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Decision Support

# Competitive analysis of the online financial lease problem

Wenqiang Dai[a], Yucheng Dong[b,*], Xiaotian Zhang[c]

[a] School of Management and Economics, University of Electronic Science and Technology of China, Chengdu 610054, China
[b] Business School, Sichuan University, Chengdu 610065, China
[c] School of Management, Xi'an Jiaotong University, Xi'an 710049, China

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ABSTRACT

The financial lease is an important financing tool by which the lessee can acquire ownership of equipment upon the expiration of the lease after making a series of rent payments for the use of the equipment. In this paper, we consider an online version of this financial lease decision problem in which the decision maker (the lessee) does not know how long he/she will use the equipment. By assuming, the lessee can use the equipment through two options: financial lease or lease; we define and solve this online financial lease decision problem using the competitive analysis method. The optimal online strategies are discussed in each financial lease case with or without down payment. Finally, the optimal strategies are summarized as simple decision rules.



## 1. Introduction

The leasing industry has significant impacts on other industries, and has catalysed economic development in different countries (Peck, 2014; Vance, 2003). To decide if leasing is a profitable financing option, the decision maker should first determine the length of time the equipment will be used. However, in practice, the exact length of time for use of some equipment is usually unavailable, so researchers explore well-established techniques in the fully distribution-free model, i.e., the online model and competitive analysis (Albers, 2003; Borodin & El-Yaniv, 1998), to evaluate their strategies. This technique compares the results obtained with an online strategy to the result that could have been obtained if one had known the exact length of time in advance, with the latter scenario represented by an optimal offline strategy. Thus, the competitive analysis aims is to design an online strategy with the best possible worst-case guarantees. Competitive analysis has been used to study many problems within the fields of finance (El-Yaniv, 1998), operations research (Chen & Wang, 2015; Liu, Chu, Xu, & Zheng, 2010; Zheng, Cheng, Xu, & Liu, 2013) and management science (Larsen & Wøhlk, 2010).

Karp first formulated the classic "leasing problem" (a.k.a. the Ski-rental problem) using the following online model and competitive analysis method (Karp, 1992): assume the decision maker has two options, to lease or purchase the equipment. In the purchase option, a one-time cost is incurred, and thereafter usage is free of charge; in the lease option, the cost is proportional to usage time, and there is no one-time cost. The solution is straightforward with competitive ratio 2, i.e., the strategy never pays more than twice the optimum. There have been many generalizations of this simple problem. For example, El-Yaniv, Kaniel, and Linial (1999) incorporated interest rates into the model and determined the optimal online strategies. Irani and Ramanathan (1998) examined a situation in which the price of the equipment fluctuates while the rental cost stays constant. More recently, Lotker, Patt-Shamir, and Rawitz (2008a, 2008b) and Zhang, Ponn, and Xu (2011) proposed the multi-slope ski-rental problem and the multiple discount option ski-rental problem, respectively. Other classic variations of the problem include the replacement problem (El-Yaniv & Karp, 1997), the capital investment problem (Azar et al., 1999) and the Bahncard problem (Fleischer, 2001).

The lease option in *all* of the above research is an operating lease. Operating leases are contracts for rent. At the end of the contract period, the ownership of the rented equipment remains with the rental company. In practice, there is another important leasing category: the financial lease. Typically, in a financial lease, the lessee will find required equipment, and then contact a leasing company (the lessor) to arrange financing. Legal ownership of the equipment remains with the lessor until lease ends, at which time the ownership is transferred to the lessee (Vance, 2003).

In this paper, the decision maker has two options, a lease or financial lease, to acquire required equipment. In the former option, the decision maker pays a rental fee for the equipment, whose ownership

* Corresponding author. Tel.: +86 2885416013.
*E-mail address:* ycdong@scu.edu.cn (Y. Dong).

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remains with the rental company at any time. In the latter option, the decision maker either makes or does not make a down payment, and then pays financial lease fees during the length of time of the lease for the equipment, whose ownership will be transferred to the lessee upon the expiration of the lease. The problem then is how to decide the time to switch to a financial lease when the decision maker does not know how long he/she will use the equipment. We will call this problem *the online financial lease problem*. In this paper, the optimal strategies to solve the online financial lease problem in cases with and without a down payment will be presented, and the optimal strategies will be summarized as simple decision rules.

The following example illustrates a typical result based on the analysis presented in this paper.

**Example**

Consider someone wants to use a repertory. There are two options: lease with a rental fee at $2000 per month, or a financial lease with a fee of $3000 per month plus a down payment of $10000 and in 24 months, the ownership will transfer to the lessee. Which policy guarantees the best performance for any possible length of time it is used in terms of a competitive analysis?

**Answer**

The following lease strategy guarantees a cost at most of 141.5 percent of the minimal possible cost, for any possible length of time: lease this repertory from the first to the 17th month and use the financial lease thereafter. On the other hand, if the lessee could negotiate with the rental company to decrease the down payment from $10000 to zero, then the strategy to lease this repertory for one year and then use the financial lease the next year can guarantee a maximum cost of 133.3 percent of the minimum possible cost. In addition, the above two percentages are the minimum for each case, i.e., the two strategies are optimal. The details are provided in Sections 5 and 6. Note that there are no assumptions about the distribution of the length of time in use.

The rest of the paper is organized as follows. Section 2 provides the precise definition of the online financial lease problem, the competitive strategy, and the competitive ratio. Section 3 then presents the competitive strategy and the matching lower bound proof for the problem of a financial lease without down payment. Thus, the competitive strategy obtained in this section is optimal. Next, Section 4 describes the optimal competitive strategies for the problem of a financial lease with down payment. We discuss this problem with different cases, and for each we obtain the optimal strategy and optimal competitive ratio. Subsequently, Section 5 summarizes all results and provides simple decision rules for different cases of the online financial leasing problem. The following Section 6 provides numerical analysis results to illustrate the proposed models and strategies. Finally, Section 7 presents the concluding remarks and areas for future research.

## 2. Problem definition and notations

In this section, we present the precise definition of and notations for the online financial lease problem.

Formally, in our problem, the decision maker (the lessee) can use the equipment by lease or financial lease. That is to say, the lessee can lease the equipment at a cost of $c$ per unit time, or can choose to use this equipment with a financial lease and pay $b \geq 0$ as a down payment, plus $r$ for each unit time.

The following basic assumptions are used in the online financial lease problem,

- In financial lease, the lessee will obtain ownership of the equipment after $z$ payments, and then no longer needs to pay the financial lease fees;
- The lessee can end a financial lease at any time, even just before obtaining ownership of the equipment. When the lessee decides to stop using the equipment within the lease period, the lease is terminated and the lessor is still the legal owner of the equipment, and the lessee does not need to pay the remaining fees;
- The length of time $t$ for the use of the equipment is unknown.

Let $A(T)$ define the strategy wherein the decision maker leases the equipment from the start to time $T$, and uses a financial lease thereafter, so $A(0)$ represents the strategy that uses a financial lease from the start, and $A(\infty)$ indicates a perpetual lease. Note that this notation defines all possible strategies.

For an (unknown) length of time to use the equipment $t \geq 0$, define $Cost_{A(T)}(t)$ as all costs paid by strategy $A(T)$ from the start to $t$, including, if any, the cost of the financial lease fees at time $t$. We define $Cost_{opt}(t)$ as the minimum cost needed to cover the period from the beginning to $t$. A strategy $A(T)$ is said to be $\alpha$-competitive if there exists two constants $\alpha$, $\beta$ such that

$$Cost_{A(T)}(t) \leq \alpha Cost_{opt}(t) + \beta \tag{1}$$

for all possible $t$, and this strategy is called an *online strategy* or a *competitive strategy*. Thus, our task is to acquire the equipment with minimal $\alpha$ in (1) for any unknown used time length $t$. All results in this paper hold under the stricter form of competitive analysis in which $\beta = 0$.

