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Highlihgts

- Analysis of a peak-load pricing framework on a network

- General conditions for existence and uniqueness of the market equilibrium

- Characterization of equilibrium investment and production

Uniqueness of Market Equilibrium on a Network: A Peak-Load Pricing Approach

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Abstract

In this paper we establish conditions under which uniqueness of market equilibrium is obtained in a setup where prior to trading of electricity, transmission capacities between different market regions are fixed. In our setup, firms facing fluctuating demand decide on the size and location of production facilities. They make production decisions constrained by the invested capacities, taking into account that market prices (partially) reflect scarce transmission capacities between the different market zones. For this type of peak-load pricing model on a network we state general conditions for existence and uniqueness of the market equilibrium and provide a characterization of equilibrium investment and production. The presented analysis covers the cases of perfect competition and monopoly—the case of strategic firms is approximated by a conjectural variations approach. Our result is a prerequisite for analyzing regulatory policy options with computational multilevel equilibrium models, since uniqueness of the equilibrium at lower levels is of key importance when solving these models. Thus, our paper contributes to an evolving strand of literature that analyzes regulatory policy based on computational multilevel equilibrium models and aims at taking into account individual objectives of various agents, among them not only generators and customers but also, e.g., the regulator deciding on network expansion.

1. Introduction

The peak-load pricing literature analyzes investment incentives in industries where demand is fluctuating and storability of the output is limited; see [6] for an overview. In such an environment firms will find it optimal to invest in a differentiated portfolio of base- and peak-load technologies. For the case of perfectly competitive markets, the unique equilibrium of this game is welfare optimal, i.e., firms take the right investment and production decisions. The

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approach of peak-load pricing is currently extensively used to analyze electricity markets, e.g., by [34] or [31], and many others.

The scope of this paper is to extend existence and uniqueness results of the peak-load pricing literature to the case where producers and consumers interact on a network. This is an important contribution to the literature on liberalized electricity markets, where typically private firms decide on investment and production, guided by incentives from spot market trading. In such an environment an adequate model of peak-load pricing on a network must account for the network constraints that the agents face at the spot markets whenever they are reflected in the spot market prices. One of the results of our analysis is that the consideration of network constraints in a model of peak-load pricing does not require additional assumptions to guarantee uniqueness of the equilibrium. That means, all assumptions on cost and demand functions that guarantee a unique solution in the absence of network considerations will always guarantee uniqueness when also considering network constraints. The ability to establish a unique solution of this game is a prerequisite to meaningfully analyze complementary decisions taken by other agents—such as the regulator's decisions on network expansion or the regulatory framework itself; see e.g., the analysis in [21].

In this paper we propose a framework that captures trading at spot markets, where market prices reflect scarce network capacities. Demand at each node is fluctuating. We analyze a setup where firms decide on size and location of production facilities and make production decisions that are constrained by the invested capacities, taking into account regionally differentiated prices reflecting network constraints. We provide general conditions that allow to establish uniqueness of the resulting market equilibrium under perfect competition, characterize this equilibrium, and provide an intuitive example. In an extension we show that our results still hold if strategic behavior of firms is approximated based on the conjectural variations approach, analogously to the approach chosen, e.g., by [43].

As a key contribution we show that uniqueness of the market outcome in our setting can be guaranteed relying on the usual assumptions used in the entire literature on modeling liberalized electricity markets. In particular, this implies that uniqueness can be obtained without strong assumptions regarding convexity of investment and production cost. The latter is convenient in theoretical modeling, but typically not easily applicable, and thus not assumed, in numerical models. When it comes to applying numerical models in order to answer questions concerning market design, uniqueness of the outcome is important for several reasons. First, comparison of market designs in models that lead to multiple predictions of the outcome is difficult. A solution could be to resort to specifically tailored equilibrium selection procedures, which are, however, controversially discussed in the literature; see, e.g., [36] for two-stage stochastic programs or [29] and [37], which apply a specific equilibrium selection mechanism. Second, a model with multiple outcomes can hardly be used to analyze interaction of the modeled environment and some complementary decisions. An example is the analysis of the interdependency of generation investment and line expansion in electricity market models; see, e.g., [30] or [21].

To the best of our knowledge, our contribution is the first to establish uniqueness of the peak-load pricing equilibrium on a network. This is an important cornerstone to the multilevel analysis of situations where competitive firms have to make production and investment decisions facing network constraints. As it is well acknowledged in the literature, multiple solutions of lower level problems hinder the solution, interpretation, and comparison of results obtained in a multilevel context; see, e.g., [8], [5], or [17]. Our result is thus important to meaningfully analyze energy policy options in computational equilibrium models, which include network expansion plans or alternative regulatory regimes.

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It should be emphasized that our approach does not cover cases where further technical constraints, such as AC or DC flow models in electricity, are reflected in spot market prices. A prominent example is the consideration of a fully-fledged physical model upon the determination of spot market prices, as it is practiced in a system with nodal pricing. Instead, our analysis captures situations where electricity is traded between different market regions with uniform electricity prices and the transmission capacities between the regions are predetermined (i.e., they are independent of realized power flows). Note that this approach of congestion management at the market stage does not perfectly capture physical network constraints but aims at reflecting the main bottlenecks within the market clearing procedure. In practice this covers both the case of regular explicit auctions as well as implicit auctions for the assignment of scarce transmission capacities. Note that from a modeling perspective where all market participants hold rational expectations and play equilibrium strategies, the outcome of explicit cross border auctions corresponds to the outcome of fully coordinated implicit auctions (see, e.g., [11] or [7]). Explicit auctions are typically introduced at early stages of interconnecting liberalized electricity markets. In the past, these procedures had been used in markets in North America and also in Australia as well as Europe. Today, while some European countries switched to more complex flow based coupling, explicit auctions are still used in Switzerland, Greece, and the Balkan countries. A very prominent example outside Europe is Latin America, where explicit trading of cross border capacities takes place among various Latin American countries (see [44]). Note that whenever transmission capacities are exogenously determined prior to the bidding process a regime of implicit auctions is fully covered by our results. However, more recent developments of flow based market coupling in some European countries or nodal pricing in Northern America are not covered.

As a summary, our study contributes to enabling a rigorous analysis of explicit and implicit auctioning of scarce transmission capacity as it is typically introduced as a first step to connect recently liberalized markets. Therefore, besides the applicability to existing systems in Europe or Latin America, our results can be helpful for the analysis of future developments in Asia and Africa, where the interconnection of electricity markets may proceed. The consideration of more complex flow models is out of the scope of this paper and topic of future research.

Our work contributes to several strands of the literature. First, it directly extends the peak-load pricing literature to peak-load pricing on a network. The seminal contributions to the analysis of peak-load pricing date back to [2] and [40]. For a more recent summary of the main findings and contributions see [6]. These contributions establish existence and uniqueness of the perfectly competitive market equilibrium in the absence of network constraints. More recently this literature has also been extended to the case of strategic firms, e.g., by [34], [28], [45], [23], or [43]. Only [45] and [23] consider specific conditions that guarantee uniqueness of the resulting market equilibrium with strategic firms. In the general case with multiple and discrete production technologies, however, uniqueness cannot be obtained in a framework with strategic firms, not even in the absence of network restrictions. In our contribution we thus chose to approximate the case of strategic interaction by a conjectural variations approach, similar to the one applied recently by [43], which allows for the establishment of a unique solution.

Our article also contributes to the literature on market interaction in the presence of network constraints. This literature dates back to early contributions by [42] and [1], who were among the first to study optimal pricing on a network with several spatially located consumers and producers. [27] or [4] build on those seminal contributions to analyze optimal transmission pricing in electricity markets under nodal pricing—a regime that nowadays is

used in various electricity markets in the US, Canada, and some other countries. European and Australian electricity markets, however, predominantly use a system of zonal prices, where only predetermined "available transfer capacities" between zones are taken into account upon trading at the spot market. For a discussion also see [35], [11], or [10]. All those studies do not focus on uniqueness of the problem under consideration and most importantly abstract from firms' endogenous choice of production capacities, which is at the heart of our analysis.

The paper is organized as follows. In Sect. 2 we introduce the notation used throughout the paper and the considered peak-load pricing model is stated. Moreover, an equivalent reformulation of this model is given, which is used in Sect. 3 to prove the uniqueness of solutions of the peak-load pricing framework. Section 4 provides an illustrative example of our findings. Finally, Sect. 5 concludes and states some topics of further research.

2. A Framework of Peak-Load Pricing on a Network

2.1. Notation and Model Formulation

We consider a general transport network modeled by a connected and directed graph $G = (N, A)$ with node set N and arc set A. Flow on arc a is denoted by f_a, which is limited by the arc capacity $\bar{f}_a \in \mathbb{R}^+$, i.e., $|f_a| \leq \bar{f}_a$. Throughout the paper we make use of the standard δ-notation, i.e., the set of in- and outgoing arcs of a node set $M \subseteq N$ is given by

$$\delta^{\text{in}}(M) := \{a = (m, n) \in A : m \notin M, n \in M\},$$
$$\delta^{\text{out}}(M) := \{a = (n, m) \in A : n \in M, m \notin M\}.$$

The time horizon (or scenario set) that we consider in our peak-load pricing framework is given as an interval $T = [t_0, t_e] \subset \mathbb{R}$ with $t_0 < t_e$. Demand $d_n(t) \geq 0$ is located at every node $n \in N$. Elastic demand at node $n \in N$ and time $t \in T$ is modeled by a continuous function $p_n(t, \cdot) : \mathbb{R}^+ \to \mathbb{R}$. For later reference we note the following additional assumption on the demand functions:

Assumption 1. *All demand functions $p_n(t, \cdot), t \in T$, are strictly decreasing, i.e., $\partial_d p_n(t, d) < 0$.*

Under Assumption 1, we can specify the definition of our demand functions to $p_n(t) : [0, \bar{d}_n(t)] \to \mathbb{R}^+$, where $\bar{d}_n(t)$ is the unique root of $p_n(t)$. Further note that the gross consumer surplus, which is defined as

$$\int_0^{d_n(t)} p_n(t, x) \, dx,$$

is concave under Assumption 1 for all $t \in T$. Note that the assumption of elastic demand is standard in economic market models (e.g., compare [33]), irrespectively of whether network constraints are considered or not.

