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Rationally triangulable automorphisms

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Abstract

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This paper provides a necessary and sufficient condition for the rational triangulability of actions of the algebraic group G_a on affine space. The criterion is used to demonstrate the rational triangulability of all G_a actions on $A^3(k)$, as well as to prove, for arbitrary n, that all G_a actions are stably rationally triangulable.

1. Introduction

A rational action of an algebraic group G, defined over the characteristic zero, algebraically-closed field k, on the affine space $A^n(k)$, is said to be *triangulable* if coordinates $x_1, \ldots, x_n$ can be chosen so that the induced automorphism on the coordinate ring has the form $x_i \mapsto \alpha_i x_i + F_i(x_1, \ldots, x_{i-1})$ with α_i in the multiplicative group of k. The action is said to be *linear* if there is a coordinate system on which it is effected by a linear change of variables, and *tame* if it lies in the group generated by the triangular and linear automorphisms.

It is known that the automorphism group of $A^2(k)$ is the amalgamated free product of the groups of linear and triangular automorphisms, but it remains unknown whether these subgroups generate the automorphism group if $n \geq 3$. Bass, in [1], and Popov, in [4], have given examples of actions of the additive group of k, denoted G_a, on $A^3(k)$ which are neither linearizable nor triangulable. The structure theory of amalgamated products thus shows that the automorphism group cannot have this structure for $n \geq 3$.

Two approximations to tameness are the notions of stable tameness and rational triangulability. An action of G on $A^n(k)$ is *stably tame* provided its extension to $A^{n+m}(k)$ by fixing the last m coordinates is tame, and *rationally* *triangulable* if there are generators $y_1, \ldots, y_n$ of the field of rational functions so that each of the subfields $k(y_1, \ldots, y_i)$ is invariant under the group of k-automorphisms of the rational function field induced by G. In [6], Smith showed that the examples of Popov are stably tame. It was asked in [1] whether every rational action of a unipotent group on affine space is rationally triangulable.

This paper provides a necessary and sufficient condition for the rational triangulability of actions of the additive group of k on affine space. The criterion can be used to demonstrate the rational triangulability of all G_a actions on $A^3(k)$, in particular those of [1] and [4], as well as to prove, for arbitrary n, that all G_a actions are stably rationally triangulable (indeed they are rationally triangulable in the extension of the action to $A_{n+1}(k)$).

2. Generation of purely transcendental extensions

We begin with a general result on pure transcendental extensions of degree one of an arbitrary field of characteristic zero.

Theorem 2.1. *Let K be a field of characteristic zero, not necessarily algebraically closed, and $K(z)$ a simple transcendental extension. An element $w \in K(z)$ satisfies $K(w) = K(z)$ if and only if there is an automorphism f of $K(z)$ fixing K and sending w to $w + c$ for some nonzero $c \in K$.*

Proof. If $K(z) = K(w)$, then $f(w) = w + 1$ is the desired automorphism. Conversely suppose that f is a K-automorphism of $K(z)$ mapping w to $w + c$ for some nonzero c in K. Then the group $\langle f \rangle$ of K-automorphisms generated by f is infinite, translates w, and leaves $K(w)$ invariant.

Since f fixes K and moves w, $K(z)$ is a finite-dimensional, separable extension of $K(w)$; in particular, only finitely many places of $K(w)$ over K ramify in $K(z)$. Since $K(w)$ is $\langle f \rangle$-invariant, those places are permuted by the $\langle f \rangle$ action and therefore an infinite subgroup H of $\langle f \rangle$ fixes them.

Let $h \in H$ with $h(w) = w + a$, for some $0 \neq a \in K$, and let $\mathcal{P}$ be a ramified place. Then $h\mathcal{P} = \mathcal{P}$ and $\mathcal{P}(w) = (h\mathcal{P})(w) = \mathcal{P}(w + a) = \mathcal{P}(w) + a$. Since $a \neq 0$, $\mathcal{P}(w) = \infty$. Thus the only places which ramify are the poles of w.

Let $[K(z) : K(w)] = n$ and let $\bar{K}$ be an algebraic closure of K. Then $[\bar{K}(z) : \bar{K}(w)] = n$ and again ramification can occur only at the places of $\bar{K}(w)$ which are poles of w. The remainder of the proof follows as in [8, p. 232]. Namely, with G (resp. g) denoting the genus of $\bar{K}(z)$ (resp. $\bar{K}(w)$), $\mathcal{D}$ the different, and $d^0(\mathcal{D})$ its degree, the Hurwitz–Zeuthen formula $2G - 2 - n(2g - 2) = d^0(\mathcal{D})$ yields $d^0(\mathcal{D}) = 2n - 2$, since $G = g = 0$. But the concentration of the ramification at the poles of w implies that $d^0(\mathcal{D}) \leq n - 1$. Thus, $n = 1$. $\square$

The theorem clearly does not hold in positive characteristic p. One simply takes $w = z^p$ and $g(z) = z + 1$.