Since strategy $A(T)$ is only determined by parameter $T$, then our problem returns to how to determine the decision variable $T$. All notations introduced above are summarized in Table 1.

## 3. Financial lease without down payment

In this section, we examine the online financial lease problem without down payment, i.e., down payment $b = 0$. The analysis here illustrates the basic ideas for the subsequent general cases. Without loss generality, we consider $r > c$, otherwise the lessee should always choose to use the equipment via financial lease. Now, the cost of the optimal offline strategy is

$$Cost_{opt}(t) = \begin{cases} ct, & t < rz/c \\ rz, & t \geq rz/c \end{cases}$$

Before providing the optimal competitive strategy result, we present Lemmas 1 and 2 as follows.

**Lemma 1.** *Let $T^* = (r/c - 1)z$, the competitive ratio of online strategy $A(T^*)$ is $2 - c/r$.*

**Proof.** Recall that $A(T^*)$ represents the strategy using a lease during time $T^*$ and then switching to a financial lease from then on. Thus, the cost of strategy $A(T^*)$ is

$$Cost_{A(T^*)}(t) = \begin{cases} ct, & t < T^* \\ cT^* + r(t - T^*), & T^* \leq t < T^* + z \\ cT^* + rz, & t \geq T^* + z \end{cases}$$

Then the ratio of costs between the online strategy $A(T^*)$ and optimal offline strategy is

$$\frac{Cost_{A(T^*)}(t)}{Cost_{opt}(t)} = \begin{cases} 1, & t < T^* \\ \frac{1}{t}(1 - \frac{r}{c})T^* + \frac{r}{c}, & T^* \leq t < z + T^*. \\ 2 - \frac{c}{r}, & t \geq z + T^* \end{cases}$$

The above ratio function is a piecewise continuous function, since $r > c$, the second function is an increasing function of $t$; thus, when $t = z + T^*$, that is $t = rz/c$, the above ratio approaches its greatest value

**Table 1**
Notations used in the online financial lease problem.

| | |
|---|---|
| $c$ | Lease cost per unit time |
| $b$ | Down payment |
| $r$ | Financial lease cost per unit time |
| $z$ | Payment duration time when the lessee obtains the equipment |
| $t$ | Length of time the equipment will be needed |
| $A(T)$ | Online strategy to lease the equipment from the start to time $T$ and switch to financial lease |
| $Cost_{A(T)}(x)$ | Total cost paid with strategy$A(T)$ from the start to moment$x$ |
| $Cost_{opt}(x)$ | Total cost paid with the optimal strategy from the start to moment$x$ |

$2-c/r$, meaning the competitive ratio of the online strategy $A(T^*)$ is $2-c/r$. □

**Lemma 2.** *The lower bound for the online financial lease problem without down payment is* $2-c/r$.

**Proof.** For all possible online strategies, set the costs of $A(T)$ as

$$Cost_{A(T)}(t)=\begin{cases} ct, & t<T \\ cT+r(t-T), & T\le t<T+z \\ cT+rz, & t\ge T+z \end{cases}$$

Let $\lambda_T(t)=\frac{Cost_{A(T)}(t)}{Cost_{opt}(t)}$ and $\lambda_T=sup_t\lambda_T(t)$. According to the definition of competitive ratio, $\lambda_T$ is the competitive ratio of online strategy $A(T)$. We divide the value of $T$ into three cases to discuss the lower bound.

**Case 1**: If $T<T^*$, then

$$\lambda_T(t)=\begin{cases} 1, & t<T \\ \frac{T(c-r)+rt}{ct}, & T\le t<T+z \\ \frac{cT+rz}{ct}, & T+z\le t<\frac{rz}{c} \\ 1+\frac{cT}{rz}, & t\ge\frac{rz}{c} \end{cases}$$

This is a piecewise continuous function. When $t=z+T$, the ratio function $\lambda_T(t)$ attains its greatest value by the monotonicity of each section. Therefore, the competitive ratio of online algorithm $A(T)$ is $\lambda_T(z+T)=\frac{cT+rz}{c(z+T)}$. If $T<T^*$, the smallest value of $\lambda_T$ approaches $\lambda_{T^*}=2-c/r$, which implies that the competitive ratios of all online strategies $A(T)$ with $T<T^*$ have a lower bound of $2-c/r$.

**Case 2**: If $T^*\le T<\frac{rz}{c}$, then the cost ratio of the online strategy over the offline strategy is

$$\lambda_T(t)=\begin{cases} 1, & t<T \\ \frac{T(c-r)+rt}{ct}, & T\le t<\frac{rz}{c} \\ \frac{T(c-r)+rt}{rz}, & \frac{rz}{c}\le t<T+z \\ 1+\frac{cT}{rz}, & t\ge T+z \end{cases}$$

According to the monotonicity of each section of this piecewise continuous function, the above ratio attains its greatest value at $t=z+T$, so the competitive ratio is $\lambda_T=\lambda_T(z+T)=1+\frac{cT}{rz}$. If $T^*\le T<rz/c$, the least value of $\lambda_T$ approaches $\lambda_{T^*}=2-c/r$, which implies that in the case of $T^*\le T<rz/c$), all online strategies $A(T)$ will have a greater competitive ratio than $2-c/r$.

**Case 3**: If $T\ge\frac{rz}{c}$, similarly,

$$\lambda_T(t)=\begin{cases} 1, & t<\frac{rz}{c} \\ \frac{ct}{rz}, & \frac{rz}{c}\le t<T \\ \frac{T(c-r)+rt}{rz}, & T\le t<T+z \\ 1+\frac{cT}{rz}, & t\ge T+z \end{cases}$$

This is also a piecewise continuous function. By analyzing the monotonicity of each section, we know that $\lambda_T(t)$ has its greatest value at $t=z+T$, and the competitive ratio of strategy $A(T)$ is $\lambda_T=\lambda_T(z+T)=1+\frac{cT}{rz}$. If $T\ge\frac{rz}{c}$, the lowest value is $\lambda_T(z+\frac{rz}{c})=2$, which implies all strategies $A(T)$ will have a greater competitive ratio than 2 when $T\ge\frac{rz}{c}$.

In summary, the lower bound of the problem in the financial lease without down payment case is $2-\frac{c}{r}$. □

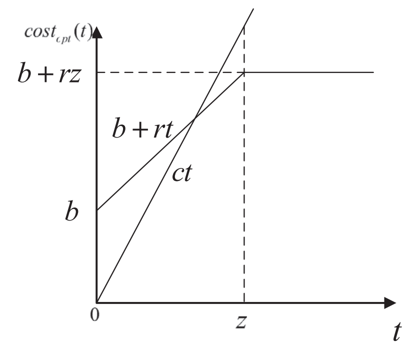


**Fig. 1.** Optimal offline cost when $b+rz<cz$.

From Lemmas 1 and 2, we obtain Theorem 1 as the optimal competitive strategy result.

**Theorem 1.** *For the online financial lease problem without down payment, let* $T^*=(r/c-1)z$ *. The strategy* $A(T^*)$ *is the optimal online strategy with competitive ratio* $(2-\frac{c}{r})$.

Theorem 1 tells us that for a financial lease without down payment, if the exact length of time $t$ the equipment will be needed is unknown, then the optimal strategy is to lease the equipment from the start to the moment $(r/c-1)z$ and use the financial lease thereafter. This strategy can guarantee a cost of at most $(2-\frac{c}{r})$ of the minimum possible cost, for any possible length of time the equipment is used.

