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Observer-based indirect adaptive fuzzy sliding mode control with state variable filters for unknown nonlinear dynamical systems

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Abstract

This paper proposes an observer-based indirect adaptive fuzzy sliding mode controller with state variable filters for a certain class of unknown nonlinear dynamic systems in which not all the states are available for measurement. To design the proposed controller, we first construct the fuzzy models to describe the input/output behavior of the nonlinear dynamic system. Then, an observer is employed to estimate the tracking error vector. Based on the observer, a fuzzy sliding model controller is developed to achieve the tracking performance. Then, a filtered observation error vector is obtained by passing the observation error vector to a set of state variable filters. Finally, on the basis of the filtered observation error vector, the adaptive laws are proposed to adjust the free parameters of the fuzzy models. The stability of the overall control system is analyzed based on the Lyapunov method. Simulation results illustrate the design procedures and demonstrate the tracking performance of the proposed controller.

Keywords: Fuzzy control; Fuzzy sliding mode control; State variable filters

1. Introduction

Recently, adaptive fuzzy control system designs have been extensively discussed in the literature [1,3,6,17]. The fundamental idea of adaptive fuzzy control is as follows: based on the universal approximation theorem [17], one first constructs a fuzzy model to describe the input/output behavior of the controlled system. After that a controller is designed based on the fuzzy model, the adaptive laws are derived to adjust the parameters of the fuzzy models.

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In an effort to improve the robustness of the adaptive fuzzy control system, many works have been published on the design of adaptive fuzzy sliding mode controller (AFSMC) [7,8,21,18,23,24], which integrates the sliding mode controller (SMC) [2,14–16,19] design technique into the adaptive fuzzy control to improve the stability and the robustness of the control system. Conventionally, AFSMC design is based on the assumption that the system states are available for measurement, so the adaptive laws of AFSMC are formulated as functions of the tracking error vector of the system [7,8,21,18,23,24]. However, in practice, not all the states of the controlled system are available for measurement. It implies that as one could not obtain all elements of the tracking error vector, the conventional adaptive laws would be difficult to realize. In order to treat this problem, several studies apply an observer to estimate the tracking error vector [9,10,12,13,20,22] and use the SPR-Lyapunov design approach [15] to design an adaptive scheme [9,10,13,20,22].

Unlike these works, this paper proposes an indirect AFSMC with state variable filters to tackle the above-mentioned problem. For a given unknown nonlinear dynamic system in which not all the states are available for measurement, we first construct the fuzzy models to describe the input/output behavior of the nonlinear dynamic system. Then, an observer is employed to estimate the tracking error vector. Based on the observer, a fuzzy sliding model controller is developed for guaranteeing the tracking performance. Subsequently, by passing the observation error vector to a set of state variable filters [4,5], we obtain a filtered observation error vector. Finally, based on the filtered observation error vector, we propose the adaptive laws to adjust the free parameters of the fuzzy models. The stability of the overall control system is analyzed based on the Lyapunov method. Simulation results illustrate the design procedures and demonstrate the tracking performance of the proposed controller.

This paper is organized as follows: Section 2 presents the fuzzy sliding mode controller (FSMC) design based on the observer. With the aid of the state variable filters, the adaptive laws for adjusting the free parameters of the proposed control strategy are presented in Section 3. In Section 4, simulations of an inverted pendulum system are given to confirm the validity of the proposed control scheme. Section 5 gives the conclusions.

2. Design of observer-based FSMC

Consider an nth-order unknown nonlinear dynamical system of the form

$$x^{(n)} = f(\mathbf{x}) + b(\mathbf{x})u + d(t),$$

$$y = x, \tag{1}$$

where $\mathbf{x} = [x, \dot{x}, \ldots, x^{(n-1)}]^{\mathrm{T}} = [x_1, x_2, \ldots, x_n]^{\mathrm{T}}$ is the state vector of the system, y is the system output, u is the control signal, $f(\mathbf{x})$ and $b(\mathbf{x})$ are unknown but continuous functions, and $d(t)$ is the external bounded disturbance. Assume that not all states x_i $(i = 1, 2, \ldots, n)$ are available for measurement, but y is measurable. Eq. (1) can be rewritten in the following form:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}[f(\mathbf{x}) + b(\mathbf{x})u + d(t)],$$

$$y = \mathbf{C}\mathbf{x}, \tag{2}$$

where

$$\mathbf{A}=\begin{bmatrix}0&1&0&\cdots&0\\0&0&1&\cdots&0\\\vdots&\vdots&\vdots&\ddots&\vdots\\0&0&0&\cdots&1\\0&0&0&\cdots&0\end{bmatrix},\quad \mathbf{B}=\begin{bmatrix}0\\0\\\vdots\\0\\1\end{bmatrix},\quad \text{and}\quad \mathbf{C}^{\mathrm{T}}=\begin{bmatrix}1\\0\\\vdots\\0\\0\end{bmatrix}.$$

Without loss of generality, we make the following assumption.

Assumption 1 (*Slotine and Li [15], Wang et al. [20], Wang [17]*). Assume that $f(\mathbf{x})$, $b(\mathbf{x})$ and $d(t)$ satisfy $|f(\mathbf{x})| \leqslant F < \infty$, $0 < b_{\min} \leqslant b(\mathbf{x}) \leqslant b_{\max} < \infty$, and $|d(t)| \leqslant D$, respectively, for all $\mathbf{x} \in U_{\mathbf{x}} \subset \Re^n$, where F, $b_{\min}$, $b_{\max}$ and D are known constants.

Control objective: Determine a control law u to force the system output y to asymptotically track a given desired output y_{d}.

Define the tracking error as $e = y - y_{\mathrm{d}}$, and let

$$\begin{aligned}\mathbf{e} &= [e, \dot{e}, \ldots, e^{(n-1)}]^{\mathrm{T}}\\ &= [y - y_{\mathrm{d}}, \dot{y} - \dot{y}_{\mathrm{d}}, \ldots, y^{(n-1)} - y_{\mathrm{d}}^{(n-1)}]^{\mathrm{T}}\\ &= [x_1 - y_{\mathrm{d}}, x_2 - \dot{y}_{\mathrm{d}}, \ldots, x_n - y_{\mathrm{d}}^{(n-1)}]^{\mathrm{T}}\end{aligned} \tag{3}$$

be the tracking error vector. Set the sliding surface H as

$$H : \{\mathbf{e} | S(\mathbf{e}) = 0\}, \tag{4}$$

$$S(\mathbf{e}) = \boldsymbol{\alpha}^{\mathrm{T}}\mathbf{e}, \tag{5}$$

where $\boldsymbol{\alpha} = [\alpha_0, \alpha_1, \ldots, \alpha_{n-1}]^{\mathrm{T}}$ is chosen such that $\alpha_{n-1} = 1$ and the polynomial $\alpha_{n-1}p^{n-1} + \alpha_{n-2}p^{n-2} + \cdots + \alpha_0$ is strictly Hurwitz [19] (here p denotes the complex *Laplace transform* variable).

To meet the control objective, it is sufficient to find a control law u so that all initial states lying off H will hit H and then remain on it. If $f(\mathbf{x})$ and $b(\mathbf{x})$ are exactly known, the control objective can be achieved by the control law designed as [15]:

$$u = u_{\mathrm{eq}} + u_{\mathrm{d}}, \tag{6}$$

where u_{eq} is the equivalent control law and is defined as

$$u_{\mathrm{eq}} = b(\mathbf{x})^{-1}\left[-\sum_{i=1}^{n-1}\alpha_{i-1}e^{(i)} - f(\mathbf{x}) + y_{\mathrm{d}}^{(n)}\right] \tag{7}$$

and

$$u_{\mathrm{d}} = -\eta\,\mathrm{sgn}(S) \tag{8}$$

is called the switching control law, in which η is a positive constant satisfying $\eta > D/b(\mathbf{x})$, with D denoting the bound of $d(t)$; i.e., $|d(t)| \leqslant D$, and

$$\operatorname{sgn}(S) = \begin{cases} 1 & \text{if } S > 0, \\ 0 & \text{if } S = 0, \\ -1 & \text{if } S < 0. \end{cases}$$

