frac{Z(t)}{Z(0)}. \tag{3.15}$$

Differentiating $Y^*(t)$, and then comparing it with (3.13), the parameters satisfy the following equations:

$$\begin{pmatrix}\alpha_2\\\alpha_3\\\alpha_4\end{pmatrix} = \frac{1}{\lambda}\begin{pmatrix}\sigma_{B_1}(K_1) & \sigma_{I_1} & \sigma_{S_1}\\ 0 & \sigma_{I_2} & \sigma_{S_2}\\ 0 & 0 & \sigma_{S_3}\end{pmatrix}^{-1}\begin{pmatrix}\lambda_{r_n}\\\lambda_I\\\lambda_S\end{pmatrix} + \left(1-\frac{1}{\gamma}\right)\begin{pmatrix}\sigma_{B_1}(K_1) & \sigma_{I_1} & \sigma_{S_1}\\ 0 & \sigma_{I_2} & \sigma_{S_2}\\ 0 & 0 & \sigma_{S_3}\end{pmatrix}^{-1}\begin{pmatrix}\sigma_{I_1}\\\sigma_{I_2}\\0\end{pmatrix}. \tag{3.16}$$

Moreover,

$$\alpha_1 = 1-\alpha_2-\alpha_3-\alpha_4,$$
$$m = \frac{1}{\gamma}\Lambda^T\Lambda + \left(1-\frac{1}{\gamma}\right)\sigma_I^T\Lambda - \alpha_2\lambda_r\sigma_{B_1}(K_1) - \frac{1}{2}\alpha_2(\alpha_2-1)\sigma_{B_1}(K_1)^2 - \alpha_3(\sigma_{I_1}\lambda_r + \sigma_{I_2}\lambda_I) - \frac{1}{2}\alpha_3(\alpha_3-1)(\sigma_{I_1}^2+\sigma_{I_2}^2) - \alpha_4(\sigma_{S_1}\lambda_{r_n} + \sigma_{S_2}\lambda_I + \sigma_{S_3}\lambda_S) - \frac{1}{2}\alpha_4(\alpha_4-1)(\sigma_{S_1}^2+\sigma_{S_2}^2+\sigma_{S_3}^2) - \alpha_2\alpha_3\sigma_{B_1}(K_1)\sigma_{I_1} - \alpha_2\alpha_4\sigma_{B_1}(K_1)\sigma_{S_1} - \alpha_3\alpha_4(\sigma_{I_1}\sigma_{S_1}+\sigma_{I_2}\sigma_{S_2}).$$

With these parameters, we can express $Y^*(t)$ in terms of $S_0(t)$, $B_{K_1}(t)$, $P_{K_2}(t)$ and $Z(t)$. Because $Z(t)$ is a fictitious asset, it does not exist in the market. However, it can be observed by the claims in the insurance market. In fact, we can arrive it by the following closed-form:

$$Z(t) = Z(0)\exp\left[-\frac{\lambda\mu_1^2\theta^2}{2\gamma^2\mu_2}t + \frac{\lambda\mu_1\theta}{\gamma\sqrt{\lambda\mu_2}}W_0(t)\right]. \tag{3.17}$$

Since the initial value of $Z(t)$ can be arbitrarily chosen, for simplicity, we set $Z(0) = 1$. As we can approximate the claims by $W_0(t)$ in the preceding research, we are also able to approximate $Z(t)$ by the claims and so $Z(t)$ can be observable in the market. Moreover, following Grandll (1991), we calculate $Z(t)$ by

$$Z(t) = \exp\left[-\frac{\lambda\mu_1^2\theta^2}{2\gamma^2\mu_2}t + \frac{\lambda\mu_1^2\theta}{\gamma\mu_2}t + \frac{\mu_1\theta}{\gamma\mu_2}\sum_{i=1}^{N_t}Y_i\right]. \tag{3.18}$$

Table 1
Values of the parameters in our model.

Text interpretation	Symbol	Value
Risk aversion	γ	2
Proportional reinsurance		
Insurance intensity	λ	3
Mean of the claim	μ_1	0.08
Second moment of the claim	μ_2	0.05
Safety loading of insurer	η	0.05
Safety loading of reinsurer	θ	0.1
Nominal interest rate		
Initial value	r_0	0.05
Mean reversion	a	0.1
Mean rate	b	0.03
Volatility of nominal interest rate	σ_{r_n}	0.01
Real interest rate	r_r	0.045
Volatility of inflation index	$(\sigma_{I_1}, \sigma_{I_2})$	(0.08, 0.05)
Maturity of rolling zero coupon bond	K_1	10
Volatility of stock	$(\sigma_{S_1}, \sigma_{S_2}, \sigma_{S_3})$	(0.1, 0.08, 0.1)
Time horizon	T	20
Initial money	x_0	1
Initial value of inflation index	$I(0)$	1

4. Sensitivity analysis

In this section, some numerical examples are given to show how the optimal strategies and optimal utility vary. In contrast to the case of self-financial problem, neither the optimal investment amounts nor the optimal proportions are deterministic. We can only study the exact amounts of mean allocations or mean proportions. First, we have the following proposition.

