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Micropolar elasticity theory: a survey of linear isotropic equations, representative notations, and experimental investigations

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Abstract
This paper is devoted to a review of the linear isotropic theory of micropolar elasticity and its development with a focus on the notation used to represent the micropolar elastic moduli and the experimental efforts taken to measure them. Notation, not only the selected symbols but also the approaches used for denoting the material elastic constants involved in the model, can play an important role in the micropolar elasticity theory especially in the context of investigating its relationship with the couple-stress and classical elasticity theories. Two categories of notation, one with coupled classical and micropolar elastic moduli and one with decoupled classical and micropolar elastic moduli, are examined and the consequences of each are addressed. The misleading nature of the former category is also discussed. Experimental investigations on the micropolar elasticity and material constants are also reviewed where one can note the questionable nature and limitations of the experimental results reported on the micropolar elasticity theory.

Keywords
Micropolar elasticity, Cosserat continuum, notations, micropolar elastic moduli, simplification, apparent inconsistencies

1. Introduction

The most popular elasticity theory is the classical theory of linear elasticity [1] based on which the internal interactions between neighbouring elements of an elastic continuum occur only by means of the force-based stress vectors (elements of the force stress tensor related to an interface form the stress vector at any point) and the macrorotation field vector (i.e. the local rotations) inside the continuum is dependent on the displacement field vector. The classical theory of linear elasticity has a long history of development and verification and produces acceptable results in numerous engineering problems with various structural materials [2]. However, for the cases with large stress gradients (e.g. in the vicinity of holes and cracks) or materials with significant microstructure contribution (e.g. composites, polymers, soil, and bone) the classical theory of elasticity fails to produce acceptable results [3]. In addition, it is not an appropriate theory for asymmetric stress-strain analysis (which is the case when dealing with an elastic continuum under the action of a volume moment distribution).

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To improve the results of the classical theory of elasticity Voigt [4] incorporated the effects of moment-based stresses known as couple stresses (in addition to the classical force stresses) and generalized the classical theory of (symmetric) elasticity to a non-symmetric elasticity theory, later named as the (indeterminate) asymmetric couple-stress theory (of elasticity) [5–11].

Voigt's theory was then extended by the Cosserat brothers [12] who considered independent (from the displacement field) local rotations inside an elastic body, known as microrotations, and suggested six degrees of freedom (DOFs) for every element or point of an elastic body: three displacements plus three microrotations. The (very general and geometrically nonlinear) Cosserat theory, formulated in an unclear manner, was further developed in restricted settings by other scholars [13–18]. Most notably, in a linear isotropic setting, the Cosserat theory of elasticity can be found in [19–23]. Those readers interested in anisotropic cases containing additional micropolar parameters (usually studied by using homogenization techniques) may refer to [24–27] and their references for a discussion on the structure of stiffness matrices and related constitutive equations, and also a review of various symmetry classes.

Eringen [23] is known as the one who extended the linear Cosserat theory to include body microinertia effects and the one who renamed it as the micropolar theory of elasticity. A micropolar elastic material can be considered as a continuum with rigid particles of infinitesimal size uniformly distributed in an elastic matrix. Nowadays, the linear theory is known as the linear theory of micropolar, Cosserat, or asymmetric elasticity. It is worth noting that the couple-stress theory is also known as the Cosserat theory with constrained rotations or briefly constrained Cosserat theory. An extensive bibliography and description of the couple-stress and micropolar elasticity theories can be found in [3, 28, 29]. Also a historical overview of the development of the micropolar continuum mechanics throughout the twentieth century can be found in [30]. During the last 40 years the micropolar elasticity model has found many applications in construction of various generalized models for beams, plates, and shells (e.g. see [31–35]).

Broadly speaking these newer, more elaborate, material models are struggling with experimental verification and the corresponding conceptualization of their meaning. They could be useful when dealing with materials that have a defined internal structure; e.g. fibrous materials such as bone, coarse granular materials, and large molecule polymers such as foams. It is noteworthy that the experimental verification of these newer models for such materials is not fully accomplished yet and one is faced with a situation where theory precedes experiment.

It is worth noting that reviewing the mathematical background behind constructing Cosserat or micropolar elasticity theory (or other generalized elasticity theories) is very deep and is far beyond the scope of the current manuscript. Those readers interested in an alternative and interesting variational approach to the mathematical background may refer to [12, 36–41].

This paper is devoted to a review of the linear isotropic micropolar theory of elasticity and its comparison against the linear isotropic classical elasticity theory with a focus on the notation used to represent the micropolar elastic moduli and the experimental efforts taken to measure them. In the following sections the linear isotropic theory of micropolar elasticity is briefly reviewed first; the central relations of the micropolar elasticity will be presented in a new notation which allows for the description of the extra micropolar features and concepts in a form parallel to the well-known classical features and concepts. Next the notation used by different authors to represent the linear theory of micropolar elasticity are surveyed and the importance of notation is discussed. Finally, a literature survey on the experimental exploration of the micropolar elasticity theory and determination of the micropolar elastic constants is presented.

Before ending this introductory section, it is worthwhile to mention the following relations for a general second-order tensor (dyadic) $\overset{d}{\leftrightarrow}$:

$$
\begin{aligned}
d_{ij} &= \frac{1}{2}\left(d_{ij} + d_{ji}\right) + \frac{1}{2}\left(d_{ij} - d_{ji}\right) = d^{\mathrm{s}}_{ij} + d^{\mathrm{a}}_{ij} \\
\epsilon_{ijk}\, d_{jk} &= \epsilon_{ijk}\left(d^{\mathrm{s}}_{jk} + d^{\mathrm{a}}_{jk}\right) = \epsilon_{ijk}\, d^{\mathrm{a}}_{jk}
\end{aligned}
\tag{1}
$$

which will be used frequently in this paper. The first relation of equation (1) points to the decomposition of a tensor $\overset{d}{\leftrightarrow}$ into its symmetric and antisymmetric (skew-symmetric) parts $\overset{d}{\leftrightarrow}^{\mathrm{s}}$ and $\overset{d}{\leftrightarrow}^{\mathrm{a}}$. Symbol ϵ_{ijk} in the second relation of equation (1) is the antisymmetric Levi-Civita (alternating or permutation) tensor.

2. Linear isotropic micropolar elasticity theory

To begin we will lay out the fundamental relations of the general linear micropolar elasticity (applied to a homogeneous, isotropic, and centrally symmetric material) in a notation inspired by, but different from, Nowacki [3]. Both Nowacki's notation and that used by the current authors allow for the description of the extra features and concepts of the micropolar elasticity model in a form parallel to the well-known classical features and concepts (which is why they are advantageous). The difference between the two notations is that in our notation we additionally made every effort to use the most frequently used symbols of the linear classical elasticity to denote or represent the concepts which are shared between the classical and micropolar elasticity theories (and simultaneously not to use the same symbol to denote more than one thing). These aspects of our notation would underline the novelties of the linear micropolar elasticity theory more effectively and would make the comparison with the linear classical elasticity theory more straightforward. Readers are referred to Table 3 for a comparison of different scholars' notation.