## 4. Financial lease with down payment

Section 3 analyzes the model for a financial lease without down payment, though there is a more realistic version of the problem in which most financial lease contracts require a down payment. For convenience, let $t_0=(b+rz)/c$ represent the moment that the lease and financial lease are equal.

First, it is apparent that the offline optimal cost is as follows (See Figs. 1 and 2):

(1) If $b+rz<cz$, we have,

$$Cost_{opt}(t)=\begin{cases} ct, & t<\frac{b}{c-r} \\ b+rt, & \frac{b}{c-r}\le t<z \\ b+rz, & t\ge z \end{cases} \tag{2}$$

(2) If $b+rz\ge cz$, we have,

$$Cost_{opt}(t)=\begin{cases} ct, & t<t_0 \\ b+rz, & t\ge t_0 \end{cases} \tag{3}$$

Second, the analysis always presumes the following:

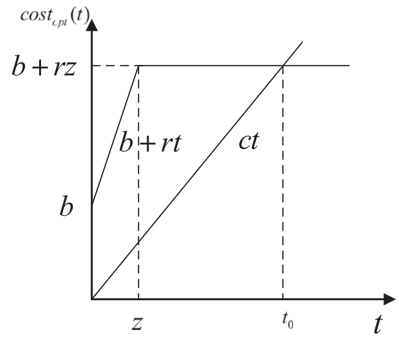


**Fig. 2.** Optimal offline cost when $b + rz \geq cz$.

**Lemma 3.** *Let* $g(x) = \max\{1 + \frac{b}{cx}, 1 + \frac{cx}{b+rz}\}$ *on a feasible set* $[a_1, a_2)$ *($a_1$, $a_2$ may be infinity), and let* $x^* = \frac{\sqrt{b(b+rz)}}{c}$*. Then, if* $x^* \in [a_1, a_2)$, $\min_{x\in[a_1,a_2)} g(x) = 1 + \sqrt{\frac{b}{b+rz}}$*; if* $x^* < a_1$, $\min_{x\in[a_1,a_2)} g(x) = 1 + \frac{ca_1}{b+rz}$*; if* $x^* \geq a_2$, $\min_{x\in[a_1,a_2)} g(x) = 1 + \frac{b}{ca_2}$.

**Proof.** It is clear that the inner function $1 + \frac{b}{cx}$ is a decreasing function of $x$, while the inner function $1 + \frac{cx}{b+rz}$ is an increasing function. To find the minimum value of $g(x)$, let $1 + \frac{b}{cx} = 1 + \frac{cx}{b+rz}$ to obtain $x^* = \frac{\sqrt{b(b+rz)}}{c}$. Thus, if $x^* \in [a_1, a_2)$, $\min_{x\in[a_1,a_2)} g(x) = g(x^*) = 1 + \sqrt{\frac{b}{b+rz}}$. By monotonicity, we know that if $x^* < a_1$, $\min_{x\in[a_1,a_2)} g(x) = 1 + \frac{ca_1}{b+rz}$; and if $x^* \geq a_2$, $\min_{x\in[a_1,a_2)} g(x) = 1 + \frac{b}{ca_2}$. This proves Lemma 3. □

Now, for any fixed $T$, the cost of online strategy $A(T)$ is described in Eq. (4):

$$Cost_{A(T)}(t) = \begin{cases} ct, & t < T \\ cT + b + r(t-T), & T \leq t < T + z \\ cT + b + rz, & t \geq T + z \end{cases} \tag{4}$$

We first analyze the case $b < (c - r)z$ and then discuss the case $b \geq (c - r)z$. Note that the case $b < (c - r)z$ implies $c > r$, $\frac{b}{c-r} < z$, and $\frac{(c-r)^2z}{2c-r} < (c - r)z$.

**Theorem 2.** *If* $b < (c - r)z$*, then*

(1) *when* $\frac{(c-r)^2z}{2c-r} < b < (c - r)z$*, the online strategy* $A(T)$ *with* $T = \frac{\sqrt{b(b+rz)}}{c}$ *is the optimal strategy with competitive ratio* $1 + \sqrt{\frac{b}{b+rz}}$;

(2) *when* $b \leq \frac{(c-r)^2z}{2c-r}$*, the online strategy* $A(T)$ *with* $T = \frac{b}{c-r}$ *is the optimal strategy with competitive ratio* $2 - \frac{r}{c}$.

**Proof.** By (2) and (4), let $\lambda_T = sup_t \lambda_T(t) = sup_t \frac{Cost_{A(T)}(t)}{Cost_{opt}(t)}$. There are three cases.

**Case 1.** $0 \leq T < \frac{b}{c-r}$. In this case, by $T < \frac{b}{c-r} < z < T + z$,

$$\lambda_T(t) = \frac{Cost_{A(T)}(t)}{Cost_{opt}(t)} = \begin{cases} 1, & t < T \\ \frac{cT+b+r(t-T)}{ct}, & T \leq t < \frac{b}{c-r} \\ \frac{cT+b+r(t-T)}{b+rt}, & \frac{b}{c-r} \leq t < z \\ \frac{cT+b+r(t-T)}{b+rz}, & z \leq t < T+z \\ \frac{cT+b+rz}{b+rz}, & t \geq T+z \end{cases}$$

For fixed $T$, this piecewise function represents the constants on $t$ in both the first and the last intervals, $\lambda_T(t)$ are decreasing functions of $t$ in both the second and the third intervals, while this function in the fourth interval is an increasing function of $t$, thus by the continuous property of this function, the maximal value of $\lambda_T(t)$, i.e., the competitive ratio of $A(T)$, is $\lambda_T = \max\{1, \lambda_T(T), \lambda_T(T + z)\} = \max\{1 + \frac{b}{cT}, 1 + \frac{cT}{b+rz}\}$.

**Case 2.** $\frac{b}{c-r} \leq T < z$. By $\frac{b}{c-r} \leq T < z < T + z$, we have:

$$\lambda_T(t) = \frac{Cost_{A(T)}(t)}{Cost_{opt}(t)} = \begin{cases} 1, & t < \frac{b}{c-r} \\ \frac{ct}{b+rt}, & \frac{b}{c-r} \leq t < T \\ \frac{cT+b+r(t-T)}{b+rt}, & T \leq t < z \\ \frac{cT+b+r(t-T)}{b+rz}, & z \leq t < T+z \\ \frac{cT+b+rz}{b+rz}, & t \geq T+z \end{cases}$$

Similarly, the piecewise function $\lambda_T(t)$ in both the first and last intervals are constants of $t$, and in both the second and fourth intervals are increasing functions of $t$, while in the third interval is a decreasing function of $t$, thus by the continuous property, the maximal value of $\lambda_T(t)$ is $\lambda_T = \max\{\frac{b+cT}{b+rT}, \frac{cT+b+rz}{b+rz}\}$.

**Case 3.** $T \geq z$; $\frac{b}{c-r} < z < T < T + z$, and thus:

$$\lambda_T(t) = \frac{Cost_{A(T)}(t)}{Cost_{opt}(t)} = \begin{cases} 1, & t < \frac{b}{c-r} \\ \frac{ct}{b+rt}, & \frac{b}{c-r} \leq t < z \\ \frac{ct}{b+rz}, & z \leq t < T \\ \frac{cT+b+r(t-T)}{b+rz}, & T \leq t < T+z \\ \frac{cT+b+rz}{b+rz}, & t \geq T+z \end{cases}$$

Similarly, in the first and last intervals, the piecewise function is the constant of $t$, and in all other intervals, $\lambda_T(t)$ are increasing functions of $t$. Thus, by the continuous property, the maximum value of $\lambda_T(t)$ is $\lambda_T = \lambda_T(T + z) = 1 + \frac{cT}{b+rz}$.

