# Author's Accepted Manuscript


www.elsevier.com/locate/omega

Coordinating Operational Policy with Financial Hedging for Risk-averse Firms

J. Ni, L.K. Chu, B.P.C. Yen

PII: S0305-0483(15)00143-7
DOI: http://dx.doi.org/10.1016/j.omega.2015.07.001
Reference: OME1567

To appear in: *Omega*

Received date: 17 January 2014
Revised date: 3 July 2015
Accepted date: 3 July 2015

Cite this article as: J. Ni, L.K. Chu, B.P.C. Yen, Coordinating Operational Policy with Financial Hedging for Risk-averse Firms, *Omega,* http://dx.doi.org/10.1016/j.omega.2015.07.001

# Coordinating Operational Policy with Financial Hedging for Risk-averse Firms

J. Ni [a], L. K. Chu [b,*], B. P.C. Yen [c]

[a] School of Finance, Southwestern University of Finance and Economics, China
[b] Department of Industrial and Manufacturing Systems Engineering, The University of Hong Kong, Hong Kong SAR
[c] School of Business, The University of Hong Kong, Hong Kong SAR

**Abstract:** A risk-averse firm's financial hedging activity can impact the decision making in its daily operations. We introduce a CE-based approach that can help the firm to simplify the procedure in making hedging-consistent decisions. A key feature of this new approach is that it allows for the existence of nonfinancial random factors, which give rise to the risk exposure that cannot be hedged in the financial market. By using a CE operator, we show that the optimal operational policy can be obtained by maximizing the CE-based value function. Although the CE operator may bring additional nonlinearity to the value function, we find that the commonly desired base-stock policy can remain optimal under specific conditions. We hope that this new approach can help pave the way for future investigation on joint operations management and financial hedging problems in dynamic settings.

**Keywords:** operations management; financial hedging; exponential utility; risk aversion



## 1. Introduction

When making procurement, inventory, and production decisions, firms are usually exposed to uncertainties such as volatile commodity price, fluctuating foreign exchange rates, as well as uncertain customer demands. Such risk exposures are undesirable for risk-averse firms, but they could be controlled by financial hedging, typically using available hedging instruments like commodity futures, options, and currency swaps from the financial market. As reported by a

[*] Corresponding author. Tel.: +852 28592590; fax: +852 28586535.

E-mail address: lkchu@hkucc.hku.hk (Dr. L.K. Chu).

recent empirical study (Bartran et al. 2009) on 7,319 nonfinancial firms across 50 countries, over half (60.3%) of the surveyed firms have implemented some form of hedging using financial derivatives. These hedging activities are found to have some significant implications on the operational decisions made in firms' daily operations (see, e.g., Ding et al. 2007, Chod et al. 2010). In particular, financial hedging can reduce at least part of the risk exposure faced by a risk-averse firm. Such reductions in risk exposure, according to Eeckhoudt et al. (1995), will then lead to an increase or decrease in the optimal purchasing quantity of the firm. As a result, there is a need to investigate how to make the optimal operational decisions that are consistent with a risk-averse firm's financial hedging activities.

Our interest in studying the aforementioned hedging-consistent operational decisions was motivated by the increasing availability and widely use of financial hedging instruments nowadays. Nevertheless, how to quantify the economic implications of financial hedging on operational decisions remains a challenging problem despite the growing academic and research effort. To simplify the analysis, some researchers have resorted to the complete market assumption; that is, assuming that the risk exposure involved in the firm's operations can be fully replicated by a "perfect" financial hedging portfolio in the market (Van Mieghem 2003). Given the existence of the replicating portfolio, the well-known risk-neutral valuation method in the finance literature can then be transplanted to "value" the operational decisions (Birge 2000). Consequently, the hedging-consistent operational decisions can be made via maximizing the expected value of the profit with the risk-neutral probability measure (Goel and Gutierrez 2011). Thus, this approach is referred to as the *EV-based* approach (expected-value-based approach) in this paper. The EV-based approach is appealing because it can help substantially reduce the number of decision variables when financial hedging is involved – the decision variables regarding the hedging positions are entirely eliminated from the Bellman equation. However, there is a major obstacle when applying this approach in practice – the complete market assumption may not be entirely justified. As Birge (2000, pp. 22-23) writes, "an investor might only be able to remove part of the market risk and then have some uncontrollable portion that still remain. This remainder would cause a limit to the extent that a market can value our decision." To appropriately account for the "remainder risk", the complete market assumption must be relaxed.

By relaxing the complete market assumption, we develop a *CE-based* approach (the certainty-equivalent-based approach) for risk-averse firms to make the hedging-consistent operational decisions in dynamic settings. The CE-based approach is a novel extension of the EV-based approach because it allows for the existence of *nonfinancial random factors* in addition to *financial random factors*. For nonfinancial firms, the distinction between the financial random factors and nonfinancial random factors is the key to differentiate the financial risk that can be hedged using derivatives from the remainder risk that cannot. On the one hand, financial random factors refer to the risk factors associated with the price processes of some financial securities/indices, such as the fluctuating commodity price and volatile currency rates. On the other hand, nonfinancial random factors represent the idiosyncratic disturbances (e.g., uncertain customer demand, random production yield) that are unrelated to the financial market. Both types of random factors can disturb a firm's operating profit in significant ways. For example, the operating profit of a multinational firm is exposed to both the volatile currency rates and uncertain global demand. The currency risk can usually be hedged using currency derivatives (Ding et al. 2007), so it should be recognized as a financial random factor. In contrast, the demand uncertainty is the remainder risk that cannot be hedged in the financial market, and thus should be treated as a nonfinancial random factor. When the nonfinancial random factor exists, the complete market assumption cannot apply, and the EV-based approach is no longer optimal. In this situation, the proposed CE-based approach can still be applied to simplify the procedure of making hedging-consistent operational decisions. The advantage is that the CE-based approach helps reduce the number of decision variables as the EV-based approach does. Moreover, we also investigate some structural properties of the CE-based value function, which allows us to prove that the commonly desired base-stock policy is optimal under a set of sufficient conditions. In addition, we present some straightforward numerical results to show that the CE-based approach dominates the EV-based approach in most of the cases.



This paper is closely related to the growing research on the interface of operations management and finance. In recent years, it is found that there exists complex interplay between the operational and financial hedging decisions. As shown by Chod et al. (2010), financial hedge and operational flexibility can be either complementary or substitutable under different situations. Another interpretation of this complicated relationship is that operational decisions can be significantly affected by financial hedging in different ways. As a consequence, many researchers have investigated how to make the hedging-consistent operational decisions for risk-averse firms. Gaur and Seshadri (2005) have shown that a risk-averse newsvendor should increase its order quantity when financial hedging is adopted to mitigate the demand risk. In another contribution, Ding et al. (2007) have investigated the implications of a global manufacturer's financial hedging activity against currency risk on its operational decisions. For other examples regarding the impacts of financial hedging on operational policies, see Caldentey and Haugh (2006), Caldentey and Haugh (2009), and references therein. However, these papers typically assume a newsvendor setting in their models (i.e., single-period problems), which may have limitations in practice. One exception is Kouvelis et al. (2012), who have analyzed the role of financial hedging in a multi-period commodity procurement and storage problem using a mean-variance utility criterion. In contrast, we present a new approach that can pave the way for investigating a class of multi-period joint operational and financial hedging problems by using the exponential utility criterion.