Proposition 4.1. *The expectation of* $Y(t)$ *is*

$$\mathbf{E}[Y^*(t)] = Y_0 \exp\Bigg\{(r_0 - b)\frac{1-\exp(-at)}{a} + bt + \frac{1}{\gamma}\Lambda^T\Lambda t + \left(1-\frac{1}{\gamma}\right)\sigma_I^T\Lambda t + \frac{\sigma_{r_n}^2}{2a^2}\left[t + \frac{2\exp(-at)}{a} - \frac{\exp(-2at)}{2a} - \frac{3}{2a}\right] - \sigma_{r_n}\left[\frac{1}{\gamma}\lambda_{r_n} + \left(1-\frac{1}{\gamma}\right)\sigma_{I_1}\right]\left[\frac{t}{a} - \frac{1}{a^2} + \frac{e^{-at}}{a^2}\right]\Bigg\}.$$

Proof. See Appendix.

Next, we analyze the sensitivity of the optimal investment and reinsurance strategies. Unless otherwise stated, the basic data we adopt for the model are presented in Table 1.

4.1. Sensitivity analysis of the optimal investment strategies

First, we reveal the evolution of the optimal reinsurance and investment strategies in some cases and study the impact of the parameters on them. Fig. 1 shows that we invest heavily in the zero-coupon bond, growing greatly over time, i.e., the zero-coupon bond curve goes fast from 0.55 at time 0 to 0.95 at time 20, and however the TIPS has been relatively stable. Contrary to Fig. 1, Fig. 2 shows that in fact the proportion of zero-coupon bond diminishes over time, while the proportion of stock increases slowly. The reinsurance proportion increases slowly to about 0.18 at time 20, which means that we are dividing less insurance risk as time goes by. It is also illustrated in Fig. 1 that the risk of inflation is not so important to us and we only need to short a few TIPS to hedge the risk of inflation. Moreover, the proportions of cash, stock and TIPS change slightly in the figure.

Fig. 3 shows the optimal mean allocation and reinsurance strategies when $\gamma = 4$. In this case, the money invested in TIPS is the largest, increasing fast from 0.41 to 0.89. In addition, the proportion of TIPS grows greatly too. The mean allocation of cash stays steady after increasing for a while via comparing with the case when $\gamma = 2$. We also have little demands of cash and stock. The difference between the above two cases is mainly due to the

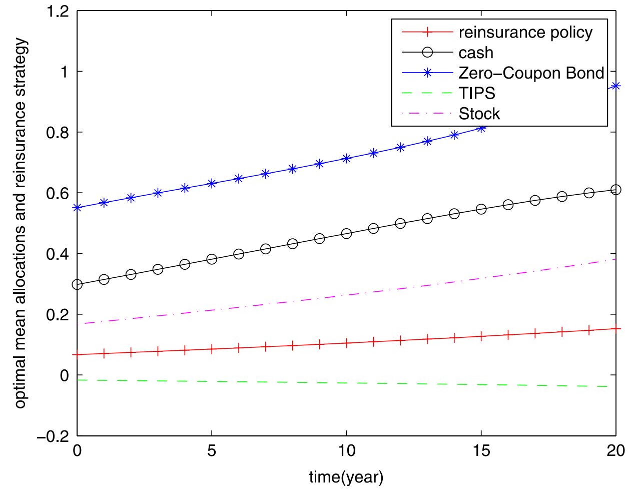

Fig. 1. $\gamma = 2$.