However, if the functions $f(\mathbf{x})$ and $b(\mathbf{x})$ are unknown, the control law (6) is generally inapplicable. Thus, we will employ the fuzzy systems $\hat{f}(\mathbf{x}|\boldsymbol{\theta}_f)$ and $\hat{b}(\mathbf{x}|\boldsymbol{\theta}_b)$ to approximate $f(\mathbf{x})$ and $b(\mathbf{x})$, respectively. Specifically, the fuzzy rule bases of $\hat{f}(\mathbf{x}|\boldsymbol{\theta}_f)$ and $\hat{b}(\mathbf{x}|\boldsymbol{\theta}_b)$, respectively, consists of rules

$$R_{\hat{f}}^{(m)}: \quad \text{IF } x_1 \text{ is } F_{x1}^m \text{ and} \ldots \text{and } x_n \text{ is } F_{xn}^m, \text{ THEN } \hat{f}(\mathbf{x}|\boldsymbol{\theta}_f) \text{ is } F_{\hat{f}}^m, \tag{9}$$

$$R_{\hat{b}}^{(m)}: \quad \text{IF } x_1 \text{ is } F_{x1}^m \text{ and} \ldots \text{and } x_n \text{ is } F_{xn}^m, \text{ THEN } \hat{b}(\mathbf{x}|\boldsymbol{\theta}_b) \text{ is } F_{\hat{b}}^m, \tag{10}$$

where $m = 1, 2, \ldots, Q$, Q is the total number of the fuzzy rules for each fuzzy model, and F_{xi}^m $(i = 1, \ldots, n)$ are the fuzzy sets associated with x_i $(i = 1, \ldots, n)$, and $F_{\hat{f}}^m$ and $F_{\hat{b}}^m$ are fuzzy singletons for $\hat{f}(\mathbf{x}|\boldsymbol{\theta}_f)$ and $\hat{b}(\mathbf{x}|\boldsymbol{\theta}_b)$, respectively. By using the singleton fuzzifier, product inference, and center average defuzzifier [17], the outputs of the fuzzy models of $\hat{f}(\mathbf{x}|\boldsymbol{\theta}_f)$ and $\hat{b}(\mathbf{x}|\boldsymbol{\theta}_b)$ can be, respectively, expressed as

$$\hat{f}(\mathbf{x}|\boldsymbol{\theta}_f) = \boldsymbol{\theta}_f^{\mathrm{T}} \boldsymbol{\xi}(\mathbf{x}), \tag{11}$$

$$\hat{b}(\mathbf{x}|\boldsymbol{\theta}_b) = \boldsymbol{\theta}_f^{\mathrm{T}} \boldsymbol{\xi}(\mathbf{x}), \tag{12}$$

where $\boldsymbol{\theta}_f = [F_{\hat{f}}^1, F_{\hat{f}}^2, \ldots, F_{\hat{f}}^Q]^{\mathrm{T}}$ and $\boldsymbol{\theta}_b = [F_{\hat{b}}^1, F_{\hat{b}}^2, \ldots, F_{\hat{b}}^Q]^{\mathrm{T}}$ are the adjustable parameter vectors, $\boldsymbol{\xi}(\mathbf{x}) = [\xi^1(\mathbf{x}), \xi^2(\mathbf{x}), \ldots, \xi^Q(\mathbf{x})]^{\mathrm{T}}$ is the vector of fuzzy basis functions [17] defined as

$$\xi^j(\mathbf{x}) = \frac{\prod_{i=1}^n F_{xi}^j(x_i)}{\sum_{j=1}^Q [\prod_{i=1}^n F_{xi}^j(x_i)]}, \quad j = 1, 2, \ldots, Q, \tag{13}$$

and $F_{xi}^j(x_i)$ represents the membership function value of x_i in F_{xi}^j. We make the following assumptions.

Assumption 2 (*Leu et al. [9], Li and Tong [10], Wang et al. [20]*). $\boldsymbol{\theta}_f$ and $\boldsymbol{\theta}_b$ belong to compact sets Ω_f and Ω_b, respectively, which are defined as $\Omega_f = \{\boldsymbol{\theta}_f \in \Re^Q | \, \|\boldsymbol{\theta}_f\| \leqslant m_f\}$ and $\Omega_b = \{\boldsymbol{\theta}_b \in \Re^Q | 0 < \|\boldsymbol{\theta}_b\| \leqslant m_b\}$, where m_f and m_b are designed finite positive constants.

From (11), (12), and by Assumption 2, we can also assume that $|\hat{f}(\mathbf{x}|\boldsymbol{\theta}_f)| \leqslant F$ and $0 < b_{\min} \leqslant \hat{b}(\mathbf{x}|\boldsymbol{\theta}_b) \leqslant b_{\max} < \infty$. Thus, we can replace (6) by the following control law [17,21]:

$$u = \hat{u}_{\mathrm{eq}} + \hat{u}_{\mathrm{d}}, \tag{14}$$

where

$$\hat{u}_{\mathrm{eq}} = \hat{b}(\mathbf{x}|\boldsymbol{\theta}_b)^{-1} \left[-\sum_{i=1}^{n-1} \alpha_{i-1} e^{(i)} - \hat{f}(\mathbf{x}|\boldsymbol{\theta}_f) + y_{\mathrm{d}}^{(n)} \right] \tag{15}$$

and

$$\hat{u}_{\rm d} = \hat{\eta}\,{\rm sgn}(S), \tag{16}$$

in which $\hat{\eta} > b_{\min}^{-1}(2F + D) + [b_{\min}^{-1} - (b_{\min}b_{\max})^{-1/2}]|\hat{u}_{\rm eq}|$, and

$$ {\rm sgn}(S) = \begin{cases} 1 & \text{if } S > 0, \\ 0 & \text{if } S = 0, \\ -1 & \text{if } S < 0. \end{cases} $$

However, this control law of (14) is still nonrealistic because not all states x_i $(i = 1, 2, \ldots, n)$ of (1) are available for measurement, and hence not all the derivative signals $e^{(i)}$ $(i = 1, 2, \ldots, n-1)$ are available for measurement. Although ideally x_i $(i = 2, 3, \ldots, n)$ and $e^{(i)}$ $(i = 1, 2, \ldots, n-1)$ can be obtained by successive differentiation of the signal x_1 and e, respectively, ideal differentiators are physically unrealizable. Thus, we have to estimate the signals x_i $(i = 2, 3, \ldots, n)$ and $e^{(i)}$ $(i = 1, 2, \ldots, n-1)$ with realizable filters. Let $\hat{x}_i$ $(i = 2, 3, \ldots, n)$ denote the estimate of x_i $(i = 2, 3, \ldots, n)$, let $\hat{\mathbf{x}} = [\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n]^{\rm T} \in U_{\hat{\mathbf{x}}} \subset \Re^n$ denote the estimate of the state vector $\mathbf{x} = [x_1, x_2, \ldots, x_n]^{\rm T}$, let $\hat{e} = \hat{y} - y_{\rm d}$ denote the estimate of $e = y - y_{\rm d}$, let $\hat{e}^{(i)}$ $(i = 1, 2, \ldots, n-1)$ denote the estimate of $e^{(i)}$ $(i = 1, 2, \ldots, n-1)$, and let $\hat{\mathbf{e}} = [\hat{e}, \dot{\hat{e}}, \ldots, \hat{e}^{(n-1)}]^{\rm T}$ denote the estimate of the tracking error vector $\mathbf{e} = [e, \dot{e}, \ldots, e^{(n-1)}]^{\rm T}$. Then, by replacing $\mathbf{x}$ of (11) and (12) by $\hat{\mathbf{x}}$, we have

$$\hat{f}(\hat{\mathbf{x}}|\boldsymbol{\theta}_f) = \boldsymbol{\theta}_f^{\rm T}\boldsymbol{\xi}(\hat{\mathbf{x}}), \tag{17}$$

$$\hat{b}(\hat{\mathbf{x}}|\boldsymbol{\theta}_b) = \boldsymbol{\theta}_f^{\rm T}\boldsymbol{\xi}(\hat{\mathbf{x}}). \tag{18}$$

By replacing $\hat{f}(\mathbf{x}|\boldsymbol{\theta}_f)$, $\hat{b}(\mathbf{x}|\boldsymbol{\theta}_b)$ and $e^{(i)}$ by $\hat{f}(\hat{\mathbf{x}}|\boldsymbol{\theta}_f)$, $\hat{b}(\hat{\mathbf{x}}|\boldsymbol{\theta}_b)$ and $\hat{e}^{(i)}$, respectively, the control law of (14) can be rewritten as

$$u = \hat{b}(\hat{\mathbf{x}}|\boldsymbol{\theta}_b)^{-1}\left[-\sum_{i=1}^{n-1}\alpha_{i-1}\hat{e}^{(i)} - \hat{f}(\hat{\mathbf{x}}|\boldsymbol{\theta}_f) + y_{\rm d}^{(n)}\right] - \hat{\eta}\,{\rm sgn}(\hat{S}), \tag{19}$$

where $\hat{S} = \boldsymbol{\alpha}^{\rm T}\hat{\mathbf{e}}$. Applying (19) to the system (1), we have

$$\begin{aligned} \dot{\mathbf{e}} &= \mathbf{A}\mathbf{e} + \mathbf{B}\{[-\boldsymbol{\alpha}_m^{\rm T}\hat{\mathbf{e}} + f(\mathbf{x}) - \hat{f}(\hat{\mathbf{x}}|\boldsymbol{\theta}_f) + [b(\mathbf{x}) - \hat{b}(\hat{\mathbf{x}}|\boldsymbol{\theta}_b)]u - \hat{\eta}\hat{b}(\hat{\mathbf{x}}|\boldsymbol{\theta}_b){\rm sgn}(\hat{S}) + d(t)\}, \\ e &= \mathbf{C}\mathbf{e}, \end{aligned} \tag{20}$$