This research is also related to the real option literature. This stream of research concerns the valuation of real options embedded in risky projects such as R&D projects (Santiago and Vakili 2005), supply chain network design (Huchzermeier and Cohen 1996), and capacity investments (Birge 2000). A central assumption in the real options theory, as analogues to ours, is that the cash flows from the real assets are correlated with the stochastic price processes of some traded securities or indices in the financial market. Then, the financial option pricing method is applied to value these risky projects due to the non-arbitrage argument or the existence of replicating portfolios (Duffie 2001). For more examples on the application of real option theory, see Dentskevich and Salkin (1991), Copeland and Antikarov (2001), Berling (2008). Our paper differs from this literature in that we do not focus on valuation. Generally speaking, valuation of real options can be performed without virtually trading financial securities in the market. In contrast, we aim at quantifying the economic implications of financial hedging on operational policies when a risk-averse firm can trade securities in an accessible financial market to construct the desired hedging portfolio.

The remainder of this paper proceeds as follows: In section 2, we introduce a general modeling framework for both the operations management and financial hedging. An illustrative example is provided to show its applicability. In section 3, the exact procedure of the CE-based approach is introduced and discussed in detail. In section 4, we numerically compare the CE-based and EV-based approaches. In section 5, we summarize the main results and present some concluding remarks. Finally, in the Online Appendix we present (i) all the missing proofs, (ii) an introduction of the EV-based approach, and (iii) a discussion on how to identify a pair of independent financial and nonfinancial random factors by transformation.

## 2. Operational Decisions and Financial Hedging - A General Modeling Framework

The planning horizon under consideration is $[0,T]$. Without loss of generality, $T$ is assumed to be a positive integer. Then, the entire planning horizon is separated evenly into $T$ periods, each of which contains a unit time and is indexed as $k = 0,1,...,T-1$. For period $k$, the associated time interval is $[k,k+1]$, whereby the index $k$ also represents the exact time instant at the beginning of that period.

### 2.1. Operational Decision Making

For each period $k$ ($0 \le k \le T-1$), the operations of the firm are characterized by the state vector $\boldsymbol{y}_k$, the operational decision (vector) $\boldsymbol{x}_k$ and the resulting operating profit $\tilde{R}_k(\boldsymbol{x}_k, \boldsymbol{y}_k)$. First, the state $\boldsymbol{y}_k$ is realized at the *beginning* of that period, and updated periodically by a transition function:

$$\boldsymbol{y}_{k+1} = \boldsymbol{y}_{k+1}(\boldsymbol{x}_k, \boldsymbol{y}_k) \tag{1}$$

Secondly, the decision $\boldsymbol{x}_k$ is made at the *beginning* of the period once the realized state vector $\boldsymbol{y}_k$ is observed (For simplicity, we assume that the decision $\boldsymbol{x}_k$ can take value in a feasible region that will not change with $k$, i.e., $\boldsymbol{x}_k \in \Omega_X$). Thirdly, $\tilde{R}_k(\boldsymbol{x}_k, \boldsymbol{y}_k)$ is realized at the *end* of period $k$, given that the state $\boldsymbol{y}_k$ has been realized and the decision $\boldsymbol{x}_k$ has been made.

The operations of the firm at period $k$ would be disturbed by both financial and nonfinancial random factors, denoted by $\omega_{k+1}$ and $\xi_{k+1}$, respectively. Note that the subscript $k+1$ here indicates that these factors are realized at the end of period $k$, or equivalently, the beginning of next period $k+1$; so they should still be regarded as random variables at the beginning of period $k$. The financial and nonfinancial random factors are distinguished from each other, in a sense that the financial random factors are associated with the random price movement in the financial market, while nonfinancial random factors are not. Intuitively, one can expect that financial hedge can only cover the risk exposure arising from the financial random factors, so the "remaining" exposure after financial hedging should arise from nonfinancial random factors. Thus, it is assumed that the financial and nonfinancial random factors are independent with each other.

**Remark 1.** There are two reasons to justify the above *independence assumption* between the financial and nonfinancial random factors. First, this assumption can avoid unnecessary complications in subsequent analysis and simplify the presentation of our main results. Second, while the assumption may appear a bit restrictive, in fact it is not. This is because the validity of the assumption relies on the choice of the financial and nonfinancial random factors used in the modeling. In many practical cases, one can "circumvent" the issue of possible dependence between financial and nonfinancial random factors by appropriately transforming the random factors and converting the profit and transition functions, as will be shown in section 3.2.

We can now formalize the above idea that the operations of the firm would be disturbed by both financial and nonfinancial random factors. First, at period $k$, the operating profit $\tilde{R}_k(\boldsymbol{x}_k, \boldsymbol{y}_k)$ as well as the transition function $\boldsymbol{y}_{k+1}(\boldsymbol{x}_k, \boldsymbol{y}_k)$ should rely on the financial and nonfinancial random factors $\boldsymbol{\omega}_{k+1}$ and $\xi_{k+1}$, i.e., $\tilde{R}_k = \tilde{R}_k(\boldsymbol{x}_k, \boldsymbol{y}_k; \boldsymbol{\omega}_{k+1}, \xi_{k+1})$ and $\boldsymbol{y}_{k+1} = \boldsymbol{y}_{k+1}(\boldsymbol{x}_k, \boldsymbol{y}_k; \omega_{k+1}, \xi_{k+1})$. Moreover, it is also possible that the operational result of the firm at period $k$ can rely on the financial and nonfinancial random factors ($\omega_t$ and $\xi_t$) for the previous periods ($0 \le t \le k$). As a result, the general functional form of the operating profit and the transition function should be written as $\tilde{R}_k = \tilde{R}_k(\boldsymbol{x}_k, \boldsymbol{y}_k; \{\omega_t\}_{0\le t\le k+1}, \{\xi_t\}_{0\le t\le k+1})$ and $\boldsymbol{y}_{k+1} = \boldsymbol{y}_{k+1}(\boldsymbol{x}_k, \boldsymbol{y}_k; \{\omega_t\}_{0\le t\le k+1}, \{\xi_t\}_{0\le t\le k+1})$. However, for notational ease, we will suppress the dependence on all these factors ($\{\omega_t\}_{0\le t\le k+1}$ and $\{\xi_t\}_{0\le t\le k+1}$) and write $\tilde{R}_k(\boldsymbol{x}_k, \boldsymbol{y}_k)$ and $\boldsymbol{y}_{k+1}(\boldsymbol{x}_k, \boldsymbol{y}_k)$ for convenience.

To show the applicability of the above modeling framework, an illustrative example is provided below. We will use this example throughout this paper to illustrate how to apply our theoretical results.

**An illustrative example: Commodity procurement and storage under financial hedging**

Commodity procurement is one of the key business activities for many manufacturers who use the commodity as production inputs. Due to the high volatility in the commodity price nowadays, manufacturers are overhauling their ways to manage the commodity supply, especially in terms of buying and hedging smartly in the commodity market (Wiggins and Blas 2008). In this context, some of the notations above are reinterpreted to provide a better description of the framework.