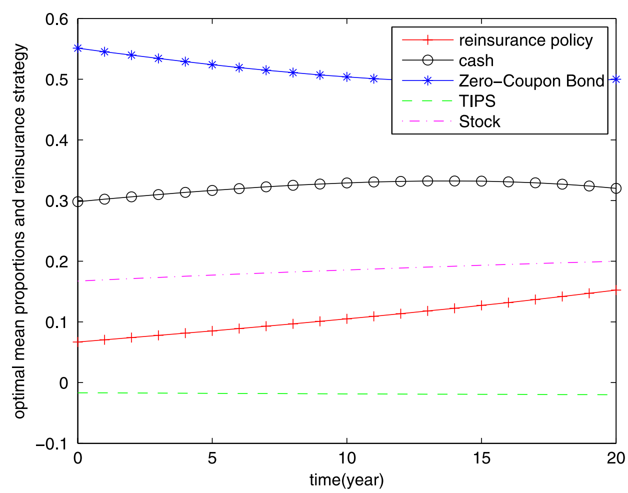

Fig. 2. $\gamma = 2$.

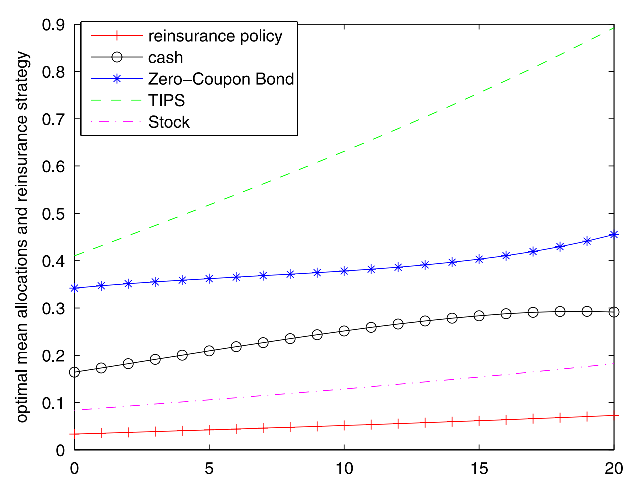

Fig. 3. $\gamma = 4$.

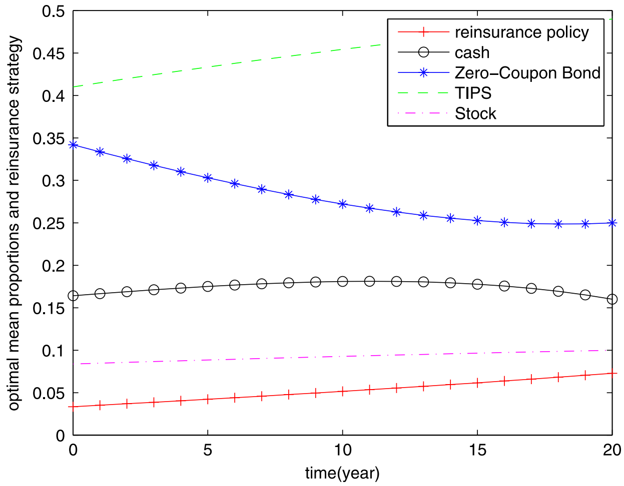

Fig. 4. $\gamma = 4$.

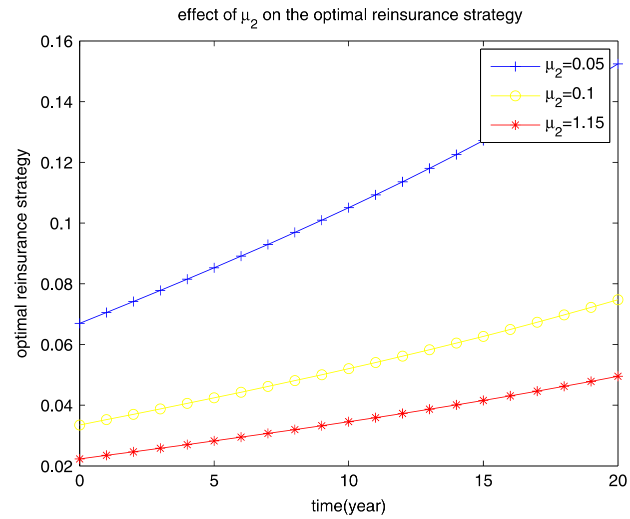

Fig. 6.

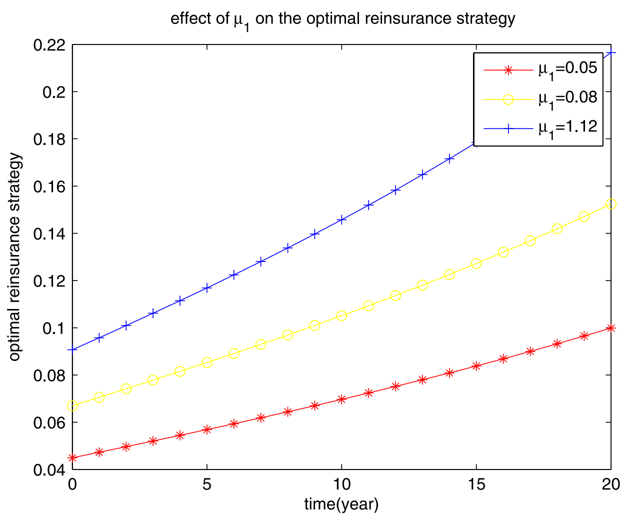

Fig. 5.