Moreover, at every node $n \in N$ a single production technology is located that is characterized by its variable production costs $c_n^{\text{var}} \in \mathbb{R}^+$ and its capacity investment costs $c_n^{\text{inv}} \in \mathbb{R}^+$. The chosen setup of a single technology per node is without loss of generality. All of our results also apply to the situation in which multiple producers with different technologies are located at the nodes. This can be easily seen by introducing an auxiliary node for every producer at the node and by connecting the auxiliary nodes with the original nodes by arcs with "infinite" capacity. Production at time $t \in T$ is denoted by $y_n(t) \in \mathbb{R}^+$ and capacity by $\bar{y}_n \in \mathbb{R}^+$, i.e., capacity is constant over time. Since actual production is nonnegative and restricted by the

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corresponding capacity, we have $0 \leq y_n(t) \leq \bar{y}_n$. It is crucial to note at this point that we do not impose strict convexity on the cost structure considered throughout this article. Both marginal cost of investment and of production of each technology is assumed to be constant. This is closely in line with the entire literature analyzing liberalized electricity markets (see the literature cited in Sect. 1), where the assumption of increasing marginal cost would be clearly unusual and unnatural. As a central contribution our results show that uniqueness in our setting obtains for the non-strict convex cost structure usually relied on in the literature. We remark that we analyze a perfectly competitive environment, i.e., all firms are price takers. Under the assumption of strategic firms it is easy to show that multiple equilibria would obtain in the present setup. Typically, papers that focus on strategic interaction analyze much simpler frameworks—and often still find multiple equilibria. Since the focus of our paper is to show uniqueness of the market game (in order to develop a basis to analyze policy proposals with computational equilibrium setups), we have to restrict attention to the case of perfect competition. In order to shed lights on a world with positive markups, in Sect. 3.3 we use a simplified approach that draws on the idea of conjectural variations; see, e.g., [18]. For later reference, we formalize an additional assumption on the variable production costs.

Assumption 2. *All variable production costs $c_n^{\mathrm{var}}, n \in N$, are pairwise distinct.*

This is a standard assumption in the peak-load-pricing literature; see, e.g., [6]. In the case without a network it directly implies that, w.l.o.g., we may assume for all producers n, n' that it holds that $c_n^{\mathrm{var}} < c_{n'}^{\mathrm{var}}$ implies $c_n^{\mathrm{inv}} > c_{n'}^{\mathrm{inv}}$. As in our case the location of a producer plays an important role, we need to use the formulation used in Assumption 2 as it generalizes to the network setting.

We now state the market model that is considered throughout the paper. To this end, we make use of our main economic assumption of perfect competition. This implies that "[. . .] all consumers and producers act as price takes. The idea behind the price-taking assumption is that if consumers and producers are small relative to the size of the market they will regard market prices as unaffected by their own actions"; see page 314 of [33]. Note that given the assumption of price taking, the specific ownership structure of generation facilities across nodes is irrelevant. Our assumption that each agent owns a single plant located at a specific node of the transport network is just made for notational convenience, which delivers equivalent results than situations where agents own several plants across nodes but still act as price takers. Note that in the absence of network or production constraints this assumption directly yields the result that under perfect competition price is equal to marginal cost of production; see, e.g., Chap. 10.C in [33]. Our results characterize the unique market equilibrium in the more general case where both production and network constraints are relevant.

Using the assumption of perfect competition we can formulate the optimization problems of all agents by using prices $\pi_n(t)$ as exogenously given data. We start with a producer (located at node $n \in N$), who maximizes its profit by solving the problem

$$\max_{y_n(\cdot), \bar{y}_n} \quad \int_T \pi_n(t) y_n(t) \, \mathrm{d}t - \int_T c_n^{\mathrm{var}} y_n(t) \, \mathrm{d}t - c_n^{\mathrm{inv}} \bar{y}_n$$
$$\text{s.t.} \quad 0 \leq y_n(t) \leq \bar{y}_n, \quad t \in T.$$

In what follows we make the technical assumption that all demand, price, and production functions are $L^2(T)$, i.e., square-integrable, functions. Thus, considering the optimization problems of the single players in continuous time leads to infinite-dimensional problems in

Banach spaces. In this setting, the optimality conditions of the producer at node n state that there exist $\beta_n^\pm(\cdot) \in L^2(T; \mathbb{R}^+)$ such that

$$0 = \pi_n(t) - c_n^{\text{var}} + \beta_n^-(t) - \beta_n^+(t) \quad \text{for almost all } t \in T,$$

$$0 = \int_T \beta_n^+(t)\, dt - c_n^{\text{inv}},$$

$$0 = \int_T y_n(t)\beta_n^-(t)\, dt = \int_T (\bar{y}_n - y_n(t))\beta_n^+(t)\, dt,$$

$$0 \leq y_n(t) \leq \bar{y}_n, \quad t \in T,$$

holds; see [24]. On the other hand, the consumer (located at node n) faces the problem

$$\max_{d_n(\cdot)} \quad \int_T \int_0^{d_n(t)} p_n(t, x)\, dx\, dt - \int_T \pi_n(t)d_n(t)\, dt$$

$$\text{s.t.} \quad d_n(t) \geq 0, \quad t \in T,$$

for which the first-order conditions state the existence of $\gamma_n(\cdot) \in L^2(T, \mathbb{R}^+)$ satisfying

$$0 = p_n(t, d_n(t)) - \pi_n(t) + \gamma_n(t) \quad \text{for almost all } t \in T,$$

$$0 = \int_T d_n(t)\gamma_n(t)\, dt,$$

$$0 \leq d_n(t).$$

Finally, the last player to model is the transmission system operator (TSO), who operates the transmission network. The TSO faces the optimization problem

$$\max_{(f_a(\cdot))_{a \in A}} \quad \int_T (\pi_m(t) - \pi_n(t))f_a(t)\, dt$$

$$\text{s.t.} \quad -\bar{f}_a \leq f_a(t) \leq \bar{f}_a, \quad t \in T,$$

for every arc $a = (n, m)$, which is equivalent to its first-order optimality conditions, i.e., the existence of $\varepsilon_a^\pm(\cdot) \in L^2(T; \mathbb{R}^+)$ with

$$0 = \pi_m(t) - \pi_n(t) + \varepsilon_a^-(t) - \varepsilon_a^+(t) \quad \text{for almost all } t \in T,$$

$$0 = \int_T (f_a(t) + \bar{f}_a)\varepsilon_a^-(t)\, dt = \int_T (\bar{f}_a - f_a(t))\varepsilon_a^+(t)\, dt,$$

$$-\bar{f}_a \leq f_a(t) \leq \bar{f}_a, \quad t \in T.$$

Taking all these optimality conditions together and adding the flow balance equations

$$\sum_{a \in \delta^{\text{in}}(n)} f_a(t) - \sum_{a \in \delta^{\text{out}}(n)} f_a(t) - d_n(t) + y_n(t) = 0, \quad t \in T, n \in N,$$

we obtain a mixed complementarity problem in which competitive prices clear the market as it is well-known for competitive equilibrium models in discrete time. Moreover, it can be seen that the presented model in continuous time is equivalent to a single welfare maximization

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problem as it is the case for discrete time models as well; see [25]. In our setting, this problem reads

$$\max_{d(\cdot),y(\cdot),\bar{y},f(\cdot)} \quad \sum_{n\in N}\int_T\int_0^{d_n(t)} p_n(t,x)\,\mathrm{d}x\,\mathrm{d}t - \sum_{n\in N}\int_T c_n^{\mathrm{var}}y_n(t)\,\mathrm{d}t - \sum_{n\in N} c_n^{\mathrm{inv}}\bar{y}_n \tag{1a}$$

$$\text{s.t.} \quad \sum_{a\in\delta^{\mathrm{in}}(n)} f_a(t) - \sum_{a\in\delta^{\mathrm{out}}(n)} f_a(t) - d_n(t) + y_n(t) = 0, \quad t\in T, n\in N, \tag{1b}$$

$$-\bar{f}_a \le f_a(t) \le \bar{f}_a, \quad t\in T, a\in A, \tag{1c}$$

$$0 \le y_n(t) \le \bar{y}_n, \quad t\in T, n\in N, \tag{1d}$$

$$0 \le d_n(t), \quad t\in T, n\in N. \tag{1e}$$

The equivalence can be shown by comparing the first-order optimality conditions of Problem (1) with the complementarity system consisting of the optimality conditions of all players.

Note that the objective function (1a) models total social welfare, which is the difference of gross consumer surplus aggregated over all scenarios (first term) and production as well as capacity investment costs (second and third term). Constraint (1b) models flow balance for every node in every scenario. It can be shown that the dual values $\alpha_n(t)$ are exactly the competitive equilibrium prices $\pi_n(t)$ of the above mentioned complementarity system which clear the market. Constraint (1d) states production restrictions according to capacity investment that is taken once for every node and which is thus independent of a specific time t. The network model covered by our analysis corresponds to a regular flow model, where flows on each line are subject to an upper capacity limit; see (1c). In all industries where firms' interaction is limited by network constraints those constraints are crucially relevant. Depending on the specifically considered industry also further constraints might be imposed on the market interaction of firms. Our results do not cover cases where such further constraints do play a central role. This is, for instance, the case when the physical transport laws need to be considered as mentioned in the introduction. In this case, e.g., the assumption of strict convexity of the cost structure or specifically tailored equilibrium selection procedures could be applied to restore uniqueness. An explicit and direct application of our results in the case of liberalized electricity markets is the case of firms' interaction in different price zones where congestion between zones is managed by implicit or explicit auction procedures; see, e.g., [10, 11].

We remark that we choose to state our peak-load pricing model (1) in continuous time for two reasons. First, it allows for a technically easier derivation and exposition of the results presented in Sect. 3; see, e.g., Assumptions 3 and 4. Second, the setting also allows for an easy extension to a nondeterministic setting. We may assume that we are given an additional probability measure on the set T; in this case the integral over T in (1a) transforms into

$$\mathbb{E}_t\left[\sum_{n\in N}\int_0^{d_n(t)} p_n(t,x)\,\mathrm{d}x - \sum_{n\in N} c_n^{\mathrm{var}}y_n(t)\right] - \sum_{n\in N} c_n^{\mathrm{inv}}\bar{y}_n,$$

where the expectation is taken over the given measure. For our results to hold in this case, one only has to make the technical assumption that the probability measure in question has a square-integrable density. Moreover, the measure is not restricted to only describe fluctuating demand over time but can also be used to model uncertainty about the demand (and possibly renewable supply) situation in the future. This argument illustrates that our approach is able to capture the incentives to install peak-load plants that respond to shortage events. In this respect, we have to note that our model captures investment incentives that origin from market prices resulting

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from market clearing at the spot market. Ceteris paribus, for smaller invested capacities market prices raise; see Assumption 1. The peak-load pricing framework considered is thus not suited to model the total breakdown of market clearing due to a shortage of installed capacities as it might occur during a blackout. The analysis of mechanisms to address such problems can thus not be in the focus of our analysis. Finally, another reason for the consideration of the model in continuous time is that it is more general than the setting of discrete time: All our assumptions and theoretical results carry over to the case of discrete time.