Consider a manufacturer who needs to procure a storable commodity from the spot market to satisfy the commodity demand in the production of the end-products. At the beginning of each period $k$, the manufacturer should determine how to adjust its commodity inventory with the commodity spot market. If the inventory level is too low, the manufacturer may replenish the inventory by purchasing from the commodity spot market. In rare cases that the inventory level is too high, the manufacturer also has the freedom to sell an amount of its stored commodity to the spot market if it is profitable to do so. Then, the adjusted commodity inventory can be used to satisfy the uncertain commodity demand $\tilde{D}_k$ at that period. Any unsatisfied demand is backlogged and any excessive inventory of the commodity is carried over to the next period. Excessive inventory for the ending period, if any, is sold to the spot market. Let $y_k$ (state variable) be the initial inventory level at period $k$ while $x_k$ (decision variable) the adjusted inventory level immediately after buying from or selling to the spot market at that period. Then, the procurement quantity is $x_k - y_k$, and the transition function of the state variable can be formulated:

$$y_{k+1} = x_k - \tilde{D}_k \tag{2}$$

Let $h$, $q$, and $r$ be the unit inventory holding cost, the backlogging cost, and the unit sales revenue of the commodity, respectively. We can write out the operating profit for period $k$:

$$\tilde{R}_k(x_k, y_k) = r\min(x_k, \tilde{D}_k) - h\left|x_k - \tilde{D}_k\right|^+ - q\left|\tilde{D}_k - x_k\right|^+ - S_k \cdot (x_k - y_k) \tag{3}$$

where $|z|^+ = \max(0, z)$, and $S_k$ is the commodity spot price.

The manufacturer is risk averse, and it seeks to mitigate its risk exposure by hedging against the volatile commodity price using commodity derivatives (futures and options), as in Kouvelis et al. (2012). In this setting, one may intuitively recognize the random variations in the commodity price as a financial random factor ($S_k$ as $\omega_k$), while the uncertain demand as a nonfinancial random factor ($\tilde{D}_k$ as $\xi_{k+1}$). See section 3.2 for a detailed discussion.

Apart from the multi-period inventory problems, the above modeling framework may also be applied to describe many other operational activities, such as production planning (Ding et al. 2007), product sourcing (Caldentey and Haugh 2009), and materials distribution (Goel and Gutierrez 2011). Moreover, it is also possible to extend our model by introducing some capacity constraints on the decision vector $\boldsymbol{x}_k$ (Birge 2000).

## 2.2. Financial Hedging Portfolio

As is standard in the finance literature, given a probability space $(\Omega, \mathcal{F}, Q)$ that describes the possible states of the financial market, the financial market is characterized by the price vector $\boldsymbol{X}_t = \boldsymbol{X}_t(\omega_t)$ of a set of securities at any time $t$. Here, we use the term "security" to denote all the relevant financial hedging instruments (e.g., commodity futures and options) available in the financial market. Let $\{\mathcal{F}_t\}_{0\le t\le T}$ be the natural filtration generated by the stochastic process $\boldsymbol{X}_t$. Without loss of generality, the probability measure $Q$ is the *risk-neutral probability measure*, which amounts to requiring that $\boldsymbol{X}_t$ is a $Q$-martingale, i.e., $E^Q\left[\boldsymbol{X}_t\right] = \boldsymbol{X}_0$ (Duffie 2001). For the moment, the risk-free interest rate is assumed to be 0, and this assumption will be relaxed in section 3.3.

To avoid unnecessary complexity and without loss of generality, the "building blocks" that we use to construct the hedging portfolio are the attainable contingent claims. A contingent claim

$G$ is said to be attainable if and only if there is a predictable self-financing strategy $\boldsymbol{\theta}$ such that $G(\boldsymbol{\theta}) = G$, where the gain process $G(\boldsymbol{\theta})$ is defined as follows:

$$G(\boldsymbol{\theta}) = \int_0^T \boldsymbol{\theta}(t) \cdot d\boldsymbol{X}_t$$

See Duffie (2001) for a detailed technical discussion of the attainable contingent claims as well as self-financing trading strategy. Further, following Caldentey and Haugh (2009), we assume that the financial market itself is complete in that any $\mathcal{F}_T$-measurable contingent claims are attainable. Such a *complete financial market* assumption relaxes the aforementioned complete market assumption – it does not exclude the existence of the nonfinancial random factors that are irrelevant to the financial market. The completeness of the financial market is also equivalent to the uniqueness of the risk-neutral probability measure $Q$ (Duffie 2001). Given the complete financial market assumption, any $\mathcal{F}_T$-measurable contingent claims must be attainable, thus allowing us to avoid the tremendous complexity of solving for the self-financing trading strategies for the financial hedging. For more discussion of the attainable contingent claim, self-financing trading strategy, and complete financial market, see Harrison and Kreps (1979), and Duffie (2001); for a detailed justification of the complete financial market assumption, see Caldentey and Haugh (2009).



### 2.3. The Joint Operational and Financial Hedging Decisions

The risk-averse firm, seeking to maximize the expected utility of terminal wealth, must choose the best operational policy in accordance with financial hedging. During the planning horizon, the operational decisions made in all periods are summarized by the operational policy $\alpha$ as follows:

$$\alpha = \{\boldsymbol{x}_k\}_{k=0}^{T-1} \tag{4}$$

Given $\alpha$, the overall operating profit $\Pi(\alpha)$ is

$$\tilde{\Pi}(\alpha) = \sum_{k=0}^{T-1} \tilde{R}_k(\boldsymbol{x}_k, \boldsymbol{y}_k) \tag{5}$$

As to financial hedging, let $G_h = G(\boldsymbol{\theta}_h)$ be the financial hedging portfolio to be constructed, where $\boldsymbol{\theta}_h$ is a self-financing strategy. To describe the firm's risk aversion in decision making, an exponential utility function is employed:

$$u(z) = -\exp(-\gamma z)$$

where the parameter $\gamma$ denotes the absolute risk aversion. A larger $\gamma$ represents a higher degree of risk aversion and vice versa.

Given the above notation, we can formulate the firm's decision-making problem as the following joint operational and financial hedging model (JOFM):

$$\max_{\alpha,\boldsymbol{\theta}_h} E\left[u_\gamma\left(\tilde{\Pi}(\alpha) + G(\boldsymbol{\theta}_h)\right)\right] \tag{6}$$

which is subject to constraints (1), (4) and (5).

Before proceeding to analyze the proposed model JOFM, three clarifying remarks are in order:

**Remark 2.** The use of the utility function in the JOFM belongs to the broad class of the so-called "interperiod" utility functions, which is described and axiomatized by Sobel (2006). It is consistent with the empirical result that firms should control the volatility of the aggregated random profits of a certain planning horizon (e.g., the overall profit of a fiscal year, see Graham and Smith 1999). Similar interperiod-utility formulation is also applied in Ding et al. (2007) and Kouvelis et al. (2012).

**Remark 3.** There is a caveat on the time consistency and applicability of the proposed model JOFM. As suggested by Sobel (2006), the interperiod utility is especially suitable to describe a firm's risk-averse behavior over a short- or medium-term horizon (e.g., one year). From the finance literature, a firm's risk aversion in decision making is mainly induced by several key determinants, including the progressivity in tax rates, the financial distress costs, and the agency costs; see Smith and Stulz (1985), and Graham and Smith (1999) for a detailed discussion. All these determinants are unlikely to change in a short run (e.g., the managerial compensation package that can incur the agency costs may change in 10 years, but it is unlikely that such a change could happen in a year). However, in the long run (e.g., 10 years), a firm might gradually become less risk averse as its financial hedging activity substantially lowers the risk exposure borne by the firm. Modeling the possible changes in a firm's risk attitude in the long run is beyond the scope of this paper, and it would be an interesting direction for future research.