where $\boldsymbol{\alpha}_m = [0, \alpha_0, \alpha_1, \ldots, \alpha_{n-2}]^{\rm T}$. For (20), we design the following observer to estimate the tracking error vector:

$$\begin{aligned} \dot{\hat{\mathbf{e}}} &= \mathbf{A}\hat{\mathbf{e}} + \mathbf{B}[-\boldsymbol{\alpha}_m^{\rm T}\hat{\mathbf{e}} - \hat{\eta}\hat{b}(\hat{\mathbf{x}}|\boldsymbol{\theta}_b){\rm sgn}(\hat{S})] + \mathbf{L}(\hat{e} - e), \\ \hat{e} &= \mathbf{C}\hat{\mathbf{e}}, \end{aligned} \tag{21}$$

where $\mathbf{L} = [l_{n-1}, l_{n-2}, \ldots, l_0]^{\rm T}$ is the observer gain vector. Define the observation error as $\tilde{e} = \hat{e} - e$, and the observation error vector by

$$\tilde{\mathbf{e}} = \hat{\mathbf{e}} - \mathbf{e}. \tag{22}$$

Subtracting (20) from (21), we obtain the observation error dynamic equation as

$$\begin{aligned} \dot{\tilde{\mathbf{e}}} &= \boldsymbol{\Lambda}_{\rm o}\tilde{\mathbf{e}} + \mathbf{B}\{\hat{f}(\hat{\mathbf{x}}|\boldsymbol{\theta}_f) - f(\mathbf{x}) + [\hat{b}(\hat{\mathbf{x}}|\boldsymbol{\theta}_b) - b(\mathbf{x})]u - d(t)\}, \\ \tilde{e} &= \mathbf{C}\tilde{\mathbf{e}}, \end{aligned} \tag{23}$$

where

$$\mathbf{\Lambda}_o = \mathbf{A} + \mathbf{LC} = \begin{bmatrix} l_{n-1} & 1 & 0 & \cdots & 0 \\ l_{n-2} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ l_1 & 0 & 0 & \cdots & 1 \\ l_0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Because $(\mathbf{C}, \mathbf{\Lambda}_o)$ pair is observable, $\mathbf{L}$ can be selected such that the characteristic polynomial of $\mathbf{\Lambda}_o$ is strictly Hurwitz and there exists a positive-definite matrix $\mathbf{P}$ satisfying

$$\mathbf{\Lambda}_o^{\mathrm{T}}\mathbf{P} + \mathbf{P}\mathbf{\Lambda}_o = -\mathbf{Q}, \tag{24}$$

where $\mathbf{Q}$ is an arbitrary symmetric positive-definite matrix [20]. Here we let $\lambda_{\min}(\mathbf{Q}) > 1$, where $\lambda_{\min}(\mathbf{Q})$ denotes the minimum eigenvalue of $\mathbf{Q}$.

To complete the controller design, we need to find $\boldsymbol{\theta}_f$ and $\boldsymbol{\theta}_b$ in (17) and (18). In the following section, we will present the adaptive laws to adjust the parameter vectors $\boldsymbol{\theta}_f$ and $\boldsymbol{\theta}_b$.

3. Observer-based indirect adaptive FSMC with state variable filters

To derive the adaptive laws for adjusting $\boldsymbol{\theta}_f$ and $\boldsymbol{\theta}_b$, we first define the optimal parameter vectors $\boldsymbol{\theta}_f^*$ and $\boldsymbol{\theta}_b^*$ as

$$\boldsymbol{\theta}_f^* = \arg \min_{\boldsymbol{\theta}_f \in \Omega_f} \left[\sup_{\mathbf{x} \in U_{\mathbf{x}},\ \hat{\mathbf{x}} \in U_{\hat{\mathbf{x}}}} |\hat{f}(\hat{\mathbf{x}}|\boldsymbol{\theta}_f) - f(\mathbf{x})| \right] \tag{25}$$

and

$$\boldsymbol{\theta}_b^* = \arg \min_{\boldsymbol{\theta}_b \in \Omega_b} \left[\sup_{\mathbf{x} \in U_{\mathbf{x}},\ \hat{\mathbf{x}} \in U_{\hat{\mathbf{x}}}} |\hat{b}(\hat{\mathbf{x}}|\boldsymbol{\theta}_b) - b(\mathbf{x})| \right]. \tag{26}$$

Define the minimum approximation error

$$w = [\hat{f}(\hat{\mathbf{x}}|\boldsymbol{\theta}_f^*) - f(\mathbf{x})] + [\hat{b}(\hat{\mathbf{x}}|\boldsymbol{\theta}_f^*) - b(\mathbf{x})]u. \tag{27}$$

Then inserting (17), (18) and (27) into (23) yields

$$\begin{aligned}
\dot{\tilde{\mathbf{e}}} = {} & \mathbf{\Lambda}_o\tilde{\mathbf{e}} + \mathbf{B}\{\hat{f}(\hat{\mathbf{x}}|\boldsymbol{\theta}_f) - \hat{f}(\hat{\mathbf{x}}|\boldsymbol{\theta}_f^*) + \hat{f}(\hat{\mathbf{x}}|\boldsymbol{\theta}_f^*) - \hat{f}(\mathbf{x}|\boldsymbol{\theta}_f^*) + \hat{f}(\mathbf{x}|\boldsymbol{\theta}_f^*) - f(\mathbf{x}) \\
& +[\hat{b}(\hat{\mathbf{x}}|\boldsymbol{\theta}_b) - \hat{b}(\hat{\mathbf{x}}|\boldsymbol{\theta}_b^*) + \hat{b}(\hat{\mathbf{x}}|\boldsymbol{\theta}_b^*) - \hat{b}(\mathbf{x}|\boldsymbol{\theta}_b^*) + \hat{b}(\mathbf{x}|\boldsymbol{\theta}_b^*) - b(\mathbf{x})]u - d(t)\}, \\
= {} & \mathbf{\Lambda}_o\tilde{\mathbf{e}} + \mathbf{B}\{\boldsymbol{\theta}_f^{\mathrm{T}}\boldsymbol{\xi}(\hat{\mathbf{x}}) - \boldsymbol{\theta}_f^{*\mathrm{T}}\boldsymbol{\xi}(\hat{\mathbf{x}}) + \boldsymbol{\theta}_f^{*\mathrm{T}}\boldsymbol{\xi}(\hat{\mathbf{x}}) - \boldsymbol{\theta}_f^{*\mathrm{T}}\boldsymbol{\xi}(\mathbf{x}) + \boldsymbol{\theta}_f^{*\mathrm{T}}\boldsymbol{\xi}(\mathbf{x}) - f(\mathbf{x}) \\
& +[\boldsymbol{\theta}_b^{\mathrm{T}}\boldsymbol{\xi}(\hat{\mathbf{x}}) - \boldsymbol{\theta}_b^{*\mathrm{T}}\boldsymbol{\xi}(\hat{\mathbf{x}}) + \boldsymbol{\theta}_b^{*\mathrm{T}}\boldsymbol{\xi}(\hat{\mathbf{x}}) - \boldsymbol{\theta}_b^{*\mathrm{T}}\boldsymbol{\xi}(\mathbf{x}) + \boldsymbol{\theta}_b^{*\mathrm{T}}\boldsymbol{\xi}(\mathbf{x}) - b(\mathbf{x})]u - d(t)\}, \\
= {} & \mathbf{\Lambda}_o\tilde{\mathbf{e}} + \mathbf{B}[w + \boldsymbol{\phi}_f^{\mathrm{T}}\boldsymbol{\xi}(\hat{\mathbf{x}}) + \boldsymbol{\phi}_b^{\mathrm{T}}\boldsymbol{\xi}(\hat{\mathbf{x}})u + \boldsymbol{\theta}_f^{*\mathrm{T}}\tilde{\boldsymbol{\xi}} + \boldsymbol{\theta}_b^{*\mathrm{T}}\tilde{\boldsymbol{\xi}}\, u - d(t)], \\
= {} & \mathbf{\Lambda}_o\tilde{\mathbf{e}} + \mathbf{B}[w + \boldsymbol{\phi}_f^{\mathrm{T}}\boldsymbol{\xi}(\hat{\mathbf{x}}) + \boldsymbol{\phi}_b^{\mathrm{T}}\boldsymbol{\xi}(\hat{\mathbf{x}})u + \upsilon],
\end{aligned} \tag{28}$$

where $\tilde{\boldsymbol{\xi}} = \boldsymbol{\xi}(\hat{\mathbf{x}}) - \boldsymbol{\xi}(\mathbf{x})$, $\upsilon = \boldsymbol{\theta}_f^{*\mathrm{T}}\tilde{\boldsymbol{\xi}} + \boldsymbol{\theta}_b^{*\mathrm{T}}\tilde{\boldsymbol{\xi}}\,u - d(t)$, $\boldsymbol{\phi}_f = \boldsymbol{\theta}_f - \boldsymbol{\theta}_f^*$ and $\boldsymbol{\phi}_b = \boldsymbol{\theta}_b - \boldsymbol{\theta}_b^*$.