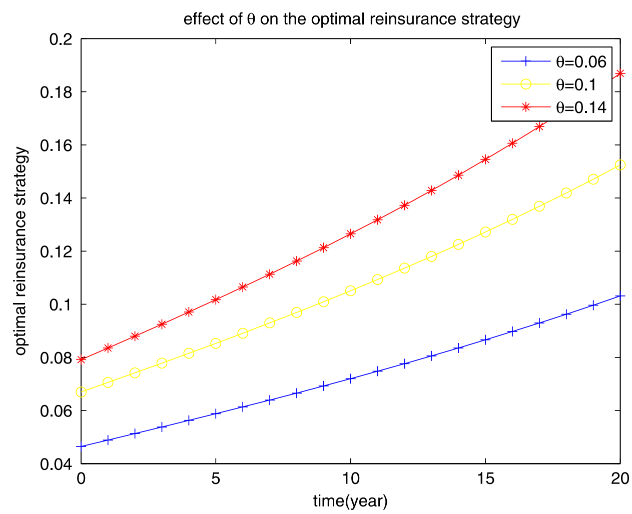

Fig. 7.

risk aversion γ. Higher γ makes the insurer more sensitive to the risk in the market. So the insurer will buy more TIPS to avoid the risk of inflation and give more insurance risk to the reinsurer. Since TIPS is correlated with the nominal interest rate, we can hedge a part of the interest risk and so the mean allocation of zero coupon bond indeed decreases when γ changes from 2 to 4. The two figures also describe the effect of γ on the optimal reinsurance (see Fig. 4). Higher γ means higher aversion of the risk, so in this situation the insurer will expect to reduce greatly its insurance risk and thus will purchase more reinsurance business.

4.2. Sensitivity analysis of the optimal reinsurance strategy

The reinsurance policy is essential in our model and we are also concerned with how the parameters affect the reinsurance strategy. Fig. 5 shows the connection between the optimal reinsurance policy and the expectation of one claim μ_1. The reinsurance strategy increases with μ_1, causing more risk of insurance for the insurer. In fact, we see from the formula of wealth that, for a larger μ_1, to cover the risk we shall take from insurance, we can get more income from the premium. Thus, we will take more insurance risk.

As the insurer controls its insurance risk by reinsurance, the optimal reinsurance strategy will also depend on the second moment μ_2 of the claim. In the market, μ_2 can be interpreted as the risk of insurance. The reinsurance strategy has a positive relationship with μ_2, which is shown in Fig. 6. In other words, if the risk of insurance becomes larger, the insurer should give more risk to the reinsurer to gain the optimal wealth. Moreover, the safety loading parameter θ is also an important factor that can affect the reinsurance strategy. θ measures the cost to hedge the risk of insurance. With higher θ, the insurer should cost more to hedge the risk of insurance, i.e., the insurer will take more risk of insurance by himself. So the reinsurance strategy is positively correlated with the safety loading of reinsurer θ, which is shown by Fig. 7. It can be also seen from Figs. 8–10 that the mean reinsurance policy is an increasing function of time t.

4.3. Sensitivity analysis of the optimal utility

This section presents how the parameters influence the optimal utility. The closed-form of optimal utility is given by Proposition 3.4. We see from the expressions of optimal investment and reinsurance strategies that the real interest rate r_r is not correlated with them. So, in the market with different real interest rates, we will adopt the same optimal strategies. However, r_r in fact affects the optimal utility. Fig. 8 states that the optimal utility is positively linked with r_r. When the real interest rate becomes bigger, the real money we own is worth more and thus we can get a bigger utility. Besides, if the initial nominal interest rate r_0 increases, we