**Remark 4.** Besides, we would like to explain why it is necessary to employ the exponential utility function, rather than other forms of utility function, to describe risk-averse behaviors in the proposed model. The reasons are two-folds. First, the exponential utility has certain appealing decision-theoretic properties – strictly increasing and concave, constant absolute risk-averse, etc. (Pratt 1964) – in describing risk-averse behaviors; so it is widely used in the literature, see Bouakiz and Sobel (1992), Eeckhoudt et al. (1995), and Chod et al. (2010). Second, the use of the exponential utility can avoid possible speculative/gambling activities in the name of "financial hedging" when making the multi-period joint financial and operational decisions. According to Henderson and Hobson (2013), for any increasing and strictly concave utility functions other than exponential, a supposedly risk-averse decision maker may exhibit counter-intuitive behaviors that are actually locally risk-seeking. Such unexpected behaviors, as explained by Henderson and Hobson, can be attributed to the local convexity in the wealth level of the decision maker's utility value at the point of indifference, as the local convexity will lead to the preference to a "fair gamble" in the financial market. Indeed, such a situation could arise if the operational decisions contain implicit timing options to defer the realization of part of operating profits/costs (e.g. deferring the procurement/production/sales of a particular item). It is this counter-intuitive risk-seeking activity that contradicts the purpose of financial hedging.

Finally, we emphasize that our main purpose is to develop a "short-cut" for operations managers to obtain the optimal hedging-consistent operational policy $\alpha^*$ in the JOFM. As a consequence, we have to limit the investigation on financial hedging. Specifically, we will only solve for the optimal financial hedging portfolio in the form of attainable contingent claims $G_h = G(\boldsymbol{\theta}_h)$, without exploring the exact form of $\boldsymbol{\theta}_h$. In fact, there would be some duplication in the broad variety of financial derivatives available in the market, e.g., a futures contract can always be replicated by two option contracts. Considering the potential duplication in constructing the hedging portfolio is beyond the scope of this paper. Therefore, in what follows, the optimal financial hedging strategy is solved only in terms of attainable contingent claims.

## 3. Coordinating Operational Decisions with Financial Hedging Activities

We firstly present the exact procedure of the CE-based approach for determining the hedging-consistent operational decisions. We then present some analytical results on the structural properties of the CE-based value function. A set of sufficient conditions are established for the base-stock-type operational policy to be optimal.

To simplify the following discussion, let us introduce some notational rules regarding the expectation operator. Let $\tilde{z}$ be a random variable dependent on both financial and nonfinancial random factors. We use $E_k\left[\tilde{z}\right]$ to denote the expected value of $\tilde{z}$ calculated at the beginning of period $k$ ($0 \le k \le T-1$), that is,

$$E_k\left[\cdot\right] = E_{\{\omega_t\}_{t=k+1}^T}\left[E_{\{\xi_t\}_{t=k+1}^T}\left[\cdot\right]\right]$$

In this way, we have $E\left[\tilde{z}\right] = E_0\left[\tilde{z}\right]$. Similarly, we use $E_k^Q\left[\cdot\right]$ to denote the expectation with respect to the risk-neutral probability measure $Q$, that is,

$$E_k^Q\left[\cdot\right] = E^Q_{\{\omega_t\}_{t=k+1}^T}\left[E_{\{\xi_t\}_{t=k+1}^T}\left[\cdot\right]\right]$$

### 3.2. The Certainty-Equivalent-Based Approach

In this section, a new approach (the CE-based approach) is established to simplify the procedure to obtain the hedging-consistent operational policy. First of all, we need to introduce a CE operator. For an arbitrary random payoff $\tilde{z}$ whose value relies on the realized value of a random variable $\eta$, the CE operator $CE_\eta\left[\cdot\right]$ is defined as follows:

$$CE_\eta\left[\tilde{z}\right] = \frac{-1}{\gamma}\log\left\{E_\eta\left[\exp\left(-\gamma\cdot\tilde{z}\right)\right]\right\} \tag{7}$$

The value of certainty equivalent $CE_\eta\left[\tilde{z}\right]$ has a deep economical meaning: it is the guaranteed amount of money that the decision maker would regard as equally desirable as the random payoff $\tilde{z}$ (see, e.g., Smith and Nau 1995). This operator is applied here to help quantify the economic impacts of nonfinancial random factors on the hedging-consistent operational policy.

As in Pratt and Zeckhauser (1987), we can use the CE operator to quantify the cash equivalent of the uncertain profit streams earned by the firm given the financial hedging. Formally, for each period $k$ ($0 \le k \le T-1$), we define the CE-based value function $V_k(\boldsymbol{x}_k, \boldsymbol{y}_k)$ – the cash equivalent of the uncertain operating profits from period $k$ onward – in an iterative manner as follows (Recall that $\{\xi_k\}$ denotes the nonfinancial random factors):

$$V_k(\boldsymbol{x}_k, \boldsymbol{y}_k) = E_k^Q\left[CE_{\xi_{k+1}}\left[V_{k+1}\left(\boldsymbol{x}_{k+1}^*(\boldsymbol{x}_k, \boldsymbol{y}_k), \boldsymbol{y}_{k+1}(\boldsymbol{x}_k, \boldsymbol{y}_k)\right) + \tilde{R}_k(\boldsymbol{x}_k, \boldsymbol{y}_k)\right]\right]$$

where $V_T = 0$, and $\boldsymbol{x}_{k+1}^*(\boldsymbol{x}_k, \boldsymbol{y}_k)$ is the optimal operational decision made at the beginning of period $k+1$. As will be shown, $\boldsymbol{x}_{k+1}^*(\boldsymbol{x}_k, \boldsymbol{y}_k)$ maximizes the CE-based value function at each period.

With the CE-based value function, we can dynamically identify the risk exposure to be hedged in the financial market at each period, that is, the equivalent financial risk exposure $J_k(\boldsymbol{x}_k, \boldsymbol{y}_k)$:

$$J_k(\boldsymbol{x}_k, \boldsymbol{y}_k) = CE_{\xi_{k+1}}\left(V_{k+1}\left(\boldsymbol{x}_{k+1}^*(\boldsymbol{x}_k, \boldsymbol{y}_k), \boldsymbol{y}_{k+1}(\boldsymbol{x}_k, \boldsymbol{y}_k)\right) + \tilde{R}_k(\boldsymbol{x}_k, \boldsymbol{y}_k)\right) - V_k(\boldsymbol{x}_k, \boldsymbol{y}_k) \qquad (8)$$

An important feature of the exposure $J_k(\boldsymbol{x}_k, \boldsymbol{y}_k)$ is that, at period $k$, it is independent of the nonfinancial random factors in the subsequent periods (i.e., $\{\xi_j\}_{j=k+1}^T$); so it is $\mathcal{F}_T$-measurable. In addition, this exposure has zero mean ($E_k^Q\left[J_k\right] = 0$). Hence, there exists an $\mathcal{F}_T$-measurable contingent claim $G_{r,k}$ that satisfies $E_k^Q\left[G_{r,k}\right] = 0$ and solves

$$G_{r,k} + J_k(\boldsymbol{x}_k, \boldsymbol{y}_k) = 0 \qquad (9)$$

In this way, a dynamic replicating portfolio $\{G_{r,k}\}_{k=0}^{T-1}$ can be constructed over the planning horizon. Then, the optimal financial hedging portfolio can be obtained by combining this replicating portfolio with a financing portfolio $G_f$, which is defined as follows:

$$\max_{G_f} E_0\left[u_\gamma\left(G_f + V_0(\boldsymbol{x}_0^*, \boldsymbol{y}_0)\right)\right] \qquad (10)$$

Given $G_f$ and $\left\{G_{r,k}\right\}_{k=0}^{T-1}$, we obtain the optimal $G_h$:

$$G_h = G_f + \sum_{k=0}^{T-1} G_{r,k} \tag{11}$$

The optimality of the above CE-based approach is guaranteed by Theorem 1.