In a linear micropolar continuum, the displacement field vector $\underset{\rightarrow}{u}$ is completed by a microrotation field vector $\underset{\rightarrow}{\vartheta}$ independent of the displacement field and a micropolar deformation is fully described by asymmetric strain and twist tensors, $\underset{\leftrightarrow}{\varepsilon}$ and $\underset{\leftrightarrow}{\tau}$, which are defined as

$$\begin{aligned} \epsilon_{ij} &= u_{j,i} - \epsilon_{ijk}\vartheta_k \\ \tau_{ij} &= \vartheta_{j,i} \end{aligned} \tag{2}$$

Based on this definition of strain the following relation for the (micropolar) microrotation vector $\underset{\rightarrow}{\vartheta}$ can be derived:

$$\vartheta_i = \frac{1}{2}\epsilon_{ijk}\left(u_{k,j} - \epsilon_{jk}\right) \tag{3}$$

It is also useful to define the (classical) macrorotation vector $\underset{\rightarrow}{\theta}$ such that

$$\theta_i = \frac{1}{2}\epsilon_{ijk}u_{k,j} \tag{4}$$

Now the strain tensor can be decomposed as

$$\begin{aligned} \varepsilon_{ij} &= \varepsilon^{\mathrm{s}}_{ij} + \varepsilon^{\mathrm{a}}_{ij} \\ \varepsilon^{\mathrm{s}}_{ij} &= \frac{1}{2}\left(u_{j,i} + u_{i,j}\right) \\ \varepsilon^{\mathrm{a}}_{ij} &= \frac{1}{2}\left(u_{j,i} - u_{i,j}\right) - \epsilon_{ijk}\vartheta_k = \epsilon_{ijk}(\theta_k - \vartheta_k) \end{aligned} \tag{5}$$

The last relation in equation (5) implies that $\underset{\leftrightarrow}{\varepsilon}^{\mathrm{a}}$ is a representation of the difference between the classical macrorotation and the micropolar microrotation.

The internal loads (between adjacent elements) in a micropolar continuum, under the action of a finite body force $\underset{\rightarrow}{f}^V$ and a finite body moment $\underset{\rightarrow}{m}^V$, are definable in terms of a classical force stress tensor $\underset{\leftrightarrow}{\sigma}$ and a micropolar couple stress tensor $\underset{\leftrightarrow}{\chi}$ which should satisfy the balance of linear and angular momenta as

$$\begin{aligned} \sigma_{ji,j} + f^V_i &= \rho^V \ddot{u}_i, \\ \chi_{ji,j} + \epsilon_{ijk}\sigma_{jk} + m^V_i &= \iota^V \ddot{\vartheta}_i, \end{aligned} \tag{6}$$

where ρ^V is the material mass density and ι^V is the material microinertia density. By decomposing the force stress tensor in equation (6) the balance of momenta relations can be rewritten as

$$\begin{aligned} \sigma^{\mathrm{s}}_{ji,j} + \sigma^{\mathrm{a}}_{ji,j} + f^V_i &= \rho^V \ddot{u}_i \\ \chi_{ji,j} + \epsilon_{ijk}\sigma^{\mathrm{a}}_{jk} + m^V_i &= \iota^V \ddot{\vartheta}_i \end{aligned} \tag{7}$$

where equation (1) is recalled. As can be concluded from equation (7), the antisymmetric part of the force stress tensor $\overset{\sigma^{\mathrm{a}}}{\leftrightarrow}$ couples the linear and angular momenta balance relations in a micropolar continuum.

The linear theory of micropolar elasticity proposes a set of two constitutive relations with six elastic constants for a homogeneous, isotropic, and centrally symmetric elastic body as

$$\begin{aligned}\sigma_{ij} &= (\mu+\kappa)\varepsilon_{ij} + (\mu-\kappa)\varepsilon_{ji} + \lambda\varepsilon_{kk}\,\delta_{ij} \\ \chi_{ij} &= (\gamma+\beta)\tau_{ij} + (\gamma-\beta)\tau_{ji} + \alpha\tau_{kk}\,\delta_{ij}\end{aligned} \tag{8}$$

where δ_{ij} is the Kronecker delta tensor (dyadic). Among the six elastic constants denoted in equation (8), μ and λ are the well-known classical Lamé coefficients (μ is the shear modulus). The other four constants, i.e. κ, γ, β, and α, are the new elastic constants usually referred to as the micropolar or Cosserat elastic constants (κ is also known as the micropolar couple modulus). By decomposing the strain and twist tensors as well as the force and couple stress tensors, the constitutive relations in equation (8) can be rewritten as

$$\begin{aligned}\sigma^{\mathrm{s}}_{ij} &= 2\mu\varepsilon^{\mathrm{s}}_{ij} + \lambda\varepsilon_{kk}\delta_{ij},\ \ \sigma^{\mathrm{a}}_{ij} = 2\,\kappa\varepsilon^{\mathrm{a}}_{ij} \\ \chi^{\mathrm{s}}_{ij} &= 2\gamma\tau^{\mathrm{s}}_{ij} + \alpha\,\tau_{kk}\delta_{ij},\ \ \chi^{\mathrm{a}}_{ij} = 2\,\beta\,\tau^{\mathrm{a}}_{ij}\end{aligned} \tag{9}$$

Simplifying the first constitutive relation given by equation (8) for the simple force stress state of uniform tension along axis ${}^{o}x_1$, where the only non-zero element of the force stress tensor is σ_{11}, results in definitions of the (classical) strain Poisson's ratio ν and the (classical) tensile or Young's modulus E as

$$\begin{aligned}\nu &= -\frac{\varepsilon_{22}}{\varepsilon_{11}} = -\frac{\varepsilon_{33}}{\varepsilon_{11}} = \frac{\lambda}{2(\mu+\lambda)} \\ E &= \frac{\sigma_{11}}{\varepsilon_{11}} = \frac{\mu(2\mu+3\lambda)}{\mu+\lambda} = 2\mu(1+\nu)\end{aligned} \tag{10}$$

Analogously, the second constitutive relation in equation (8) can be simplified to account for the simple couple stress state of uniform torsion along axis ${}^{o}x_1$, where the only non-zero element of the couple stress tensor is χ_{11}. Then the (micropolar) twist Poisson's ratio ξ and the (micropolar) tortile or torsional modulus $\mathcal{E}$ can be defined as

$$\begin{aligned}\xi &= -\frac{\tau_{22}}{\tau_{11}} = -\frac{\tau_{33}}{\tau_{11}} = \frac{\alpha}{2(\gamma+\alpha)} \\ \mathcal{E} &= \frac{\chi_{11}}{\tau_{11}} = \frac{\gamma(2\gamma+3\alpha)}{\gamma+\alpha} = 2\gamma(1+\xi)\end{aligned} \tag{11}$$

Note the parallel structure of the relations in equations (10) and (11).