The following lemma is required in the stability analysis.

Lemma. *If Assumptions* 1 *and* 2 *are satisfied, then* $w \in L_\infty$ *and* $\upsilon \in L_\infty$.

Proof. We first prove $w \in L_\infty$. From Assumptions 1 and 2, we have

$$\begin{aligned}|w| &\leqslant |\hat{f}(\hat{\mathbf{x}}|\boldsymbol{\theta}_f^*) - f(\mathbf{x})| + |\hat{b}(\hat{\mathbf{x}}|\boldsymbol{\theta}_b^*) - b(\mathbf{x})|\,|u(t)| \\ &\leqslant |\hat{f}(\hat{\mathbf{x}}|\boldsymbol{\theta}_f^*)| + |f(\mathbf{x})| + [|\hat{b}(\hat{\mathbf{x}}|\boldsymbol{\theta}_b^*)| + |b(\mathbf{x})|]|u(t)| \\ &\leqslant \|\boldsymbol{\theta}_f^{*\mathrm{T}}\|\,\|\boldsymbol{\xi}(\hat{\mathbf{x}})\| + |f(\mathbf{x})| + [\|\boldsymbol{\theta}_b^{*\mathrm{T}}\|\,\|\boldsymbol{\xi}(\hat{\mathbf{x}})\| + |b(\mathbf{x})|]|u(t)| \\ &\leqslant m_f + F + [m_b + b_{\max}]\left(\max_t\,|u(t)|\right).\end{aligned} \tag{29}$$

Since the control signal u is designed as a bounded signal, we obtained that w is bounded; i.e., $w \in L_\infty$.
Next, we prove that $w \in L_\infty$. Based on Assumptions 1 and 2, we have

$$\begin{aligned}|\upsilon| &\leqslant |\boldsymbol{\theta}_f^{*\mathrm{T}}\tilde{\boldsymbol{\xi}}\,| + |\boldsymbol{\theta}_b^{*\mathrm{T}}\tilde{\boldsymbol{\xi}}\,u| + |d(t)| \\ &\leqslant \|\boldsymbol{\theta}_f^{*\mathrm{T}}\|\,\|\tilde{\boldsymbol{\xi}}\,\| + \|\boldsymbol{\theta}_b^{*\mathrm{T}}\|\,\|\tilde{\boldsymbol{\xi}}\,\|\left(\max_t\,|u(t)|\right) + D \\ &\leqslant m_f + m_b\left(\max_t\,|u(t)|\right) + D.\end{aligned}$$

Since the control signal u is designed as a bounded signal, we obtained that υ is bounded; i.e., $\upsilon \in L_\infty$. This completes the proof. □

Regarding $[w + \boldsymbol{\phi}_f^{\mathrm{T}}\boldsymbol{\xi}(\hat{\mathbf{x}}) + \boldsymbol{\phi}_b^{\mathrm{T}}\boldsymbol{\xi}(\hat{\mathbf{x}})u + \upsilon]$ as the input for (28), we can obtain

$$\tilde{e} = \Xi(p)\{w + \boldsymbol{\phi}_f^{\mathrm{T}}\boldsymbol{\xi}(\hat{\mathbf{x}}) + \boldsymbol{\phi}_b^{\mathrm{T}}\boldsymbol{\xi}(\hat{\mathbf{x}})u + \upsilon\}, \tag{30}$$

where

$$\Xi(p) = \frac{1}{p^n - l_{n-1}p^{n-1} - l_{n-2}p^{n-2} - \cdots - l_0} \tag{31}$$

and p denotes the complex *Laplace transform* variable. As has been discussed, because not all the states are available for measurement, we could not obtain all the elements of $\mathbf{e}$; as a result, we could not obtain all the elements of $\tilde{\mathbf{e}}$. To deal with this problem, the state variable filters [4,5] will be employed. First, we introduce a stable filter $\Omega(p)$ to (30), and obtain the steady-state equation

$$[(p^n - l_{n-1}p^{n-1} - l_{n-2}p^{n-2} - \cdots - l_0)\Omega(p)]\{\tilde{e}\} = \Omega(p)\{w + \boldsymbol{\phi}_f^{\mathrm{T}}\boldsymbol{\xi}(\hat{\mathbf{x}}) + \boldsymbol{\phi}_b^{\mathrm{T}}\boldsymbol{\xi}(\hat{\mathbf{x}})u + \upsilon\}, \tag{32}$$

where

$$\Omega(p) = \frac{1}{p^n + \omega_{n-1}p^{n-1} + \cdots + \omega_0}. \tag{33}$$

Define a set of state variable filters $T_i(p)$ by

$$T_i(p) = \Omega(p)p^i, \quad i = 0, 1, 2, \ldots, n-1, \tag{34}$$

and the corresponding filtered signals $e_{\mathbf{F}i},\ \varepsilon_{\mathbf{F}},\ \boldsymbol{\xi}_{\mathbf{F}}$ and $\boldsymbol{\psi}_{\mathbf{F}}$ by

$$e_{\mathbf{F}i} = T_i(p)\{\tilde{e}\}, \quad i = 0, 1, 2, \ldots, n-1, \tag{35a}$$

$$\varepsilon_{\mathbf{F}} = T_0(p)\{w + \upsilon\}, \tag{35b}$$

$$\boldsymbol{\xi}_{\mathbf{F}} = T_0(p)\{\boldsymbol{\xi}(\hat{\mathbf{x}})\}, \tag{35c}$$

$$\boldsymbol{\psi}_{\mathbf{F}} = T_0(p)\{\boldsymbol{\xi}(\hat{\mathbf{x}})u\}. \tag{35d}$$

Then (32) yields a filtered equation as follows:

$$\begin{aligned}\dot{\mathbf{e}}_{\mathbf{F}} &= \boldsymbol{\Lambda}_{\mathrm{o}}\mathbf{e}_{\mathbf{F}} + \mathbf{B}(\varepsilon_{\mathbf{F}} + \boldsymbol{\phi}_f^{\mathrm{T}}\boldsymbol{\xi}_{\mathbf{F}} + \boldsymbol{\phi}_b^{\mathrm{T}}\boldsymbol{\psi}_{\mathbf{F}}),\\ e_{\mathbf{F}0} &= \mathbf{C}\mathbf{e}_{\mathbf{F}},\end{aligned} \tag{36}$$

in which $\mathbf{e}_{\mathbf{F}} = [e_{\mathbf{F}0}, e_{\mathbf{F}1}, \ldots, e_{\mathbf{F}n-1}]^{\mathrm{T}}$. In order to derive the adaptive laws for adjusting $\boldsymbol{\theta}_f$ and $\boldsymbol{\theta}_b$, we consider the following Lyapunov function candidate:

$$V = \frac{1}{2}\,\mathbf{e}_{\mathbf{F}}^{\mathrm{T}}\mathbf{P}\mathbf{e}_{\mathbf{F}} + \frac{1}{2\gamma_f}\,\boldsymbol{\phi}_f^{\mathrm{T}}\boldsymbol{\phi}_f + \frac{1}{2\gamma_b}\,\boldsymbol{\phi}_b^{\mathrm{T}}\boldsymbol{\phi}_b, \tag{37}$$

which $\mathbf{P}$ is given by (24), and γ_f and γ_b are positive constants. The time derivative of (37) along the trajectory (36) is

