Probabilistic Fuzzy Logic and Probabilistic Fuzzy Systems

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Abstract

A novel concept of *probabilistic fuzzy logic* is introduced as a way of representing and/or modeling existing randomness in many real world systems and natural language propositions. The approach is actually based on combining both the concepts of probability of truth and degree of truth in a unique framework. This combination is carried out in both the fuzzy sets and fuzzy rules resulting in the new concepts of *probabilistic fuzzy sets* and *probabilistic fuzzy rules*, respectively. Having one of these probabilistic elements, a *probabilistic fuzzy system* is then introduced as a fuzzy-probabilistic model of a complex non-deterministic system. In a simple example, human skepticism about the optimal fuzzy rule base is modeled through substituting a probabilistic fuzzy rule base for a conventional one. The closed loop response of the resulting controller for tank level control is shown through simulation and is compared with a conventional fuzzy controller.

1. Introduction and Motivation

Introduction

One of the main advantages of fuzzy logic systems has been their ability for handling and representing one class of uncertainties, the *non-statistical* uncertainty. Moreover, fuzzy logic is a framework for representing and manipulating linguistic variables and sentences in *natural language*. This feature enables us to incorporate human expert knowledge in the form of fuzzy if-then rules and fuzzy membership functions. Furthermore, the universal approximation property of fuzzy systems guarantees their ability for modeling deterministic complex and uncertain systems. These superior traits however can be degraded by the existence of randomness and probabilistic elements. Randomness is another type of uncertainty named *statistical* uncertainty. In this paper, the shortcomings of conventional fuzzy logic systems in some particular situations will be first discussed leading to the motivations for integrating fuzziness and probability. Consequently, a new concept of *probabilistic fuzzy logic* is developed and used to enhance the universal applicability of fuzzy systems by bridging the gap between fuzziness and probability. The approach is mainly different from the well-established concept of *fuzzy probabilities*. Fuzzy probability is a fuzzy approach to probability theory whereas the proposed probabilistic fuzzy logic is a combined framework in which both probability as well as fuzzy theories co-exist.

Probability and Fuzziness

Statistical and *non-statistical* uncertainties are two conceptually different kinds of uncertainty. *Non-statistical* uncertainty is best represented with the concept of fuzziness where fuzzy logic is used to describe partial truth and approximate reasoning. This type of uncertainty is indeed an ambiguity in assigning the degree of compatibility of an instance with a semantic concept. *Statistical uncertainty*, on the other hand, may be viewed as a kind of uncertainty concerning the occurrence of an event in the future. Statistical uncertainty is best represented with probability, which gives us the likelihood of an outcome that may or may not happen. Probability gives the likelihood of the outcome in a statistical manner and tells us about populations not instances.

The relationship between fuzziness and probability has been discussed in literature frequently [1][2][3]. Most of the related arguments lead us to the final conclusion that fuzziness and probability are distinct phenomena and should be treated differently. Although it is now commonly accepted that they are complementary rather than competitive [1], there is not yet a unifying framework for their integration. We believe that they should indeed be treated *differently* but not necessarily *separately*. The need for an integrated framework is best revealed in situations where both types of uncertainty exist concurrently and where each of the fuzziness and probability concepts alone are necessary but not sufficient. As one example of such situations, consider the following example:

Example 1

Suppose there is a black box containing 100 balls having different colors. The color of the balls may vary in a broad range from completely white to completely black. There are two types of uncertainty in description of the actual color of a sampled ball randomly picked up from the box. One type of uncertainty is the one associated with random selection of the ball and can be represented with the *probability* of selection of a desired color (for example black) and the other one is the uncertainty or ambiguity in describing the color of the ball even after the ball has been selected and is known. Therefore, the probability of selection of a black ball as an outcome and degree of blackness of that ball may both be used to describe the uncertainty about the color of a selected ball.

1.1 *Natural language*

Fuzzy logic systems have a linguistic structure capable of handling and representing linguistic propositions in human natural language. However, there are often semantics and propositions which cannot be completely represented with fuzzy logic alone. Consider the following sentence:

It is very likely that tomorrow it will rain heavily

Although *"heavily"* is a fuzzy concept and can be represented with a fuzzy set, the fuzziness is no longer enough to describe the weather conditions. A probabilistic element also exists in prediction of the weather conditions. In this particular example, the probability itself is also specified with a fuzzy concept and is, therefore, an example of fuzzy probability as well.

