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Note

# A note on the relationship between quasi-symmetric mappings and $\varphi$-uniform domains

Xiantao Wang [a], Qingshan Zhou [a], Tiantian Guan [b], Manzi Huang [b], Antti Rasila [c,*]

[a] *Department of Mathematics, Shantou University, Shantou, Guangdong 515063, People's Republic of China*
[b] *Department of Mathematics, Hunan Normal University, Changsha, Hunan 410081, People's Republic of China*
[c] *Department of Mathematics and Systems Analysis, P.O. Box 11100, Aalto University, FI-00076 Aalto, Finland*

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ABSTRACT

The aim of this note is to construct a $\psi$-uniform domain $G$ in the complex plane $\mathbb{C}$ such that the identity mapping id: $(G, j_G) \to (G, k_G)$ is not an $\eta$-quasi-symmetric mapping for any homeomorphism $\eta$: $[0, \infty) \to [0, \infty)$. This result shows that the answer to the related open problem, posed by Hästö, Klén, Sahoo and Vuorinen, is negative.



## 1. Introduction

For a proper subdomain $G$ of $\mathbb{R}^n$ and $z_1$, $z_2 \in G$, the distance ratio metric $j_G$ is defined by

$$j_G(z_1, z_2) = \log\left(1 + \frac{|z_1 - z_2|}{\min\{\delta_G(z_1), \delta_G(z_2)\}}\right),$$

where $\delta_G(z_1)$ denotes the Euclidean distance from $z_1$ to the boundary $\partial G$ of $G$. We remark that the above form of $j_G$, introduced in [10], is obtained by a slight modification of a metric that was studied in [2,3].

For a rectifiable arc or a path $\gamma$ in $G$, its *quasihyperbolic length* of $\gamma$ in $G$ is the number:

$$\ell_{k_G}(\gamma) = \int_\gamma \frac{|dz|}{\delta_G(z)}.$$

* Corresponding author.
*E-mail addresses:* xtwang@stu.edu.cn (X. Wang), q476308142@qq.com (Q. Zhou), 1548643494@qq.com (T. Guan), mzhuang79@163.com (M. Huang), antti.rasila@iki.fi (A. Rasila).

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The *quasihyperbolic metric* $k_G(z_1, z_2)$ between $z_1$ and $z_2$ is defined by

$$k_G(z_1, z_2) = \inf\{\ell_{k_G}(\gamma)\},$$

where the infimum is taken over all rectifiable arcs $\gamma$ joining $z_1$ and $z_2$ in $G$. It is well-known that for $z_1$ and $z_2 \in G$, we have $k_G(z_1, z_2) \geq j_G(z_1, z_2)$ (cf. [3]).

The class of uniform domains was introduced by Martio and Sarvas in 1979 [6]. The precise definition is as follows.

**Definition 1.1.** Given $c \geq 1$, a domain $G$ in $\mathbb{R}^n$ is called *c-uniform* provided that each pair of points $z_1$, $z_2$ in $G$ can be joined by a rectifiable arc $\gamma$ in $G$ satisfying

(1) $\min\{\ell(\gamma[z_1, z]), \ell(\gamma[z_2, z])\} \leq c\, \delta_G(z)$ for all $z \in \gamma$;
(2) $\ell(\gamma) \leq c|z_1 - z_2|$,

where $\ell(\gamma)$ denotes the length of $\gamma$ and $\gamma[z_j, z]$ stands for the part of $\gamma$ between $z_j$ and $z$. An arc $\gamma$ with the above properties is called a *double c-cone* arc. A domain is called *uniform* if it is $c$-uniform for some constant $c \geq 1$.