$$\begin{aligned}\dot{V} &= \frac{1}{2}\,\dot{\mathbf{e}}_{\mathbf{F}}^{\mathrm{T}}\mathbf{P}\mathbf{e}_{\mathbf{F}} + \frac{1}{2}\,\mathbf{e}_{\mathbf{F}}^{\mathrm{T}}\mathbf{P}\dot{\mathbf{e}}_{\mathbf{F}} + \frac{1}{\gamma_f}\,\boldsymbol{\phi}_f^{\mathrm{T}}\dot{\boldsymbol{\phi}}_f + \frac{1}{\gamma_b}\,\boldsymbol{\phi}_b^{\mathrm{T}}\dot{\boldsymbol{\phi}}_b\\ &= \frac{1}{2}\,\mathbf{e}_{\mathbf{F}}^{\mathrm{T}}\boldsymbol{\Lambda}_{\mathrm{o}}^{\mathrm{T}}\mathbf{P}\mathbf{e}_{\mathbf{F}} + \frac{1}{2}\,\mathbf{e}_{\mathbf{F}}^{\mathrm{T}}\mathbf{P}\boldsymbol{\Lambda}_{\mathrm{o}}\mathbf{e}_{\mathbf{F}} + \mathbf{e}_{\mathbf{F}}^{\mathrm{T}}\mathbf{P}\mathbf{B}(\varepsilon_{\mathbf{F}} + \boldsymbol{\phi}_f^{\mathrm{T}}\boldsymbol{\xi}_{\mathbf{F}} + \boldsymbol{\phi}_b^{\mathrm{T}}\boldsymbol{\psi}_{\mathbf{F}}) + \frac{1}{\gamma_f}\,\boldsymbol{\phi}_f^{\mathrm{T}}\dot{\boldsymbol{\phi}}_f + \frac{1}{\gamma_b}\,\boldsymbol{\phi}_b^{\mathrm{T}}\dot{\boldsymbol{\phi}}_b\\ &= -\frac{1}{2}\,\mathbf{e}_{\mathbf{F}}^{\mathrm{T}}\mathbf{Q}\mathbf{e}_{\mathbf{F}} + \mathbf{e}_{\mathbf{F}}^{\mathrm{T}}\mathbf{P}\mathbf{B}(\varepsilon_{\mathbf{F}} + \boldsymbol{\phi}_f^{\mathrm{T}}\boldsymbol{\xi}_{\mathbf{F}} + \boldsymbol{\phi}_b^{\mathrm{T}}\boldsymbol{\psi}_{\mathbf{F}}) + \frac{1}{\gamma_f}\,\boldsymbol{\phi}_f^{\mathrm{T}}\dot{\boldsymbol{\phi}}_f + \frac{1}{\gamma_b}\,\boldsymbol{\phi}_b^{\mathrm{T}}\dot{\boldsymbol{\phi}}_b\\ &= -\frac{1}{2}\,\mathbf{e}_{\mathbf{F}}^{\mathrm{T}}\mathbf{Q}\mathbf{e}_{\mathbf{F}} + \boldsymbol{\phi}_f^{\mathrm{T}}\left(\mathbf{e}_{\mathbf{F}}^{\mathrm{T}}\mathbf{P}\mathbf{B}\boldsymbol{\xi}_{\mathbf{F}} + \frac{1}{\gamma_f}\,\dot{\boldsymbol{\theta}}_f\right) + \boldsymbol{\phi}_b^{\mathrm{T}}\left(\mathbf{e}_{\mathbf{F}}^{\mathrm{T}}\mathbf{P}\mathbf{B}\boldsymbol{\psi}_{\mathbf{F}} + \frac{1}{\gamma_b}\,\dot{\boldsymbol{\theta}}_b\right) + \mathbf{e}_{\mathbf{F}}^{\mathrm{T}}\mathbf{P}\mathbf{B}\varepsilon_{\mathbf{F}}.\end{aligned} \tag{38}$$

By the above equation, we choose the adaptive laws as follows:

$$\dot{\boldsymbol{\theta}}_f = -\gamma_f\mathbf{e}_{\mathbf{F}}^{\mathrm{T}}\mathbf{P}\mathbf{B}\boldsymbol{\xi}_{\mathbf{F}}, \tag{39a}$$

$$\dot{\boldsymbol{\theta}}_b = -\gamma_b\mathbf{e}_{\mathbf{F}}^{\mathrm{T}}\mathbf{P}\mathbf{B}\boldsymbol{\psi}_{\mathbf{F}}. \tag{39b}$$

Therefore, we obtain

$$\dot{V} = -\tfrac{1}{2}\,\mathbf{e}_{\mathbf{F}}^{\mathrm{T}}\mathbf{Q}\mathbf{e}_{\mathbf{F}} + \mathbf{e}_{\mathbf{F}}^{\mathrm{T}}\mathbf{P}\mathbf{B}\varepsilon_{\mathbf{F}}. \tag{40}$$

Let $\lambda_{\min}(\mathbf{Q})$ be the minimum eigenvalue of $\mathbf{Q}$ satisfying $\lambda_{\min}(\mathbf{Q}) > 1$, then

$$\begin{aligned}\dot{V} &\leqslant -\frac{\lambda_{\min}(\mathbf{Q}) - 1}{2}\,\|\mathbf{e}_{\mathbf{F}}\|^2 - \frac{1}{2}(\|\mathbf{e}_{\mathbf{F}}\|^2 - 2\mathbf{e}_{\mathbf{F}}^{\mathrm{T}}\mathbf{P}\mathbf{B}\varepsilon_{\mathbf{F}} + \|\mathbf{P}\mathbf{B}\varepsilon_{\mathbf{F}}\|^2) + \frac{1}{2}\|\mathbf{P}\mathbf{B}\varepsilon_{\mathbf{F}}\|^2\\ &\leqslant -\frac{\lambda_{\min}(\mathbf{Q}) - 1}{2}\,\|\mathbf{e}_{\mathbf{F}}\|^2 + \frac{1}{2}\|\mathbf{P}\mathbf{B}\|^2\|\varepsilon_{\mathbf{F}}\|^2.\end{aligned} \tag{41}$$

Integrating both sides of (41) yields

$$V(\infty) - V(0) \leqslant -\frac{\lambda_{\min}(\mathbf{Q}) - 1}{2}\int_0^\infty \|\mathbf{e}_{\mathbf{F}}\|^2\,\mathrm{d}t + \frac{1}{2}\,\|\mathbf{P}\mathbf{B}\|^2\|\int_0^\infty \|\varepsilon_{\mathbf{F}}\|^2\,\mathrm{d}t. \tag{42}$$

After some manipulations, we have

$$\int_0^\infty \|\mathbf{e_F}\|^2\,\mathrm{d}t \leqslant \frac{2}{\lambda_{\min}(\mathbf{Q})-1}\,[V(0)-V(\infty)] + \frac{2}{\lambda_{\min}(\mathbf{Q})-1}\,\|\mathbf{PB}\|^2\int_0^\infty \|\varepsilon_\mathbf{F}\|^2\,\mathrm{d}t. \tag{43}$$

By (33) and (34), we know $T_0(p)$ is stable, and from the Lemma, we have $w \in L_\infty$ and $\upsilon \in L_\infty$, so $\varepsilon_\mathbf{F} = T_0(p)\{w+\upsilon\} \in L_2 \cap L_\infty$. Hence, the right-hand side of (43) is bounded and we obtain $\mathbf{e_F} \in L_2 \cap L_\infty$. Also, because all the variables in the right-hand side of (36) are bounded, we obtain $\dot{\mathbf{e}}_\mathbf{F} \in L_\infty$. By using Barbalat's lemma [15], we have $\lim_{t\to\infty} \|\mathbf{e_F}(t)\| = 0$. Consequently, $\lim_{t\to\infty} e_{\mathbf{F}0}(t) = 0$.

Recall that

$$\begin{aligned} e_{\mathbf{F}1} &= T_1(p)\{\tilde{e}\}, \\ &= \frac{p}{p^n + \omega_{n-1}p^{n-1} + \cdots + \omega_0}\{\tilde{e}\}. \end{aligned} \tag{44}$$

Hence,

$$e_{\mathbf{F}n+1} + \omega_{n-1}e_{\mathbf{F}n} + \cdots + \omega_0 e_{\mathbf{F}1} = \dot{\tilde{e}}. \tag{45}$$

Since $\mathbf{e_F} \in L_\infty$, $\dot{\mathbf{e}}_\mathbf{F} \in L_\infty$ and $\dot{\tilde{e}} \in L_\infty$, it implies $e_{\mathbf{F}n+1} \in L_\infty$. Moreover, since $\int_0^\infty e_{\mathbf{F}n}\,\mathrm{d}t = \lim_{t\to\infty} e_{\mathbf{F}n-1} = 0$, it implies $\lim_{t\to\infty} e_{\mathbf{F}n} = 0$.

Also, because

$$\begin{aligned} e_{\mathbf{F}0} &= T_0(p)\{\tilde{e}\}, \\ &= \frac{1}{p^n + \omega_{n-1}p^{n-1} + \cdots + \omega_0}\{\tilde{e}\} \end{aligned} \tag{46}$$

can be rewritten as

$$\tilde{e} = e_{\mathbf{F}n} + \omega_{n-1}e_{\mathbf{F}n-1} + \cdots + \omega_0 e_{\mathbf{F}0}, \tag{47}$$

and consequently

$$\lim_{t\to\infty} \tilde{e} = \lim_{t\to\infty}(e_{\mathbf{F}n} + \omega_{n-1}e_{\mathbf{F}n-1} + \cdots + \omega_0 e_{\mathbf{F}0}). \tag{48}$$

From the facts that $\lim_{t\to\infty} \|\mathbf{e_F}\| = 0$ and $\lim_{t\to\infty} e_{\mathbf{F}n} = 0$, we obtain $\lim_{t\to\infty} \tilde{e} = 0$. Moreover, because $\lim_{t\to\infty} \|\hat{\mathbf{e}}\| = 0$, it implies $\lim_{t\to\infty} e = \lim_{t\to\infty}(\hat{e} - \tilde{e}) = 0$. Thus, the control objective can be achieved by the control law (19) with the adaptive laws (39).