Also by considering the original constitutive relations in equation (8) and applying the Einstein summation convention on the force and couple stress tensors, $\overset{\sigma}{\leftrightarrow}$ and $\overset{\chi}{\leftrightarrow}$, it can be shown that

$$\begin{aligned}\sigma_{kk} &= 3\,B\,\varepsilon_{kk},\, B = \lambda + \frac{2}{3}\mu \\ \chi_{kk} &= 3\,\mathcal{B}\,\tau_{kk},\, \mathcal{B} = \alpha + \frac{2}{3}\gamma\end{aligned} \tag{12}$$

where B is known as the (classical) tensile bulk modulus, and $\mathcal{B}$ as dual of the tensile bulk modulus can be called the (micropolar) tortile or torsional bulk modulus.

By substituting from equations (2) and (8) into equation (6) the equations of motion for a micropolar continuum will be obtained as

$$\begin{aligned}&(\mu+\kappa)u_{i,jj} + (\mu-\kappa+\lambda)u_{j,ji} + 2\,\kappa\,\epsilon_{ijk}\,\vartheta_{k,j} + f^V_i = \rho^V\ddot{u}_i \\ &(\gamma+\beta)\vartheta_{i,jj} + (\gamma-\beta+\alpha)\vartheta_{j,ji} + 2\,\kappa\left(\epsilon_{ijk}\,u_{k,j} - 2\vartheta_i\right) + m^V_i = \iota^V\ddot{\vartheta}_i\end{aligned} \tag{13}$$

Table 1. A summary comparison of the classical and micropolar theories of elasticity.

Subject	Classical theory of elasticity	Micropolar theory of elasticity
Assumptions	$\vartheta_i = \theta_i = -\frac{1}{2}\epsilon_{ijk}u_{j,k}$	independent u_i and ϑ_i
	existence of just σ_{ij}, and $\chi_{ij} = 0$	existence of both σ_{ij} and χ_{ij}
Deformation Tensor(s)	$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$	$\varepsilon_{ij} = u_{j,i} - \epsilon_{ijk}\vartheta_k$ $\tau_{ij} = \vartheta_{j,i}$
Constitutive Relation(s)	$\frac{1}{2}\left(\sigma_{ij} + \sigma_{ji}\right) = 2\mu\varepsilon_{ij} + \lambda\varepsilon_{kk}\delta_{ij}$	$\sigma_{ij} = (\mu+\kappa)\varepsilon_{ij} + (\mu-\kappa)\varepsilon_{ji} + \lambda\varepsilon_{kk}\delta_{ij}$ $\chi_{ij} = (\gamma+\beta)\tau_{ij} + (\gamma-\beta)\tau_{ji} + \alpha\tau_{kk}\delta_{ij}$
Equations of Motion	$\sigma_{ji,j} + f_i^V = \rho^V \ddot{u}_i$ $\epsilon_{ijk}\sigma_{jk} + m_i^V = 0$	$\sigma_{ji,j} + f_i^V = \rho^V \ddot{u}_i$ $\epsilon_{ijk}\sigma_{jk} + \chi_{ji,j} + m_i^V = \iota^V \ddot{\vartheta}_i$
	$\mu\, u_{i,jj} + (\mu+\lambda)u_{j,ji} + f_i^V - \frac{1}{2}\epsilon_{ijk}m_{j,k}^V =$ $\rho^V \ddot{u}_i$	$(\mu+\kappa)u_{i,jj} + (\mu-\kappa+\lambda)u_{j,ji} -$ $2\,\kappa\,\epsilon_{ijk}\vartheta_{j,k} + f_i^V = \rho^V\ddot{u}_i$ $(\gamma+\beta)\vartheta_{i,jj} - 4\,\kappa\,\vartheta_i + (\gamma-\beta+\alpha)\vartheta_{j,ji} -$ $2\,\kappa\,\epsilon_{ijk}\,u_{j,k} + m_i^V = \iota^V\ddot{\vartheta}_i$
Force System	combined (or effective) $f_i^V - \frac{1}{2}\epsilon_{ijk}\, m_{j,k}^V$	separate f_i^V and m_i^V
Elastic Energy	$\mathcal{U}_e^V = \frac{1}{2}\sigma_{ij}\varepsilon_{ij}$	$\mathcal{U}_e^V = \frac{1}{2}\sigma_{ij}\varepsilon_{ij} + \frac{1}{2}\chi_{ij}\tau_{ij}$

or corresponding to equation (7) one can derive

$$\begin{aligned} &\mu\, u_{i,jj} + (\mu+\lambda)u_{j,ji} - 2\,\kappa\,\epsilon_{ijk}\left(\theta_{k,j} - \vartheta_{k,j}\right) + f_i^V = \rho^V \ddot{u}_i \\ &(\gamma+\beta)\vartheta_{i,jj} + (\gamma-\beta+\alpha)\vartheta_{j,ji} + 4\,\kappa(\theta_i - \vartheta_i) + m_i^V = \iota^V\ddot{\vartheta}_i \end{aligned} \tag{14}$$

In the linear theory of micropolar elasticity the strain energy density $\mathcal{U}_e^V$ is

$$2\mathcal{U}_e^V = \sigma_{ij}\varepsilon_{ij} + \chi_{ij}\tau_{ij} = \sigma_{ij}^s\varepsilon_{ij}^s + \sigma_{ij}^a\varepsilon_{ij}^a + \chi_{ij}^s\tau_{ij}^s + \chi_{ij}^a\tau_{ij}^a \tag{15}$$

which by substitution from equation (9) can be rewritten as

$$\begin{aligned} 2\mathcal{U}_e^V = 2\,\mu\,\varepsilon_{ij}^s\,\varepsilon_{ij}^s + \lambda\,\varepsilon_{ii}^s\,\varepsilon_{jj}^s + 2\,\kappa\varepsilon_{ij}^a\,\varepsilon_{ij}^a \\ 2\,\gamma\,\tau_{ij}^s\tau_{ij}^s + \alpha\,\tau_{ii}^s\tau_{jj}^s + 2\,\beta\tau_{ij}^a\,\tau_{ij}^a \end{aligned} \tag{16}$$

Now a positive-definite quadratic form for the strain energy density $\mathcal{U}_e^V$ implies the following restrictions on the material elastic constants:

$$\begin{aligned} \mu>0,\ \kappa>0,\ 2\,\mu + 3\,\lambda>0 \\ \gamma>0,\ \beta>0,\ 2\gamma + 3\,\alpha>0 \end{aligned} \tag{17}$$

A summary of the equations of the micropolar elasticity model, mentioned in this section, along with their comparison with the equations of the classical elasticity model can be found in Table 1. Also the

Table 2. Duality between the terminologies in the classical and micropolar theories of elasticity.