In summary, given $T$, the competitive ratio for strategy $A(T)$ is

$$\lambda_T = \begin{cases} \max\{1 + \frac{b}{cT}, 1 + \frac{cT}{b+rz}\}, & 0 \leq T < \frac{b}{c-r} \\ \max\{\frac{b+cT}{b+rT}, 1 + \frac{cT}{b+rz}\}, & \frac{b}{c-r} \leq T < z \\ 1 + \frac{cT}{b+rz}, & T \geq z \end{cases} \tag{5}$$

To determine the optimal competitive strategy, we must compute the minimum value of $\lambda_T$. In the first interval $0 \leq T < \frac{b}{c-r}$ of (5), by Lemma 3, if $0 \leq \frac{\sqrt{b(b+rz)}}{c} < \frac{b}{c-r}$, which is equivalent to $b > \frac{(c-r)^2z}{2c-r}$, the minimum value in this interval is $1 + \sqrt{\frac{b}{b+rz}}$, while if $\frac{\sqrt{b(b+rz)}}{c} \geq \frac{b}{c-r}$, which is equivalent to $b \leq \frac{(c-r)^2z}{2c-r}$, the minimum value in this interval is $1 + \frac{b}{c\frac{b}{c-r}} = 2 - \frac{r}{c}$.

In the second interval $\frac{b}{c-r} \leq T < z$ of (5), both inner functions are increasing functions of $T$. Thus, their minimal solutions must be the same (left) point $\frac{b}{c-r}$, and the minimum value in this interval is $\max\{\frac{b+c\frac{b}{c-r}}{b+r\frac{b}{c-r}}, 1 + \frac{c\frac{b}{c-r}}{b+rz}\} = \max\{2 - \frac{r}{c}, 1 + \frac{bc}{(b+rz)(c-r)}\}$. That is, if $2 - \frac{r}{c} < 1 + \frac{bc}{(b+rz)(c-r)}$, which is equivalently to $b > \frac{(c-r)^2z}{2c-r}$, the minimum value is $1 + \frac{bc}{(b+rz)(c-r)}$, while if $2 - \frac{r}{c} \geq 1 + \frac{bc}{(b+rz)(c-r)}$, which is equivalently to $b \leq \frac{(c-r)^2z}{2c-r}$, the minimum value is $2 - \frac{r}{c}$.

In the last interval $T \geq z$ of (5), the function $1 + \frac{cT}{b+rz}$ is an increasing function, so the minimum value in this interval is $1 + \frac{cz}{b+rz}$.

Above all, we have obtained the minimum value in each interval for (5). Thus, the global minimum value for $\lambda_T$, denoted by $\lambda_{opt}$, is:

(1) if $\frac{(c-r)^2z}{2c-r} < b < (c - r)z$, then

$$\lambda_{opt} = \min\left\{1 + \sqrt{\frac{b}{b+rz}}, 1 + \frac{bc}{(b+rz)(c-r)}, 1 + \frac{cz}{b+rz}\right\}$$

It is easy to confirm this using the conditions $\frac{(c-r)^2z}{2c-r} < b < (c - r)z$, $1 + \sqrt{\frac{b}{b+rz}} < 1 + \frac{bc}{(b+rz)(c-r)} < 1 + \frac{cz}{b+rz}$. Thus, $\lambda_{opt} =$

$1+\sqrt{\frac{b}{b+rz}}$ and the optimal competitive strategy $A(T)$ is to take $T=\frac{\sqrt{b(b+rz)}}{c}$.

(2) if $b \le \frac{z(c-r)^2}{2c-r}$, then

$$\lambda_{opt} = \min\left\{2-\frac{r}{c}, 2-\frac{r}{c}, 1+\frac{cz}{b+rz}\right\}$$

Note that by $b \le \frac{z(c-r)^2}{2c-r}$, we have $\frac{cz}{b+rz} \ge \frac{cz}{\frac{z(c-r)^2}{2c-r}+rz} = 2-\frac{r}{c}$, thus $\lambda_{opt} = 2-\frac{r}{c}$ and the optimal competitive strategy $A(T)$ is to take $T=\frac{b}{c-r}$.

This completes the proof of Theorem 2. □

Theorem 2 tells us the optimal competitive strategies for a financial lease with down payment $0 < b < (c-r)z$. In the following, we analyze the case when $b \ge (c-r)z$. First, we have the following closed-form competitive ratio for strategy $A(T)$.

**Lemma 4.** *If $b+rz \ge cz$, then the competitive ratio for strategy $A(T)$ is:*

$$\lambda_T = \begin{cases} 1+\frac{cT}{b+rz}, & T > t_0 \\ \max\{1+\frac{b}{cT}, 1+\frac{cT}{b+rz}\}, & t_0-z < T \le t_0 \text{ and } b+(c-r)T \ge 0 \\ 1+\frac{cT}{b+rz}, & t_0-z < T \le t_0 \text{ and } b+(c-r)T < 0 \\ 1+\frac{b}{cT}, & T \le t_0 - z \text{ and } b+(c-r)T \ge 0 \\ \frac{cT+b+rz}{c(T+z)}, & T \le t_0-z \text{ and } b+(c-r)T < 0 \end{cases}$$

**Proof.** There are three cases in which to discuss $\lambda_T = sup_t \lambda_T(t) = sup_t \frac{Cost_{A(T)}(t)}{Cost_{opt}(t)}$.

**Case 1.** $t_0 < T < T+z$. The cost ratio of the online strategy $A(T)$ to the offline optimal strategy is:

$$\lambda_T(t) = \frac{Cost_{A(T)}(t)}{Cost_{opt}(t)} = \begin{cases} 1, & t < t_0 \\ \frac{ct}{b+rz}, & t_0 \le t < T \\ \frac{cT+b+r(t-T)}{b+rz}, & T \le t < T+z \\ \frac{cT+b+rz}{b+rz}, & t \ge T+z \end{cases}$$

For a fixed $T$, the piecewise functions in both the first and last intervals are constant, and in both the second and third intervals are increasing functions; thus, the maximal value of $\lambda_T(t)$, i.e., the competitive ratio of $A(T)$ in this case, is $\lambda_T = \frac{cT+b+rz}{b+rz} = 1+\frac{cT}{b+rz}$.

**Case 2.** $T \le t_0 < T+z$, or $t_0 - z < T \le t_0$. Now the cost ratio of the online strategy $A(T)$ to the offline optimal strategy is:

$$\lambda_T(t) = \frac{Cost_{A(T)}(t)}{Cost_{opt}(t)} = \begin{cases} 1, & t < T \\ \frac{cT+b+r(t-T)}{ct}, & T \le t < t_0 \\ \frac{cT+b+r(t-T)}{b+rz}, & t_0 \le t < T+z \\ \frac{cT+b+rz}{b+rz}, & t \ge T+z \end{cases}$$

In both the first and last intervals, these piecewise functions are constants, and in the third interval, the function is an increasing function, while in the second interval, note that $\frac{cT+b+r(t-T)}{ct} = \frac{b+(c-r)T}{ct} + \frac{r}{c}$, the monotonicity of $\lambda_T(t)$ depends on the sign of $b+(c-r)T$. If $b+(c-r)T \ge 0$, the function in the second interval is a decreasing function, thus $\lambda_T(t)$ has a maximal value at $t=T$ or $t=T+z$, and the competitive ratio is $\max\{\lambda_T(T), \lambda_T(T+z)\} = \max\{1+\frac{b}{cT}, 1+\frac{cT}{b+rz}\}$. However, if $b+(c-r)T < 0$, the function in the second interval is an increasing function, and $\lambda_T(t)$ has a maximal value at $t=T+z$, with competitive ratio factor $\lambda_T(T+z) = 1+\frac{cT}{b+rz}$. Hence, the competitive ratio of $A(T)$ in this case is $\lambda_T = \max\{1+\frac{b}{cT}, 1+\frac{cT}{b+rz}\}$ if $b+(c-r)T \ge 0$ and $\lambda_T = 1+\frac{cT}{b+rz}$ if $b+(c-r)T < 0$.