The following convenient characterization of uniform domains, by means of the quasihyperbolic and distance ratio metrics, was given by Gehring and Osgood [2]: a proper subdomain $G$ of $\mathbb{R}^n$ is uniform if and only if there exists a constant $\mu \geq 1$, depending only on $c$, such that for all $z_1$ and $z_2$ in $G$,

$$k_G(z_1, z_2) \leq \mu j_G(z_1, z_2).$$

We remark that the above characterization is again slightly different from the one given in [2], as the original result had an additive constant on the right hand side. Later, it was shown by Vuorinen [10] that this constant is not necessary. Motivated by this observation, Vuorinen [10] gave the following more general definition of *φ-uniform* domains:

**Definition 1.2.** Let $\varphi$: $[0, \infty) \to [0, \infty)$ be a homeomorphism. A domain $G \subset \mathbb{R}^n$ is said to be *φ-uniform* if for all $z_1$, $z_2 \in G$,

$$k_G(z_1, z_2) \leq \varphi\Big(\frac{|z_1 - z_2|}{\min \delta_G(z_1), \delta_G(z_2)}\Big).$$

Obviously, uniformity implies $\varphi$-uniformity with $\varphi(t) = \mu \log(1+t)$ for $t > 0$ with $\mu \geq 1$. It is easy to see that the converse is not true.

Interesting results on the above classes of domains have been obtained by Väisälä [7] (see also [8]). In particular, he observed that the class of $\varphi$-uniform domains coincides with the class of uniform domains if $\varphi$ is a *slow function*, i.e.,

$$\lim_{t\to\infty} \frac{\varphi(t)}{t} = 0.$$

Recently, the geometric properties of this class of domains have been investigated in [4]. The stability of $\varphi$-uniform domains has been established [5].

**Definition 1.3.** A homeomorphism $f$: $\mathbb{R}^n \to \mathbb{R}^n$ is said to be *η-quasi-symmetric* if there is a homeomorphism $\eta : [0, \infty) \to [0, \infty)$ such that

$$|x - a| \leq t|x - b| \text{ implies } |f(x) - f(a)| \leq \eta(t)|f(x) - f(b)|$$

for each $t > 0$ and for all points $x$, $a$ and $b$ in $\mathbb{R}^n$.

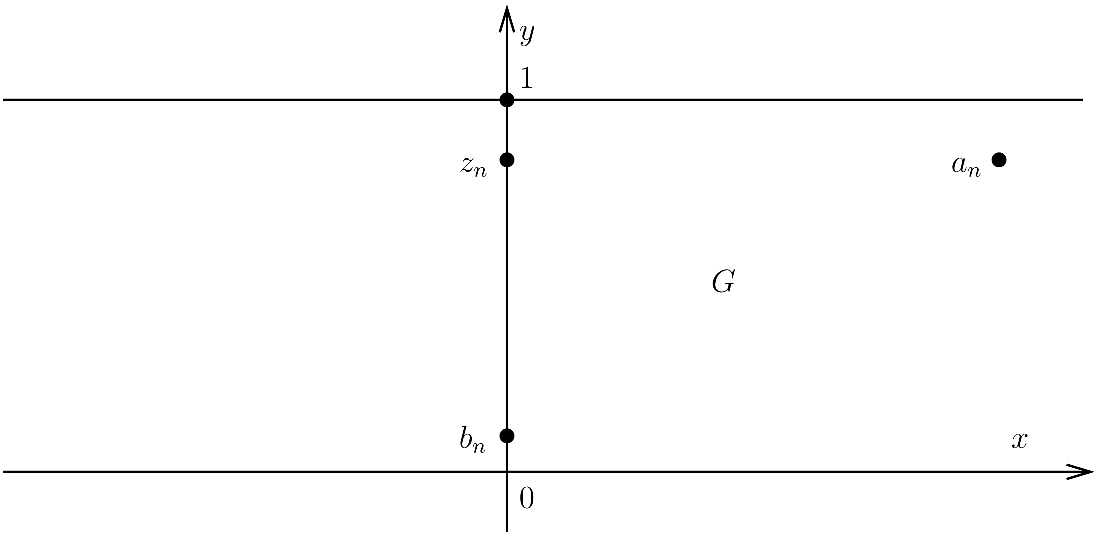


**Fig. 1.** The points $a_n, z_n$ and $b_n$ in $G$.

With the aid of quasi-symmetric mappings, the authors in [4] provided a sufficient condition for a domain in $\mathbb{R}^n$ to be $\varphi$-uniform, whose precise statement is as follows.

**Theorem A.** ([4, Proposition 2.5]) *If the identity mapping* id*:* $(G, j_G) \to (G, k_G)$ *is* $\eta$*-quasi-symmetric, then* $G$ *is* $\varphi$*-uniform for some homeomorphism* $\varphi$*:* $[0, \infty) \to [0, \infty)$ *depending only on* $\eta$*.*

In the same paper, the following open problem was presented:

**Open problem 1.1.** ([4, Question 2.6]) *Is the converse of Theorem A true?*

In the next section, we shall construct an example to show that the answer to the question of Open Problem 1.1 is negative.