Term in classical elasticity	Dual term in micropolar elasticity
Macrorotation θ	Microrotation ϑ
(Mass) density ρ^V	Microinertia density ι^V
(Force) stress σ	Couple stress χ
Classical or Lamé constants μ and λ	Micropolar or Cosserat constants κ, γ, β, and α
Strain ε	Twist or wryness τ
Normal strain	Normal twist or torsion
Tension	Torsion
Tensile	Tortile or torsional
Tensile or Young's modulus E	Tortile or torsional modulus $\mathcal{E}$
(Tensile) bulk modulus B	Tortile or torsional bulk modulus $\mathcal{B}$
(Strain) Poisson's ratio v	Twist Poisson's ratio ξ

parallel terminologies used for denotation of analogous subjects in the classical and micropolar theories of elasticity are listed in Table 2. Note that the similarities between the subjects related to the force stresses and displacements (in the classical or micropolar theory of elasticity) and those related to the couple stresses and microrotations (in the micropolar theory of elasticity), which have appeared in this section as parallel or analogous relations and definitions, appeal for a parallel terminology to be used for denotation of these subjects.

3. Comparison of notation and notation-induced discrepancies

Despite its simple nature and common usage, notation can play an important role in developing physical theories; there is notion in the notation. Through developing the elasticity models, notation and especially the way in which the material elastic constants are represented, can be very important to the understanding of the concepts. In the classical theory of elasticity, for example, the two material constants can be symbolized as the pair of Lamé coefficients μ and λ, or the pair of Young's modulus and Poisson's ratio E and v, or the pair of bulk modulus and Poisson's ratio B and v. Whereas these notations are related and describe the same physical model, i.e. the classical elasticity model, they result in different interpretations about the physics of the model.

The notation is even more important in the context of studying micropolar elasticity because throughout its development micropolar elasticity theory has been presented using several different notations and one has to be very careful about the notation used by different authors. Indeed, in this regard, it has been noted by Neff [42] that "as often the case, notation is a nightmare". Cowin [43], for example, compared the notations used by some authors. A similar comparison containing the notation used herein can be found in Table 3 where the reader's attention is directed to the different representations of the material elastic constants. It is worthwhile to note that, whereas in Eringen's notation the classical and micropolar elastic constants are coupled together (i.e.the Lame shear modulus is connected to the Cosserat couple modulus), the rest of the notations, including the notation of the present text, completely decouple the two classical Lame parameters from the four micropolar elastic constants and utilize single separate (or uncoupled) symbols to denote them.

Eringen is known as one of the main developers of the micropolar (Cosserat) theory of elasticity and the person from whom the name "micropolar elasticity" has originated. His derivation of the linear theory of micropolar elasticity [23] is probably one of the most cited works in the field of micropolar elasticity. Therefore, it is not a surprise that Eringen's notation, summarized in Table 3, is also the most popular one used by other authors on micropolar elasticity.

Despite its popularity, Eringen's notation, especially his representation of the micropolar elastic constants, is not the best. In effect, it is a misleading notation as it may result in mistaking Eringen's symbol μ for the classical Lamé shear modulus often denoted as μ in the classical theory of elasticity. Actually, this has been a frequent mistake appearing in the literature where Eringen's notation was employed [31, 44–49]. The difference between Eringen's symbol μ and the Lamé coefficient μ, which can be understood from a study of Eringen's preliminary work on the linear micropolar elasticity [23], is clarified in Table 3.

Table 3. Comparison of notations used for representation of micropolar elasticity.

Notation group/description		Representative symbol used by				
		Present authors	Nowacki [3]	Eringen [23]	Cowin [43]	Neff [56]
Kinematics/kinetics	Displacement vector	u_i	u_i	u_i	u_i	u
	Microrotation vector	ϑ_i	φ_i	φ_i	$\hat{\psi}_i$	ϕ
	Macrorotation vector	θ_i	$\frac{1}{2}\in_{ijk}u_{k,j}$ †	r_i	$-\hat{\omega}_i$	$\frac{1}{2}\mathrm{curl}\,u$ †
	Strain tensor	ε_{ij}	γ_{ij}	ε_{ji} ‡	$\varepsilon_{ij}+\gamma_{ij}$ †	$\bar{\varepsilon}$
	Twist tensor	τ_{ij}	$\varkappa_{ij}$	$\varphi_{j,i}$	$\hat{\kappa}_{ij}$	$\nabla\phi$
	Force stress tensor	σ_{ij}	σ_{ij}	t_{ij}	$\tau_{ij}+\sigma_{ij}$ †	σ
	Couple stress tensor	χ_{ij}	μ_{ij}	m_{ij}	$\hat{\mu}_{ij}$	m
Material constants	Lamé shear modulus	μ	μ	$\mu+\frac{1}{2}\kappa$	μ	μ
	Lamé coefficient	λ	λ	λ	λ	λ
	Cosserat couple modulus	κ	α	$\frac{1}{2}\kappa$	τ	μ_c
	Cosserat twist coefficient	γ	γ	$\frac{1}{2}(\gamma+\beta)$	$\eta+\eta'$	$\frac{1}{2}(\gamma+\beta)$
	Cosserat twist coefficient	β	ε	$\frac{1}{2}(\gamma-\beta)$	$\eta-\eta'$	$\frac{1}{2}(\gamma-\beta)$
	Cosserat twist coefficient	α	β	α	$\frac{1}{2}\alpha$	α
	Mass density	ρ^v	ϱ	ρ	†	†
	Microinertia density	ι^v	J	ρj	†	†

†: No (explicit) symbol assigned.
‡: Note that the subscript is *ji* and not *ij*.

The first erroneous conclusion following Eringen's notation appeared in his own preliminary work on the derivation of the linear micropolar elasticity [23]. As mentioned by Cowin [50], mistaking his symbol μ for the Lamé shear modulus, Eringen has deduced an incorrect thermodynamic inequality in [23]. This incorrect inequality further misled some other authors to incorrectly compare the couple-stress theory and the micropolar elasticity theory as two independent theories (without realizing that the couple-stress theory is a special case of the micropolar elasticity theory) and also to wrongly remark that the couple-stress theory contradicts thermodynamic restrictions [50].