Model (1) is a concave optimization problem over a polytopal feasible set, where the boundedness follows from the production constraints (1d). Note again that in the considered setting the competitive market outcome can also be obtained equivalently from the welfare maximization problem. Next to the assumption of price taking behavior (this assumption is dropped in Sect. 3.3) this requires the absence of externalities and the completeness of markets. In the context of investment in electricity markets the assumption of complete markets has been challenged recently by observing that different agents typically tend to have different discount rates which gives rise to inefficiencies. For a recent analysis of this issue see, e.g., [13, 14]. The present results do not include the case of diverging discount rates, however.

2.2. Model Reformulation

Our goal is to show that the presented peak-load pricing framework has a unique solution. To this end, we equivalently reformulate Model (1) as

$$\max_{\bar{y}} \quad \psi(\bar{y}) := \int_T \phi(t, \bar{y}) \, \mathrm{d}t - \sum_{n \in N} c_n^{\mathrm{inv}} \bar{y}_n, \tag{2}$$

where $\phi(t, \bar{y})$ is defined as the optimal value function of the subproblem for fixed time t:

$$\phi(t, \bar{y}) := \max_{d_t, y_t, f_t} \quad \sum_{n \in N} \int_0^{d_n(t)} p_n(t, x) \, \mathrm{d}x - \sum_{n \in N} c_n^{\mathrm{var}} y_n(t) \tag{3a}$$

$$\text{s.t.} \quad \sum_{a \in \delta^{\mathrm{in}}(n)} f_a(t) - \sum_{a \in \delta^{\mathrm{out}}(n)} f_a(t) - d_n(t) + y_n(t) = 0, \quad n \in N, \tag{3b}$$

$$-\bar{f}_a \leq f_a(t) \leq \bar{f}_a, \quad a \in A, \tag{3c}$$

$$0 \leq y_n(t) \leq \bar{y}_n, \quad n \in N, \tag{3d}$$

$$0 \leq d_n(t), \quad n \in N. \tag{3e}$$

Here and in what follows, quantities without node or arc indices denote the vector of the corresponding quantities; e.g., $d := (d_n(\cdot))_{n \in N}$ is the vector of demand functions at all nodes $n \in N$.

Note that the master problem (2) is an unconstrained optimization problem and does not explicitly depend on the network flow model. Subproblem (3) is again a concave maximization problem over a polytopal feasible set in which the capacity investments are fixed.

This reformulated model has a strong similarity to a two-stage stochastic program. If we interpret the time integral (after normalization) as the expected welfare we see that in the first stage we choose long-term capacity investments which then parameterize the second stage, in which production and demand realize in dependence on the scenarios.

3. Existence and Uniqueness

Since existence of solutions is trivial (e.g., $(d, y, f) = (0, 0, 0)$ is always feasible), we focus on uniqueness of the solution. To this end, we exploit the decomposition into a master- and

a subproblem introduced in Sect. 2.2. First, we prove uniqueness of the Subproblem (3) in Sect. 3.1 and then show, using this result, the uniqueness of the master problem (2) in Sect. 3.2. By this, it directly follows that the original model (1) has a unique solution.

We remark that we choose a proof strategy that heavily relies on the specific formulation of the problem at hand. The reason is that this way it is possible to gain more insight into the structure of the solution compared to proof strategies that rely on more general results from the literature: For instance, the results from [32] could be used to prove uniqueness of demands in the subproblem but it would not allow for the insight of the existence of price clusters within every solution.

Before we start by proving the uniqueness of the solution of Problem (1), we note that replacing the linear cost functions

$$\sum_{n \in N} \int_T c_n^{\text{var}} y_n(t) \, dt, \quad \sum_{n \in N} c_n^{\text{inv}} \bar{y}_n,$$

in the Objective (1a) by convex cost functions

$$\sum_{n \in N} \int_T \hat{c}_n^{\text{var}}(y_n(t)) \, dt, \quad \sum_{n \in N} \hat{c}_n^{\text{inv}}(\bar{y}_n) \tag{4}$$

would yield a maximization problem that is strictly concave in d, y, and $\bar{y}$ and that thus obviously has a unique solution:

Theorem 1. *Consider Problem* (1) *with strictly convex cost functions* (4) *and suppose that Assumption 1 holds. Then, Problem* (1) *has a unique solution in* $(d, y, \bar{y})$.

Proof. See, e.g., the results from [32]. □

Note that the solution does not have to be unique w.r.t. to flows.

3.1. The Subproblem

We begin our considerations about the subproblem with the repetition of the simple observation that the subproblem is a concave maximization problem over a flow polyhedron with additional restrictions on the production variables y. The latter implies that the feasible set is a polytope.

For the sake of simplicity, we drop the argument t in this section. That is, e.g., $y = (y_n(t))_{n \in N} \in \mathbb{R}^N$ denotes the finite vector of production at all nodes for the considered $t \in T$.

The first step is to show that it is sufficient to prove that there is a unique solution if we fix the binding inequalities. For this, we define sets of active indices in dependence of a feasible point $z := (d, y, f)$:

$$\mathcal{A}_f^-(z) := \{a \in A \colon f_a = -\bar{f}_a\}, \qquad \mathcal{A}_f^+(z) := \{a \in A \colon f_a = \bar{f}_a\},$$
$$\mathcal{A}_y^-(z) := \{n \in N \colon y_n = 0\}, \qquad \mathcal{A}_y^+(z) := \{n \in N \colon y_n = \bar{y}_n\},$$
$$\mathcal{A}_d^-(z) := \{n \in N \colon d_n = 0\}.$$

We can now state the following lemma:

Lemma 1. *Suppose Assumption 1 holds. Then, exactly one of the two following cases occurs:*

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1. *There exist demands d^* and productions y^* such that every optimal solution of Subproblem (3) is of the form (d^*, y^*, f) for some flow f.*

2. *There exist two optimal solutions $z' := (d', y', f')$ and $z'' := (d'', y'', f'')$ of Subproblem (3) with $(d', y') \neq (d'', y'')$ and*

$$
\begin{aligned}
\mathcal{A}_y^-(z') &= \mathcal{A}_y^-(z''), & \mathcal{A}_y^+(z') &= \mathcal{A}_y^+(z''), \\
\mathcal{A}_f^-(z') &= \mathcal{A}_f^-(z''), & \mathcal{A}_f^+(z') &= \mathcal{A}_f^+(z''), \\
\mathcal{A}_d^-(z') &= \mathcal{A}_d^-(z'').
\end{aligned}
$$

As with every two distinct solutions the whole segment between them lies in the feasible set, the lemma is a consequence of the following observation: In the interior of this segment, the binding patterns coincide. Hence we can always choose suitable solutions. More formally, the lemma can be deduced from the following proposition.

Proposition 1. *Let $z^\lambda := (d^\lambda, y^\lambda, f^\lambda)$ be an infinite family of optimal solutions for $\lambda \in [0, 1]$ of the form*

$$
(d^\lambda, y^\lambda, f^\lambda) := \lambda(d^1, y^1, f^1) + (1 - \lambda)(d^0, y^0, f^0).
$$

Let $c^T z \leq r$ be a linear inequality such that $c^T z^\lambda \leq r$ holds for all $\lambda \in [0, 1]$. Then, exactly one of the following cases occurs:

1. *$c^T z^\lambda = r$ for all $\lambda \in [0, 1]$,*

2. *$c^T z^\lambda < r$ for all $\lambda \in (0, 1)$.*

Proof. By the definition of z^λ we can write $c^T z^\lambda = \lambda c^T z^1 + (1 - \lambda) c^T z^0$. This leads to the following observations: If $c^T z^0 = c^T z^1 = r$, we are in Case 1 and if $c^T z^0 < r$ and $c^T z^1 < r$ both hold, we are in Case 2. Hence, it remains to treat the case where exactly one of $c^T z^0 = r$ or $c^T z^1 = r$ holds. Without loss of generality we assume that $c^T z^0 = r$ and $c^T z^1 < r$ hold. Then, for $\lambda > 0$ we have

$$
c^T z^\lambda = \lambda c^T z^1 + (1 - \lambda) c^T z^0 = \lambda c^T z^1 + (1 - \lambda) r < \lambda r + (1 - \lambda) r = r.
$$

Thus, we are in Case 2. □

For the following it is advantageous to use the concept of price clusters.

Definition 1. Given a solution z of Subproblem (3), we say that a partition $C = \{C_i\}_{i=1}^I$ partitions the node set N into *price clusters*, if for all $C \in C$ holds, that for all nodes in the cluster C the shadow prices of the flow conservation constraints (i.e., the dual variables of Constraints (3b)) are equal. We also write $C(z)$ to emphasize the dependence on the solution z. An arc $a = (n, m)$ is called an *inter-cluster arc*, if $n \in C_i$ and $m \in C_j$ with $i \neq j$ and we denote the set of inter-cluster arcs by A^{inter}.

We now want to use a result shown by [38] in a slightly different situation; namely that price clusters of the network are characterized by the binding constraints in (3c). For this we introduce another partition.

Definition 2. Given a solution z of Subproblem (3), we say that the partition $C = \{C_i\}_{i=1}^I$ of the node set N is the *flow-induced partition*, if each C_i is a connected component of the graph $\tilde{G}(z) = (V, A \setminus A^{\text{sat}})$, where $A^{\text{sat}} := \{a \in A : |f_a| = \bar{f}_a\}$.