**Case 3.** $T < T+z \le t_0$. Now, the cost ratio of the online strategy $A(T)$ to the offline optimal strategy is:

$$\lambda_T(t) = \frac{Cost_{A(T)}(t)}{Cost_{opt}(t)} = \begin{cases} 1, & t < T \\ \frac{cT+b+r(t-T)}{ct}, & T \le t < T+z \\ \frac{cT+b+rz}{ct}, & T+z \le t < t_0 \\ \frac{cT+b+rz}{b+rz}, & t \ge t_0 \end{cases}$$

Similarly, in the first and last intervals, these piecewise functions are constants, and in the third interval, the function is a decreasing function, while in the second interval, similar to the discussion of Case 2, this piecewise function is a decreasing function if $b+(c-r)T \ge 0$ and is an increasing function if $b+(c-r)T < 0$. Thus, by the continuous property of $\lambda_T(t)$, if $b+(c-r)T \ge 0$, $\lambda_T(t)$ arrives its maximal value at $t=T$ and if $b+(c-r)T < 0$, and $\lambda_T(t)$ arrives its maximal value at $t=T+z$. That is, $\lambda_T = 1+\frac{b}{cT}$ if $b+(c-r)T \ge 0$ and $\lambda_T = \frac{cT+b+rz}{c(T+z)}$ if $b+(c-r)T < 0$. These cases together conclude the proof of Lemma 4. □

To determine the optimal competitive strategy, we must analyze the minimum value of $\lambda_T$ in Lemma 4, which depends on $c$ and $r$. We analyze $c \ge r$ and present Theorem 3.

**Theorem 3.** *If $b \ge (c-r)z$ and $c \ge r$, then the optimal competitive strategy is $A(T)$ with $T=\frac{\sqrt{b(b+rz)}}{c}$, and its competitive ratio is $1+\sqrt{\frac{b}{b+rz}}$.*

**Proof.** If $c \ge r$, $b+(c-r)T > 0$ always holds. Thus by Lemma 4, given $T$ and $b \ge (c-r)z$, the competitive ratio for strategy $A(T)$ can be simplified by:

$$\lambda_T = \begin{cases} 1+\frac{cT}{b+rz}, & T > t_0 \\ \max\{1+\frac{b}{cT}, 1+\frac{cT}{b+rz}\}, & t_0-z < T \le t_0 \\ 1+\frac{b}{cT}, & T \le t_0-z \end{cases}$$

In the first interval $T > t_0$, the minimal value is $1+\frac{ct_0}{b+rz} = 2$ when $T$ approaches $t_0$, and in the third interval $T \le t_0 - z$, the minimal value is $1+\frac{b}{c(t_0-z)} = 1+\frac{b}{b+rz-cz}$. In the second interval, note that by $b \ge (c-r)z > \frac{(c-r)^2 z}{2c-r}$, we have $\frac{\sqrt{b(b+rz)}}{c} > \frac{b+rz-cz}{c} = t_0 - z$, and $\frac{\sqrt{b(b+rz)}}{c} \le \frac{b+rz}{c} = t_0$ always holds, thus by Lemma 3, the minimum value in the second interval is $1+\sqrt{\frac{b}{b+rz}}$.

Thus, the global minimum competitive ratio is:

$$\min\left\{2, 1+\sqrt{\frac{b}{b+rz}}, 1+\frac{b}{b+rz-cz}\right\}$$

Note that if $c > r$, and then by $b \ge (c-r)z > \frac{(c-r)^2 z}{2c-r}$, we have $\sqrt{\frac{b}{b+rz}} \le \frac{b}{b+rz-cz}$, and the above value is $1+\sqrt{\frac{b}{b+rz}}$, the optimal competitive strategy is $A(T)$ with $T=\frac{\sqrt{b(b+rz)}}{c}$.

This completes the proof of Theorem 3. □

We now analyze the case where $b+rz \ge cz$ and $c < r$. In this case, the condition $c < r$ autonomous guarantees the condition $b+rz \ge cz$. To simplify the following analysis, we highlight a simple algebraic fact here in Lemma 5.

**Lemma 5.** *If $c < r$, then $b(2c-r) < (c-r)^2 z$ is equivalent to $\frac{b}{r-c} < \frac{\sqrt{b(b+rz)}}{c} < t_0 - z$.*

**Proof.** If $c < r$, the identical relationship can be easily verified: $\frac{b}{r-c} < \frac{\sqrt{b(b+rz)}}{c} \iff \frac{c}{r-c} < \sqrt{1+\frac{rz}{b}} \iff \frac{rz}{b} > \frac{c^2-(r-c)^2}{(r-c)^2} = \frac{r(2c-r)}{(r-c)^2} \iff b(2c-r) < (r-c)^2 z$. Additionally, $\frac{\sqrt{b(b+rz)}}{c} < t_0 - z = \frac{b+rz-cz}{c} \iff b(b+rz) < (b+rz-cz)^2 \iff b(2c-r) < (r-c)^2 z$. □

**Table 2**
The optimal online strategy and competitive ratio for all cases.

| | $b < (c-r)z$ | | $b \geq (c-r)z$ | | |
|---|---|---|---|---|---|
| | $b > \frac{(c-r)^2 z}{2c-r}$ | $b \leq \frac{(c-r)^2 z}{2c-r}$ | $c \geq r$ | $c<r$ | |
| | | | | $b(2c-r)>(c-r)^2 z$ | $b(2c-r)\geq(c-r)^2 z$ |
| Optimal online strategy $A(T)$ with $T$ | $\frac{\sqrt{(b+rz)}}{c}$ | $\frac{b}{c-r}$ | $\frac{\sqrt{(b+rz)}}{c}$ | $\frac{\sqrt{(b+rz)}}{c}$ | $\frac{b+rz-cz}{c}$ |
| Optimal competitive ratio | $1+\sqrt{\frac{b}{b+rz}}$ | $2-\frac{r}{c}$ | $1+\sqrt{\frac{b}{b+rz}}$ | $1+\sqrt{\frac{b}{b+rz}}$ | $2-\frac{cz}{b+rz}$ |

**Theorem 4.** *If $c < r$, then*

(1) *when $b(2c-r) \leq (c-r)^2 z$, the optimal competitive strategy is $A(T)$ with $T = t_0 - z = \frac{b+rz-cz}{c}$, and its competitive ratio is $2-\frac{cz}{b+rz}$;*
(2) *when $b(2c-r) > (c-r)^2 z$, the optimal competitive strategy is $A(T)$ with $T = \frac{\sqrt{b(b+rz)}}{c}$, and its competitive ratio is $1+\sqrt{\frac{b}{b+rz}}$.*

**Proof.** If $c < r$, then $b + rz > cz$ is satisfied. Note that for $T \geq 0$, $b + (c-r)T \geq 0$ is equivalent to $T \leq \frac{b}{r-c}$. This combined with Lemma 4 indicates that for any given $T$, the competitive ratio of $A(T)$ is:

$$\lambda_T = \begin{cases} 1+\frac{cT}{b+rz}, & T > t_0 \\ \max\{1+\frac{b}{cT}, 1+\frac{cT}{b+rz}\}, & t_0 - z < T \leq t_0 \text{ and } T \leq \frac{b}{r-c} \\ 1+\frac{cT}{b+rz}, & t_0 - z < T \leq t_0 \text{ and } T > \frac{b}{r-c} \\ 1+\frac{b}{cT}, & T \leq t_0 - z \text{ and } T \leq \frac{b}{r-c} \\ \frac{cT+b+rz}{c(T+z)}, & T \leq t_0 - z \text{ and } T > \frac{b}{r-c} \end{cases} \quad (6)$$

We now discuss the piecewise function $\lambda_T$ by analyzing the case $\frac{b}{r-c}$, $t_0$ and $t_0 - z$.