1.2. *Statistical diversity in human expert knowledge*

Fuzzy systems are knowledge-based systems which utilize human expert opinion as one of its primary resources of acquiring this knowledge-base. However, there is not a systematic way of ensuring optimality and uniqueness of a human expert knowledge. Consequently, it is often necessary to collect many experts' knowledge about the system. The

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differing and/or conflicting experts' opinions cannot be represented simultaneously within a conventional fuzzy framework and are better to be considered in a more general statistical framework. For example suppose different experts are asked to define the grade of membership of a 26 years old person to the fuzzy set *old*. Since the grade of membership of an object in a fuzzy set is a subjective matter, we will get different answers from different experts.

Conventional fuzzy logic has to consider all of the answers to reach a fixed and deterministic membership value between zero and one. Later in this paper a new concept of *probabilistic fuzzy set* is introduced in which the membership value for each object x, denoted by $\mu(x)$ is a random variable defined with its probability density function. Thus, in probabilistic fuzzy logic, $\mu_{old}(26)$ is defined as a random variable with a known probability density function.

1.3. *Probabilistic Versus Deterministic Modeling:*

Many of the real world complex systems may exhibit randomness in their behavior. Human control strategy [4] is one example of such non-deterministic system. This means a human may not necessarily repeat the same action or response in various times under the same conditions. Modeling these complex systems can be more realistic if their statistical properties are also modeled using a probabilistic modeling strategy.

Fuzzy modeling techniques are well established and (due to their universal approximation property [5]) are extensively used for modeling complex and nonlinear *deterministic* systems [6][7][8]. They are not suitable however, in their conventional form, for *probabilistic modeling* of randomized and stochastic systems. Although it is possible to modify existing fuzzy modeling techniques to increase modeling accuracy of randomized systems [9], the need for a *probabilistic fuzzy modeling* approach is inevitable. The proposed *probabilistic fuzzy logic* in this paper leads to definition of *probabilistic fuzzy systems*, which can be used as a framework for *probabilistic fuzzy modeling*.

The proposed framework is expected to have more capabilities in handling both the statistical and non-statistical uncertainty concurrently.

2. Probabilistic Fuzzy Logic

Fuzzy logic, similar to some other brilliant scientific theories, is not simply a new theory but also is an extension to previous theories, and in particular the conventional Boolean logic. Fuzzy logic concerns with the general concept of "*degree of truth.*" It is an extension to conventional crisp logic in the sense that *degree of truth* is no longer limited to zero and one. Probability, on the other hand concerns with the concept of "*probability of truth*" and gives information about the likelihood of an event in the future. As stated before, probability and fuzziness are representing two different kinds of uncertainty, statistical (or random) and non-statistical uncertainty.

Consequently, the new concept of probabilistic fuzzy logic is introduced as an extension to conventional fuzzy logic such that the truth value is not only specified with a degree of truth between zero and one, but also with a probability of truth in the form of a probability number or a probability distribution function (pdf). As illustrated in Figure 1, *degree of truth* and *probability of truth* are simultaneously considered in a probabilistic fuzzy logic. Probabilistic fuzzy logic can be also interpreted as a special kind of fuzzy logic in which the degree of truth is a random variable taking values between zero and one (this is the basic issue in probabilistic fuzzy sets). The above extension principle also holds here because the conventional fuzzy logic is a special case of probabilistic fuzzy logic where the probability of truth is either zero or one.

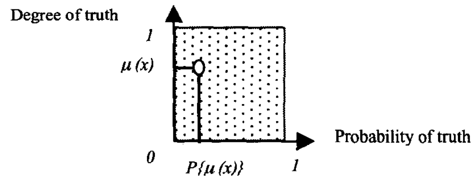

Fig. 1. Probabilistic Fuzzy Logic

Example 2

Consider the following sentence.

I think it will rain *heavily* with probability 90%.

There is a non-statistical uncertainty about the exact amount of rain and "*heavily*" is used as a fuzzy concept to represent this uncertainty. However, the probability of truth is also included in the sentence to represent human skepticism about its truth value (the probability of heavy rain).