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With this definition, the required result reads as follows:

Theorem 2. *Let $z^* := (d^*, y^*, f^*)$ be an optimal solution of Subproblem* (3) *and let $C(z^*)$ be the corresponding flow-induced partition. Then,*

$$\phi(\bar{y}) = \max_{d,y} \quad \sum_{n \in N} \int_0^{d_n} p_n(x)\,\mathrm{d}x - \sum_{n \in N} c_n^{\mathrm{var}} y_n \tag{5a}$$

$$s.t. \quad \sum_{n \in C} d_n - \sum_{n \in C} y_n = \hat{f}_C, \quad C \in C(z^*), \tag{5b}$$

$$0 \le y_n \le \bar{y}_n, \quad n \in N, \tag{5c}$$

$$0 \le d_n, \quad n \in N, \tag{5d}$$

where $\hat{f}_C = \sum_{a \in \delta^{\mathrm{in}}(C)} f_a^* - \sum_{a \in \delta^{\mathrm{out}}(C)} f_a^*$ is the total in- or outflow of zone C. This implies that $C(z^*)$ is a partition into price clusters.

Proof. The proof is given in AppendixA. □

Thus, Lemma 1 combined with the cited result states that whenever there exist two different optimal solutions, there also exist two different solutions with the same price clusters. Moreover, the flows between these clusters are unique since they are at their bounds.

Lemma 2. *Suppose Assumption 1 holds. Then, exactly one of the two following cases occurs:*

1. *There exist demands d^* and productions y^* such that every optimal solution of Subproblem* (3) *is of the form (d^*, y^*, f) for some flow f.*

2. *There exist two optimal solutions $z' := (d', y', f')$ and $z'' := (d'', y'', f'')$ of Subproblem* (3) *with $(d', y') \ne (d'', y'')$ such that*

   (a) *$C(z') = C(z'')$ and*

   (b) *for z' and z'' it holds that Constraint* (3c) *is tight for an arc a if and only if a is an inter-cluster arc.*

Proof. The lemma follows directly from Lemma 1 with the following additional argument: Assume there exists an arc $a = (n, m)$ with $a \in \mathcal{A}_f^+(z')$ and a is not an inter-cluster arc, i.e., $n, m \in C$ for some $C \in C$. We show that we can modify solution z' so that we obtain an optimal solution $\tilde{z}'$ with the same activity pattern with the exception that $\mathcal{A}_f^+(\tilde{z}') = \mathcal{A}_f^+(z') \setminus \{a\}$. As a is not an inter-cluster arc, there must exist a path P connecting n and m completely lying in cluster C such that for all $a \in P$ it holds that $a \notin \mathcal{A}_f^+(z') \cup \mathcal{A}_f^-(z')$, i.e., no flow bound on P is active. That means it must be possible to send an additional amount of flow ε along P without violating any bounds. Hence, we can reduce the amount of flow sent along a by $\varepsilon/2$ and send the same amount along path P. This gives us a new flow $\tilde{f}'$. Set $\tilde{z}' := (d', y', \tilde{f}')$, then the flow bound for arc a is no longer active. As a was an arbitrary non-inter-cluster arc, we can iterate this procedure until only flow bounds on inter-cluster arcs are attained. This can be done with both z' and z'' and thus we obtain the desired result. □

The last lemma implies that the ambiguity of solutions has to be "inside" the price clusters. Thus, we only have to consider these clusters in the following. Since the network constraints

do not play a role within the price clusters, Subproblem (3) for a single cluster reduces to the concave maximization problem

$$\max_{d,y} \quad \sum_{n \in C} \int_0^{d_n} p_n(x) \, dx - \sum_{n \in C} c_n^{\text{var}} y_n \tag{6a}$$

$$\text{s.t.} \quad \sum_{n \in C} d_n - \sum_{n \in C} y_n = \hat{f}_C, \tag{6b}$$

$$0 \le y_n \le \bar{y}_n, \quad n \in C, \tag{6c}$$

$$0 \le d_n, \quad n \in C, \tag{6d}$$

where $C \subseteq N$ is the set of nodes of the considered price cluster and $\hat{f}_C$ is total in- or outflow of this cluster; see Theorem 2. The KKT conditions of this problem comprise the dual feasibility conditions

$$p_n(d_n) + \alpha + \gamma_n = 0 \quad \text{for all } n \in C,$$
$$-c_n^{\text{var}} - \alpha + \beta_n^- - \beta_n^+ = 0 \quad \text{for all } n \in C,$$

where $\alpha \in \mathbb{R}$ is the dual variable of Constraint (6b), $\beta_n^-, \beta_n^+, n \in N$, are the dual variables of the lower and upper production bounds in (6c), and γ_n is the dual variable of the demand bounds (6d). This immediately implies a single price $p_C := -\alpha$ with $p_C = p_n(d_n)$ for all $n \in C$ with $d_n > 0$. Nodes n with $d_n = 0$ do not contribute to the objective value and hence their price can be ignored. Moreover,

$$p_C - c_n^{\text{var}} + \beta_n^- - \beta_n^+ = 0 \tag{7}$$

holds for all $n \in C$ with $d_n > 0$.

Our goal is now to show that productions and demands inside a cluster are uniquely determined. The flow values within the price clusters, however, are not unique, since we can always modify a solution with a flow along a cycle as long as we stay inside the bounds. Since we do not consider, e.g., transportation costs, these ambiguous flows do not interfere with the optimal demand and production values and thus do not influence the objective function value. We summarize our findings in the following theorem:

Theorem 3. *Suppose Assumptions 1 and 2 hold. Then, there are unique demands d_C^* and production y_C^* such that every optimal solution of Model (6) has the form (d_C^*, y_C^*, f_C) for some f_C.*

Proof. Assume that the price inside the price cluster is given by p_C. As the demand functions p_n are strictly decreasing and thus bijective, there is a unique demand d_n for every $n \in C$. Hence, there exists a function $d_C(p)$ that maps every price p to the unique aggregate demand at that price point. We define $\hat{d}_C(p) := d_C(p) - \hat{f}_C$. As the demand function for each node is strictly decreasing, the aggregated function $\hat{d}_C(p)$ is strictly decreasing as well.

On the production side we can see that given a p_C we can immediately determine (by using Condition (7)) which nodes $n \in C$ are definitely not producing ($c_n^{\text{var}} > p_C$), the ones definitely producing at maximum capacity ($c_n^{\text{var}} < p_C$), and the ones where the production amount is indeterminate, i.e., between 0 and $\bar{y}_n$ ($c_n^{\text{var}} = p_C$). Under Assumption 2 there exists at most one node such that $c_n^{\text{var}} = p_C$. Hence for all nodes except at most one, the price p_C uniquely determines the production values of the nodes. Moreover, we obtain two functions $y_C^{\min}(p)$ and $y_C^{\max}(p)$ which are the minimal, resp. maximal, production in the price cluster at a given price p.

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Both of these functions are monotonically increasing. If we intersect the functions $\hat{d}_C$ and $y_C^{\min}$, we observe that they have at most one intersection point and analogously for the functions $\hat{d}_C$ and $y_C^{\max}$. From the construction of $y_C^{\min}$ and $y_C^{\max}$ it then follows that there is exactly one price p_C^* such that $y_C^{\min} \leq \hat{d}_C(p_C^*) \leq y_C^{\max}$. Hence, every optimal solution of our problem yields the same price p_C^*. From the discussion of the first paragraph the uniqueness of the demands then follows directly. For the production the uniqueness is also clear for all nodes except at most one. The production of this last node, however, is also uniquely determined by the market clearing constraint. □

The proof allows us also to conclude that the dual variables β are unique as well; see Condition (7).

Corollary 1. *Suppose Assumption 1 and 2 hold. Then, the difference $\beta_n^+ - \beta_n^-$ is unique for all nodes $n \in N$. If $\bar{y}_n > 0$, the values of the dual variables $\beta_n^\pm$ themselves are unique as well.*

All in all, we have the following result concerning Subproblem (3):

Theorem 4. *Suppose Assumption 1 and 2 hold. Furthermore, let $C = \{C_i\}_{i=1}^I$ be the unique partition of the node set into price clusters, let $A^{\text{inter}} := \{a = (n, m) \in A : n \in C_i, m \in C_j, i \neq j\}$ be the set of inter-cluster arcs. Then, the solution (d, y, f) of Subproblem (3) is unique in $(d, y, f_{A^{\text{inter}}})$.*

Proof. By Lemma 2 we need to consider two cases. In the first case we are done. We need to show that the second case cannot occur. This, however, follows directly from Theorem 3. □

3.2. The Master Problem

In this section we prove that—given the results of the preceding section—the master problem (2) has a unique solution. To this end, we prove that the Hessian $H(\bar{y})$ of ψ is negative definite. Since the linear terms $\sum_{n \in N} c_n^{\text{inv}} \bar{y}_n$ in (2) vanish in second order, the Hessian of ψ is completely given by the Hessian of the integral terms. Thus, we have to compute the second derivative $H(\bar{y})$ w.r.t. $\bar{y}$ of

$$\int_T \phi(t, \bar{y}) \, \mathrm{d}t. \tag{8}$$

We split this section into two parts: In Sect. 3.2.1, we determine the second derivative w.r.t. $\bar{y}$ of $\phi(t, \bar{y})$ for a fixed time t. The subsequent Sect. 3.2.2 then considers the second derivative of (8).