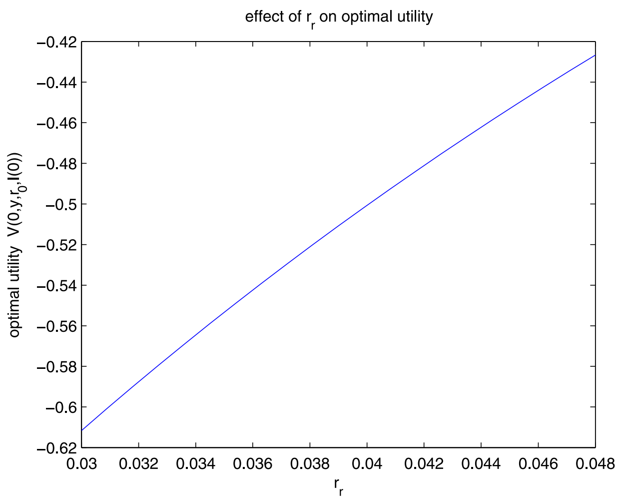

Fig. 8.

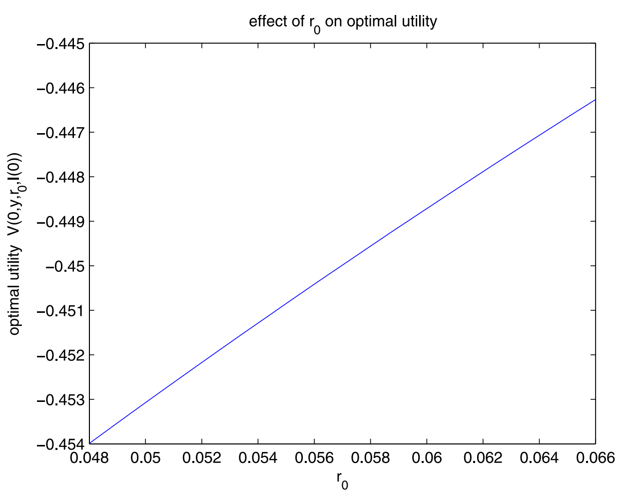

Fig. 9.

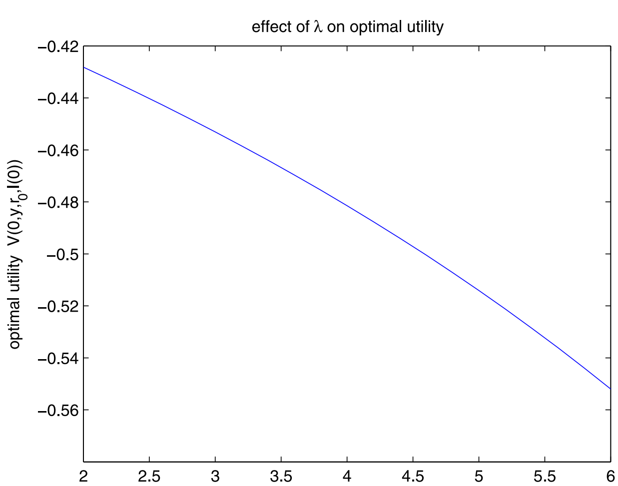

Fig. 10.

will make more money by investment, and also get greater utility, as shown in Fig. 9. The parameter λ means the intensity of claims, so if λ increases, then we may have more claims in a fixed time horizon, i.e., we will lose more money and the utility will decrease.

5. Conclusion

In this paper, we consider the optimal reinsurance and investment problems under stochastic nominal interest rate and stochastic inflation index. The nominal interest rate is modeled by the Ornstein–Uhlenbeck process and the inflation index is derived through the Fisher equation. The surplus process of the insurer is given by the classical Lundberg model first and approximated by a diffusion process. We can invest in the cash, zero coupon bonds, TIPS and a stock to hedge the risk. Because the original problem is not self-financing, we introduce an auxiliary self-financing problem and solve it by the stochastic dynamic programming. Finally, we get the optimal reinsurance and investment strategies under maximizing CRRA utility in Section 3. The optimal strategies consist of a strategy to gain the optimal utility, the optimal investment to hedge the risk of inflation and an investment in zero coupon bonds to counteract the effect of outcome of the wealth. We also find that the real interest rate has no effect on the optimal reinsurance and investment strategies. Moreover, we present sensitivity analysis at the end of this paper to show the economic behavior of the optimal strategies and optimal utility.