To facilitate addressing the problems related to Eringen's notation, from now on, Eringen's symbols representing the elastic constants will be illustrated with a subscript E. Cowin [50] has shown that the Lamé shear modulus μ and the micropolar couple modulus κ are related to Eringen's symbols μ_{E} and κ_{E} according to

$$\mu = \mu_{\mathrm{E}} + \frac{1}{2}\kappa_{\mathrm{E}}, \qquad \kappa = \frac{1}{2}\kappa_{\mathrm{E}} \tag{18}$$

He has also shown that the incorrect inequalities presented by Eringen as [23]

$$\mu_{\mathrm{E}} \geq 0, \qquad \kappa_{\mathrm{E}} \geq 0$$

should be replaced with the following correct inequalities [50]:

$$\mu = \mu_{\mathrm{E}} + \frac{1}{2}\kappa_{\mathrm{E}} \geq 0, \qquad \kappa = \frac{1}{2}\kappa_{\mathrm{E}} \geq 0 \tag{19}$$

Whereas Eringen's incorrect inequality has been replaced by the correct form as early as its discovery by Cowin, the use of Eringen's notation continued in other works on micropolar elasticity. In fact, to the best of the current authors' knowledge, no one has paid attention to the origin of the incorrect inequality, and the misleading nature of Eringen's notation has not been discussed.

Probably the most important consequence of using Eringen's notation has been the loose conclusion (without a thorough examination of it or study of other options or possibilities) that the classical theory of elasticity is a special case of the micropolar theory of elasticity when Eringen's elastic constant κ_{E} tends to zero. In other words, the relationship between the Lamé shear modulus μ and Eringen's symbols μ_E and κ_E as given by equation (18) has deceived authors using Eringen's notation to simply think of the classical theory of elasticity as a special case of the micropolar theory elasticity when Eringen's elastic constant κ_E tends to zero. This conclusion has been accepted, without any further study, by authors using the notations with decoupled symbols for the classical and micropolar elastic coefficients [50–54].

There are however two controversies around this conclusion. First, in Eringen's notation the classical Lamé shear modulus μ is dependent on both μ_E and κ_E as given by equation (18). As a consequence, by varying κ_E both classical and micropolar elastic properties are being changed simultaneously. Simplification of the micropolar elasticity model to the classical elasticity model by letting κ_E go to zero is not therefore appropriate because the classical properties of the model are being varied too. Second, although simplifying the micropolar elasticity for the case of a zero κ_E (and therefore a zero couple modulus κ) is more straightforward, there are authors who observed some inconsistencies in a micropolar elasticity model with a zero κ_E (or κ) [55, 56] and there are scholars who believe this is an oversimplified approach and the connection between the two theories is more involved [57].

It is worth noting that in a micropolar boundary-value problem with coupled displacements and microrotations inside the problem domain, the displacements and microrotations on the boundaries are most likely coupled. This coupling in the boundary conditions may not be eliminated even if the governing equilibrium equations are decoupled (by letting κ_E or κ go to zero). A more thorough examination of the boundary conditions would be necessary provided the reduction of the micropolar boundary value problem to the classical boundary value problem by letting κ_E or κ go to zero is considered.

The unfortunate coupling of the classical and micropolar elastic properties of a material, induced by Eringen's notation, has not been remarked upon by scholars on this field. There are works considering the cases where κ_E is varied to study the effects of this coefficient on the micropolar elasticity behaviour or where κ_E is respectively set to zero and infinity to recover the classical and couple-stress elasticity models (the couple-stress theory is known to be a special case of the micropolar elasticity when κ_E tends to infinity) [23, 31, 58–63]; in these works however it has not been noticed that the cases will also be different in terms of their classical properties and this has caused some difficulties or confusions [58–60]. In this context, notations utilizing single symbols for the classical Lamé parameters and representing the extra micropolar elastic constants with completely separate (uncoupled) symbols, such as the notation used by Nowacki [3] as summarized in Table 3, are to be preferred to Eringen's notation because they are less likely to result in any of these confusions.

Here we would like to mention that an alternative approach for simplifying the micropolar elasticity theory to the classical elasticity theory without necessitating a zero couple modulus has been suggested recently [64–66]. The suggested step-by-step approach simplifies the micropolar elasticity model to the classical elasticity model while the couple-stress elasticity model is derived as an intermediate model. This approach is illustrated by the flowchart shown in Figure 1 demonstrating that the micropolar elasticity model with the six elastic constants μ, κ, λ, γ, β, and α will be simplified to the couple-stress elasticity model with the four elastic constants μ, λ, γ, and β provided κ, $\alpha \to \infty$ (see [51]); then letting γ, $\beta \to$ 0, will further simplify the couple-stress elasticity model to the classical elasticity model with the two elastic constants μ and λ. Note that since this simplification approach does not eliminate the coupling between the displacements and microrotations (the coupling is in fact strengthened), it does not require any special treatment of the coupled boundary conditions when simplifying a micropolar boundary value problem to a classical one.

The flowchart shown in Figure 1 can be compared against the commonly accepted parallel structure shown in Figure 2 which implies that the couple-stress and classical elasticity theories are two different (separate and independent) special cases of the micropolar elasticity theory with the six elastic constants μ, κ, λ, γ, β, and γ, respectively obtained by letting κ, $\alpha \to \infty$ and κ, γ, β, $\alpha \to 0$; note that this parallel simplification covers the fact that the classical elasticity model is a special case of the couple-stress theory too.

4. Experimental investigations

Since the appearance of the extended continuum theories, such as the couple-stress and micropolar elasticity theories, a great deal of effort has been put into determining the material coefficients and the

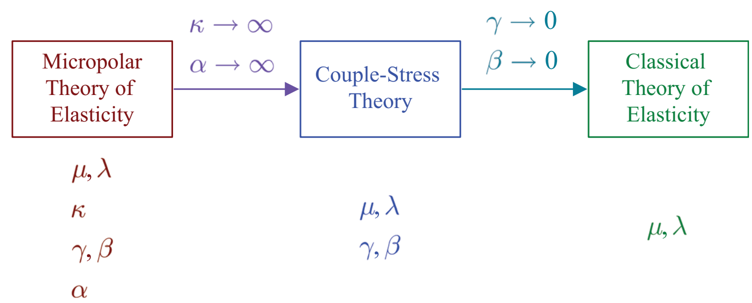

Figure 1. The recently suggested alternative sequential approach for simplification of the micropolar theory of elasticity to the couple-stress theory and the classical theory of elasticity [66].