3.2.1. The Single-Scenario Case

In this section we compute the Hessian for a fixed time t, i.e., the Hessian

$$H(t, \bar{y}) = \nabla_{\bar{y}\bar{y}}^2 \phi(t, \bar{y})$$

of $\phi(t, \bar{y})$. The first-order partial derivatives are known from standard sensitivity analysis (see, e.g., [3]) of convex optimization:

$$\frac{\partial}{\partial \bar{y}_n} \phi(t, \bar{y}) = \beta_n^+(t), \quad n \in N,$$

where $\beta_n^+(t)$ is the dual variable corresponding to the upper bound in Constraint (3d). We note that this condition only holds for $\bar{y}_n > 0$. We will later, however, make an assumption

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Table 1: Subsets of the node set and time horizon as well as (blocks of) considered Hessian matrices (all these sets depend on $\bar{y}$)

Set	Explanation
$C_i(t) \subseteq N$	ith price cluster at time t
$C(t) = \{C_i(t)\}_{i=1}^{I_t}$	Partition of the node set into price clusters for time t
$A_i(t) \subseteq C_i(t)$	Nodes of price cluster $i \in \{1, \ldots, I_\tau\}$ in time t with $\beta_n(t) > 0$
$\hat{T}$	Times t where solutions of Problem (9) do not satisfy strict complementarity
$T_\tau \subseteq T$	Times t with equal price clusters $C_i(t)$
$\mathcal{T} = \{T_\tau\}_\tau$	Price cluster specific time horizon partition
$T_{\tau,j} \subseteq T_\tau$	times t with equal price clusters and equal binding production nodes
$\{T_{\tau,j}\}_j$	Price cluster and active production nodes specific time horizon subset partition
H	Hessian of ψ
$H(t)$	Hessian of ψ for a single time t
H_τ	Hessians of ψ for the time $t \in T_\tau$
$H_{\tau,i}$	Submatrix (block) of H_τ induced by price cluster i
$H_{\tau,i,j}$	Submatrix (block) of $H_{\tau,i}$ induced by active production nodes

(Assumption 4) that implies this for all nodes $n \in N$. Thus, we now have to compute the derivative of $\beta_n^+(t)$ with respect to $\bar{y}_m$ for all $n, m \in N$. In the following we require a series of partitions of the node set and the time horizon. An overview over all partitions and subsets is given in Table 1. For a fixed time t, we obtain a partition $C(t, \bar{y}) = \{C_i(t, \bar{y})\}_{i=1}^{I_t}$ of the node set N into price clusters as described in the last section. Now, we consider a single price cluster $C_i(t, \bar{y})$, i.e., we fix some $i \in \{1, \ldots, I_t\}$ for the moment. It can be easily verified that the first-order conditions of Subproblem (3) imply

$$\beta_n^+(t) = \begin{cases} p_i(t) - c_n^{\text{var}}, & \text{if } y_n(t) = \bar{y}_n, \\ 0, & \text{if } y_n(t) < \bar{y}_n, \end{cases}$$

where $p_i(t)$ is the price of cluster $C_i(t, \bar{y})$. The derivative of $\beta_n^+(t)$ w.r.t. $\bar{y}_m$ is obviously zero for every node $m \in N$ in the second case. The first case, i.e., the case in which $y_n(t) = \bar{y}_n$ with $n \in C_i(t, \bar{y})$ holds, is more complicated. Let $A_i(t, \bar{y}) \subseteq C_i(t, \bar{y}) \subseteq N$ be the set of nodes of the price cluster $C_i(t, \bar{y})$ that are strictly active, i.e., all nodes $m \in N$ with $\beta_m^+(t) > 0$, which implies $y_m(t) = \bar{y}_m$. As an auxiliary result we first need to compute the derivative of the total demand of a single cluster with respect to the capacity of a single node of that cluster. To this end, we first rewrite Model (6) for cluster $C_i(t, \bar{y})$ using the aggregated demand function $P_i(t, \cdot)$ and the

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total demand $D_i(t)$:

$$\phi_{C_i(t)}(\bar{y}) := \max_{D_i(t), y_i(t)} \quad \int_0^{D_i(t)} P_i(t, x) \, \mathrm{d}x - \sum_{n \in C_i(t)} c_n^{\mathrm{var}} y_n(t) \tag{9a}$$

$$\text{s.t.} \quad D_i(t) - \sum_{n \in C_i(t)} y_n(t) = \hat{f}_i(t), \tag{9b}$$

$$0 \le y_n(t) \le \bar{y}_n, \quad n \in C_i(t), \tag{9c}$$

$$D_i(t) \ge 0. \tag{9d}$$

Proposition 2. *Let $t \in T$ and let $(D(t), y(t); \alpha(t), \beta^{\pm}(t), \gamma^-(t))$ be an optimal primal-dual solution of Problem* (9) *such that strict complementarity holds. Let $n^* \in C_i(t, \bar{y})$ be the node with largest variable costs in cluster $C_i(t, \bar{y})$ with $y_{n^*}(t) > 0$. If $\gamma^-(t) > 0$ or $\beta_{n^*}^+(t) = 0$ then*

$$\frac{\partial D(t)}{\partial \bar{y}_n} = 0, \quad n \in C_i(t, \bar{y}).$$

If, however, $\gamma^-(t) = 0$ and $\beta_{n^*}^+(t) > 0$ holds, then for all $n \in C_i(t, \bar{y})$, we have

$$\frac{\partial D(t)}{\partial \bar{y}_n} = \begin{cases} 1, & \text{if } y_n(t) > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. After elimination of the dual variables of Constraint (9b), the KKT conditions of Problem (9) contain the following equations:

$$P(t, D(t)) - c_n^{\mathrm{var}} + \beta_n^-(t) - \beta_n^+(t) + \gamma^-(t) = 0, \quad n \in C_i(t, \bar{y}),$$
$$D(t) - \sum_{n \in C} y_n(t) - \hat{f}_i(t) = 0,$$
$$\beta_n^-(t) y_n(t) = 0, \quad n \in C_i(t, \bar{y}),$$
$$\beta_n^+(t)(\bar{y}_n - y_n(t)) = 0, \quad n \in C_i(t, \bar{y}),$$
$$\gamma^-(t) D(t) = 0.$$

This is a system $F(x; \bar{y}_n) = 0$ of equations with $x = (D(t), y(t), \beta^{\pm}(t), \gamma^-(t))$. Since strict complementarity holds we may apply the implicit function theorem, yielding

$$J_x F \cdot J_{\bar{y}_n} x = -J_{\bar{y}_n} F,$$

where, e.g., $J_x F$ denotes the Jacobian of F with respect to x. Solving this system of equations yields the claim. □

We observe that

$$\phi(t, \bar{y}) = \sum_{i=1}^{I_t} \phi_i(t, \bar{y}) \tag{10}$$

holds. Now we are able to compute the second partial derivatives of $\phi(t, \bar{y})$.

Lemma 3. *Let $\bar{y}$ and t be given and assume that the solutions of Problem* (9) *fulfill strict complementarity for all $i \in \{1, \ldots, I_t\}$. If $n \in C_i(t, \bar{y})$ and $m \in C_j(t, \bar{y})$ with $i \ne j$, then*

$$\frac{\partial}{\partial \bar{y}_m} \frac{\partial}{\partial \bar{y}_n} \phi(t, \bar{y}) = 0. \tag{11}$$
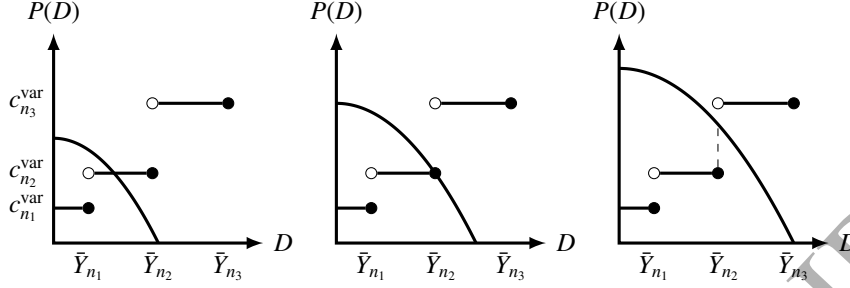
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Figure 1: Illustration of Proposition 2 and Assumption 3; $\bar{Y}_i := \sum_{k=1}^{i} \bar{y}_k$. Strict complementarity holds in the left and right figure, whereas it is violated in the middle case.

If $n, m \in C_i(t, \bar{y})$ and $\gamma^-(t) > 0$ or $\beta_{n^*}^+(t) = 0$, where $\gamma^-(t), \beta_{n^*}^+(t)$ are the respective dual variables of Problem (9) for cluster $C_i(t, \bar{y})$ and n^* is defined as in Proposition 2, then

$$\frac{\partial}{\partial \bar{y}_m} \frac{\partial}{\partial \bar{y}_n} \phi(t, \bar{y}) = 0. \tag{12}$$

Otherwise, i.e., $\gamma^-(t) = 0$ and $\beta_{n^*}^+(t) > 0$, we have

$$\frac{\partial}{\partial \bar{y}_m} \frac{\partial}{\partial \bar{y}_n} \phi(t, \bar{y}) = \begin{cases} B_i(t, \bar{y}), & \text{if } n, m \in A_i(t, \bar{y}), \\ 0, & \text{otherwise,} \end{cases} \tag{13}$$

where $B_i(t, \bar{y}) < 0$ is the negative slope of the aggregated demand function $P_i(t, \bar{y})$ at the total demand $D_i(t, \bar{y})$ of price cluster $C_i(t, \bar{y})$.

Proof. Equation (11) follows directly from Equation (10). For the remaining cases we make the following observation:

$$\frac{\partial}{\partial \bar{y}_m} \frac{\partial}{\partial \bar{y}_n} \phi(t, \bar{y}) = \frac{\partial}{\partial \bar{y}_m} \beta_n^+(t).$$

The KKT conditions of Problem (9) imply

$$\beta_n^+(t) = P_i(t, D_i(t, \bar{y})) - c_n^{\text{var}}, \quad n \in C_i(t, \bar{y}) \text{ with } y_n(t) > 0.$$

Thus, for $n \in C_i(t)$ with $y_n(t) > 0$ we can write

$$\frac{\partial}{\partial \bar{y}_m} \beta_n^+(t) = \frac{\partial}{\partial \bar{y}_m} P_i(t, D_i(t, \bar{y}))$$

$$= \frac{\partial}{\partial D_i(t)} P_i(t, D_i(t, \bar{y})) \frac{\partial}{\partial \bar{y}_m} D_i(t, \bar{y}) = B_i(t, \bar{y}) \frac{\partial}{\partial \bar{y}_m} D_i(t, \bar{y}),$$

where $D_i(t, \bar{y})$ is the (unique) total demand in an optimal solution of Model (9) for price cluster $C_i(t, \bar{y})$ in dependence on $\bar{y}$.

The remaining Equations (12) and (13) follow directly from Proposition 2. □

We write down the necessary property from the preceding lemma.