**Theorem 1**. *The optimal solution of JOFM can be obtained by the CE-based approach. Specifically, the optimal operational policy* $\alpha^* = \left\{\boldsymbol{x}_k^*\right\}_{k=0}^{T-1}$ *can be obtained by solving the following dynamic program (*$V_T^* = 0$*):*

$$V_k^*(\boldsymbol{y}_k) = \max_{\boldsymbol{x}_k} E_k^Q\left[CE_{\xi_{k+1}}\left[V_{k+1}^*\left(\boldsymbol{y}_{k+1}(\boldsymbol{x}_k, \boldsymbol{y}_k)\right) + \tilde{R}_k(\boldsymbol{x}_k, \boldsymbol{y}_k)\right]\right] \tag{12}$$

*Once the optimal operational policy is obtained, the optimal financial hedging portfolio can then be constructed using (9), (10) and (11).*

The Bellman equation (12) reveals an important feature of the CE-based approach: we can obtain the optimal hedging-consistent operational policy $\alpha^*$ without knowing the exact composition of the optimal hedging portfolio. As a result, the CE-based approach can help reducing the number of decision variables. To understand this, consider the commodity procurement example. In practice, there are at least 12 commodity futures contracts (with different maturity dates) available in the market, not to mention the commodity options. Hence, we need a twelve-dimensional vector $\boldsymbol{\theta}_h$ to represent the hedging positions for these commodity futures. Together with the inventory decision variable $x_k$, we have totally 13 decision variables in the original JOFM, which makes the problem extremely difficult to solve. Fortunately, we can use (12) to simplify the problem, in order to obtain the hedging-consistent procurement policy $\alpha^* = \left\{x_k^*\right\}_{k=0}^{T-1}$ as follows:

$$\begin{aligned} V_k^*(y_k) = \max_{x_k} E_{S_{k+1}|S_k}^Q \Big[ & -S_k(x_k - y_k) + CE_{\tilde{D}_k}\Big[V_{k+1}^*\left(x_k - \tilde{D}_k\right) \\ & + r\min(x_k, \tilde{D}_k) - h\left|x_k - \tilde{D}_k\right|^+ - q\left|\tilde{D}_k - x_k\right|^+\Big]\Big] \end{aligned}$$

From Theorem 1, we can summarize exact procedure of the CE-based approach in Figure 1.

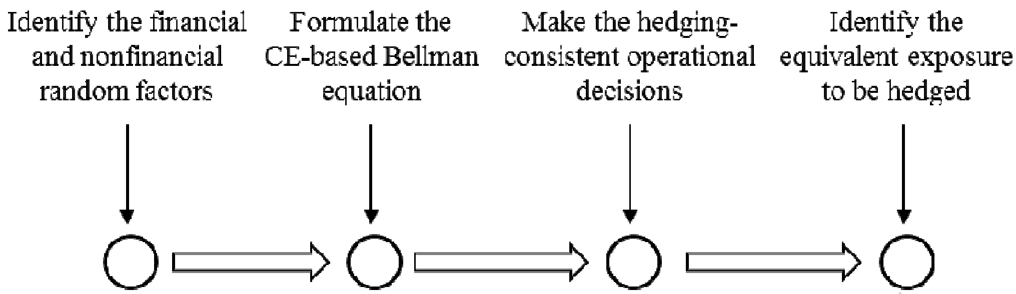


Figure 1. Procedure of the CE-based approach

To apply the CE-based approach, one must appropriately identify the financial and nonfinancial random factors, as shown in Figure 1. In our framework, the distinction between the financial and nonfinancial random factors is somewhat "technical". We make two key points for the identification of these factors in practice.

The first point is that one should always perceive the nonfinancial random factor as the remaining "uncertainty" after accounting for the effect of financial random factor. This point is equivalent to the previous independence assumption between financial and nonfinancial random factors. Such independence relies on the technical choice of financial and nonfinancial random factors used in the model. Even if the original financial and nonfinancial random factors are correlated, in many cases (e.g., normally or log-normally distributed random factors) we can still identify a new set of independent financial and nonfinancial random factors by applying a transformation of the nonfinancial random factor; see the Online Appendix for more discussion.

Given the transformation of the nonfinancial random factor, we can convert the profit and transition functions so as to accommodate these new random factors in our model. To illustrate this, suppose that in the commodity procurement example, the commodity demand is related to the price as follows: $\tilde{D}_k = f_k(S_k, \tilde{m}_k)$, where $\tilde{m}_k$ is a random variable independent of the commodity price. We can simply use $\tilde{m}_k$ as the new nonfinancial random factor instead of the commodity demand $\tilde{D}_k$, and rewrite the transition function (2) and profit function (3) as follows:

$$y_{k+1} = x_k - f_k(S_k, \tilde{m}_k)$$

$$\tilde{R}_k(x_k, y_k) = r\min[x_k, f_k(S_k, \tilde{m}_k)] - h\left|x_k - f_k(S_k, \tilde{m}_k)\right|^+ - q\left|f_k(S_k, \tilde{m}_k) - x_k\right|^+ - S_k \cdot (x_k - y_k)$$

Given the conversion, all else being equal, the Bellman equation (12) applies.

The second point is about the recognition of financial random factor, which is restricted to the firm's access to the financial market. An uncertain factor should be recognized as a financial random factor if and only if the risk exposure, associated with this uncertain factor, can be readily hedged with the related financial instruments in the accessible financial market. If a random factor is just correlated with the price process of a financial security, but this security is not available for the firm to use it, then the random factor should also be treated technically as nonfinancial. For instance, consider a local manufacturer who needs to procure two different commodities, namely, A and B (e.g., copper and nickel), from the local commodity market to serve its production of the end products. The price volatilities of both commodities can be regarded as market risks in the finance literature. However, the manufacturer may have limited access to the commodity derivatives market. Specifically, the manufacturer can trade the derivatives for the commodity A in the local market, but not for B, perhaps because commodity derivatives for B are only traded in some remote foreign markets. In such a case, the manufacturer can construct a market portfolio to hedge against the variations in the price of A, which should be treated as a financial random factor. However, the price volatility of commodity B should be treated as a nonfinancial random factor. A more complicated situation is that the random prices of A and B may be correlated, but this can be resolved by using our first point above.



### 3.3. Further Extension and Analysis

To further facilitate the application of the CE-based approach, we present several extensive results in this section. First, we relax the former assumption that the interest rate is zero. Next, we investigate the structural properties of the CE-based value function, and show that the base-stock-type policy is optimal under a set of sufficient conditions.