## 2. An example

**Example 2.1.** Let $G = \{z = x + iy \in \mathbb{C} : \ 0 < y < 1\}$ (see Fig. 1). Then

(1) $G$ is $\varphi$-uniform with $\varphi(t) = t$;
(2) the identity mapping id: $(G, j_G) \to (G, k_G)$ is not $\eta$-quasi-symmetric for any homeomorphism $\eta$: $[0, \infty) \to [0, \infty)$.

**Proof.** The proof of the assertion (1) in the example easily follows from [9, Remarks 2.19(2)] or [8, Remark 6.17]. In the following, we prove the assertion (2). We shall show this assertion by contradiction. Suppose the identity mapping id: $(G, j_G) \to (G, k_G)$ is $\eta$-quasi-symmetric for some homeomorphism $\eta$: $[0, \infty) \to [0, \infty)$. It follows that for all $a$, $z$ and $b$ in $G$,

$$\frac{k_G(a,z)}{k_G(z,b)} \le \eta\Big(\frac{j_G(a,z)}{j_G(z,b)}\Big). \tag{2.1}$$

In order to get a contradiction, for any integer $n \ge 16$, we let (see Fig. 1)

$$a_n = \Big(n, 1 - \frac{1}{n}\Big), \ \ z_n = \Big(0, 1 - \frac{1}{n}\Big) \ \text{and} \ \ b_n = \Big(0, \frac{1}{n^3}\Big).$$

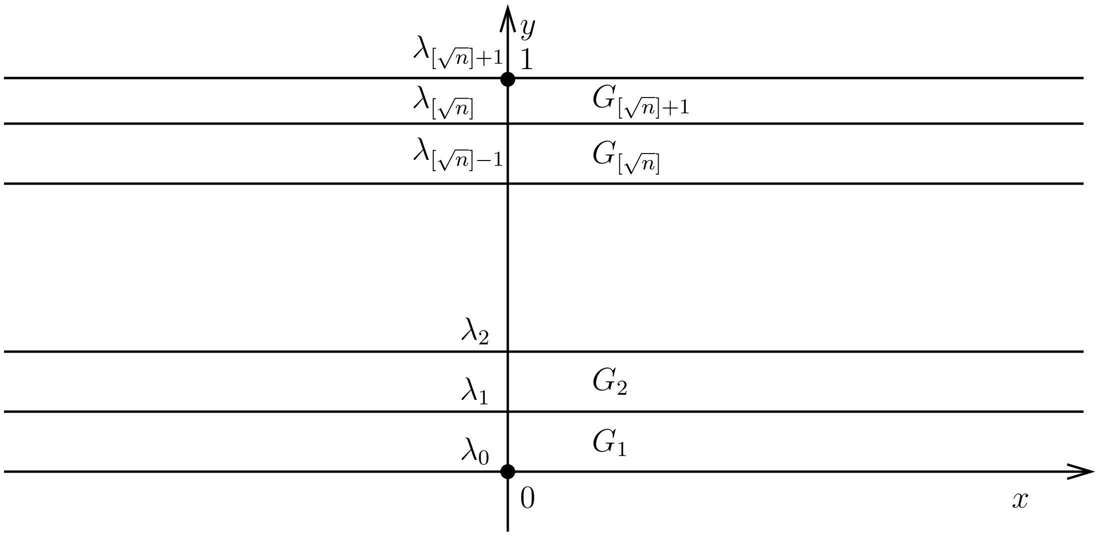


**Fig. 2.** The partition of $G$.

Then $a_n$, $z_n$ and $b_n \in G$, and further, we have the following:

**Claim 2.1.** $\frac{j_G(a_n,z_n)}{j_G(z_n,b_n)} < 1$.

Because $n > 3$, the proof of this claim easily follows from the following two facts:

$$j_G(a_n, z_n) = \log\left(1 + \frac{|a_n - z_n|}{\min\{\delta_G(a_n), \delta_G(z_n)\}}\right) = \log(1 + n^2)$$

and

$$j_G(z_n, b_n) = \log\left(1 + \frac{|z_n - b_n|}{\min\{\delta_G(z_n), \delta_G(b_n)\}}\right) = \log(n^3 - n^2).$$

