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Solving Nonlinear Programming Problem in Fuzzy Environment

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Abstract

In this paper, the fuzzy nonlinear programming problem is discussed. In order to obtain more accurate solution, the properties of fuzzy set and fuzzy number with linear membership function and fuzzy maximum decision maker is utilized to fuzzifying the crisp problem . An example is provided to show the effectiveness of the proposed method.

Keywords: Fuzzy numbers, Nonlinear programming, Optimization problem, Fuzzy maximum decision making

1 Introduction

Fuzzy nonlinear programming problem (FNLPP) is useful in solving problems which are difficult, impossible to solve due to the imprecise, subjective nature of the problem formulation or have an accurate solution. In this paper we will discuss the concepts of fuzzy decision making introduced by [1] and the maximum decision [15] that is used in NLPP to find the optimal decision (solution). This decision making used in fuzzy linear programming problem [8] and [7]. Furthermore, this problems has fuzzy objective function and fuzzy variables in the constraints [13], [10] and [5] where the fuzzy left and right hand side coefficients on constraints [14]. In addition, the fuzzy NLPP is used in quadratic programming [9, 11] which has fuzzy multi objective function and fuzzy parameters on constraints so in our NLPP that have fuzzy properties on

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the inequality $(\widetilde{\leq}, \widetilde{\geq})$ and have fuzzy linear membership function. The outline of this paper is as follows: In Section 2 we introduce some essential definitions that are useful in our problem. Section 3 we state the nonlinear programming in fuzzy environment by transform the crisp problem. A numerical example will be given in Section 4 and the conclusions will be presented in Section5.

2 Preliminary

In this section several necessary basic concepts are recalled.

Definition 2.1: [15] If x is a collection of objects denoted generally by X, then a fuzzy set $\widetilde{A}$ in X is a set of order pairs

$$\widetilde{A} = \{(x, \mu_{\widetilde{A}}) : x \in X\} \tag{2.1}$$

where $\mu_{\widetilde{A}} : x \to [0,1]$ is called the membership function or grade of membership of x in $\widetilde{A}$ which maps x to the membership range M (when M contains only the two points 0 and 1), $\widetilde{A}$ is non fuzzy and $\mu_{\widetilde{A}}$ is identical to the characteristic function of crisp set. It should be emphasized that the range of membership function is a subset of the non-negative real numbers. The elements with a zero degree of membership are normally not listed.

Definition 2.2: [2] The function $L : X \to [0,1]$ is a function with two parameters defined as:

$$L(x;\alpha,\beta) = \begin{cases} 1, & \text{if} \quad x < \alpha \\ \frac{\alpha+\beta-x}{\beta} & \text{if} \quad \alpha \leq x < \alpha+\beta \\ 0, & \text{if} \quad x > \beta \end{cases} \tag{2.2}$$

It is called the trapezoidal linear membership function. This type of fuzzy number is very useful which has a large non convex fuzzy rejoin set. Clearly, it gives us the high degree of $\mu_{\widetilde{A}}$. (See the Fig. 1)

Definition 2.3: [1] Given a fuzzy goal (fuzzy objective function) $\widetilde{G}$ and fuzzy constraints $\widetilde{C}$ in a space of alternatives X. The $\widetilde{G}$ and $\widetilde{C}$ combine to form a decision, $\widetilde{D}$, which is a fuzzy set resulting from intersection of $\widetilde{G}$ and

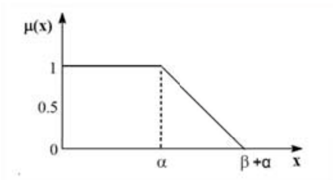

Figure 1: L-function

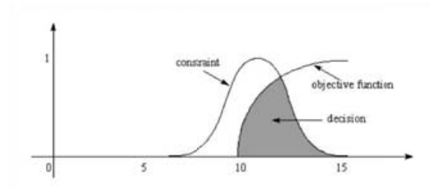

Figure 2: Fuzzy decision

$\widetilde{C}$. Furthermore, $\widetilde{D} = \widetilde{G} \cap \widetilde{C}$ is the membership function of $\widetilde{D}$ can be defined as $\mu_{\widetilde{D}} = \min\{\mu_{\widetilde{G}}, \mu_{\widetilde{C}}\}$. In general, if we have n goals $\widetilde{G}_1, \ldots, \widetilde{G}_n$ and m constraints $\widetilde{C}_1, \ldots, \widetilde{C}_m$,then, the resultant decision can be defined as

$$\widetilde{D} = \widetilde{G}_1 \cap \ldots \cap \widetilde{G}_n \cap \widetilde{C}_1 \cap \ldots \cap \widetilde{C}_m \tag{2.3}$$