Probabilistic Fuzzy Set

Probabilistic fuzzy logic leads us to definition of some other new concepts such as *probabilistic fuzzy set*. Unlike the conventional fuzzy set, the degree of membership (membership value) in a probabilistic fuzzy set $\mu(x)$ is no longer specified with a fixed known value for each x. In fact, the membership value is now a random variable and should be specified with its discontinuous (or continuous) probability distribution function (pdf). Regardless of its distribution, $\mu(x)$ should still remain between zero and one. Mathematically, a probabilistic fuzzy set in the universe of discourse U is defined as follows:

$$A=\{x, f_{\mu(x)}(\mu(x)) \mid x \in U\} \tag{1}$$

Where $f_{\mu(x)}(\mu(x))$ *is the pdf of the random variable* $\mu(x)$ such that:

$$P(a < \mu(x) < b) = \int_a^b f_{\mu(x)}(\mu(x))\, d\mu(x) \tag{2}$$

$f_{\mu(x)}(\mu(x))$ is the continuous probability distribution function of the random variable $\mu(x)$. In the general case, this pdf can be a function of both the x and $\mu(x)$, (i.e. $f=f_{\mu(x)}(\mu(x),x)$).

Example 3

A probabilistic fuzzy set A has been defined such that its membership value is a normal random variable with its mean values ($E\{\mu_A(x)\}$) on a triangular function centered at zero and spanned over [-4,4] and its variance defined as in (4). Furthermore, $\mu_A(x)$ is limited to zero and one.

$$E\{\mu_A(x)\} = \overline{\mu}_A(x) \tag{3}$$

$$\sigma\{\mu_A(x)\} = k(0.5 - |\overline{\mu}_A(x) - 0.5|)^2 \tag{4}$$

Where $\overline{\mu}_A(x)$ is the conventional triangular membership function and $k > 0$ is a weighting factor for variance.

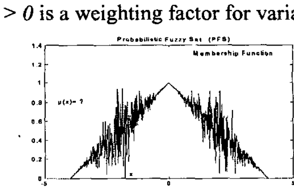

Fig. 2. An example of a probabilistic fuzzy set

3. Probabilistic Fuzzy Reasoning

Natural language (NL) has been a good source of inspiration for invention of the fuzzy logic and fuzzy reasoning. The common aspect of both the reasoning in natural language and in fuzzy logic is their approximate nature. This approximate nature is because of the *generalized modes ponens* law in fuzzy logic. Consider the following sentence:

If it is cloudy today, then it will rain this afternoon

In classical logic, the weather is either cloudy or not cloudy and the truth-value of the whole sentence can be inferred using the *modes ponens* law of inference. In fuzzy logic, on the other hand, cloudy and rainy are fuzzy concepts and *generalized modes ponens* law states that the cloudier the sky is, the heavier it will rain this afternoon. This is an example of the approximate reasoning in fuzzy logic where partial truth can be tolerated. In this way, fuzzy logic extends the scope of inference laws from the crisp world of classical logic to the real world of uncertainty and fuzziness.

But the scope of uncertainties in real world is much broader. Statistical uncertainty and randomness is another type of uncertainty that may not be represented with fuzzy logic. Indeed, it has been represented with probability theory and random processes. This type of uncertainty exists in natural language as well. Consider the following sentence for example:

If it is cloudy today, then it will probably rain this afternoon

Although the exact degree of cloudiness may be known in a fuzzy manner, we may not be certain about the result of inference because of another existing uncertainty in the form of probability and randomness. Since the statistical uncertainty is inherently associated with time, we may remain in doubt until the afternoon arrives when the statistical uncertainty about the result of inference disappears but the uncertainty about the compatibility of the weather with the concept "rainy" remains. The above reasoning and the associated if-then rule are named *probabilistic fuzzy reasoning* and *probabilistic fuzzy rule*, respectively. In a probabilistic fuzzy rule, quantitative representation of both types of uncertainties is possible through *membership function* and *probability distribution function*, respectively.

Probabilistic fuzzy logic thus involves two types of reasoning about the truth-value of a proposition, the reasoning about degree of truth and about probability of truth, and hence is an extension to both the fuzzy reasoning and probabilistic reasoning.

Probabilistic Fuzzy Rule (PFR)

A conventional fuzzy if-then rule with multiple inputs and single output can be represented in the following form:

$$If\ x_1\ is\ A\ and\ x_2\ is\ B\ \ldots\ and\ x_n\ is\ C\ then\ y\ is\ OMF(k) \tag{5}$$

Where A, B, ... and C stand for input membership functions (fuzzy sets) and $OMF(k)$ is the k^{th} output membership function (output fuzzy set). Generalization to MIMO case is simple and straightforward.