Assumption 3. *For $\bar{y}$ let $\hat{T}(\bar{y})$ be the set of all $t \in T$ such that there exists a price cluster $i \in \{1, \ldots, I_t\}$ where the unique solution of Problem (9) does not satisfy strict complementarity. We assume that $\hat{T}(\bar{y})$ has measure zero for all $\bar{y}$.*

Before we turn to the multi-scenario case, we briefly discuss the mathematical necessity of Assumption 3 and illustrate the economic interpretation of Proposition 2 and strict complementarity (or its violation) using the example of the production constraints $y_n \leq \bar{y}_n$ and their dual variables $\beta_n^+ \geq 0$. We again drop the time index for better readability. Figure 1 illustrates three possible aggregated demand functions (continuous and strictly decreasing curves) and a single aggregated supply function for a price cluster. Total demand is positive in all three cases. The price cluster equilibrium in the first case (left figure) is characterized by the intersection of the aggregated demand curve and the variable production costs of the second cheapest producer, say n_2. In this case the production of n_2 fulfills $y_{n_2} \in (0, \bar{y}_{n_2})$, i.e., $\beta_n^- = \beta_n^+ = 0$, and strict complementarity holds. Dual feasibility then yields $P(D) = c_{n_2}^{\text{var}}$, which can also be seen in the left figure. Moreover, it can be seen that $\partial_{\bar{y}_n} D = 0$ for all nodes n. The other case satisfying strict complementarity is illustrated in the right figure: For all producing nodes m it holds that $y_m = \bar{y}_m$. Moreover, $\beta_{n_2}^+ = P(D) - c_{n_2}^{\text{var}} > 0$ (dashed line) is the earning of node n_2. The right figure also illustrates that $\partial_{\bar{y}_{n_k}} D = 1$ for all $k \leq 2$ and $\partial_{\bar{y}_{n_k}} D = 0$ for all $k > 2$ holds; see Proposition 2. The only problematic case is shown in the middle figure: Aggregated demand intersects aggregated supply at the rightmost point ($\bar{Y}_{n_2}$) of producer n_2 thus yielding $y_{n_2} = \bar{y}_{n_2}$ and $\beta_{n_2}^+ = 0$, i.e., strict complementarity does not hold. The mathematical severity of this case is that $\partial_{\bar{y}_{n_k}} D$ does not exist; only directional derivatives exist and equal $c_{n_3}^{\text{var}} - c_{n_2}^{\text{var}} > 0$ and 0, respectively. Finally, the middle figure suggests that this is a rare event because it only appears if the aggregated demand curve intersects the supply curve in a finite number of special points, i.e., $\bar{Y}_{n_i}, i = 1, 2, \ldots$, out of a continuum of points.

To further illustrate Ass. 3 we consider the case without network constraints. Then, for every fixed capacity investment there are only finitely many total demand values for which strict complementarity does not hold. The assumption now states that the set of scenarios in which these total demand values realize must have zero measure. This is for instance the case if all scenario sets with equal total demand have zero measure, which is, e.g., the case if $D'_t \neq D_t$ for all $t \neq t'$.

3.2.2. The Multi-Scenario Case

Up to this point, we have computed the second derivative for a fixed time t. We now show that the complete Hessian

$$H(\bar{y}) = \int_T H(t, \bar{y}) \, dt$$

of (2) is negative definite. To this end, we partition the time horizon T in

$$\mathcal{T}(\bar{y}) = \{T_\tau(\bar{y})\}_\tau \cup \hat{T}(\bar{y})$$

such that for all τ all times $t \in T_\tau(\bar{y})$ have the same price clusters $C(t, \bar{y})$. We remark that there only exist finitely many τ since there also exist only finitely many price cluster configurations. This allows us to state the following proposition:

Proposition 3. *Suppose Assumption 3 holds. Then, the Hessian $H(\bar{y})$ can be written as*

$$H(\bar{y}) = \int_T H(t, \bar{y}) \, dt = \sum_\tau \int_{T_\tau(\bar{y})} H(t, \bar{y}) \, dt.$$

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Note that the definition of $\mathcal{T}(\bar{y})$ requires that the sets $T_\tau(\bar{y})$ in T are measurable. Under this assumption the definition permits the notations $H_\tau(\bar{y})$ and $C_\tau(\bar{y})$. The following proposition readily follows from (11) and states that an entry of Hessian $H(t, \bar{y})$ corresponding to two nodes n, m is zero for all nodes in different price clusters and $t \in T_\tau(\bar{y})$.

Proposition 4. *Let $(H(t, \bar{y}))_{n,m}$ denote the entry in row n and column m of the matrix $H(t, \bar{y})$. Then for all $n, m \in N$ and for all τ we have that*

$$(H(t, \bar{y}))_{n,m} = 0, \quad t \in T_\tau(\bar{y}),$$

if $n \in C_{\tau,i}(\bar{y})$ and $m \in C_{\tau,j}(\bar{y})$ with $i \neq j$.

Note that this proposition yields a block structure of $H(t, \bar{y}), t \in T_\tau(\bar{y})$, induced by the price clusters $C_\tau(\bar{y}) = \{C_{\tau,i}(\bar{y})\}_{i=1}^{I_\tau}$ at these times. The corresponding matrix block is denoted by $H_{\tau,i}(\bar{y})$ and, after re-ordering of the nodes, we obtain

$$H(t, \bar{y}) = \mathrm{diag}(H_{\tau,i}(\bar{y}))_{i=1}^{I_\tau}.$$

We now partition the times $T_\tau(\bar{y})$ further into $\{T_{\tau,j}(\bar{y})\}_j$ such that $A_i(t, \bar{y}) = A_i(t', \bar{y})$ holds for all $t, t' \in T_{\tau,j}(\bar{y})$. We denote the corresponding activity patterns by $A_{i,j}(\bar{y})$. The following proposition is a direct consequence of these partitions.

Proposition 5. *For all $t, t' \in T_{\tau,j}(\bar{y})$ it holds that*

$$H(t, \bar{y})\big|_{C_{\tau,i}} = H(t', \bar{y})\big|_{C_{\tau,i}},$$

where $H(t, \bar{y})\big|_{C_{\tau,i}}$ denotes the restriction of $H(t, \bar{y})$ to the block corresponding to $C_{\tau,i}$.

This proposition allows us to introduce the notation

$$H_{\tau,i,j}(\bar{y}) := H(t, \bar{y})\big|_{C_{\tau,i}}$$

for all $t \in T_{\tau,j}(\bar{y})$. Moreover, note that $H_{\tau,i,j}(\bar{y})$ is a matrix with a left-upper block with values $B_{\tau,i,j}(\bar{y}) < 0$ of size $|A_{\tau,i,j}(\bar{y})|$ and zeros elsewhere.

The rest of the proof is split up into two parts. First, we show that all Hessians $H_{\tau,i}(\bar{y})$ are negative semi-definite. Second, we show that under additional assumptions, there exist some $H_{\tau,i}(\bar{y})$ that are negative definite. Both results together finally imply the negative definiteness of the overall Hessian for all $\bar{y}$ and thus that the peak-load pricing model (1) has a unique solution.

Proposition 6. *For all τ and all i, the corresponding block $H_{\tau,i}(\bar{y})$ is negative semi-definite.*

Proof. Let τ and i be given. Then, by Proposition 5

$$H_{\tau,i}(\bar{y}) = \sum_j \int_{T_{\tau,j}(\bar{y})} H(t, \bar{y})\big|_{C_{\tau,i}} \, \mathrm{d}t = \sum_j \mu(T_{\tau,j}(\bar{y})) H_{\tau,i,j}(\bar{y})$$

holds with $H_{\tau,i,j}(\bar{y})$ being rank-1-matrices in which all non-vanishing entries equal $B_{\tau,i,j}(\bar{y}) < 0$. Here, $\mu(T_{\tau,j}(\bar{y}))$ is the Lebesgue measure of $T_{\tau,j}(\bar{y})$ in T. Since $H_{\tau,i}(\bar{y})$ is now shown to be a sum of negative semi-definite matrices, this shows that $H_{\tau,i}(\bar{y})$ itself is negative semi-definite. $\square$

Note that from the latter proposition directly follows that $H_\tau(\bar{y})$ is negative semi-definite for all τ, since $H_\tau(\bar{y})$ is a block-diagonal matrix with blocks $H_{\tau,i}(\bar{y})$

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Proposition 7. *Let τ and i be given. If the partition $\{T_{\tau,j}(\bar{y})\}_{j=1}^{J}$, $J = |C_{\tau,i}(\bar{y})|$, of $T_\tau(\bar{y})$ can be chosen so that*

$$A_{i,j+1}(\bar{y}) = A_{i,j}(\bar{y}) \cup \{n_{j+1}\}, \quad A_{i,1}(\bar{y}) = \{n_1\}$$

holds, where the nodes $n_1, \ldots, n_J$ are ordered in such a way that $c_{n_k}^{\mathrm{var}} < c_{n_\ell}^{\mathrm{var}}$ if and only if $k < \ell$ for all $1 \le k, \ell \le J$, and if $\mu(T_{\tau,j}(\bar{y})) > 0$ holds for all j, then $H_{\tau,i}(\bar{y})$ is negative definite.

Proof. The partition of the set of times and nodes readily implies

$$H_{\tau,i}(\bar{y}) = \sum_j \mu(T_{\tau,j}(\bar{y})) H_{\tau,i,j}(\bar{y}) =: \sum_j \tilde{H}_{\tau,i,j}(\bar{y}).$$

We now define $\tilde{B}_{\tau,i,j} := \mu(T_{\tau,j}(\bar{y})) B_{\tau,i,j}(\bar{y})$. With this notation the following holds:

$$(\tilde{H}_{\tau,i,j}(\bar{y}))_{\nu,\xi} = \begin{cases} \tilde{B}_{\tau,i,j}, & \text{if } \nu, \xi \le j, \\ 0, & \text{otherwise.} \end{cases}$$

We now apply Gaussian elimination: In the kth step we subtract row $J - k + 1$ from all rows 1 to $J - k + 2$. After $J - 1$ steps this yields the matrix

$$\begin{bmatrix} \tilde{B}_{\tau,i,1} & & & \\ \tilde{B}_{\tau,i,2} & \tilde{B}_{\tau,i,2} & & \\ \vdots & & \ddots & \\ \tilde{B}_{\tau,i,J-1} & \cdots & \tilde{B}_{\tau,i,J-1} & \\ \tilde{B}_{\tau,i,J} & \cdots & & \tilde{B}_{\tau,i,J} \end{bmatrix}.$$

Since all diagonal elements $\tilde{B}_{\tau,i,j}$ of the resulting matrix are strictly negative the matrix is negative definite. $\qquad\square$

The last proposition leads us to the following assumption:

Assumption 4. *There exists a time partition index τ such that for all $i = 1, \ldots, I_t$, $T_\tau(\bar{y})$ can be partitioned as $\{T_{\tau,j}(\bar{y})\}_{j=1}^{J}$, $J = |C_{\tau,i}(\bar{y})|$, with*

$$A_{i,j+1}(\bar{y}) = A_{i,j}(\bar{y}) \cup \{n_{j+1}\}, \quad A_{i,1}(\bar{y}) = \{n_1\},$$

where the nodes $n_1, \ldots, n_J$ are ordered in such a way that $c_{n_k}^{\mathrm{var}} < c_{n_\ell}^{\mathrm{var}}$ if and only if $k < \ell$ for all $1 \le k, \ell \le J$ and $\mu(T_{\tau,j}(\bar{y})) > 0$ holds for all j.