Therefore, for $j = 1, 2, \ldots, n$ and $i = 1, 2, \ldots, m$ it can be written as follows

$$\begin{aligned} \mu_{\widetilde{D}} &= \min\{\min\{\mu_{\widetilde{G}_j}\}, \min\{\mu_{\widetilde{C}_i}\}\} \\ &= \min\{\mu_{\widetilde{G}_1}, \ldots, \mu_{\widetilde{G}_n}, \mu_{\widetilde{C}_1}, \ldots, \mu_{\widetilde{C}_m}\} \\ &= \min\{\mu_{\widetilde{G}_j}, \ldots, \mu_{\widetilde{C}_i}\} \end{aligned} \tag{2.4}$$

(See the Fig. 2)

Remarks: If the decision-maker wants to have "crisp" decision proposal, it is appropriate to suggest which of them have the highest degree of membership in the fuzzy set "decision". The maximizing decision is defined by

$$X_{\max} = \max_x M_{\widetilde{D}}(x) = \max_x\{\mu_{\widetilde{D}_j}(x), \mu_{\widetilde{C}_i}(x)\} \tag{2.5}$$

Where $\widetilde{D}_j$ and $\widetilde{C}_i$ are in Definition 2.3, for $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$ [15].

3 Fuzzy nonlinear programming problems

In this section we discuss the optimization problem with nonlinear fuzzy objective function and fuzzy flexible nonlinear constraints. Consider the following non linear programming problem:

Min/Max$f(x)$

Subject to

$$g_i(x) \geq (\leq) b_i, \qquad i = 1, 2, \ldots, m \tag{3.1}$$

For all $x \in \mathbb{R}^n$ and $x \geq 0$. The fuzzy version of the problem (3.1) is

Min/Max$f(x)$
Subject to

$$g_i(x)\widetilde{\geq}(\widetilde{\leq})b_i, \qquad i = 1, 2, \dots, m \tag{3.2}$$

For all $x \in \mathbb{R}^n$ and $x \geq 0$. In problem (3.2), the tilde sign denotes a fuzzy satisfaction of the constraints. It is clear that these constraints are flexible constrains. The fuzzy max (min) corresponds to achieving the highest (lowest) possible aspiration level for the general $f(x)$. This problem can be solved by using the properties of fuzzy decision making problems (3.1) and (3.2) as follows:

Step 1: Fuzzify the objective function by calculating the lower and the upper bounds of the optimal values. The bounds of optimal values z_l and z_u can be obtained by solving the standard crisp NLPP as follows:
$z_1 =$ **Min/Max**$f(x)$
Subject to

$$g_i(x) \geq (\leq)b_i, \quad i = 1, 2, \dots, m \tag{3.3}$$

For all $x \in \mathbb{R}^n$ and $x \geq 0$ and
$z_2 =$ **Min/Max**$f(x)$
Subject to

$$g_i(x) \geq (\leq)b_i + p_i, \quad i = 1, 2, \dots, m \tag{3.4}$$

For all $x \in \mathbb{R}^n$ and $x \geq 0$. Where the objective function take the values between z_1 and z_2. Let $z_l = \min(z_1, z_2)$ and $z_u = max(z_1, z_2)$, where z_l and z_u are the lower and upper bounds of the optimal values. Suppose $\widetilde{M}$ is the fuzzy set representing the objective function $f(x)$ such that $\widetilde{M} = \{(x, \mu_{\widetilde{M}}(x)) : x \in \mathbb{R}^n\}$, where

$$\mu_{\widetilde{M}}(x) = \begin{cases} 1, & \text{if} \quad z_u < f(x) \\ \frac{f(x)-z_l}{z_u-z_l} & \text{if} \quad z_l < f(x) < z_u \\ 0, & \text{if} \quad z_l > f(x) \end{cases} \tag{3.5}$$