In a conventional fuzzy if-then rule structure, for a given output variable y, a combination of input fuzzy sets is mapped into a fixed and known output fuzzy set (linguistic value) and thus resulting a deterministic input-output mapping. This mapping can be illustrated in a look up table. As a result, the consequent part of a conventional fuzzy if-then rule refers to a unique output fuzzy set for the given output variable.

In a *probabilistic fuzzy rule* on the other hand, each possible combination of the input sets may be mapped into each of the possible output fuzzy sets. The selection probability for all output fuzzy sets is known a priori while the actual selected output fuzzy set remains unknown until selected by a random mechanism based on the associated probabilities of the sets. Therefore the consequent part of each probabilistic fuzzy rule (PFR) is no longer an index to a particular fuzzy set but is a vector of probabilities, P for various output fuzzy sets, each element of the vector is the likelihood of selection of that particular output fuzzy set as the consequent. The mechanism of input-output mapping for each rule is shown in the following example.

Example 4

Consider a 2-input, 1-output fuzzy system having membership functions $\{A_1, A_2, A_3\}$ and $\{B_1, B_2, B_3\}$ for the first and second input variables, respectively. Also having membership functions $\{OMF(1),\ OMF(2),\ OMF(3),\ OMF(4),\ OMF(5)\}$ for the output variable.

A sample probabilistic fuzzy rule may be in the following form:

$If\ x_1\ is\ A_1\ and\ x_2\ is\ B_2\ \ then\ \ y\ is\ OMF_1$ (with probability P_1) & $y\ is\ OMF_2$ (with probability P_2) & $y\ is\ OMF_3$ (with probability P_3) ... & $y\ is\ OMF_5$ (with probability P_5) (6)

$$P = [P_1, P_2, P_3, \ldots, P_5],\ \ P_1 + P_2 + P_3 + \ldots + P_5 = 1 \tag{7}$$

To investigate the implication and aggregation process in a *probabilistic fuzzy system*, consider a simple *max_min Mamdani* inference engine. Selection of the proper output membership function (fuzzy set) for implication is performed using a random selection mechanism such as roulette wheel. The below figure illustrates the implication mechanism.

The output probabilities comprise the weighting coefficients in the roulette wheel selection mechanism, which operates each time the outputs are evaluated and selects an output membership function based on its probability (likelihood) of selection

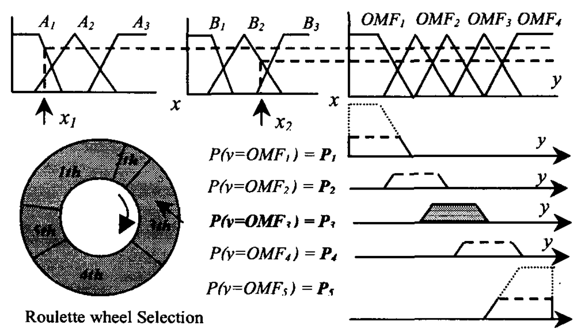

Fig. 3. Implication process in a probabilistic fuzzy rule

4. Probabilistic Fuzzy Systems and Probabilistic Fuzzy Modeling

Probabilistic fuzzy sets, probabilistic fuzzy rules or both can be used as building blocks of the proposed probabilistic fuzzy system resulting a probabilistic framework which exhibit a randomized nonlinear fuzzy mapping between inputs and outputs. Since a fuzzy system is more dependent on fuzzy rules rather than fuzzy membership functions, the random nature of the mapping is more significant when probabilistic fuzzy rules are used. To more investigate probabilistic fuzzy systems, consider the following example of a probabilistic system with probabilistic fuzzy rules defined by a human expert.

Example 5

Consider designing a fuzzy controller for liquid level control in a tank through its input valve position. A simple fuzzy controller may employ Δh and dh/dt as inputs and $d\alpha/dt$ (changes of input valve position) as the output. Where h is the actual liquid level, h_d is desired value of the level, $\Delta h=h_d -h$ is the error in level and α is the valve position, $\alpha\in[0,1]$

Three Gaussian input membership functions (*negative, zero, positive*) and five triangular output membership functions (*close-fast, close-slow, no-change, open-slow, open-fast*) are used for each variable. The following fuzzy rules may be obtained using a human expert's knowledge.