**Case 1.** If $\frac{b}{r-c} \leq t_0 - z$, then it is equivalent to $b(2c-r) \leq (c-r)^2 z$ by Lemma 5.

In this case, the second interval in the piecewise function of (6) is omitted, the third interval is simplified as $t_0 - z < T \leq t_0$, the fourth interval is simplified as $T \leq \frac{b}{r-c}$, and the last interval is $\frac{b}{r-c} < T \leq t_0 - z$. Thus, $\lambda_T$ is:

$$\lambda_T = \begin{cases} 1+\frac{cT}{b+rz}, & T > t_0 \\ 1+\frac{cT}{b+rz}, & t_0 - z < T \leq t_0 \\ 1+\frac{b}{cT}, & T \leq \frac{b}{r-c} \\ \frac{cT+b+rz}{c(T+z)}, & \frac{b}{r-c} < T \leq t_0 - z \end{cases}$$

For this new simplified formulation, in the first interval $T > t_0$, the minimal value is $1+\frac{ct_0}{b+rz} = 2$; in the second interval $t_0 - z < T \leq t_0$, the minimal value is $1+\frac{c(t_0-z)}{b+rz} = 2-\frac{cz}{b+rz}$; in the third interval $T \leq \frac{b}{r-c}$, the minimal value is $1+\frac{b}{c(\frac{b}{r-c})} = \frac{r}{c}$; in the last interval $\frac{b}{r-c} < T \leq t_0 - z$, the function $\frac{cT+b+rz}{c(T+z)} = 1+\frac{b+rz-cz}{cT+cz}$ is a decreasing function of $T$, and the minimal value is $1+\frac{b+rz-cz}{c(t_0-z)+cz} = 2-\frac{cz}{b+rz}$. Thus, the global minimum value is $\min\{2, 2-\frac{cz}{b+rz}, \frac{r}{c}, 2-\frac{cz}{b+rz}\}$, note that $2-\frac{cz}{b+rz} \leq \frac{r}{c}$ by $b(2c-r) \leq (c-r)^2 z$. Therefore, in this case, the optimal competitive ratio is $2-\frac{cz}{b+rz}$, and the optimal competitive strategy is $A(T)$ with $T = t_0 - z = \frac{b+rz-cz}{c}$.

**Case 2.** If $t_0 - z < \frac{b}{r-c} \leq t_0$, it is equivalent to $(c-r)^2 z < b(2c-r) < r(r-c)z$ by Lemma 5 and simple algebra.

Here, in the piecewise function of (6), the second interval is $t_0 - z < T \leq \frac{b}{r-c}$, the third interval is simplified to $\frac{b}{r-c} < T \leq t_0$, the fourth interval is simplified to $T \leq t_0 - z$, and the last interval is omitted. Thus, $\lambda_T$ is:

$$\lambda_T = \begin{cases} 1+\frac{cT}{b+rz}, & T > t_0 \\ \max\{1+\frac{b}{cT}, 1+\frac{cT}{b+rz}\}, & t_0 - z < T \leq \frac{b}{r-c} \\ 1+\frac{cT}{b+rz}, & \frac{b}{r-c} < T \leq t_0 \\ 1+\frac{b}{cT}, & T \leq t_0 - z \end{cases}$$

In this simplified formulation, in the first interval $T > t_0$, the minimal value is $1+\frac{ct_0}{b+rz} = 2$; in the second interval $t_0 - z < T \leq \frac{b}{r-c}$, by $b(2c-r) > (c-r)^2 z$ and Lemma 5, we have $t_0 - z < \frac{\sqrt{b(b+rz)}}{c} \leq \frac{b}{r-c}$; thus, by Lemma 3, the minimal value is $1+\sqrt{\frac{b}{b+rz}}$. In the third interval $\frac{b}{r-c} < T \leq t_0$, the minimal value is $1+\frac{c(\frac{b}{r-c})}{b+rz} = 1+\frac{bc}{(b+rz)(r-c)}$; and in the last interval $T \leq t_0 - z$, the minimal value is $1+\frac{b}{c(t_0-z)} = 1+\frac{b}{b+rz-cz}$. Thus, the global minimum value is $\min\{2, 1+\sqrt{\frac{b}{b+rz}}, 1+\frac{bc}{(b+rz)(r-c)}, 1+\frac{b}{b+rz-cz}\}$. Note that $\frac{bc}{(b+rz)(r-c)} > \frac{b}{b+rz-cz} > \sqrt{\frac{b}{b+rz}}$ by $b(2c-r) > (c-r)^2 z$, so in this case, the optimal competitive ratio is $1+\sqrt{\frac{b}{b+rz}}$ with optimal strategy $A(T)$ with $T = \frac{\sqrt{b(b+rz)}}{c}$.

**Case 3.** $\frac{b}{r-c} > t_0 > t_0 - z$, which is equivalent to $b(2c-r) > r(r-c)z$ by Lemma 5. We will prove that both the optimal competitive strategy and competitive ratio are the same as in Case 2.

In case 3, in the piecewise function of (6), the second interval is $t_0 - z < T \leq t_0 < \frac{b}{r-c}$, both third and last intervals are omitted, the fourth interval is simplified to $T \leq t_0 - z$, that is, $\lambda_T$ is:

$$\lambda_T = \begin{cases} 1+\frac{cT}{b+rz}, & T > t_0 \\ \max\{1+\frac{b}{cT}, 1+\frac{cT}{b+rz}\}, & t_0 - z < T \leq t_0 \\ 1+\frac{b}{cT}, & T \leq t_0 - z \end{cases}$$

In this new formulation, in the first interval $T > t_0$, the minimal value is $1+\frac{ct_0}{b+rz} = 2$. In the second interval $t_0 - z < T \leq t_0$, by $b(2c-r) > (c-r)^2 z$ and by Lemma 5, $t_0 - z \leq \frac{\sqrt{b(b+rz)}}{c} < t_0$ always holds, thus, by Lemma 3, the minimal value is $1+\sqrt{\frac{b}{b+rz}}$. In the third interval $T \leq t_0 - z$, the minimal value is $1+\frac{b}{c(t_0-z)} = 1+\frac{b}{b+rz-cz}$. Thus, the global minimum value is $\min\{2, 1+\sqrt{\frac{b}{b+rz}}, 1+\frac{b}{b+rz-cz}\}$. Note that $\frac{b}{b+rz-cz} > \sqrt{\frac{b}{b+rz}}$ by $b(2c-r) > r(r-c)z$, thus, in this case, the optimal competitive strategy is $A(T)$ with $T = \frac{\sqrt{b(b+rz)}}{c}$, and the optimal competitive ratio is $1+\sqrt{\frac{b}{b+rz}}$.

This completes the proof of Theorem 4. □

Note that case 1 in Theorem 4 reduces to Theorem 1 when $b = 0$.

## 5. Decision rules for the online financial lease problem

We have obtained the optimal online strategy and competitive ratio for all cases from Theorems 1–4, which can be summarized in Table 2.