Remark 1. Obviously, it is difficult to apply the adaptive laws (39) to satisfy Assumption 2. Therefore, the Projection algorithm [11] is adopted to treat this problem. The adaptive laws for adjusting $\boldsymbol{\theta}_f$ and $\boldsymbol{\theta}_b$ are redesigned as

$$\dot{\boldsymbol{\theta}}_f = \begin{cases} -\gamma_f \mathbf{e}_\mathbf{F}^\mathrm{T}\mathbf{PB}\boldsymbol{\xi}_\mathbf{F} & \text{if } \{\|\boldsymbol{\theta}_f\| < m_f\} \\ & \quad \text{or } \{\|\boldsymbol{\theta}_f\| = m_f \text{ and } \mathbf{e}_\mathbf{F}^\mathrm{T}\mathbf{PB}\boldsymbol{\theta}_f^\mathrm{T}\boldsymbol{\xi}_\mathbf{F} \geqslant 0\}, \\ -\gamma_f \mathbf{e}_\mathbf{F}^\mathrm{T}\mathbf{PB}\boldsymbol{\xi}_\mathbf{F} + \gamma_f \mathbf{e}_\mathbf{F}^\mathrm{T}\mathbf{PB}\,\dfrac{\boldsymbol{\theta}_f\boldsymbol{\theta}_f^\mathrm{T}}{\boldsymbol{\theta}_f^\mathrm{T}\boldsymbol{\theta}_f}\,\boldsymbol{\xi}_\mathbf{F} & \text{if } \{\|\boldsymbol{\theta}_f\| - m_f \text{ and } \mathbf{e}_\mathbf{F}^\mathrm{T}\mathbf{PB}\boldsymbol{\theta}_f^\mathrm{T}\boldsymbol{\xi}_\mathbf{F} < 0\}, \end{cases} \tag{49a}$$

$$\dot{\boldsymbol{\theta}}_b = \begin{cases} -\gamma_b \mathbf{e}_\mathbf{F}^\mathrm{T}\mathbf{PB}\boldsymbol{\psi}_\mathbf{F} & \text{if}\{\|\boldsymbol{\theta}_b\| < m_b\} \\ & \text{or}\{\|\boldsymbol{\theta}_b\| = m_b \text{ and } \mathbf{e}_\mathbf{F}^\mathrm{T}\mathbf{PB}\boldsymbol{\theta}_b^\mathrm{T}\boldsymbol{\psi}_\mathbf{F} \geqslant 0\}, \\ -\gamma_b \mathbf{e}_\mathbf{F}^\mathrm{T}\mathbf{PB}\boldsymbol{\psi}_\mathbf{F} + \gamma_b \mathbf{e}_\mathbf{F}^\mathrm{T}\mathbf{PB}\dfrac{\boldsymbol{\theta}_b\boldsymbol{\theta}_b^\mathrm{T}}{\boldsymbol{\theta}_b^\mathrm{T}\boldsymbol{\theta}_b}\boldsymbol{\psi}_\mathbf{F} & \text{if } \{\|\boldsymbol{\theta}_b\| = m_b \text{ and } \mathbf{e}_\mathbf{F}^\mathrm{T}\mathbf{PB}\boldsymbol{\theta}_b^\mathrm{T}\boldsymbol{\psi}_\mathbf{F} < 0\}. \end{cases} \tag{49b}$$

On the basis of the above discussions, the following theorem can be obtained.

Theorem. *Consider the nonlinear dynamical system* (1) *with the control law given by* (19) *and the observer given by* (21) *to estimate the tracking error vector. Let the parameter vector* $\boldsymbol{\theta}_f$ *and* $\boldsymbol{\theta}_b$ *be adjusted by the adaptive laws* (49). *If Assumptions* 1 *and* 2 *are satisfied, then the tracking error* $e(t)$ *converges to zero as* $t \to \infty$; *i.e.*, $\lim_{t\to\infty} e(t) = 0$.

Remark 2. In practice, because the control law given by (19) contains $\mathrm{sgn}(\hat{S})$, a discontinuous term, applying (19) will cause a chattering problem. Thus, we may replace $\mathrm{sgn}(\hat{S})$ by a saturation function of the form [15]:

$$\mathrm{sat}(\hat{S}/\kappa) = \begin{cases} \mathrm{sgn}(\hat{S}/\kappa) & \text{if } |\hat{S}| \geqslant \kappa, \\ \hat{S}/\kappa & \text{if } |\hat{S}| < \kappa, \end{cases}$$

where κ is a positive constant. The control law of (19) will be modified as

$$u = \hat{b}(\hat{\mathbf{x}}|\boldsymbol{\theta}_b)^{-1}\left[-\sum_{i=1}^{n-1}\alpha_i \hat{e}^{(i)} - \hat{f}(\hat{\mathbf{x}}|\boldsymbol{\theta}_f) + y_\mathrm{d}^{(n)}\right] - \hat{\eta}\,\mathrm{sat}(\hat{S}/\kappa). \tag{50}$$

So, if $|\hat{S}| \geqslant \kappa$, the control law of (50) is equivalent to (19), which guarantees that the sliding condition is still satisfied. While $|\hat{S}| < \kappa$, the control law of (50) becomes a smooth function. This leads to tracking within a guaranteed precision κ while allowing the alleviation of the chattering phenomenon.

The overall design procedure can be summarized in the following steps:

Step 1: Construct two fuzzy systems, $\hat{f}(\hat{\mathbf{x}}|\boldsymbol{\theta}_f)$ and $\hat{b}(\hat{\mathbf{x}}|\boldsymbol{\theta}_b)$ as given in (17) and (18), respectively, to describe the input/output behavior of the unknown dynamic system. Next, solve the vector of fuzzy basis functions $\boldsymbol{\xi}(\hat{\mathbf{x}}) = [\xi^1(\hat{\mathbf{x}}), \xi^2(\hat{\mathbf{x}}), \ldots, \xi^Q(\hat{\mathbf{x}})]^\mathrm{T}$ by (13).

Step 2: Specify the observer gain vector $\mathbf{L}$ and choose a symmetric positive-definite matrix $\mathbf{Q}$ so that $\lambda_{\min}(\mathbf{Q}) > 1$. Then solve the positive-definite matrix $\mathbf{P}$ by (24). Subsequently, design an observer as given in (21) to estimate the tracking error vector.

Step 3: Choose the suitable sliding surface as given in (5) and choose the suitable $\hat{\eta}$ such that $\hat{\eta} > b_{\min}^{-1}(2F + D) + [b_{\min}^{-1} - (b_{\min}b_{\max})^{-1/2}]|\hat{u}_{\mathrm{eq}}|$. If necessary, choose $\kappa > 0$ for the controller given by (50).

Step 4: Choose the filter $\Omega(p)$ as given in (33) and a set of filters $T_i(p)(i = 0, 1, 2, \ldots, n-1)$ as given in (34). Then, solve $e_{\mathbf{F}i}$ $(i = 0, 1, 2, \ldots, n-1)$, $\boldsymbol{\xi}_\mathbf{F}$ and $\boldsymbol{\Psi}_\mathbf{F}$ by (35). Set γ_f and γ_b. Design the adaptive laws for adjusting $\boldsymbol{\theta}_f$ and $\boldsymbol{\theta}_b$ by (49). Then apply the controller as given by (19) (or by (50)) to control the nonlinear dynamic system.

Fig. 1 illustrates the architecture of the observer-based indirect adaptive fuzzy sliding mode controller.