Let's consider a non-zero interest rate, or equivalently, a nontrivial discounting factor $\beta$ ($0 < \beta < 1$). In this situation, one can replace the original operating profit $\tilde{R}_k(\boldsymbol{x}_k, \boldsymbol{y}_k)$ and the security price vector $\boldsymbol{X}_t$ with their discounted versions, namely, the discounted operating profit and the discounted security price vector defined as follows: $\tilde{R}_k'(\boldsymbol{x}_k, \boldsymbol{y}_k) = \beta^k \tilde{R}_k(\boldsymbol{x}_k, \boldsymbol{y}_k)$ and $\boldsymbol{X}_t' = \beta^t \boldsymbol{X}_t$. Then, we have an important result of Harrison and Kreps (1979): any trading strategy $\boldsymbol{\theta}$ is self-financing with respect to the original security price vector $\boldsymbol{X}_t$ if and only if it is self-financing with respect to the discounted security price vector $\boldsymbol{X}_t'$. Accordingly, we can establish a generalized version of the CE-based approach, which is summarized in Theorem 2.

**Theorem 2**. *Given a nontrivial discounting factor $\beta$ ($0 < \beta < 1$), the optimal operational policy $\boldsymbol{\alpha}^* = \left\{\boldsymbol{x}_k^*\right\}_{k=0}^{T-1}$ can be obtained by solving the following dynamic program ($V_T^* = 0$):*

$$V_k^*(\boldsymbol{y}_k) = \max_{\boldsymbol{x}_k} E_k^Q\left[CE_{\xi_{k+1}}\left[V_{k+1}^*\left(\boldsymbol{y}_{k+1}(\boldsymbol{x}_k, \boldsymbol{y}_k)\right) + \beta^k \tilde{R}_k(\boldsymbol{x}_k, \boldsymbol{y}_k)\right]\right] \tag{13}$$

*Once the optimal operational policy is obtained, the optimal financial hedging portfolio can then be constructed using (9), (10) and (11), but the security price vector $\boldsymbol{X}_t$ must be replaced by its discounted version $\boldsymbol{X}_t' = \beta^t \boldsymbol{X}_t$.*

Next, we embark on a discussion on the structural properties of the CE-based value function defined in the Bellman equation (13). As shown by Smith and McCardle (2001), one can usually characterize the optimal solution of a Bellman equation by deriving several desired structural properties of the value function (e.g., monotonicity and concavity), and this can be done easily if the Bellman equation involves only linear operators. Unfortunately, the CE operator is nonlinear, which makes it difficult to analytically characterize the optimal solution of the Bellman equation (13). However, the difficulty can partially be relieved by the following result of Proposition 1, which shows that the CE operator preserves concavity – a commonly desired structural property in characterizing optimal solutions.

**Proposition 1.** *The CE operator $CE_\eta\left[\cdot\right]$ preserves concavity. Specifically, if a function $W(\boldsymbol{y};\eta)$ is concave in $\boldsymbol{y}$ for any realized value of the random variable $\eta$, so is the value function $V(\boldsymbol{y}) = CE_\eta\left[W(\boldsymbol{y};\eta)\right]$.*

From Proposition 1, one can expect that the CE-based value function $V_k^*(\boldsymbol{y}_k)$ is concave in $\boldsymbol{y}_k$ under appropriate conditions. In particular, we can establish a set of sufficient conditions which restrict the functional forms of the operating profit function $\tilde{R}_k(\boldsymbol{x}_k, \boldsymbol{y}_k)$ and the transition function $\boldsymbol{y}_{k+1}(\boldsymbol{x}_k, \boldsymbol{y}_k)$:

(**A1**) The transition function is affine, i.e. $\boldsymbol{y}_{k+1}(\boldsymbol{x}_k, \boldsymbol{y}_k) = \tilde{A}_k \boldsymbol{x}_k + \tilde{B}_k \boldsymbol{y}_k + \tilde{c}_k$, where the coefficients $\tilde{A}_k$ and $\tilde{B}_k$ are (random) matrices while $\tilde{c}_k$ a (random) vector.

(**A2**) The condition (A1) holds and $\tilde{B}_k = 0$, i.e. the transition function does not rely on the current state vector $\boldsymbol{y}_k$.

(**A3**) The profit function is separable: $\tilde{R}_k(\boldsymbol{x}_k, \boldsymbol{y}_k) = \tilde{J}_k(\boldsymbol{x}_k) + \tilde{L}_k(\boldsymbol{y}_k)$, where $\tilde{J}_k(\cdot)$ is concave while $\tilde{L}_k(\cdot)$ is both concave and independent of the nonfinancial random factor $\xi_{k+1}$.

In general, condition (A1) ensures that the value function $V_k^*(\boldsymbol{y}_k)$ inherits the concavity from the operating profit function $\tilde{R}_k(\boldsymbol{x}_k, \boldsymbol{y}_k)$; conditions (A2) and (A3) then ensures that a base-stock-type policy is optimal. These results are summarized in Theorem 3.



**Theorem 3.** (i) *Suppose that condition (A1) is true. Then, if the operating profit* $\tilde{R}_k(\boldsymbol{x}_k, \boldsymbol{y}_k)$ *is concave in* $(\boldsymbol{x}_k, \boldsymbol{y}_k)$ *for each period* $k$*, the CE-based value function* $V_k^*(\boldsymbol{y}_k)$ *will also be concave in* $\boldsymbol{y}_k$.

(ii) *Suppose that conditions (A2) and (A3) are both true. Then, for the Bellman equation (13), a base-stock-type policy is optimal, that is,*

$$x_k^* = \arg\max E_k^Q\left[CE_{\xi_{k+1}}\left[V_{k+1}^*\left(\tilde{A}_k \boldsymbol{x}_k + \tilde{c}_k\right) + \beta^k \tilde{J}_k(\boldsymbol{x}_k)\right]\right] \tag{14}$$

As a direct application of Theorem 3, we can show that a base-stock policy is optimal for the commodity procurement example presented in section 2. From equation (3), we know that the transition function is affine and does not rely on the current inventory level; so, conditions (A1) and (A2) are satisfied. Then, from equation (2) it is straightforward to verify that condition (A3) is also satisfied. From Theorem 3, we know that a base-stock policy is optimal; see Corollary 1. We note that this result is analogous to the Theorem 1 of Bouakiz and Sobel (1992), who have also concluded that a base-stock policy is optimal for a finite horizon inventory model under the exponential utility criterion. However, Bouakiz and Sobel made the assumption that the purchasing price of the inventory item is deterministic; therefore, financial hedging is not considered in their model.

**Corollary 1.** *For the commodity procurement and storage problem formulated in (2) and (3), the CE-based value function* $V_k^*(y_k)$ *is concave in* $y_k$*. Moreover, the optimal hedging-consistent operational policy is of the base-stock type.*

## 4. Numerical Experiment

So far, we have shown that for a risk-averse firm who uses financial derivatives to hedge against its risk exposure, its optimal operational policy can be obtained through the CE-based approach. While the CE-based approach is optimal, the EV-based approach is simpler (see the Online Appendix for a discussion). Thus, a natural question arises: is it possible to use the simpler EV-based approach to achieve a near-optimal result? To answer this question, a "mini" numerical experiment is conducted, and some simple yet straightforward computational results are presented to illustrate the effectiveness of the CE-based approach.