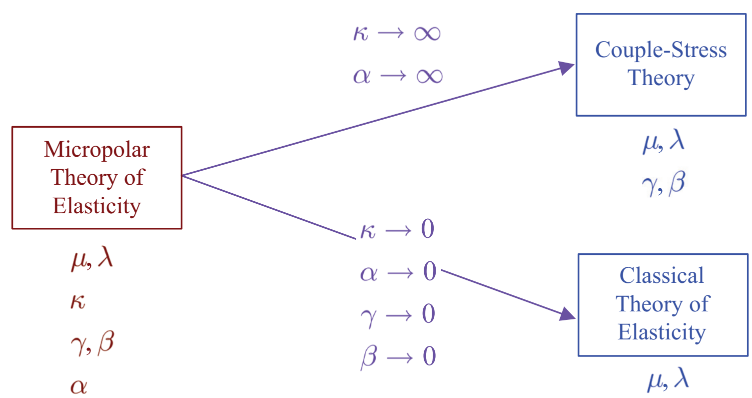

Figure 2. The commonly used parallel approach for simplification of the micropolar theory of elasticity to the couple-stress theory and the classical theory of elasticity.

constitutive equations in such theories (or suggesting a consistent approach for finding them). According to Truesdell [67] the constitutive equations are "the heart of continuum mechanics". In this context both theoretical (via homogenization or numerical techniques) and experimental investigations have been undertaken from the very beginning of the development of these continuum theories. Although this section is mainly devoted to the experimental investigations of the material elastic moduli in the linear micropolar theory of elasticity (and to some extent in the linear couple-stress theory of elasticity), it is worth mentioning a few of the early theoretical investigations.

In the context of theoretical (non-experimental) efforts for characterization of the constitutive equations corresponding to generalized continuum theories one can generally distinguish two types of approaches; homogenization methods used to replace composite materials, granular media, lattices, and cellular structures by an effective generalized continuum model and numerical methods in which the elastic moduli are determined by the need to regularize solutions of problems of elasticity or elastoplasticity or to accelerate their convergence [68, 69].

Adomeit [70] considered a three-dimensional honeycomb structure built of cubical cells and calculated the elastic coefficients of its equivalent couple-stress continuum from a structural view point. Employing the couple-stress theory to describe the dynamics of a composite with a laminated structure, Herrmann and Achenbach [71] obtained the non-classical material constants as functions of the geometry and the classical material constants of the composite's constituents. Banks and Sokolowski [72] showed an analogy between the equations governing the behaviour of a planar lattice structure built up from elastic beams and the equations of the two-dimensional couple-stress theory and related the elastic moduli of the couple-stress continuum to the elastic properties of the beam elements in the lattice structure. In a similar work, Askar and Cakmak [73] proposed an equivalent micropolar continuum model for a two-dimensional lattice structure composed of orientable mass points joined by massless beam elements and represented the micropolar elastic moduli of the equivalent continuum in terms of the properties of the connecting beams. In these works, the characteristic length (of the couple-stress or micropolar elasticity theory) was found to be related to the size of the structural elements.

There have been many other scholars trying to theoretically (usually based on an equivalent continuum approach) calculate the material coefficients from structural considerations [74–88]. Recently, Ganghoffer et al. [89–93] have put a lot of effort into using the homogenization techniques to establish non-classical constitutive equations and determine the elastic moduli corresponding to repetitive lattices and trabecular structures (e.g. bone).

In addition to these theoretical examinations there have been experimental investigations for determining the properties of materials which can be understood via couple-stress or micropolar elasticity models. The first experimental attempts were to determine (the order of magnitude of) the characteristic length, as found in the couple-stress theory or the micropolar elasticity theory, for common materials such as aluminum and steel. Schijve [94] discussed the physical significance of couple stresses and carried out some bending tests on sheet specimens of an aluminum alloy with varying thickness. However, he did not observe any dependencies between the flexural rigidity and the sheet thickness as is expected from the couple-stress theory. Ellis and Smith [95] tried to use the results of the cylindrical bending tests, carried out on steel and aluminum plates of different thicknesses, for determining the order of magnitude of the new material constants as found in the couple-stress theory. They discovered that the couple stress effects in steel and aluminum, should there be any at all, are not significant (or measurable in their experiments). It was concluded from these works that the characteristic length of common solid metals should be on the order of the atomic spacing.

The aforementioned unsuccessful investigations have lead researchers to look for the non-classical phenomena in the materials with more distinguishable internal microstructures rather than solid (alloy) metals. Askar [96] suggested a method to calculate the (micropolar) material coefficients of the molecular crystals from the optically measured rotational and ionic frequencies. In particular, numerical values were obtained for KNO_3 crystals. Most importantly, he reported a couple modulus to shear modulus ratio of 0.1 (not a very small or negligible ratio) which he noted “should bring rather important coupling of the displacement field with the rotational field”. Askar’s work [96] could be considered as the first partially successful attempt for determination of some of the micropolar material coefficients. Perkins and Thompson [97] used the couple-stress theory to interpret the thickness-dependency of the apparent shear stiffness of an elastic layer of material embedded between two rigid planes. Their dynamic test results, based on measuring the resonance frequency of a single-DOF rotary system which contained an elastic layer of closed-cell polyvinyl chloride foam plastic, proved the qualitative expediency of a couple-stress elasticity model with a characteristic length equal to the foam’s average cell size for description of the phenomenon.

The first direct attempt to determine all six elastic constants of the linear isotropic micropolar elasticity theory was carried out by Gauthier and Jahsman [44, 46, 98]. First, they showed theoretically that in the cases of torsion of a circular cylinder and cylindrical bending of a rectangular plate micropolar elastic effects raise the apparent stiffness over the classical values and that this was most in evidence when the cylinder diameter and plate thickness were small, on the order of the dimension of the microstructure. Then they prepared an artificial composite in which uniformly distributed rigid aluminum shot was cast in an elastic epoxy matrix. Making rectangular plate and circular cylinder specimens from this material, they performed both static and dynamic tests to determine all six elastic constants of the material. Although in the static tests the micropolar behaviour of the material was masked by the material inhomogeneity and no conclusion was drawn [44], the dynamic tests proved to be useful (with some assumptions and limitations) to calculate four (out of six) elastic constants [46].

The procedure suggested by Gauthier and Jahsman, i.e. using the test results of specimens with different sizes to calculate the micropolar elastic moduli, was further developed and extensively used by Lakes et al. [99] to determine the micropolar elastic constants of different materials with internal microstructures such as bone [47, 100–105], polymeric foams [48, 106–110], and metallic foams [111]. The approach, which they called “the method of size effects” [112–114], is based on the size effects (stiffening) predicted to occur in thin specimens. In this method the analytical solutions obtained for simple three-dimensional micropolar boundary value problems (predicting size effects in different geometries) are investigated with respect to the experimental results for specimens with different sizes to determine the micropolar elastic moduli. The method of size effects is a recurrent theme in the experimental determination of micropolar parameter values and is also attracting attention in conjunction with micro and nano devices [115].