Note that, p_i is vector of relaxation and can be determined by fuzzifying b_i (denoted by $\widetilde{b}_i$) by using the definition of L-function of the membership function as follows

$$\widetilde{b}_i = \{(x, \mu_{\widetilde{b}_i}(x)) : x \in \mathbb{R}^n\}$$

where

$$\mu_{\widetilde{b}_i}(x) = \begin{cases} 1, & \text{if} \quad x < b_i \\ \frac{b_i + p_i - x}{p_i} & \text{if} \quad b_i < x < b_i + p_i \\ 0, & \text{if} \quad x > b_i + p_i \end{cases} \tag{3.6}$$

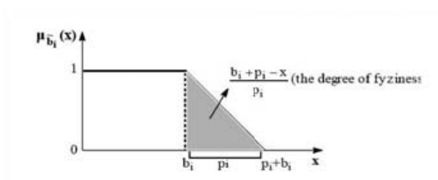

Figure 3: Represent $\mu_{\widetilde{b}_i}(x)$ membership function

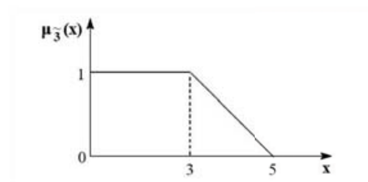

Figure 4: membership function of $\mu_{\widetilde{3}}(x)$

Step 2: Fuzzify the constraint $g_i(x), \quad i = 1, 2, \ldots, m$. Let $\widetilde{C}_i$ is the fuzzy set for ith constraints such that $\widetilde{C}_i = \{(x, \mu_{\widetilde{C}_i}(x)) : x \in \mathbb{R}^n\}$, where

$$\mu_{\widetilde{C}_i}(x) = \begin{cases} 1, & \text{if} \quad g_i(x) < b_i \\ \frac{b_i + p_i - g_i(x)}{p_i} & \text{if} \quad b_i < g_i(x) < b_i + p_i \\ 0, & \text{if} \quad g_i(x) > b_i + p_i \end{cases} \tag{3.7}$$

Let $\widetilde{D}$ be the fuzzy decision set, where

$$\widetilde{D} = \widetilde{M} \cap \widetilde{C}_i, \quad i = 1, 2, \ldots, m \tag{3.8}$$

Therefore, $\widetilde{D} = \widetilde{M} \cap \widetilde{C}_1 \cap \ldots \cap \widetilde{C}_m$, and $\widetilde{D} = \{(x, \mu_{\widetilde{M}}(x)) : x \in \mathbb{R}^n\}$. Then we have

$$\mu_{\widetilde{D}}(x) = \min\{\mu_{\widetilde{M}}(x), \min\{\mu_{\widetilde{C}_1}(x), \ldots, \mu_{\widetilde{C}_m}(x)\}\}.$$

Now, if we suppose

$$\lambda = \min\{\mu_{\widetilde{M}}(x), \min\{\mu_{\widetilde{C}_1}(x), \ldots, \mu_{\widetilde{C}_m}(x)\}\} \tag{3.9}$$

then we have the optimal decision:

$$x^* = \max \lambda, \quad x^* \in \mathbb{R}^n$$

Now, the problem (3.2) becomes the following crisp NLPP

Max λ
Subject to

$$\begin{aligned}
\overline{g}_1 &: \lambda - \mu_{\widetilde{M}}(x) \leq 0 \\
\overline{g}_2 &: \lambda - \mu_{\widetilde{C}_1}(x) \leq 0 \\
&\vdots \\
\overline{g}_m &: \lambda - \mu_{\widetilde{C}_{m-1}}(x) \leq 0 \\
\overline{g}_{m+1} &: \lambda - \mu_{\widetilde{C}_m}(x) \leq 0
\end{aligned} \tag{3.10}$$

where $0 \leq \lambda \leq 1, x \geq 0$ and $x \in \mathbb{R}^n$. This is equivalent to the problem

Maxλ
Subject to

$$\begin{aligned}
\overline{g}_1 &: \lambda - \left(\frac{f(z) - z_l}{z_u - z_l} \right) \leq 0 \\
\overline{g}_2 &: \lambda - \left(\frac{b_1 + p_1 - g_l(x)}{p_1} \right) \leq 0 \\
&\vdots \\
\overline{g}_{m+1} &: \lambda - \left(\frac{b_m + p_m - g_m(x)}{p_m} \right) \leq 0
\end{aligned} \tag{3.11}$$

where $0 \leq \lambda \leq 1, x \geq 0$ and $x \in \mathbb{R}^n$. After that, we can obtain the optimal solution $x^* \in \mathbb{R}^n$ and substitute problem (3.1) in the objective function. It can be easily seen that

$$z_l < z_{AF} < z_u$$

where z_{AF} is the objective function after fuzziness.