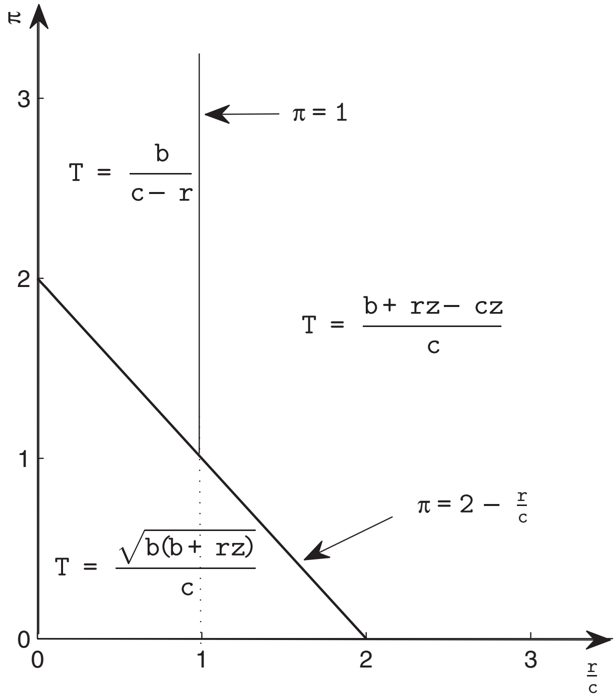


**Fig. 3.** Optimal online strategies.

To give a simpler decision rule for the optimal online strategy in different cases, we need to discuss the constraints further, as summarized in Theorem 5.

**Theorem 5.** *Let* $\pi = \frac{cz}{b+rz}$ *. The optimal online strategy* $A(T)$ *with its competitive ratio for the online financial lease problem is*

(1) *When* $\frac{r}{c} \geq 2$, *then* $T = \frac{b+rz-cz}{c}$, *and the competitive ratio is* $2 - \frac{cz}{b+rz}$;

(2) *When* $\frac{r}{c} < 2$ *and* $0 < \pi \leq 2 - \frac{r}{c}$, *then* $T = \frac{\sqrt{b(b+rz)}}{c}$, *and the competitive ratio is* $1 + \sqrt{\frac{b}{b+rz}}$;

(3) *When* $0 < \frac{r}{c} < 1$ *and* $\pi > 2 - \frac{r}{c}$, *then* $T = \frac{b}{c-r}$, *and the competitive ratio is* $2 - \frac{r}{c}$;

(4) *When* $1 < \frac{r}{c} < 2$ *and* $\pi > 2 - \frac{r}{c}$, *then* $T = \frac{b+rz-cz}{c}$, *the competitive ratio is* $2 - \frac{cz}{b+rz}$.

**Proof.** In Table 2, there are three optional strategies: $A(T)$ with $T = \frac{\sqrt{b(b+rz)}}{c}$, $\frac{b}{c-r}$, and $\frac{b+rz-cz}{c}$. First, for optimal strategy $T = \frac{b+rz-cz}{c}$, the constraints are $b > (c-r)z$, $c < r$, and $b(2c-r) \leq (c-r)^2 z$. $b \geq 0$ and $c < r$ implies $b > (c-r)z$, conditions equivalent to (i) $c < r$ and $2\,c \leq r$ or (ii) $2\,c > r$, $b \leq \frac{(c-r)^2 z}{2c-r}$, and $c < r$. A crucial note here is that $b \leq \frac{(c-r)^2 z}{2c-r}$ is equivalent to $\frac{cz}{b+rz} \geq 2 - \frac{r}{c}$. This leads to $\pi = \frac{cz}{b+rz}$, and thus, the constraints to ensure the optimal strategy $A(T)$ with $T = \frac{\sqrt{b(b+rz)}}{c}$ are (i) $\frac{r}{c} \geq 2$ or (ii) $1 < \frac{r}{c} < 2$ and $\pi \geq 2 - \frac{r}{c}$.

To ensure that the optimal strategy $A(T)$ takes $T = \frac{b}{c-r}$, the conditions are $b \leq \frac{(c-r)^2 z}{2c-r}$ and $b < (c-r)z$. Note that $b \geq 0$ implies $c > r$, so the constraints of this case are $\frac{r}{c} < 1$ and $b \leq \min\{\frac{(c-r)^2 z}{2c-r}, (c-r)z\} = \frac{(c-r)^2 z}{2c-r}$. Thus, we find that the optimal strategy $A(T)$ will take $T = \frac{b}{c-r}$ if the constraints are $\frac{r}{c} < 1$ and $\pi \geq 2 - \frac{r}{c}$.

Finally, the conditions for the optimal strategy $A(T)$ takes $T = \frac{\sqrt{b(b+rz)}}{c}$ are either (i) $b < (c-r)z$ and $b > \frac{(c-r)^2 z}{2c-r}$, (ii) $b \geq (c-r)z$ and $c \geq r$, or (iii) $b \geq (c-r)z$, $c < r$ and $b(2c-r) > (c-r)^2 z$. Note that the constraints $b \geq (c-r)z$ and $c \geq r$ imply $b(2c-r) > (c-r)^2 z$, so the constraint for this optimal strategy in all cases is $b(2c-r) > (c-r)^2 z$, which can be simplified to $\frac{r}{c} \leq 2$ and $0 < \pi \leq 2 - \frac{r}{c}$.

Theorem 5 thus summarizes the simple decision. □

We call the index $\pi = \frac{cz}{b+rz}$ the *rent-to-value index*, which presents the ratios of the lease cost and financial lease cost. In addition, the decision rules in Theorem 5 can be illustrated as in Fig. 3, with three areas in the first quadrant denoting various corresponding optimal strategies.

## 6. Numerical analysis

This section performs the numerical analysis to validate our theoretical results. We take the example used in Section 1: consider someone with two options to use a repertory: lease with rental cost of \$2000 per month, or use a financial lease with a cost of \$ 3000 per month, with the ownership transferring to the lessee after 24 months. Here, $c = 2000$, $r = 3000$, and $z = 24$.

We consider a financial lease without down payment, ($b = 0$). In Table 3, we present the numerical results for $T = 10, 11, \ldots, 14$. For each $T$, assume the length of time the equipment will be used is $t = 30, 31, \ldots, 40$, we compute the optimal offline cost and the cost of strategy $A(T)$. From Table 3, we see that, given $T$, the cost of the offline strategy and $A(T)$ are non-decreasing functions with $t$. However, they increase at different rates. In Fig. 4 we illustrate the ratio functions

**Table 3**
Numerical examples of financial lease without down payment (expressed in thousands of units).

| | t | 30 | 31 | 32 | 33 | 34 | 35 | 36 | 37 | 38 | 39 | 40 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | $Cost_opt(t)$ | 60 | 62 | 64 | 66 | 68 | 70 | 72 | 72 | 72 | 72 | 72 |
| $T = 10$ | $Cost_{A(10)}(t)$ | 80 | 83 | 86 | 89 | 92 | 92 | 92 | 92 | 92 | 92 | 92 |
| $T = 11$ | $Cost_{A(11)}(t)$ | 79 | 82 | 85 | 88 | 91 | 94 | 94 | 94 | 94 | 94 | 94 |
| $T = 12$ | $Cost_{A(12)}(t)$ | 78 | 81 | 84 | 87 | 90 | 93 | 96 | 96 | 96 | 96 | 96 |
| $T = 13$ | $Cost_{A(13)}(t)$ | 77 | 80 | 83 | 86 | 89 | 92 | 95Z | 98 | 98 | 98 | 98 |
| $T = 14$ | $Cost_{A(14)}(t)$ | 76 | 79 | 82 | 85 | 88 | 91 | 94 | 97 | 100 | 100 | 100 |