R1. If Δh is zero then valve is *no-change*
R2. If Δh is *positive* then valve is *open-fast*
R3. If Δh is *negative* then valve is *close-fast*
R4. If Δh is *zero* and *dh/dt* is *positive* then valve is *close-slow*
R5. If Δh is *zero* and *dh/dt* is *negative* then valve is *open-slow*

In order to model the existing skepticism of humans' opinion in defining the optimal rule set, we may substitute each conventional rule with a probabilistic fuzzy rule with output probability vector P defined such that the only output sets of the conventional fuzzy rules are the most probable output sets of the probabilistic fuzzy rules. Also the neighboring fuzzy sets in the PFR have smaller probabilities and the other fuzzy sets have zero probabilities. For example rule R1 in the above rule set may be modified as follows:

R1. If Δh is zero then valve is *no-change* with probability 80% and valve is *close-slow* with probability 10% and valve is *open-slow* with probability 10%

The consequent part of the PFR can be thus expressed in a compact form using the output probabilities vector P. Here is a sample probabilistic fuzzy rule set:

R1. If Δh is zero $P=$ [0.0 0.1 0.8 0.1 0.0]
R2. If Δh is *positive* $P=$ [0.0 0.0 0.0 0.2 0.8]
R3. If Δh is *negative* $P=$ [0.8 0.2 0.0 0.0 0.0]
R4. If Δh is *zero* and *dh/dt* is *positive,* $P=$ [0.1 0.8 0.1 0.0 0.0]
R5. If Δh is *zero* and *dh/dt* is *negative,* $P=$ [0.0 0.0 0.1 0.8 0.1]

Step responses of both the initial conventional and the new probabilistic controllers are shown in Figure 4. It should be noted that this is only a model of the human skepticism and is not an optimal controller for the system.

The extra parameters of a probabilistic fuzzy system such as output probability vector in probabilistic fuzzy rules and variance factor in the previously defined probabilistic fuzzy set are classified as *statistical parameters*. Statistical parameters determine degree of randomness in a probabilistic fuzzy system. It is always possible to interpret a conventional fuzzy system as a probabilistic one in which the statistical parameters are selected such that degree of randomness tends to zero (For example zero probabilities for all except one output membership functions in a PFR).

Another possible way of designing a probabilistic fuzzy system may be to use an automatic learning algorithm such as evolutionary learning methods to determine the parameters of the system. This learning may be done through observation while using a similarity measure to measure the fidelity between a set of observed input-output data-pairs of the real system to be learned and the probabilistic fuzzy model.

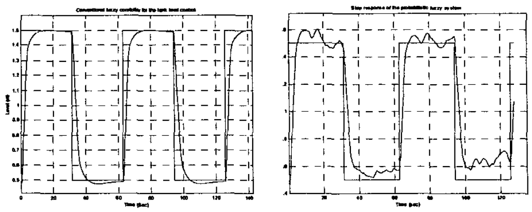

Fig. 4. Conventional (a) and probabilistic (b) fuzzy controller step responses

5. SUMMARY

- There is very little absolute knowledge in the real world. Inquiry must be a process of removal of doubt and skepticism or representing it properly. One of the main aspects of this skepticism has been identified as fuzziness, which is well represented with conventional fuzzy logic.

Another aspect of the existing skepticism is uncertainty about the future of random processes and probabilistic events. Many of the real world systems, although may not have a random nature, may seem random to us due to insufficient knowledge and should be modeled accordingly. Although these two aspects have, up until now, been studied separately, the new concept of probabilistic fuzzy logic tries to merge them in a unifying framework.

- *Probabilistic fuzzy logic* is a new approach for incorporating probability in fuzzy logic in order to better represent non-deterministic real world systems. It has not only the advantages of the approximate reasoning property of fuzzy systems, but also can be regarded as an extension to conventional one in the sense that the latter is a special case of the former with zero degree of randomness.

- The existing randomness in a probabilistic fuzzy system can be interpreted as:

1. Probabilistic nature in some of the natural language propositions and human reasoning as well as the statistical differences and variety of different human experts' knowledge.
2. Existing randomness in many of the real world systems, which is required to be modeled, including human skepticism in defining the fuzzy knowledge base.

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