This assumption is violated if there exist two nodes n, n' for which the following holds: For almost all time periods in which the nodes are part of the same price cluster C either both $y_{t,n} = \bar{y}_n$ and $y_{t,n'} = \bar{y}_{n'}$ or both $y_{t,n} = 0$ and $y_{t,n'} = 0$ hold. As such, this assumption can be seen as a natural extension of Assumption 2. Not only do the variable costs need to be distinct, but there must exist enough scenarios where this matters. If our scenario set does not fulfill the assumption, i.e., informally speaking, that given two different nodes the following situation occurs: In almost all scenarios where they are part of the same price cluster they are always both producing at full capacity or both do not produce at all. In other words, the scenario set is not large enough to distinguish between these two nodes. Then, it is clear that the solution may not be unique. With realistic data, however, this should not occur as producers are sufficiently different and scenario sets are sufficiently large to ensure this condition. An assumption like

this is also needed in the classical peak-load-pricing setting without consideration of network constraints: The difference between the variable costs of two producers needs to be large enough such that there actually exist scenarios where this matters. Assumption 4 is a weaker formulation since we only need to consider nodes that share price clusters. Summing up all results of the last sections, we obtain the following main theorem:

Theorem 5. *Suppose Assumptions 1–4 hold. Then, the matrix $H(\bar{y})$ is negative definite and, thus, Model (1) has a unique solution in $(d, y, f_{A^{inter}})$.*

3.3. The Case of Market Power

As we have argued earlier, it is impossible to meaningfully analyze the proposed framework using a rigorous game theoretic approach to strategic interaction among firms. Various papers have shown that multiple equilibria already arise in a setup with strategic interaction in the absence of networks; see, e.g., [45]. In our contribution we thus choose to approximate the case of strategic interaction by a conjectural variations approach, similar to the one applied recently by [43], which allows to establish a unique solution. While this approach and its outcome cannot be related to a proper game structure, it nevertheless might be suitable to capture important aspects of an environment where firms manage to charge significant markups.

To this end, we replace objective function (1a)

$$\psi^1 := \sum_{n \in N} \int_T \int_0^{d_n(t)} p_n(t, x) \, dx \, dt - \sum_{n \in N} \int_T c_n^{var} y_n(t) \, dt - \sum_{n \in N} c_n^{inv} \bar{y}_n$$

by

$$\psi^\lambda := \lambda \psi^1 + (1 - \lambda) \psi^0, \quad \lambda \in [0, 1], \tag{14}$$

where

$$\psi^0 := \sum_{n \in N} \int_T p_n(t, d_n(t)) d_n(t) \, dt - \sum_{n \in N} \int_T c_n^{var} y_n(t) \, dt - \sum_{n \in N} c_n^{inv} \bar{y}_n.$$

Note that this extension is a convex combination of the situation, in which competitive firms trade on a market (ψ^1) and the case of a monopoly (ψ^0). It is easily seen that this extension only affects the demand terms, i.e.,

$$\psi^\lambda = \lambda \sum_{n \in N} \int_T \int_0^{d_n(t)} p_n(t, x) \, dx \, dt + (1 - \lambda) \sum_{n \in N} \int_T p_n(t, d_n(t)) d_n(t) \, dt$$
$$- \sum_{n \in N} \int_T c_n^{var} y_n(t) \, dt - \sum_{n \in N} c_n^{inv} \bar{y}_n$$

holds. In the following, we show that all results presented so far are also valid for the case of Objective (14) under the following additional assumption:

Assumption 5. *All demand functions $p_n(t, \cdot)$ fulfill Assumption 1 and the additional condition $p'_n(t, d) + p''_n(t, d)d < 0$, where the derivatives are taken with respect to d.*

We note that in the common case where $p_n(t)$ is a linear function, Assumption 1 directly implies Assumption 5.

Lemma 4. *It holds that*

$$\psi^\lambda = \sum_{n \in N} \int_T \int_0^{d_n(t)} p_n^\lambda(t, x) \, \mathrm{d}x \, \mathrm{d}t - \sum_{n \in N} \int_T c_n^{\mathrm{var}} y_n(t) \, \mathrm{d}t - \sum_{n \in N} c_n^{\mathrm{inv}} \bar{y}_n,$$

where

$$p_n^\lambda(t, x) := p_n(t, x) + (1 - \lambda) p_n'(t, x) x.$$

If $p_n(t)$ fulfills Assumption 5, then $p_n^\lambda(t)$ fulfills Assumption 1.

Proof. We only have to consider the demand terms for fixed time t and node n separately. Then, the proof is a straight forward application of integration by parts:

$$\begin{aligned}
\int_0^{d_n(t)} p_n^\lambda(t, x) \, \mathrm{d}x &= \int_0^{d_n(t)} p_n(t, x) + (1 - \lambda) p_n'(t, x) x \, \mathrm{d}x \\
&= \int_0^{d_n(t)} p_n(t, x) \, \mathrm{d}x + (1 - \lambda) \int_0^{d_n(t)} p_n'(t, x) x \, \mathrm{d}x \\
&= \int_0^{d_n(t)} p_n(t, x) \, \mathrm{d}x \\
&\quad + (1 - \lambda) \left([p_n(t, x) x]_0^{d_n(t)} - \int_0^{d_n(t)} p_n(t, x) \, \mathrm{d}x \right) \\
&= \lambda \int_0^{d_n(t)} p_n(t, x) \, \mathrm{d}x + (1 - \lambda) p_n(t, d_n(t)) d_n(t).
\end{aligned}$$

The second claim is immediately clear. □

This lemma shows that the model using the modified objective (14) is simply the basic peak-load pricing model (1) with demand functions $p_n(t)$ replaced by $p_n^\lambda(t)$, which are again strictly decreasing. Thus, all results from Section 3 also apply to the model using Objective (14).

3.4. Characterization and Discussion

We now discuss how the optimal solution of Problem (1) can be characterized. If we analyze the situation of a single scenario, we observe that prices in neighboring clusters differ by the shadow price of their saturated connecting arcs. Assume we are given two clusters C_P and C_C, where C_P supplies more than it demands and C_C demands more than it supplies. Then, the first-order conditions of Problem (3) directly imply that on all arcs connecting C_P with C_C the flow direction is from C_P to C_C and that $p_{C_P} < p_{C_C}$. Thus, flow goes from the lower to the higher price.

Focusing on the full problem, we are interested in how investments are taken. We observe from the first-order conditions of Problem (2) that for the optimal solution holds that

$$c_n^{\mathrm{inv}} = \int_T \beta_n^+(t, \bar{y}) \, \mathrm{d}t. \tag{15}$$

This means that only those scenarios contribute to the investment costs of a node, in which the node has variable costs that are strictly lower than the price in its price cluster, i.e., $\beta_n^+(t, \bar{y}) > 0$.

Combining these observations we see that the network structure induces investment incentives to install capacity close to consumers: The prices for nodes that consume in many

scenarios are high and thus it is interesting to invest there. If the network exhibits a persistent bottleneck that manifests itself in most scenarios then investment in capacity on the demand side of that bottleneck will be efficient even if variable costs for the respective technology is higher.

It is also interesting to contrast the possibility of firms to earn money in the different discussed settings. In the case of perfect competition, one can directly deduce from the KKT conditions that the firms completely recover their investment and variable costs. The firms are, however, not able to make a profit. Note that this also applies to multiple technologies per node; see Sect. 2.1. Intuitively, in the market equilibrium under perfect competition, whenever a production unit would make positive profits, then it is profitable to increase investment in that unit such that positive profits finally disappear in equilibrium. This logic applies independently whether technologies are located at different nodes of the network or at the same node. In the case of a monopoly, however, the firms are able to make a profit. For firm n the profit is then given by

$$\int_T -p'(t, d_n(t))d_n(t)^2 \, \mathrm{d}t.$$

In the case where the parameter λ in our conjectural variations approach is between 0 and 1, the profit is scaled by the factor $1 - \lambda$ accordingly.

The comparison to the classical peak-load pricing settings without a network is instructive. Despite the difference in the respective subproblems (without network the subproblem reduces to Problem (5) with a single cluster for all times t), the overall structure (15) of the investment solution is similar. If we have only one price cluster for all times (i.e., we have "no network"), the investment solution will strictly prefer nodes with low variable costs irrespective of their position in the network. This can lead to wildly different investment solutions and may especially lead to over- resp. underinvestment in the case of persistent bottlenecks.

This immediately suggests that zonal pricing could be used in order to solve the trade-off between the local distribution of capacity investments and network expansion. The issue of investment incentives and in particular the interdependence with the congestion management regime has received increasing attention in recent years (see, e.g., [41]). Up to now, the literature, however, has focused mainly on important issues arising in the short run; see, e.g., [10, 12, 19, 26]. The present contribution now helps to link congestion management and generation investment. Contributions that use the uniqueness result derived in this paper show that price clusters might adjust incentives in the right direction (see, e.g., [21] and the references therein). In this context, policy makers and stakeholders are particularly interested in the proper configuration of price clusters to achieve a welfare improvement (see, for instance, [9, 15, 16, 41]). In this respect, [22] shows that price clusters need to be configured carefully in order to actually achieve a welfare improvement. A multilevel mixed-integer model for the computation of optimal price zone configurations has been proposed very recently in [20].

The proper derivation of an ideal zone configuration with a limited number of zones is still an open research problem, however.