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Appendix

A.1. Proof of Proposition 3.4

The boundary condition for $V(t, y, r_n, I)$ is $V(T, y, r_n, I) = \frac{1}{1-\gamma}(\frac{y}{I})^{1-\gamma}$. We guess that the term y in $V(t, y, r_n, I)$ can be separated and $V(t, y, r_n, I)$ has the following form:

$$V(t, y, r_n, I) = \frac{1}{1-\gamma}\left(\frac{y}{I}\right)^{1-\gamma} h(t, r_n), \quad \text{and} \quad h(T, r_n) = 1. \tag{A.1}$$

Substituting (A.1) into (3.9), we see that the optimal strategy $u^*(t)$ is

$$u^*(t) = \frac{y}{\gamma}\Sigma^{-1}\sigma\Lambda + \left(1 - \frac{1}{\gamma}\right) y\Sigma^{-1}\sigma\sigma_I + \frac{1}{\gamma}\frac{h_{r_n}}{h} y\Sigma^{-1}\sigma\sigma_r. \tag{A.2}$$

Next, we substitute the last formula into (3.4), and then find that $h(t, r_n)$ satisfies the following equation:

$$\begin{aligned}
&\frac{h_t}{h} + \frac{h_{r_n}}{h}[ab - ar_n + (\gamma - 1)\sigma_r^T\sigma_I] - \frac{\gamma - 1}{2\gamma}\frac{h_{r_n}^2}{h^2}\sigma_r^T\sigma_r \\
&- \frac{h_{r_n}}{h}\left[\frac{\gamma - 1}{\gamma}\Lambda^T\sigma_r + \frac{(1-\gamma)^2}{\gamma}\sigma_I^T\sigma_r\right] \\
&+ \frac{1}{2}\frac{h_{r_nr_n}h}{h}\sigma_r^T\sigma_r + (\gamma - 1)(-r_r + \sigma_{I_1}\lambda_r + \sigma_{I_2}\lambda_I) \\
&+ \frac{1}{2}(\gamma - 1)(\gamma - 2)\sigma_I^T\sigma_I - (\gamma - 1) \\
&\times \left[\frac{1}{2\gamma}\Lambda^T\Lambda + \frac{(1-\gamma)^2}{2\gamma}\sigma_I^T\sigma_I + \left(1 - \frac{1}{\gamma}\right)\Lambda^T\sigma_I\right] = 0.
\end{aligned}$$

The solution $h(t, r_n)$ must be the following form:

$$h(t, r_n) = \exp[q_1(t) r_n + q_2(t)], \tag{A.3}$$

and the $q_1(t)$ and $q_2(t)$ satisfy the boundary conditions $q_1(T) = q_2(T) = 0$. Thus we derive the explicit forms of $q_1(t)$ and $q_2(t)$:

$$q_1(t) = 0,$$
$$q_2(t) = \int_t^T (\gamma - 1)\left[-r_r(s) + \sigma_{I_1}\lambda_r + \sigma_{I_2}\lambda_I - \frac{1}{2\gamma}\Lambda^T\Lambda - \left(1 - \frac{1}{\gamma}\right)\Lambda^T\sigma_I - \frac{1}{2\gamma}\sigma_I^T\sigma_I\right]\mathrm{d}s.$$

Once we get the explicit form of $h(t, r_n)$, the explicit forms of $u^*(t)$ and $V(t, y, r_n, I)$ can be easily derived, thus the proposition follows.

A.2. Proof of Proposition 4.1

In order to calculate $\mathbf{E}[Y^*(t)]$, first we have the following observations about $r_n(t)$:

Lemma A.1. *The Ornstein–Uhlenbeck process satisfied by $r_n(t)$ is a solvable equation and the explicit form of $r_n(t)$ is*

$$r_n(t) = (r_0 - b)\exp(-at) + b - \sigma_{r_n}\exp(-at) \times \int_0^t \exp(as)\mathrm{d}W_{r_n}(s). \tag{A.4}$$

Furthermore, the integral of $r_n(t)$ has the following expression,

$$\int_0^t r_n(s)\mathrm{d}s = (r_0 - b)\frac{1 - \exp(-at)}{a} + bt - \int_0^t \sigma_{B_1}(t - s)\mathrm{d}W_{r_n}(s). \tag{A.5}$$

Hence, the $\int_0^t r_n(s)\mathrm{d}s$ is a random variable with normal distribution, i.e., $\int_0^t r_n(s)\mathrm{d}s \sim N[(r_0 - b)\frac{1-\exp(-at)}{a} + bt, \frac{\sigma_{r_n}^2}{a^2}[t + \frac{2\exp(-at)}{a} - \frac{\exp(-2at)}{2a} - \frac{3}{2a}]]$.