The following result shows that rationally triangulable actions of G_a have a particularly simple form.

Theorem 2.2. *If $G = G_a$, acting rationally and nontrivially on $A^n(k)$, is rationally triangulable, then $k(x_1, \ldots, x_n) = k(z_1, \ldots, z_n)$ where $k(z_1, \ldots, z_{n-1})$ is fixed by G, and for all $\sigma \in G$, $\sigma(z_n) = z_n + t_\sigma$ for $t_\sigma \in k(z_1, \ldots, z_{n-1})$.*

Proof. Let $y_1, \ldots, y_n$ generate $k(x_1, \ldots, x_n)$ with the fields $k(y_1, \ldots, y_i)$ invariant under the given G_a action. By Rosenlicht's cross-section theorem, $k(y_1, y_2) = K^G(w)$ where K^G is the fixed field of G in its restriction to $k(y_1, y_2)$, and w is transcendental over k [2, p. 152]. By the generalized Luroth theorem [3], $K^G = k(z_1)$. Thus $k(x_1, \ldots, x_n) = k(z_1)(w_1, y_3, \ldots, y_n)$. By induction, it follows that $k(y_1, \ldots, y_n) = k(z_1, \ldots, z_{n-1})(w_n)$ and that $k(z_1, \ldots, z_{n-1})$ is fixed by G.

Since G acts rationally and nontrivially on $k[x_1, \ldots, x_n]$, there is a finite-dimensional generating subspace V on which the action can be represented by unipotent matrices, and an element $w \in V$ for which $\sigma(w) = w + t_\sigma$ for all $\sigma \in G$. It follows from Theorem 2.1 that $k(z_1, \ldots, z_{n-1})(w_n) = k(z_1, \ldots, z_n)(w)$. □

3. Rationally and stably rationally triangular actions

A rational action of an algebraic group G on an affine domain A over k has a unique extension to an action on the field of fractions K. The subfield of K fixed elementwise by the extended action will be denoted K^G.

Theorem 3.1. *Every rational action of G_a on $k[x_1, \ldots, x_n]$ is stably rationally triangulable. An action is rationally triangulable if and only if $k(x_1, \ldots, x_n)^{G_a}$ is a pure transcendental extension of k.*

Proof. Let $k^{(n)} = k(x_1, \ldots, x_n)$ and $F = k^{(n)G_a}$. Then $k^{(n)} = F(w)$ and G_a acts as translations on w. The second assertion is therefore obvious. However, $F(w)$ is a pure transcendental extension of k, and so therefore is $F(x_{n+1})$ for a new variable x_{n+1}. The action of G_a, extended to $k^{(n)}(x_{n+1})$ by fixing this variable, is therefore rationally triangulable. □

Corollary 3.2. *Every rational G_a action on $k[x_1, x_2, x_3]$ is rationally triangulable.*

Proof. According to Castlenuovo's theorem, a unirational field of transcendence degree 2 over an algebraically-closed field of characteristic zero is pure transcendental. This applied to F of the previous theorem yields the result. □

A finite-dimensional linear representation of G_a in $\mathrm{GL}(V)$ induces an action on the affine space spec $S(V)$, where $S(V)$ is the symmetric algebra of V. Such an action is called a *linear G_a action*. Alternatively, a nilpotent endomorphism of V extends to a locally nilpotent derivation of $S(V)$, which can be exponentiated to yield a one-parameter group of automorphisms of $S(V)$ isomorphic to G_a. Indeed, all linear G_a actions arise in this way. If δ is such a derivation and $f \in S(V)$ one of its constants, then $t \mapsto \exp(tf\delta)$ is a G_a action since $f\delta$ is at least locally nilpotent on $S(V)$. The examples of Bass [1] and Popov [4] are precisely of this form, and so will be called *Popov G_a actions*.

Corollary 3.3. *All Popov G_a actions are rationally triangulable.*

Proof. The Jordan normal form of a nilpotent endomorphism of V shows that a linear G_a action is triangulable (a fortiori rationally so). As such the fixed field is pure transcendental over k. However, the fixed field of a Popov action is identical to that of the linear action from which it was derived. □

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