**Claim 2.2.** $\frac{k_G(a_n,z_n)}{k_G(z_n,b_n)} > \frac{\sqrt{n}}{8\log n}$.

To prove the inequality in the claim, let $\gamma_n$ be a quasihyperbolic geodesic in $G$ connecting $a_n$ and $z_n$, i.e.

$$\ell_{k_G}(\gamma_n) = k_G(a_n, z_n). \tag{2.2}$$

Note that the existence of such a $\gamma_n$ follows from Lemma 1 in [2]. Obviously,

$$\ell(\gamma_n) \geq n. \tag{2.3}$$

To continue the proof, we need a partition of $G$. For each $m \in \{1, \ldots, [\sqrt{n}]\}$, we let (see Fig. 2)

$$G_m = \{z = x + iy \in \mathbb{C} : \ \lambda_{m-1} < y \leq \lambda_m\}$$

and

$$G_{[\sqrt{n}]+1} = \{z = x + iy \in \mathbb{C} : \ \lambda_{[\sqrt{n}]} < y < \lambda_{[\sqrt{n}]+1} = 1\},$$

where $[\sqrt{n}]$ denotes the integer part of $\sqrt{n}$ and $\lambda_m = (1 - \frac{1}{n})\frac{m}{[\sqrt{n}]}$. Clearly,

$$G = \bigcup_{m=1}^{[\sqrt{n}]+1} G_m,$$

and then there is at least an $m \in \{1, \ldots, [\sqrt{n}] + 1\}$ such that

$$\ell(\gamma_n \cap G_m) \geq m,$$

because otherwise, we get

$$\ell(\gamma_n) = \sum_{m=1}^{[\sqrt{n}]+1} \ell(\gamma_n \cap G_m) < \sum_{m=1}^{[\sqrt{n}]+1} m < n,$$

since $n \geq 16$, which contradicts (2.3).

Since for any $z \in G_m$,

$$\delta_G(z) \leq \begin{cases} \lambda_m, & \text{if } m \in \{1, \ldots, [\sqrt{n}]\}, \\ \frac{1}{n}, & \text{if } m = [\sqrt{n}] + 1, \end{cases}$$

it follows from (2.2) that

$$k_G(a_n, z_n) = \ell_{k_G}(\gamma_n) \geq \ell_{k_G}(\gamma_n \cap G_m) > \frac{1}{2}\sqrt{n}. \tag{2.4}$$

Moreover, we have

$$k_G(z_n, b_n) \leq \int_{[b_n, z_n]} \frac{|dz|}{\delta_G(z)} = \int_{\frac{1}{n^3}}^{\frac{1}{2}} \frac{dt}{t} + \int_{\frac{1}{2}}^{1-\frac{1}{n}} \frac{dt}{1-t} = 4\log n - 2\log 2, \tag{2.5}$$

where $[b_n, z_n]$ stands for the segment in $G$ with the endpoints $b_n$ and $z_n$. Then we can easily conclude the inequality in Claim 2.2 from (2.4) and (2.5).

Now, we are ready to reach a contradiction. It follows from (2.1) together with Claims 2.1 and 2.2 that

$$\frac{\sqrt{n}}{8\log n} < \frac{k_G(a_n, z_n)}{k_G(z_n, b_n)} \leq \eta\Big(\frac{j_G(a_n, z_n)}{j_G(z_n, b_n)}\Big) \leq \eta(1),$$

which is impossible since $\frac{\sqrt{n}}{8\log n} \to \infty$ as $n \to \infty$, and thus, the proof is complete.

It is well-known that simply connected domains in plane are quasidisks [6] (or [1]), and then the complement of such a uniform domain also is uniform. Naturally, we propose the following problem [11].

Suppose $G \subsetneq \mathbb{R}^n$ is a $\varphi$-uniform domain. Find the condition on $\varphi$ such that the complement $\mathbb{R}^n \backslash \overline{G}$ of $G$ in $\mathbb{R}^n$ is also a $\varphi_1$-uniform domain for some $\varphi_1$.

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