4.1 Lakes' experiments on bone

By noting the effect of size on the apparent torsion and bending stiffness of a circular cylinder based on the couple-stress and micropolar elasticity theories, Yang and Lakes [47, 100] used the quasi-static torsion and bending tests performed on compact bone specimens with different sizes to determine the elastic coefficients of bone including its torsion and bending characteristic lengths. Torsional resonance experiments on bone were used by Lakes [101] to study the size effects on the effective shear modulus of circular cylinders of bone, from which the torsion characteristic length for the bone was determined. In these works it has been shown that the characteristic lengths are of the order of an osteon diameter, i.e. 0.15–0.25 (mm), as the dominant structural element of bone (osteons are the parallel fibres constructing the bone as a natural composite). To complement their results, Lakes and Yang [102] further studied the value of the couple modulus in bone and, while confirming their prior results, were able to measure that.

Park and Lakes [103] examined micromechanical effects in prismatic bars of wet and dry bones under torsion. They showed that, for specimens of the wet bone, shear strain distributions across lateral surfaces are in agreement with the prediction of the micropolar elasticity model built using the elastic coefficients measured previously for the bone [47, 100, 101]. Lakes et al. [104] used the micropolar elasticity theory to explain the tensile and torsional behaviour of notched specimens of the bone and studied the fracture mechanics of the bone. They were able to interpret the higher (than expected classically) toughness, the lower stress concentration, and the strain redistribution (from high-strain regions to low-strain regions) effects, which were observed experimentally in notched specimens of the bone, based on the micropolar theory of elasticity.

Lakes [105] also explained the higher shear moduli in tests upon single osteons (more than four times higher than moduli reported for macroscopic specimens of bone), as observed by Ascenzi et al. [116], based on a micropolar elasticity model with the elastic constants picked from his prior studies [47, 100, 102].

4.2 Lakes' experiments on foams

Lakes [106] experimentally studied the size effects on torsion and bending stiffness of rods of a low-density polymeric foam (i.e. polystyrene foam). Using both quasi-static and dynamic tests, on circular cylinders and rectangular bars of the foam, he was able to, for the first time, measure all six micropolar material constants for a material, i.e. the polymeric foam. In an analogous study, Lakes [107] used the quasi-static torsion and bending test results obtained for circular cylinder specimens of two high-density polymeric foams (i.e. polyurethane and syntactic foams) to determine the six micropolar elastic moduli of each. Both foams exhibited size effects which is understandable based on a micropolar elasticity model. In these experiments the characteristic lengths were found to be comparable to the average cell sizes in the foams.

In another study Lakes et al. [108] introduced a new method, i.e. the holographic screening method, in which the double exposure holography of a single specimen (i.e. a prismatic bar with a crack in the corner under torsion) of the material is used to quickly (quick compared with the commonly used tedious method of size effects) judge whether a material behaves as a non-classical material or not. Chen and Lakes [109] investigated the dispersion of standing waves and cut-off frequency effects observed in polymer foam materials in torsional vibration. They attributed their observations to the micro-vibrations of the cell ribs in the foams. In another work, Chen and Lakes [111] suggested the use of the double-exposure holographic interferometry technique to determine the elastic moduli of metallic foams. In particular, they have used the method to determine the classical elastic properties, i.e. Young's moduli, Poisson's ratio, and yield strength, of conventional and re-entrant (negative Poisson's ratio) copper foams along with their micro-deformation characteristic lengths associated with the non-affine deformations (i.e. inhomogeneous deformations associated with the microstructure) observed in bending.

Anderson and Lakes [48] used the method of size effects to experimentally investigate the micropolar elasticity model and the surface damage effects in closed-cell polymethacrylimide foams of three different grades (i.e. WF51, WF110, WF300). By putting both cylindrical and square cross-section specimens of different foams under bending and torsion tests, they determined the micropolar elastic constants for polymethacrylimide foams. Anderson et al. [110] used the double exposure holographic method to study the micropolar effects on torsion of the square cross-section specimens of a closed-cell polymethacrylimide foam (grade WF300). The cross-section warp, evaluated from the observed fringe patterns, was shown to be in agreement with the corresponding theoretical warp, predicted by a micropolar elasticity model (with the elastic coefficients determined in a prior study [48]).

4.3 Other experiments

There were a few experimental works on the couple-stress and micropolar elasticity theories done by other scholars contemporaneously or after Lakes. Tang [117] performed uniaxial tensile and four-point bend tests on cylindrical and square (prismatic) specimens of graphite of three different grades to find their couple-stress constants. He found that for each graphite a characteristic length equal to the maximum grain size provides a consistent correlation between the couple-stress theory and experimental data.

Chiroiu and Munteanu [49] measured the natural frequencies of a rectangular plate of the Gauthier micropolar material [44, 46, 98] and by inverting the vibrational data through a two-stage minimization algorithm (minimizing the differences between computed and measured natural frequencies) determined all six elastic moduli for the Gauthier micropolar material. They did not however discussed any details about the measurement procedure and instruments which they used or the uniqueness of the elastic moduli which they calculated.

Mora and Waas [118] carried out uniaxial compression tests on plate specimens of a circular-cell polycarbonate honeycomb with a circular hole and a circular rigid inclusion. Comparing the test results against the available analytical solutions they found a range for the couple-stress characteristic length of the circular-cell honeycomb. In a series of related works, Beveridge et al. [119], Waseem et al. [120], and McGregor and Wheel [121] extensively studied a few heterogeneous materials (i.e. materials consisting of a repeated pattern of circular voids within an polymeric or metallic elastic matrix) in the context of the micropolar elasticity theory. Using both experimental testing and finite element analysis they determined the characteristic length and the micropolar couple modulus for their heterogeneous materials. A notable observation in these works was the analogy between the characteristic lengths identified by their approach and the intrinsic dimensions defining the heterogeneous structure of the materials.

4.4 Experimental results

Despite relatively numerous publications on the experimental investigation of the micropolar (or couple-stress) elasticity theory, there are only a few successful works with a full set of micropolar (or couple-stress) elastic constants measured for a material. Indeed, experimental difficulties and limitations have lead different researches more towards justification of the micropolar elasticity model or prior measurement results rather than directly measuring a new material constant or putting a new material under the test. The results of the few available comprehensive works are summarized in Table 4.