4 Numerical example

In the following example, we will illustrate presented theory. Our computation is carried out by utilizing MATHLAB 7.2. Suppose we are going to make a box for airport shipping [3]. The box is made of two materials; the top of the box is made of a material costing \$17 per square foot and the rest of the box is made of material that cost \$3 per square foot. Then, the baggage restrictions require that the dimensions of the box end of it must sum to at most 3 foot. The dimensions of the box of should be a maximum volume costing and not more than \$108. Mathematical description x_1, x_2 and x_3 are present the length, width and height of the box, respectively. The NLP optimization problem is

$\mathbf{Max} x_1 x_2 x_3 = z$
Subject to

$$g_1 : x_2 + x_3 \leq 3$$
$$g_2 : 17x_1x_2 + 3(x_1x_2 + 2x_1x_3 + 2x_2x_3) \leq 108$$
$$x_1, x_2, x_3 \geq 0 \tag{4.1}$$

The optimal solution is $x_1^* = 2.99526551$, $x_1^* = 1.00316204$, and $x_1^* = 1.99687905$. Therefore $z^* = 5.99996738$ satisfies the constraints of problem (4.1). Now, the fuzzy version of the problem is

$\mathbf{Max} x_1 x_2 x_3 = z$
Subject to

$$g_1 : x_2 + x_3 \widetilde{\leq} 3$$
$$g_2 : 17x_1x_2 + 3(x_1x_2 + 2x_1x_3 + 2x_2x_3) \widetilde{\leq} 108$$
$$x_1, x_2, x_3 \geq 0 \tag{4.2}$$

Therefore, $b_1 = 3$ and $b_2 = 108$. In order to obtain p_1 and p_2, we have

$$\widetilde{3} = \{(x, \mu_{\widetilde{3}}(x)) : x \in \mathbb{R}^n\}$$

where

$$\mu_{\widetilde{3}}(x) = \begin{cases} 1, & \text{if} \quad x < 3 \\ \frac{5-x}{2} & \text{if} \quad 3 \leq x \leq 5 \\ 0, & \text{if} \quad x \geq 5 \end{cases}$$

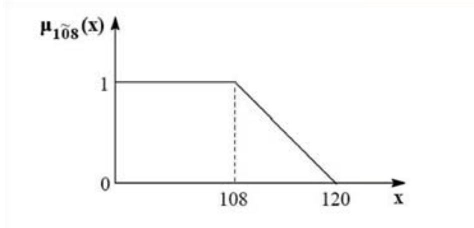

Figure 5: membership function of $\mu_{\widetilde{108}}(x)$

It is shown in the Figure (4). Hence $p_1 = 2$. Similarly p_2 can be obtained by

$$\widetilde{108} = \{(x, \mu_{\widetilde{108}}(x)) : x \in \mathbb{R}^n\}$$

where

$$\mu_{\widetilde{108}}(x) = \begin{cases} 1, & \text{if} \quad x < 108 \\ \frac{120-x}{12} & \text{if} \quad 108 \leq x \leq 120 \\ 0, & \text{if} \quad x \geq 108 \end{cases}$$

We get $p_2 = 12$. It is shown in the Figure (5). Now, we can find z_l and z_u by solving the two crisp NLPP as follows:

1. $z_1 = z$. Since the problem is the same first problem, and they have the same solution, therfore $z_1 = 5.99996738\$$.

2. $z_2 = \mathbf{Max} x_1 x_2 x_3$
Subject to

$$x_1^* = 1.77424067, \quad x_2^* = 1.44655028, \quad x_3^* = 3.55344549.$$

which is satisfy the constraints. Finally, $z_2 = 9.12001855$. Let $\widetilde{M}$ be the fuzzy set of all objective function such that

$$\widetilde{M} = \{(x, \mu_{\widetilde{M}}(x)) : x \in \mathbb{R}\}$$

and

$$\mu_{\widetilde{M}}(x) = \begin{cases} 1, & \text{if} \quad 5.999987738 < x_1x_2x_3 \\ \frac{x_1x_2x_3 - 5.999987738}{9.12001855 - 5.999987738}, & \text{if} \quad 5.9999 < x_1x_2x_3 < 9.1200 \\ 0, & \text{if} \quad x_1x_2x_3 < 9.12001855 \end{cases}$$