Proof. We easily find that the solution of the Ornstein–Uhlenbeck equation is the first formula. For the second formula, we have

$$\begin{aligned}\int_0^t r_n(s)\mathrm{d}(s) &= \int_0^t \left[(r_0 - b)\exp(-as) + b - \sigma_{r_n}\exp(-as)\int_0^s \exp(au)\mathrm{d}W_{r_n}(u)\right]\mathrm{d}s \\ &= (r_0 - b)\frac{1 - \exp(-at)}{a} + bt - \sigma_{r_n}\int_0^t \exp(-as)\int_0^s \exp(au)\mathrm{d}W_{r_n}(u)\mathrm{d}s \\ &= (r_0 - b)\frac{1 - \exp(-at)}{a} + bt - \sigma_{r_n}\int_0^t \frac{1 - \exp(-a(t-s))}{a}\mathrm{d}W_{r_n}(s) \\ &= (r_0 - b)\frac{1 - \exp(-at)}{a} + bt - \int_0^t \sigma_{B_1}(t - s)\mathrm{d}W_{r_n}(s).\end{aligned} \tag{A.6}$$

And so the distribution of $\int_0^t r_n(s)\mathrm{d}s$ follows.

Next, we derive the mean of $Y^*(t)$. In fact, $Y^*(t)$ has the following expression:

$$Y^*(t) = Y_0 \exp\left\{\frac{1}{\gamma}\Lambda^T\Lambda t + \left(1 - \frac{1}{\gamma}\right)\sigma_I^T\Lambda t - \frac{1}{2}\left[\frac{1}{\gamma}\Lambda^T + \left(1 - \frac{1}{\gamma}\right)\sigma_I^T\right]\left[\frac{1}{\gamma}\Lambda + \left(1 - \frac{1}{\gamma}\right)\sigma_I\right]t + \int_0^t r_n(s)\mathrm{d}s + \left[\frac{1}{\gamma}\Lambda^T + \left(1 - \frac{1}{\gamma}\right)\sigma_I^T\right]W(t)\right\}.$$

So

$$\begin{aligned}\mathbf{E}[Y^*(t)] &= Y_0 \exp\left\{\frac{1}{\gamma}\Lambda^T\Lambda t + \left(1 - \frac{1}{\gamma}\right)\sigma_I^T\Lambda t - \frac{1}{2}\left[\frac{1}{\gamma}\Lambda^T + \left(1 - \frac{1}{\gamma}\right)\sigma_I^T\right]\left[\frac{1}{\gamma}\Lambda + \left(1 - \frac{1}{\gamma}\right)\sigma_I\right]t\right\} \\ &\quad \cdot \mathbf{E}\exp\left\{\int_0^t r_n(s)\mathrm{d}s + \left[\frac{1}{\gamma}\Lambda^T + \left(1 - \frac{1}{\gamma}\right)\sigma_I^T\right]W(t)\right\} \\ &= Y_0 \exp\left\{\left(\frac{1}{\gamma} - \frac{1}{2\gamma^2}\right)\Lambda^T\Lambda t + \left(1 - \frac{1}{\gamma}\right)^2\Lambda^T\Lambda t - \frac{1}{2}\left(1 - \frac{1}{\gamma}\right)^2\sigma_I^T\sigma_I t\right\} \\ &\quad \cdot \mathbf{E}\exp\left\{\int_0^t r_n(s)\mathrm{d}s + \left[\frac{1}{\gamma}\Lambda^T + \left(1 - \frac{1}{\gamma}\right)\sigma_I^T\right]W(t)\right\}.\end{aligned} \tag{A.7}$$

We only need to calculate $\mathbf{E}\{\exp\{\int_0^t r_n(s)\mathrm{d}s + [\frac{1}{\gamma}\Lambda^T + (1 - \frac{1}{\gamma})\sigma_I^T]W(t)\}\}$.

Denote $Q_t = \int_0^t r_n(s)\mathrm{d}s + [\frac{1}{\gamma}\Lambda^T + (1 - \frac{1}{\gamma})\sigma_I^T]W(t)$. Then Q_t is a normally distributed random variable and $\exp\left\{\int_0^t r_n(s)\mathrm{d}s + [\frac{1}{\gamma}\Lambda^T + (1 - \frac{1}{\gamma})\sigma_I^T]W(t)\right\}$ has a lognormal distribution. It follows

$$\mathbf{E}\exp\left\{\int_0^t r_n(s)\mathrm{d}s + \left[\frac{1}{\gamma}\Lambda^T + \left(1 - \frac{1}{\gamma}\right)\sigma_I^T\right]W(t)\right\} = \exp\left[\mathbf{E}(Q_t) + \frac{1}{2}\mathrm{Var}(Q_t)\right], \tag{A.8}$$