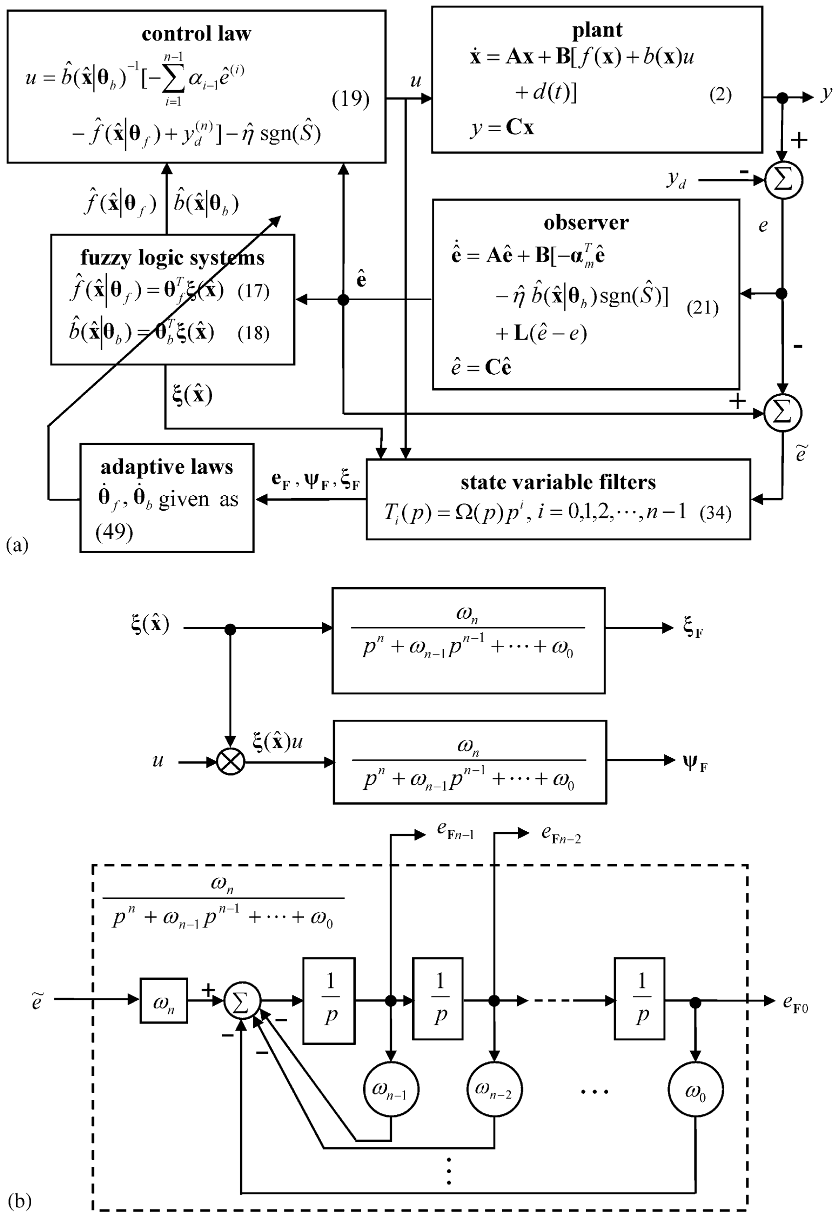

Fig. 1. The overall scheme of the proposed controller. (a) The block diagram of the overall system. (b) Internal structure of the block “state variable filters”.

4. Simulation examples

This section presents the simulation results of the proposed control strategy for an inverted pendulum system. Consider the dynamic equations of the inverted pendulum system as follows

[8,9,17,20–22]:

$$\dot{x}_1 = x_2,$$
$$\dot{x}_2 = \frac{g \sin x_1 - \cos x_1 (m_p l/(m_c + m_p)\, x_2^2 \sin x_1 - 1/(m_c + m_p)\, u)}{l(4/3 - (m_p \cos^2 x_1)/(m_c + m_p))} + d(t),$$
$$y = x_1, \tag{51}$$

where g is the acceleration due to gravity ($g = 9.8\,\mathrm{m/s^2}$), m_c is the mass of the cart, m_p is the mass of the pole, l is the half length of the pole, x_1 is the angular position of the pole, x_2 is the angular velocity of the pole, y is the system output, u is the applied force (the control signal), and $d(t)$ is the external disturbance. In this simulation, we let $m_c = 1\,\mathrm{kg}$, $m_p = 0.1\,\mathrm{kg}$, $l = 0.5\,\mathrm{m}$, and the sampling period be 0.001 s.

If we require that $|x_1| \leqslant \pi/6$ and $|x_2| \leqslant \pi/6$, then the bounds $b_{\max}$ and $b_{\min}$ can be calculated as $|b(x_1, x_2)| \leqslant 1.46 = b_{\max}$ and $|b(x_1, x_2)| \geqslant 1.12 = b_{\min}$. Also, since $|f(x_1, x_2)| \leqslant 15.78 + 0.0366 x_2^2$, we can set $F = 16$.

The following cases are simulated:

Case 1: The desired trajectory $y_d = 0$, the initial values $\mathbf{x}(0) = [0.2, 0.3]^T$ and $\hat{\mathbf{e}}(0) = [-0.2, -0.1]^T$, and $d(t)$ is an independently random noise uniformly distributed in the interval $[-0.1, 0.1]$.

Case 2: The desired trajectory $y_d = \pi \sin(t)/30$, the initial values $\mathbf{x}(0) = [-0.15, -0.15]^T$ and $\hat{\mathbf{e}}(0) = [0.15, 0.15]^T$, and $d(t)$ is an independently random noise uniformly distributed in the interval $[-0.1, 0.1]$.

According to the design procedure, the controller can be designed in the following steps:

Step 1: To construct two fuzzy logic systems, $\hat{f}(\hat{\mathbf{x}}|\boldsymbol{\theta}_f)$ and $\hat{b}(\hat{\mathbf{x}}|\boldsymbol{\theta}_b)$ as given in (17) and (18), respectively, we select the membership functions for $\hat{x}_i$ $(i = 1, 2)$ from the following fuzzy sets:

$$\exp\{-[(\hat{x}_i + \pi/6)/(\pi/24)]^2\}, \quad \exp\{-[(\hat{x}_i + \pi/12)/(\pi/24)]^2\}, \quad \exp\{-[\hat{x}_i/(\pi/24)]^2\},$$
$$\exp\{-[(\hat{x}_i - \pi/12)/(\pi/24)]^2\},$$

and

$$\exp\{-[(\hat{x}_i - (\pi/6)/(\pi/24)]^2\}.$$

Therefore, to cover the whole case, we apply 25 rules for each of the $\hat{f}(\hat{\mathbf{x}}|\boldsymbol{\theta}_f)$ and $\hat{b}(\hat{\mathbf{x}}|\boldsymbol{\theta}_b)$. We choose the initial $\boldsymbol{\theta}_f(0)$ and $\boldsymbol{\theta}_b(0)$ randomly in the intervals $[-5, 5]$ and $[1.12, 1.46]$, respectively. Next, solve the vector of fuzzy basis functions $\boldsymbol{\xi}(\hat{\mathbf{x}}) = [\xi^1(\hat{\mathbf{x}}), \xi^2(\hat{\mathbf{x}}), \ldots, \xi^Q(\hat{\mathbf{x}})]^T$ by (13).

Step 2: Select the observer gain vector $\mathbf{L} = [-200, -600]^T$ and choose

$$\mathbf{Q} = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix},$$

which satisfies $\lambda_{\min}(\mathbf{Q}) > 1$. After solving (24), we obtain

$$\mathbf{P} = \begin{bmatrix} 0.67 & -2 \\ -2 & 6.0034 \end{bmatrix}.$$

We then design an observer as given in (21) to estimate the tracking error vector.

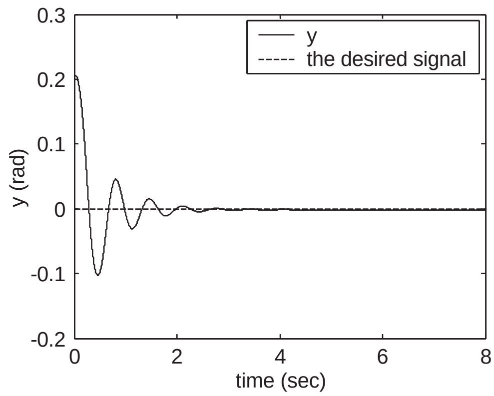

Fig. 2. The trajectories of y and y_{d} for Case 1.

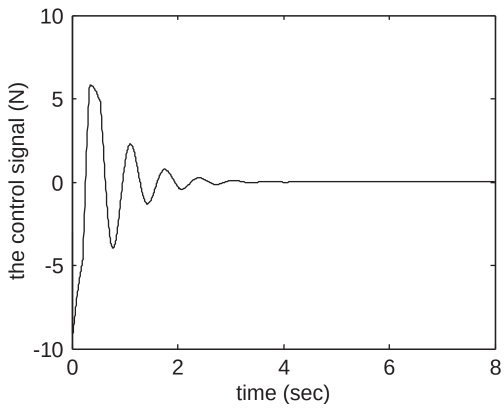

Fig. 3. Trajectory of the control signal for Case 1.

Step 3: Choose the sliding surface as $S(\mathbf{e}) = \boldsymbol{\alpha}^{\mathrm{T}}\mathbf{e} = 0$, where $\mathbf{e} = [e, \dot{e}]^{\mathrm{T}}$, $\boldsymbol{\alpha} = [5, 1]^{\mathrm{T}}$. Select $\hat{\eta} = b_{\min}^{-1}(2F + D) + [b_{\min}^{-1} - (b_{\min}b_{\max})^{-1/2}]|\hat{u}_{\mathrm{eq}}|$ and $\kappa = 0.5$ for Case 1 and $\kappa = 0.1$ for Case 2.

Step 4: Choose the filter $\Omega(p) = 50/(p^2 + 20p + 100)$ and a set of filters T_i $(i = 0, 1)$ as given in (34). Solve $e_{\mathbf{F}i}$ $(i = 0, 1)$, $\boldsymbol{\xi}_{\mathbf{F}}$ and $\boldsymbol{\psi}_{\mathbf{F}}$ by (35). Set $\gamma_f = 100$ and $\gamma_b = 5$, and adjust $\boldsymbol{\theta}_f$ and $\boldsymbol{\theta}_b$ by the adaptive laws (49). Next, apply the controller as given by (50) to control the nonlinear dynamic system.