Note that in the literature on micropolar elastic parameters the results are frequently illustrated in terms of the shear modulus, strain Poisson's ratio, Young's modulus, and the following technical parameters [112]:

$$\begin{array}{ll} \text{polar ratio :} & \Psi = \frac{2\gamma}{2\gamma + \alpha} \\ \text{characteristic length for torsion :} & \ell_t^2 = \frac{\gamma}{\mu} \\ \text{characteristic length for bending :} & \ell_b^2 = \frac{\gamma + \beta}{4\mu} \\ \text{coupling number :} & N^2 = \frac{\kappa}{\mu + \kappa} \end{array} \tag{20}$$

However, in Table 4 the results are translated into the more fundamental elastic parameters appearing in the constitutive relations given in equation (8) and the supplementary elastic parameters defined by

Table 4. The measured classical and micropolar properties for some materials.

Property group/name		Material†								
		Al-Ep	Gr-H237	HB	PSF	PUF	SyF	PMIF-WF51	PMIF-WF110	PMIF-WF300
Classical	Lamé shear modulus μ (MPa)	1890	2123	4000	0.607	104.0	1033	30.0	75.00	290.0
	Lamé coefficient λ (MPa)	7590	289.5	††	0.099	762.7	2097	6.00	487.4	102.1
	Tensile modulus E (MPa)	5293	4500	14000	1.30	299.5	2758	65.0	215.0	655.5
	Strain Poisson's ratio ν	0.40	0.06	††	0.07	0.44	0.34	0.08	0.43	0.13
	Tensile bulk modulus B (MPa)	8850	1705	††	0.504	832.0	2786	26.0	537.4	295.4
	Mass density ρ^v (kg/m^3)	††	1800	2000	37	340	585	60	110	380
	Cosserat couple modulus κ (MPa)	7450	∞ ‡	12000	0.060	4.333	114.8	1.25	0.7576	21.83
Micropolar	Cosserat twist coefficient γ (N)	2640	1605	193.6	8.77	39.98	4.364	8.75	20.28	185.6
	Cosserat twist coefficient β (N)	§	19790	2904	52.0	4.504	−0.133	27.6	16.47	502.2
	Cosserat twist coefficient α (N)	‡‡	∞ ‡	−129.1	−5.84	−26.65	−2.910	−5.83	−13.52	−123.7
	Tortile modulus $\mathcal{E}$ (N)	‡‡	∞ ‡	0	0	0	0	0	0	0
	Twist Poisson's ratio ξ	‡‡	$\frac{1}{2}$‡	−1	−1	−1	−1	−1	−1	−1
	Tortile bulk modulus $\mathcal{B}$ (N)	‡‡	∞ ‡	0	0	0	0	0	0	0
	Microinertia density ι^v (g/m)	0.429	0‡	‡‡	‡‡	‡‡	‡‡	‡‡	‡‡	‡‡

†:Al-Ep: Composite aluminum shot in epoxy matrix [46,122]; Gr-H237: Graphite grade H237 [117]; HB: Human bone [47,100,102,105]; PSF: Polystyrene foam [106]; PUF: Polyurethane foam [107]; SyF: Syntactic foam [107]; PMIF-WF51: Polymethacrylimide foam grade WF51 [48]; PMIF-WF110: Polymethacrylimide foam grade WF110 [48]; PMIF-WF300: Polymethacrylimide foam grade WF300 [48,110].

§:The values of γ and β are not measured explicitly, instead the value of $\gamma + \beta = 2640$ is given as a whole.

‡:Interpreted based on the couple-stress elasticity model for which $\kappa = \infty$, $\alpha = \infty$, and $\iota^v = 0$.

††:Not calculable due to anisotropy.

‡‡:Not measured or not calculable based on the measured parameters.

equations (10), (11), and (12). This translation is done by using the following relations (the shear modulus, strain Poisson's ratio, and Young's modulus are retained):

$$\begin{array}{lll} \kappa = \frac{N^2}{1-N^2}\mu, & \gamma = \ell_t^2\mu, & \beta = 4\ell_b^2\mu - \gamma \\ \lambda = \frac{2\nu}{1-2\nu}\mu, & B = \lambda + \frac{2}{3}\mu & \\ \alpha = 2\left(\frac{1}{\Psi} - 1\right)\gamma, & \mathcal{B} = \alpha + \frac{2}{3}\gamma & \\ \xi = \frac{\alpha}{2(\gamma+\alpha)}, & \mathcal{E} = 2(1+\xi)\gamma & \end{array} \tag{21}$$

The results summarized in Table 4 are important as they illustrate how far from being complete the works accomplished so far are. There are experimental values for only nine materials (or more exactly four different materials being aluminum-epoxy composite, graphite, human bone, and foam) and yet these results are questionable. To be more specific, one can discuss the zero values obtained for the tortile modulus $\mathcal{E}$ and the tortile bulk modulus $\mathcal{B}$ or the -1 values obtained for the twist Poisson's ratio ξ. We believe that the recent advances in measurement equipment and sample preparation methods call for new research on developing new procedures for measuring the micropolar elastic constants, verifying previously measured micropolar material constants, and putting new materials under test. For example, there have been many efforts recently put into establishing a theoretical foundation for using dynamic tests and wave propagation experiments to measure the elastic moduli in the micropolar elasticity theory (or the elastic moduli appearing in other generalized continuum theories) [122–124].

The other point in Table 4 is the fact that (based on the measurements accomplished so far) the ratio of micropolar couple modulus over the Lamé shear modulus is not very small for all micropolar materials as suggested and assumed by many authors [31, 46, 62, 63].

5. Summary

A thorough survey of the notational systems used in micropolar elasticity has been presented. This survey should make the existing literature more accessible to new researchers. The notation introduced here for the representation of the micropolar elasticity model allows for the description of the extra micropolar elasticity features and concepts in a form parallel to the well-known classical elasticity features and concepts. There are micropolar technical constants defined as duals of the technical moduli defined in the classical elasticity model.

Despite its simple nature and common usage, notation can be very important to the understanding of the micropolar elasticity model. The notations used by different authors to represent the linear theory of micropolar elasticity can be categorized into two groups: those where classical and micropolar elastic moduli are denoted by coupled symbols and those where classical and micropolar elastic moduli are represented with separate symbols.

The notations of the first category, however, can be confusing (e.g. mistaking one of the elastic coefficients for the classical Lame shear modulus) and may result in a loose conclusion about the relationship between the micropolar and classical elasticity models. In particular, the traditional approach for simplification of the micropolar elasticity model to the classical elasticity model, suggested based on a coupled notation, is controversial and can be replaced with an alternative simplification approach which has recently been suggested based on a decoupled notation.

The literature survey of the experimental examination of the micropolar elasticity theory and determination of the micropolar elastic constants reveals the limitations of these experimental results. In particular, the micropolar couple modulus is not very small for all materials as suggested and used in the literature on the micropolar elasticity theory.

Conflict of interest

None declared.

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