### 4.1 Experimental Setup

We continue to use the commodity procurement example, which is introduced in section 2. Our choice of the base-case parameters closely follows Kouvelis et al. (2012). Specifically, following Kouvelis et al., both the spot price and demand are assumed to follow the geometric Brownian motion (GBM) in the planning horizon, i.e. $S_{k+1} = S_k e^{\sigma_S B_{k+1} - \sigma_S^2/2} / \chi$ and $D_{k+1} = D_k e^{\sigma_D W_{k+1} - \sigma_D^2/2}$, where $B_{k+1}$ and $W_{k+1}$ both follows standard normal distribution. To account for the potential price elasticity of demand, $B_{k+1}$ and $W_{k+1}$ may be negatively correlated with each other, with a correlation coefficient $-\rho$, where $\rho \in [0,1]$. In the base case, we set

$\rho = 0$, and other parameters in the GBMs are $\chi = 0.9758$ and $\sigma_S = \sigma_D = 0.114$. Moreover, the sales revenue is set to $15 per unit and the inventory holding cost (backlogging cost) is set to $3 per unit ($5 per unit). In addition, the initial inventory level is set to zero. Finally, according to Chod et al. (2010), a moderate degree of risk aversion is set at $\gamma = 2$.

We then introduce a measure to facilitate the comparison of performance between the CE-based and EV-based approaches. In general, it is cumbersome to directly compare the value of the negative exponential utility function. Instead, from Theorem 2, it is more convenient to directly compare the CE-based value function $V_k^*(\boldsymbol{y}_k)$. Moreover, the CE-based value has an economic meaning: the cash equivalent of the risky operating profit earned by the manufacturer considering the financial hedging; see Pratt (1964) and Pratt and Zeckhauser (1987) for more details. Accordingly, a *performance gap* $\Gamma$ is defined to measure the percentage difference in the performance between the CE-based and EV-based approaches:

$$\Gamma = \frac{V_0^* - \widehat{V}_0^*}{V_0^*} \times 100\%$$

where $V_0^*$ is the maximized CE-based value when the operational policy is solved by the CE-based approach, i.e., by using the Bellman equation (13); and $\widehat{V}_0^*$ is the CE-based value if the operational policy is just approximately solved by the EV-based approach. From the definition, the performance gap measures by how much the suboptimal CE-based value obtained by the EV-based approach falls short of the optimal CE-based value obtained by the CE-based approach. Equivalently, the ratio $1 - \Gamma$ measures the effectiveness of the EV-based approximation to the optimal solution. Hence, a lower $\Gamma$ indicates a better approximation produced by the EV-based approach.

### 4.2 Numerical Results

We compute the performance gap $\Gamma$ under different key parameter values, namely, $\gamma$, $\sigma_S$, $\sigma_D$, and $\rho$. As shown in (7), $\gamma$ plays an important role in the CE operator. Besides, $\sigma_S$ and $\sigma_D$, respectively, capture the significances of the financial and nonfinancial random factors. If $\sigma_D = 0$, the complete market assumption will apply, and the CE-based approach will be degenerated to the EV-based approach ($\Gamma = 0$). If, alternatively, $\sigma_S = 0$, the proposed model is then reduced to the one considered in Bouakiz and Sobel (1992). Further, if there exist a significant correlation between the price and demand factors ($\rho < 0$), one should apply the conversion introduced at the end of section 3.2 to formulate the Bellman equation. See Figure 2 for the related sensitivity analysis regarding these key parameters.

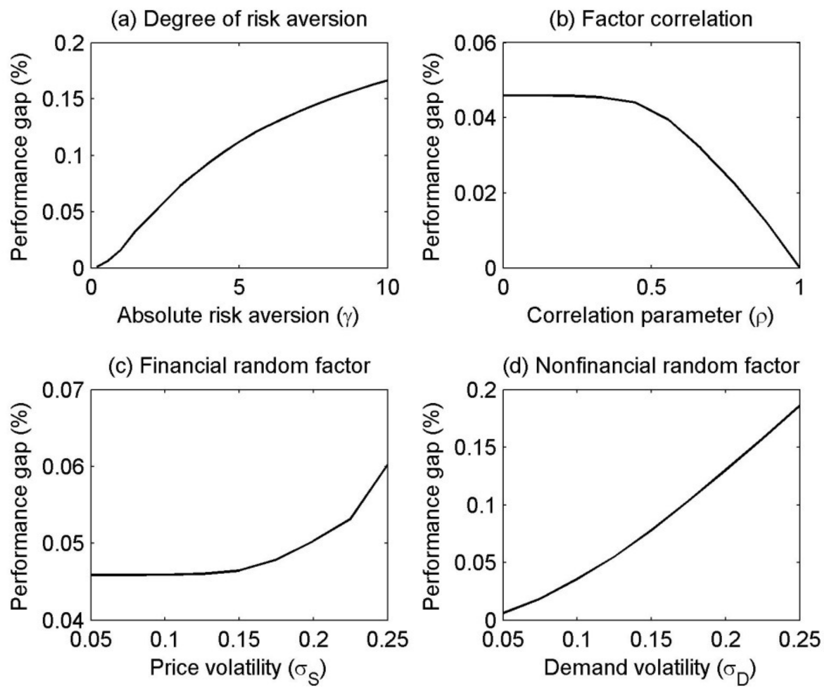


Figure 2. Summary of the sensitivity analysis



We first look at Figure 2 (a), where the performance gap $\Gamma$ is plotted against the degree of risk aversion $\gamma$, which takes value ranging from 0 to 10. We observe a clear trend as $\Gamma$ grows with $\gamma$. Such growing trend corroborates the common intuition that a firm's risk aversion motivates the firm to hedge against risk. Then, we observe two extreme scenarios in this plot: (i) the performance gap $\Gamma$ becomes remarkably large (more than 15%) when $\gamma$ is sufficient high ($\gamma \geq 8$); (ii) $\Gamma$ diminishes quickly as $\gamma$ approaches zero. This observation suggests that the CE-based approach is especially valuable for firms with relatively high degree of risk aversion, but it will degenerate to the EV-based approach in the risk-neutral case. For firms having moderate risk aversion ( $\gamma = 2$ ), the CE-based approach still dominates the EV-based approach ( $\Gamma \approx 5\%$ ).

Next, let us look at Figure 1 (b), where the effect of the price-demand correlation is examined. Recall that the correlation coefficient is $-\rho$, the range of $\rho$ is set to $[0, 1]$ to reflect the typical negative price elasticity of demand in reality. We observe that $\Gamma$ is a monotonically decreasing function of $\rho$, implying that the EV-based approach becomes more effective for a deeper price-demand correlation. Moreover, $\Gamma$ reduces to zero for a perfect correlation ( $\rho = 1$ ). This result is expected because, under a perfect correlation, the complete market assumption that underpins the EV-based approach applies.

We then look at Figure 2 (c), where the performance gap $\Gamma$ is charted as a function of the price volatility $\sigma_S$, which takes value ranging from 0.05 to 0.25. Surprisingly, we can see that the performance gap increases monotonically as the price volatility grows, though the increase appears moderate (from 4.5% to 6%). This result implies that when nonfinancial random factor exists, the EV-based approach can become less effective as the financial random factor (i.e., the price volatility) becomes more significant, despite the intention that the EV-based approach is designed to incorporate financial random factors and financial hedging (Birge 2000).

Finally, let us look at Figure 2 (d) to examine the effects of demand volatility $\sigma_D$ on the performance gap $\Gamma$. Similarly, the range of $\sigma_D$ is set to $[0.05, 0.25]$. We observe that as $\sigma_D$ grows in this range, $\Gamma$ increases substantially from nearly zero to nearly 20%. This observation suggests that the EV-based approach can deteriorate quickly as the nonfinancial random factor (demand volatility) becomes more significant, which is consistent with the fact that nonfinancial random factor is the "culprit" to flaw the complete market assumption. In addition, we see that $\Gamma$ can become quite small when the demand volatility shrinks to nearly zero, implying that the EV-based approach can produce near-optimal results when the nonfinancial random factor is negligible.