where

$$\mathbf{E}\{Q_t\} = \mathbf{E}\left[\int_0^t r_n(s)\mathrm{d}s\right] = (r_0 - b)\frac{1 - \exp(-at)}{a} + bt,$$

$$\begin{aligned}\mathrm{Var}\{Q_t\} &= \mathrm{Var}\left[\int_0^t r_n(s)\mathrm{d}s + \left[\frac{1}{\gamma}\Lambda^T + \left(1 - \frac{1}{\gamma}\right)\sigma_I^T\right]W(t)\right] \\ &= \mathrm{Var}\left[\int_0^t r_n(s)\mathrm{d}s\right] + \mathrm{Var}\left[\left[\frac{1}{\gamma}\Lambda^T + \left(1 - \frac{1}{\gamma}\right)\sigma_I^T\right]W(t)\right] + 2\mathrm{Cov}\left[\int_0^t r_n(s)\mathrm{d}s, \left[\frac{1}{\gamma}\Lambda^T + \left(1 - \frac{1}{\gamma}\right)\sigma_I^T\right]W(t)\right].\end{aligned}$$

Because

$$\mathrm{Var}\left[\left[\frac{1}{\gamma}\Lambda^T + \left(1 - \frac{1}{\gamma}\right)\sigma_I^T\right]W(t)\right] = \left[\frac{1}{\gamma}\Lambda^T + \left(1 - \frac{1}{\gamma}\right)\sigma_I^T\right]\left[\frac{1}{\gamma}\Lambda + \left(1 - \frac{1}{\gamma}\right)\sigma_I\right]t,$$

$$\mathrm{Var}\left[\int_0^t r_n(s)\mathrm{d}s\right] = \frac{\sigma_{r_n}^2}{a^2}\left[t + \frac{2\exp(-at)}{a} - \frac{\exp(-2at)}{2a} - \frac{3}{2a}\right]$$

and

$$\begin{aligned}
&\mathrm{Cov}\left[\int_0^t r_n(s)\mathrm{d}s, \left[\frac{1}{\gamma}\Lambda^T + \left(1-\frac{1}{\gamma}\right)\sigma_I^T\right]W(t)\right] \\
&= \mathbf{E}\left[\int_0^t r_n(s)\mathrm{d}s \cdot \left[\frac{1}{\gamma}\Lambda^T + \left(1-\frac{1}{\gamma}\right)\sigma_I^T\right]W(t)\right] \\
&= \left[\frac{1}{\gamma}\lambda_{r_n} + \left(1-\frac{1}{\gamma}\right)\sigma_{I_1}\right]\mathbf{E}\left[\int_0^t r_n(s)\mathrm{d}s \cdot W_{r_n}(t)\right] \\
&= -\left[\frac{1}{\gamma}\lambda_{r_n} + \left(1-\frac{1}{\gamma}\right)\sigma_{I_1}\right] \\
&\quad \times \mathbf{E}\left[\int_0^t \sigma_{B_1}(t-s)\mathrm{d}W_{r_n}(s)\int_0^t \mathrm{d}W_{r_n}(s)\right] \\
&= -\left[\frac{1}{\gamma}\lambda_{r_n} + \left(1-\frac{1}{\gamma}\right)\sigma_{I_1}\right]\int_0^t \sigma_{B_1}(t-s)\mathrm{d}s \\
&= -\sigma_{r_n}\left[\frac{1}{\gamma}\lambda_{r_n} + \left(1-\frac{1}{\gamma}\right)\sigma_{I_1}\right]\left[\frac{t}{a} - \frac{1}{a^2} + \frac{e^{-at}}{a^2}\right],
\end{aligned}$$

we obtain

$$\begin{aligned}
\mathbf{E}[Y^*(t)] = Y_0\exp\Bigg\{&(r_0-b)\frac{1-\exp(-at)}{a} + bt + \frac{1}{\gamma}\Lambda^T\Lambda t \\
&+ \left(1-\frac{1}{\gamma}\right)\sigma_I^T\Lambda t + \frac{\sigma_{r_n}^2}{2a^2}\left[t + \frac{2\exp(-at)}{a}\right. \\
&\left. - \frac{\exp(-2at)}{2a} - \frac{3}{2a}\right] \\
&- \sigma_{r_n}\left[\frac{1}{\gamma}\lambda_{r_n} + \left(1-\frac{1}{\gamma}\right)\sigma_{I_1}\right]\left[\frac{t}{a} - \frac{1}{a^2} + \frac{e^{-at}}{a^2}\right]\Bigg\}.
\end{aligned}$$

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