For Case 1, the simulation results are shown in Figs. 2–4. Fig. 2 shows the trajectories of y (solid line) and of y_{d} (dashed line). Fig. 3 shows the control signal. Fig. 4 shows the trajectory of the estimated error $\hat{e}$ (solid line) and the trajectory of the actual tracking error e (dotted line). From these simulation results, we see that the estimated error can asymptotically track the actual error, and the system output can asymptotically track the desired output. That is, the proposed controller can attain the control objective and is robust against the external noise.

In Case 2, to compare the control performance, we also control the inverted pendulum system with the same parameters given by the direct adaptive fuzzy-neural control with the state observer (SO-DAFC) presented in [20] and the observer-based indirect adaptive fuzzy control (O-IAFC) presented in [10]. The simulation results are shown in Figs. 5–10. Figs. 5, 6 and 7 show the trajectories of the system output y with the proposed controller, SO-DAFC and O-IAFC, respectively. Figs. 8, 9 and 10 show the control

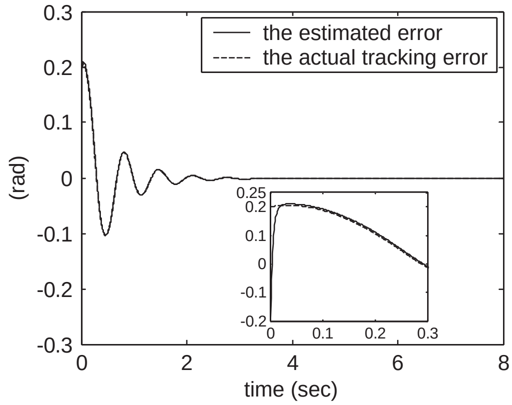

Fig. 4. The trajectories of $\hat{e}$ and e for Case 1.

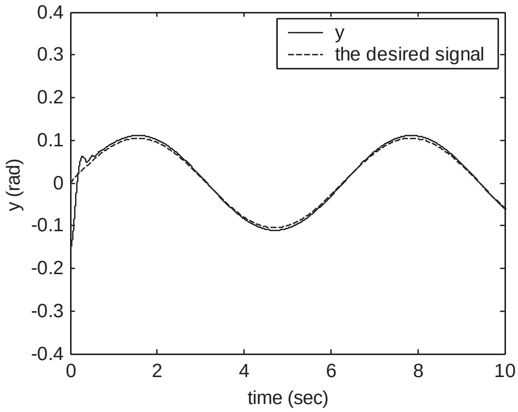

Fig. 5. The trajectory of y with the proposed controller for Case 2.

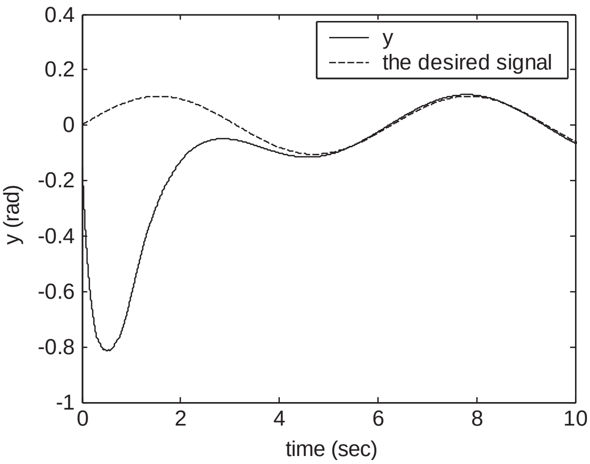

Fig. 6. The trajectory of y with SO-DAFC [20] for Case 2.

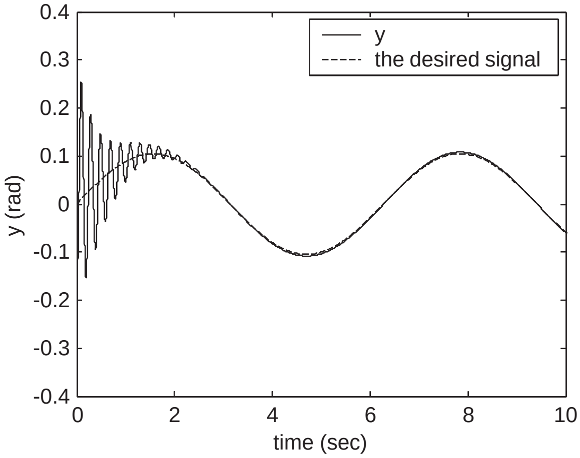

Fig. 7. The trajectory of y with O-IAFC [10] for Case 2.

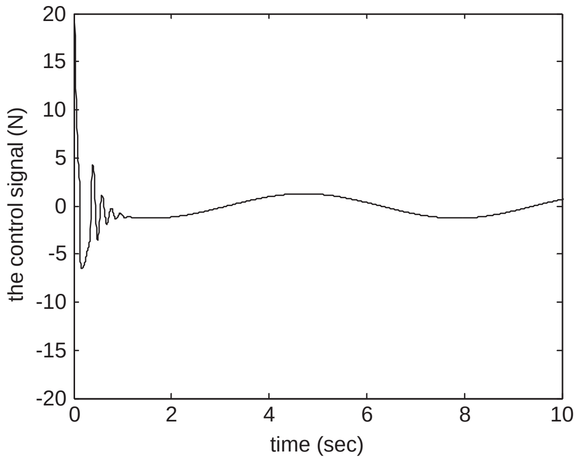

Fig. 8. The trajectory of the control signal with the proposed controller for Case 2.

signals with the proposed controller, SO-DAFC and O-IAFC, respectively. Comparing these simulation results, we see that the proposed controller can use the smallest magnitude of the control signal, taking the shortest time to track the desired output y_d to achieve the control objective. It implies that the proposed controller can achieve the better performance with a smaller control signal compare to that of SO-DAFC and O-IAFC.

5. Conclusions

In this paper, we have proposed a method for designing an observer-based indirect adaptive fuzzy sliding mode controller with state variable filters for the control of a certain class of unknown nonlinear

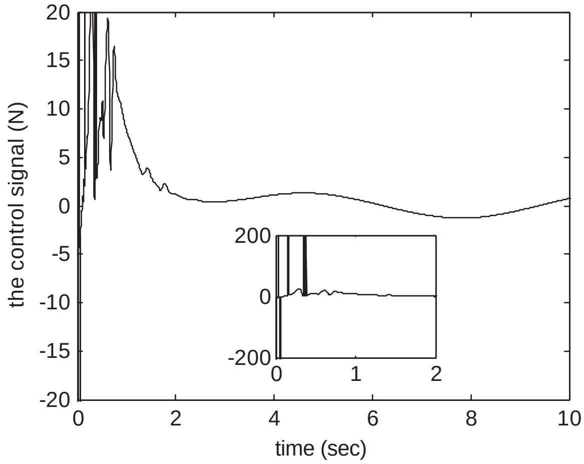

Fig. 9. The trajectory of the control signal with SO-DAFC [20] for Case 2.

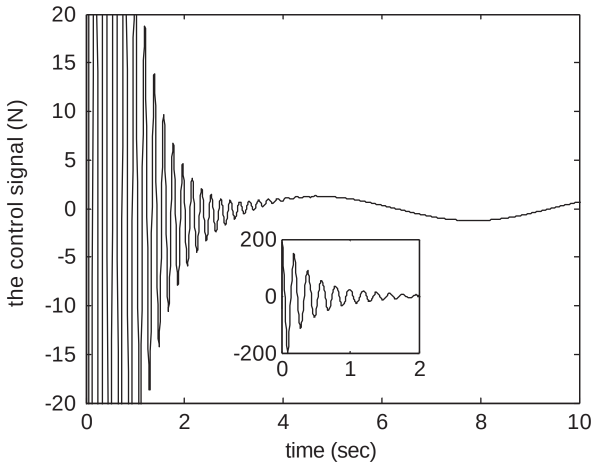

Fig. 10. The trajectory of the control signal with O-IAFC [10] for Case 2.

dynamic systems, in which not all the states are available for measurement. We first construct the fuzzy models to describe the input/output behavior of the nonlinear dynamic system. Then, an observer is employed to estimate the tracking error vector. Based on the observer, a fuzzy sliding model controller is developed to achieve the tracking performance. By passing the observation error vector to a set of state variable filters, a filtered observation error vector is obtained. The free parameters of the fuzzy models can be adjusted by the adaptive laws, based on the filtered observation error vector and the Lyapunov synthesis method. With the proposed control strategy, the stability of the overall control system can be guaranteed. The simulation results show that the proposed control strategy can turn in a good tracking performance and is robust against the external noise.

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