In addition, let $\widetilde{C}_1$ be the fuzzy set for g_1 such that

$$\widetilde{C}_1 = \{(x, \mu_{\widetilde{C}_1}(x)) : x \in \mathbb{R}\}$$

where

$$\mu_{\widetilde{C}_1}(x) = \begin{cases} 1, & \text{if} \quad x_2 + x_3 < 3 \\ \frac{5-(x_1+x_3)}{2} & \text{if} \quad 3 < x_2 + x_3 < 5 \\ 0, & \text{if} \quad 5 < x_2 + x_3 \end{cases}$$

and $\widetilde{C}_2$ be the fuzzy set for g_2 such that

$$\widetilde{C}_2 = \{(x, \mu_{\widetilde{C}_2}(x)) : x \in \mathbb{R}\}$$

where

$$\mu_{\widetilde{C}_1}(x) = \begin{cases} 1, & \text{if} \quad 17x_1x_2 + 3(x_1x_2 + 2x_1x_3 + 2x_2x_3) < 108 \\ \frac{120-(17x_1x_2+3(x_1x_2+2x_1x_3+2x_2x_3))}{12} & \text{if} \quad 108 < 17x_1x_2 + 3(x_1x_2 + 2x_1x_3 + 2x_2x_3) < 120 \\ 0, & \text{if} \quad 120 < 17x_1x_2 + 3(x_1x_2 + 2x_1x_3 + 2x_2x_3) \end{cases}$$

The fuzzy decision making for this problem is

$$\mu_{\widetilde{D}}(x) = \min\{\mu_{\widetilde{M}}(x), \min\{\mu_{\widetilde{C}_1}(x), \mu_{\widetilde{C}_2}(x)\}\}.$$

For $\lambda = \min\{\mu_{\widetilde{M}}(x), \min\{\mu_{\widetilde{C}_1}(x), \mu_{\widetilde{C}_2}(x)\}\}$, with optimal decision $x^* = \mathbf{Max}\lambda$.
Finally, the crisp NLP corresponding with the fuzzy NLP is given by

$\mathbf{Max}\lambda$

Subject to

$$\overline{g}_1 : \lambda - \mu_{\widetilde{M}}(x) \leq 0$$

$$\overline{g}_2 : \lambda - \mu_{\widetilde{C}_1}(x) \leq 0$$

$$\overline{g}_3 : \lambda - \mu_{\widetilde{C}_2}(x) \leq 0$$

where $x \geq 0$ and $0 \leq \lambda \leq 1$ which is equivalent to the following problem
$\textbf{Max}X^* = \lambda$
Subject to

$$\overline{g}_1 : \lambda - \left(\frac{x_1x_2x_3 - 5.999987738}{9.12001855 - 5.999987738}\right) \leq 0$$

$$\overline{g}_2 : \lambda - \left(\frac{5 - (x_1x_2)}{2}\right) \leq 0$$

$$\overline{g}_3 : \lambda - \left(\frac{120 - (17x_1x_2 + 3(x_1x_2 + 2x_1x_3 + 2x_2x_3))}{12}\right) \leq 0 \qquad (4.3)$$

Where $x_1, x_2, x_3 \geq 0$ and $0 \leq \lambda \leq 1$. Therefore, the solution of problem (4.3) is

$$x_1^* = 2.3047, \quad x_2^* = 1.2152, \quad x_3^* = 2.7300,$$

and $\lambda^* = 0.5274$ which satisfy in the constraints while the result of crisp problem before fuzziness is

$$x_1^* = 2.99526551, \quad x_2^* = 1.00316204, \quad x_3^* = 1.99687905.$$

Now, we can submit , x_1^*, x_2^* and x_3^* in the objective function of the crisp NLP. It can be obtained

$$z_{AF} = x_1^*x_2^*x_3^* = 7.64583302.$$

where $z_l < z_{AF} < z_u$. Clearly, in comparison the crisp problem we have more accurate solution.

5 Conclusion

In this work, the fuzzy solution of optimization problems and insensitive solution to the optimization problems are presented. Furthermore, it is proposed that the results solution of fuzzy optimization is a generalization of the solution of the crisp optimization problem. In addition, the numerical experiment show us the solution in fuzzified problems are more accurate than results in crisp problems.

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