We finally close this section with some technical remarks on the proven results. The results are also valid for the case in which we replace the continuous time horizon $T = [t_0, t_e] \subset \mathbb{R}$ with a discrete set of time periods $T = \{t_0, t_1, \ldots, t_e\}$ with $t_i < t_{i+1}$. However, some of the assumptions have to be adjusted accordingly.
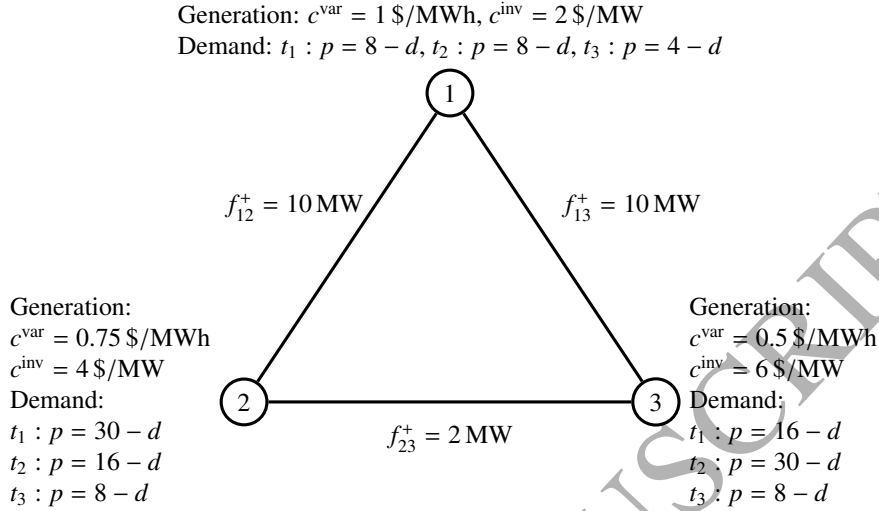
Generation: $c^{\mathrm{var}} = 1\,\$/\mathrm{MWh}$, $c^{\mathrm{inv}} = 2\,\$/\mathrm{MW}$
Demand: $t_1 : p = 8 - d$, $t_2 : p = 8 - d$, $t_3 : p = 4 - d$

Figure 2: Three-Node Network

4. Illustrative Example: Three-Node Network

In this section we consider a three-node network that illustrates our concepts and theoretical results. The network and the scenario set are chosen such that they allow us to discuss all relevant structural effects that are outcome of our theoretical analysis, but still have manageable size. Important features of this example are that the price clusters change over time and one scenario has non-unique flows. The changing price clusters can be directly observed in the structure of the Hessians corresponding to the different scenarios. As depicted in Fig. 2, the three nodes are connected by three arcs. At the three nodes investment in production capacity can take place with investment and production costs as shown in the figure. It can be directly seen that Assumption 2 holds, i.e., variable costs are pairwise distinct. We consider three scenarios and, for the ease of presentation, use linear demand functions that vary across these scenarios and fulfill Assumption 1. The scenarios last 1 h and all data of the corresponding scenarios are constant during that time. Observe that demand at node 1 is relatively low in all scenarios. Scenario 1 (scenario 2) is characterized by the high (low) demand at node 2 and a comparatively low (high) demand at node 3. In scenario 3 overall demand is low.

Table 2 and 3 list parts of the primal and dual solutions. The optimal solution shows that it is efficient to install 18.12 MW of new capacity at node 1, 14.29 MW at node 2, and 13.46 MW at node 3. Thus, the amount of installed capacity is ordered with increasing investment costs. In scenario 1, arcs $(1, 2)$ and $(2, 3)$ are saturated. Therefore, in scenario 1 there are two different price clusters, which are formed by a flow-induced partition (see Definition 2): $C_{1,1} = \{2\}$ and $C_{1,2} = \{1, 3\}$. As it can be seen in Table 3, both prices (p) and dual variables of corresponding flow balance constraints (α) are identical for nodes 1 and 3 (see Definition 1). In analogy to scenario 1, in scenario 2 we also have two price clusters given the saturated lines $(1, 3)$ and $(2, 3)$. Thus, we have $C_{2,1} = \{3\}$ and $C_{2,2} = \{1, 2\}$.

Now consider the last scenario 3, in which no line is saturated. This yields a single price cluster $C_{3,1} = N$. It can be easily seen that the intra-cluster flows are not unique (see Theorem 3)

Table 2: Parts of the Primal Solution of the Three-Node Network

Scen.	d_1	d_2	d_3	y_1	y_2	y_3	f_{12}	f_{23}	f_{13}
1	5.79	26.29	13.79	18.12	14.29	13.46	10	−2	2.33
2	6.21	14.21	25.46	18.12	14.29	13.46	1.91	2	10
3	3.25	7.25	7.25	0	4.29	13.46	2.26	−0.7	−5.51

Table 3: Prices and Parts of the Dual Solution of the Three-Node Network

Scen.	p_1	p_2	p_3	α_1	α_2	α_3	β_1^+	β_2^+	β_3^+
1	2.21	3.71	2.21	2.21	3.71	2.21	1.21	2.96	1.71
2	1.79	1.79	4.54	1.79	1.79	4.54	0.79	1.04	4.04
3	0.75	0.75	0.75	0.75	0.75	0.75	0	0	0.25

since adding a small cycle flow is still feasible and does not change the objective function value.

It can be also seen that the solution satisfies Assumption 3 since it is strictly complementary—the case of the middle part of Figure 1 does not occur. Moreover, we see that every node is the most expensive production node in its zone in at least one scenario: For node 2 this holds in scenario 1, for node 3 in scenario 2, and for node 1 this holds in each of the three scenarios. Thus, also the last Assumption 4 holds.

Regarding the profits of the firms one can easily check that all of them are zero; see Section 3.4.

To show that optimal capacity investment is unique, we next compute the Hessian of the master problem for the considered example. As stated in Proposition 3, the Hessian

$$H = \begin{bmatrix} -1 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{3}{2} & 0 \\ -\frac{1}{2} & 0 & -\frac{11}{6} \end{bmatrix}$$

can be expressed as the sum of the Hessians $H_t, t = 1, 2, 3$:

$$H_1 = \begin{bmatrix} -\frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & -1 & 0 \\ -\frac{1}{2} & 0 & -\frac{1}{2} \end{bmatrix}, \quad H_2 = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad H_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{3} \end{bmatrix}.$$

The Hessian H is negative definite, i.e., Proposition 7 holds, and thus optimal production and capacity investment is unique (see Theorem 5).

5. Conclusion

In this paper we have analyzed a framework of peak-load pricing on a network where competitive firms take investment and production decisions facing network constraints expressed by fixed inter-zonal capacities. We have shown existence and uniqueness of the solution and characterized equilibrium investments. We also presented an approach that sheds light on a

market where markups can be charged—although a full-fledged analysis of strategic interaction is not possible in our setup.

As one of the results of our analysis we show that the consideration of network constraints does not require any additional assumptions compared to those guaranteeing uniqueness of the equilibrium in a standard peak-load pricing model that disregards network constraints. Our results are an important prerequisite for the analysis of energy policy proposals using multilevel computational equilibrium frameworks. These approaches can only be meaningfully used if lower-level problems have unique solutions that restrict feasible solutions at higher levels. This has been emphasized by various authors, e.g., [8], [5], or [17]. Our contribution provides such a result for electricity market analyses. In [21] the result is already used in order to analyze optimal transmission expansion in liberalized electricity markets under different regulatory regimes.

However, there are still open issues for future research. For instance, it would be of impact to establish comparable uniqueness results for different extensions of our model like the case of transportation costs or a DC load flow model.

Acknowledgements

AppendixA. Proof of Theorem 2

For completeness, we sketch a self-contained proof of Theorem 2, which is a variant of a result given in [38]. For this we need two observations: First, the max-flow-min-cut theorem immediately allows to us to write down an alternative characterization of the feasibility of Subproblem (3). This is a restatement of the well-known theorems of Gale and Hoffman for our particular case, see [39, Chapter 11].

Proposition 8. *The vector (d, y, f) is a feasible solution of Problem (3) if and only if $\sum_{n \in N} d_n = \sum_{n \in N} y_n$ and for all $C \subseteq N$, it holds that*

$$\sum_{n \in C} d_n - \sum_{n \in C} y_n \leq \hat{f}_C,$$

where $\hat{f}_C = \sum_{a \in \delta^{\text{in}}(C)} \bar{f}_a + \sum_{a \in \delta^{\text{out}}(C)} \bar{f}_a$.

The second observation states that if two non-disjoint sets are tight at the upper bound, its union and intersection are as well. As a byproduct we obtain that in this case the connecting arcs between these two sets are unused. This result is also well-known, see again [39].

Proposition 9. *Let (d, y, f) be a feasible solution of Problem* (3) *and let $X, Y \subseteq N$ with $X \cap Y \neq \emptyset$ such that*

$$\sum_{n \in X}(d_n - y_n) = \hat{f}_X, \quad \sum_{n \in Y}(d_n - y_n) = \hat{f}_Y.$$

Then the following equations hold as well:

$$\sum_{n \in X \cap Y}(d_n - y_n) = \hat{f}_{X \cap Y}, \quad \sum_{n \in X \cup Y}(d_n - y_n) = \hat{f}_{X \cup Y}.$$

Now, we are ready to prove Theorem 2.

Proof of Theorem 2 . Set $z^0 := (d^*, y^*)$ and let $z^1 = (d^1, y^1)$ be a solution of Problem (5). As z^0 is feasible for Problem (5), $\phi(z^0) \leq \phi(z^1)$ holds. It remains to show that $\phi(z^0) \geq \phi(z^1)$ holds. If there exists a flow f^1 such that (d^1, y^1, f^1) is feasible for Problem (3), $\phi(z^1) \leq \phi(z^0)$ holds and we obtain the desired inequality. Assume no such flow exists. Then it follows from Proposition 8 that there exists a set $U \subset N$ with $U \neq \emptyset$ such that

$$\sum_{n \in U} d_n^1 - \sum_{n \in U} y_n^1 > \hat{f}_U.$$

From the construction of C it follows from Proposition 9 that we may assume that $U \subset C$ for a $C \in C$. Set $\mathcal{U}$ to be the set of all such sets U and define $z^\lambda := (1 - \lambda)z^0 + \lambda z^1$ for $\lambda \in [0, 1]$. It now follows that there exists a $\lambda_U > 0$ for each $U \in \mathcal{U}$ such that z^λ satisfies

$$\sum_{n \in U} d_n^\lambda - \sum_{n \in U} y_n^\lambda \leq \hat{f}_U.$$

Set $\rho := \min_{U \in \mathcal{U}} \lambda_U$. Then, it again follows from Proposition 8 that there exists a flow f such that (z^ρ, f) is feasible for Problem (3). Since $\rho > 0$, it also follows $\phi(z^\rho) > \phi(z^0)$, which is a contradiction to the optimality of z^*. Hence, $\phi(z^0) \geq \phi(z^1)$ holds. $\qquad\square$

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