**Table 4**
Numerical examples of financial lease without down payment (expressed in thousands of units).

| | t | 36 | 37 | 38 | 39 | 40 | 41 | 42 | 43 | 44 | 45 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | $Cost_opt(t)$ | 72 | 74 | 76 | 78 | 80 | 82 | 82 | 82 | 82 | 82 |
| $T = 15$ | $Cost_{A(15)}(t)$ | 103 | 106 | 109 | 112 | 112 | 112 | 112 | 112 | 112 | 112 |
| $T = 16$ | $Cost_{A(16)}(t)$ | 102 | 105 | 108 | 111 | 114 | 114 | 114 | 114 | 114 | 114 |
| $T = 17$ | $Cost_{A(17)}(t)$ | 101 | 104 | 107 | 110 | 113 | 116 | 116 | 116 | 116 | 116 |
| $T = 18$ | $Cost_{A(18)}(t)$ | 100 | 103 | 106 | 109 | 112 | 115 | 118 | 118 | 118 | 118 |
| $T = 19$ | $Cost_{A(19)}(t)$ | 99 | 102 | 105 | 108 | 111 | 114 | 117 | 120 | 120 | 120 |

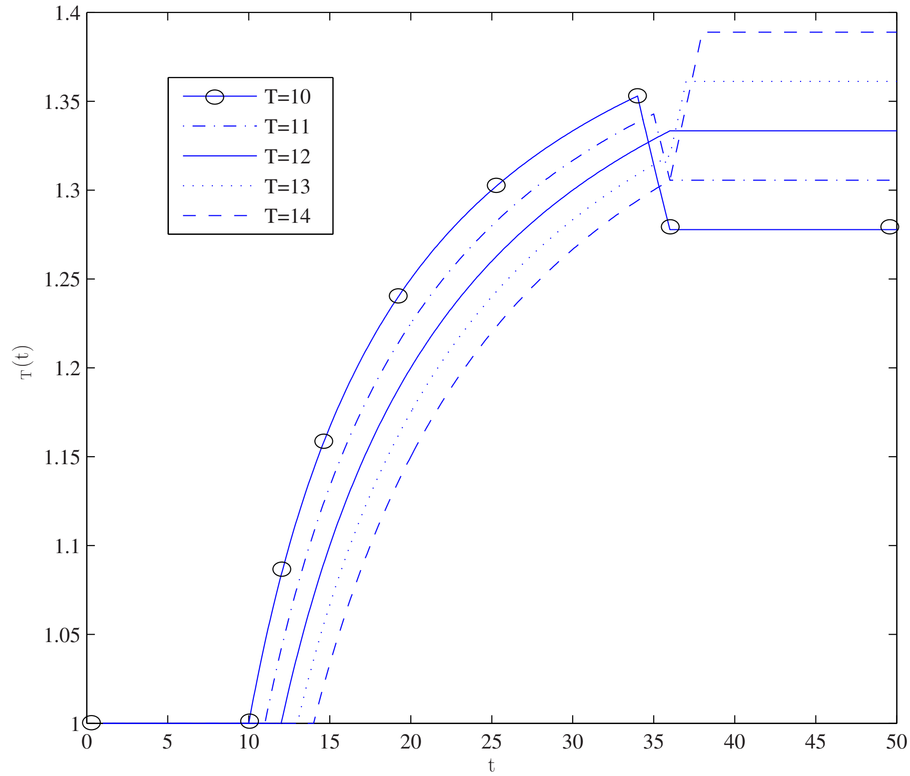


**Fig. 4.** Ratio function $\lambda_T(t)$ for a financial lease without down payment.

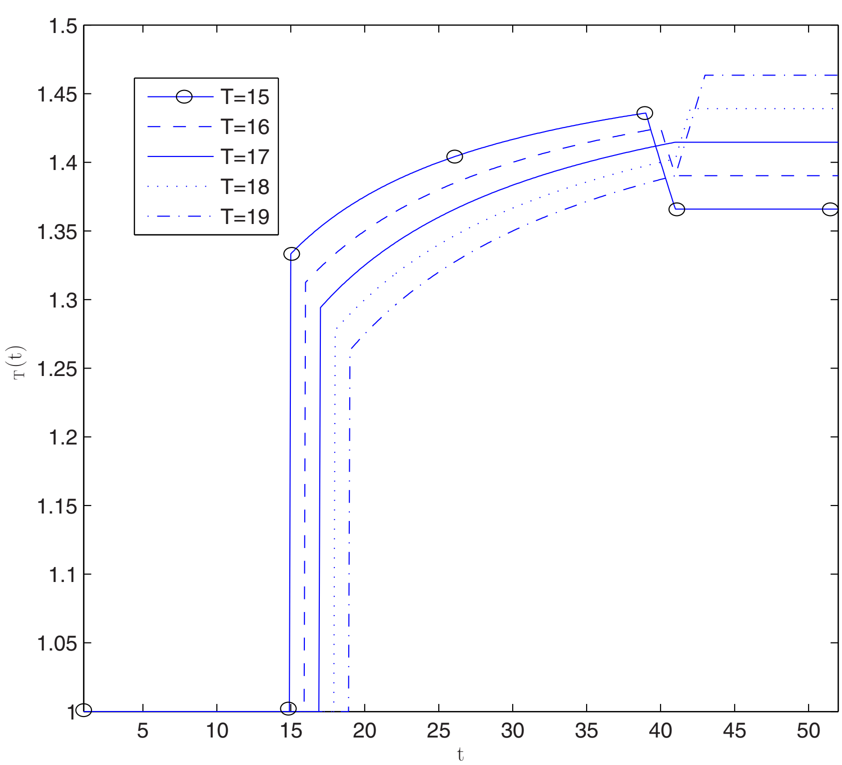


**Fig. 5.** Ratio function $\lambda_T(t)$ for financial lease with down payment.

$\lambda_T(t) = \frac{Cost_{A(T)}(t)}{Cost_{opt}(t)}$ for $t \geq 0$. According to the competitive ratio definition, given $T$, $\lambda_T = \max_t \lambda_T(t)$ is the competitive ratio for the online strategy $A(T)$. Fig. 4 shows that the optimal strategy is the one with the minimal maximum value 1.333, that is, $A(T)$ with $T = 12$. Thus, we see in our example of a financial lease without down payment, the optimal online strategy $A(T)$ is $T = 12$ with competitive ratio 1.333. It is easy to verify that this solution is consistent with Theorem 1, as well as the generality formulation of Theorems 4.1 and 5.

We next consider the case of a financial lease with down payment. Assume $b = \$10000$. Table 4 presents the numerical results, and Fig. 5 illustrates the ratio functions $\lambda_T(t)$ for $t \geq 0$. As in the previous discussion, the results show that the optimal strategy is the one with the minimal competitive ratio 1.415, that is, the optimal online strategy $A(T)$ with $T = 17$. This solution is also consistent with Theorems 4.2 and 5.

## 7. Conclusion

As the modern leasing industry develops, leasing grows in popularity. Since the decision maker usually cannot determine the exact length of time the equipment will be needed, we explore the well-established techniques of the online competitive analysis method to find the optimal strategy. However, all of previous research into the lease option considers only operating leases. In practice, there is another important type of lease: the financial lease. The key feature of a financial lease is that the lessee may obtain ownership of the equipment after the lease period. This paper develops the optimal strategy between leasing and financial leasing using competitive analysis. We divide financial leasing according to whether there is a down payment and present the optimal competitive strategy for each case. Finally we summarize and present the optimal strategies with simple decision rules. In our future research, we plan to introduce the rate factor and consider more general models.

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