From the above results, we can infer that the EV-based approach can produce near-optimal results only in two special cases: (i) the firm is just slightly risk-averse, and (ii) the nonfinancial random factors are nearly negligible. In most of the cases, however, the CE-based approach dominates the EV-based approach.

## 5. Concluding Remarks

We have developed a CE-based approach for a risk-averse firm to make hedging-consistent operational decisions in a simplified way. This new approach overcomes some of the shortcomings embedded in the existing EV-based approach while retaining its major advantage. In particular, the complete market assumption that underpins the EV-based approach is relaxed by allowing for the existence of nonfinancial random factors, which enables the CE-based approach to be applied in a much broader risky environment. Although the CE operator may introduce additional nonlinearity into the Bellman equation, the commonly desired base-stock-type policy can remain optimal under certain conditions. Besides, the CE-based value function can also help identify the equivalent financial risk exposure that should be hedged in the financial market. Therefore, this paper is a contribution to the growing literature on the interface of operations management and finance.

For risk-averse firms, the procedure of the CE-based approach has an interesting managerial implication. Specifically, this approach allows us to eliminate the financial hedging decision variables in the Bellman equation when making hedging-consistent operational decisions, despite our initial intention to integrate the two decision-making processes in JOFM. This implies that it is still optimal for a risk-averse firm to make the operational decisions at first, perhaps by an operations manager, followed by the financial hedging decisions made by the firm's finance department. The only necessary change is the use of CE-based value function as the objective function to determine the operational policy. Hence, our results support the separation of the operational decision-making process from the financial hedging, in a sense that the operations manager does not need to know the exact formation of the financial hedging portfolio.

This paper has some limitations and can be extended by future research. First, it would be interesting to further incorporate some capacity constraints (e.g., $\psi(x_k, y_k) \geq 0$) into our model (JOFM). A key difficulty in analyzing such kind of constraints is that the solution space of the decision variable $x_k$ will change with $y_k$, which can significantly complicate the problem. In some special cases, we may circumvent this difficulty by introducing a new decision variable. For example, consider a linear constraint: $x_k - y_k \geq 0$ (a constraint that can be found in most multi-period inventory models when there is no spot market). This constraint amounts to requiring that the solution space of $x_k$ depends on $y_k$, i.e., $x_k \in [y_k, \infty)$. Thus, we can define a new decision variable: $\hat{x}_k = x_k - y_k$. Its solution space is $\hat{x}_k \in [0, \infty)$, which implies that our CE-based approach can be applied again by using $\hat{x}_k$. Unfortunately, this method (to circumvent the complexity of capacity constraints by using new decision variables) may not be extended to general cases, especially when the constraint $\psi(x_k, y_k) \geq 0$ is nonlinear. Nevertheless, our CE-based approach can still serve as starting point for future research on joint operations management and financial hedging problems with complex capacity constraints.

Besides, it is also possible to further incorporate the potential changes in the firm's degree of risk aversion in decision making over time, which is especially relevant for operational problems with a long planning horizon (e.g., 10 years). Our model is limited to consider only the short or medium term cases, which allow us to employ the interperiod utility in modeling and assume that the firm's risk attitude is unchanged. To extend the present model to allow for possible changes in risk attitude, one may need to explore an appropriate combination of the interperiod and intraperiod utility functions (Sobel 2006) in modeling, which deserves future investigation. Moreover, a third possible extension is to explore the monotonic properties of the base-stock policy in Theorem 3. As suggested by Smith and McCardle (2001), this may involve exploring new sufficient conditions and structural properties of the Bellman equation to ensure that the value function preserves supermodularity, which is challenging.



## References

Bartram, S. M., G. W. Brown, and F. R. Fehle. 2009. International evidence on financial derivatives usage. *Financial Management* 185-206.

Berling, P. 2008. Real options valuation principle in the multi-period base-stock problem. *Omega* 36 1086-1095.

Birge, J. R. 2000. Option methods for incorporating risk into linear capacity planning models. *Manufacturing & Service Operations Management* 2(1) 19-31.

Bouakiz, M., M. J. Sobel. 1992. Inventory control with an exponential utility criterion. *Operations Research* 40(3) 603-608.

Caldentey, R., M. B. Haugh. 2006. Optimal control and hedging of operations in the presence of financial markets. *Mathematics of Operations Research* 31(2): 285-304.

Caldentey, R., M. B. Haugh. 2009. Supply contracts with financial hedging. *Operations Research* 57(1) 47-65.

Chod, J., N. Rudi, J. A. Van Mieghem. 2010. Operational flexibility and financial hedging: complements or substitute? *Management Science* 56(6) 1030-1045.

Copeland, T., V. Antikarov. 2001. *Real options: A practitioner's Guide*. Texere Publisher, London, UK.

Dentskevich, P., G. Salkin. 1991. Valuation of real projects using option pricing techniques. *Omega* 19(4) 207-222.

Ding, Q., L. Dong, P. Kouvelis. 2007. On the integration of production and financial hedging decisions in global markets. *Operations Research* 55(3) 470-489.

Duffie, D. 2001. *Dynamic Asset Pricing Theory*, 3rd ed. Princeton University Press, Princeton, NJ.

Eeckhoudt, L., C. Gollier, H. Schlesinger. 1995. The risk-averse (and prudent) newsboy. *Management Science* 41(5) 786-794.

Gaur, V., Seshadri, S., 2005. Hedging inventory risk through market instruments. *Manufacturing & Service Operations Management* 7(2), 103-120.

Goel, A., G. J. Gutierrez. 2011. Multiechelon procurement and distribution policies for traded commodities. *Management Science* 57(12) 2228-2244.

Graham, J. R., C. W. Smith. 1999. Tax incentives to hedge. *Journal of Finance* 54(6): 2241-2261.

Harrison, J. M., D. M. Kreps. 1979. Martingales and arbitrage in multi-period securities markets. *Journal of Economic Theory* 20 381-408.

Henderson, V., D. Hobson. 2013. Risk aversion, indivisible timing options, and gambling. *Operations Research* 61(1) 126-137.

Huchzermeier, A., M. Cohen. 1996. Valuing operational flexibility under exchange rate risk. *Operations Research* 44(1) 100-113.

Kouvelis, P., R. Li, D. Ding. 2012. Managing storable commodity risks: role of inventories and financial hedges. In: P. Kouvelis, L. Dong, O. Boyabatli, R. Li, (eds), *Handbook of Integrated Risk Management in Global Supply Chains*. John Wiley & Sons, Inc., Hoboken, NJ, USA.

Pratt, J. W. 1964. Risk Aversion in the small and in the large. *Econometrica* 32 122-136.

Pratt, J. W., R. J. Zeckhauser. 1987. Proper risk aversion. *Econometrica* 55 143-154.

Renardy, M., R. C. Rogers. 2004. *An introduction to partial differential equations*, 2nd ed. Springer, New York.

Santiago, L. P., P. Vakili. 2005. On the value of flexibility in R&D projects. *Management Science* 51(8) 1206-1218.

Smith, J. E., K. F. McCardle. 2001. Structural properties of stochastic dynamic programs. *Operations Research* 50(5) 796-809.

Smith, J. E., R. F. Nau. 1995. Valuing risky projects: option pricing theory and decision analysis. *Management Science* 41(5) 795-816.

Smith, C. W., R. M. Stulz. 1985. The determinants of firms' hedging policies. *Journal of Financial and Quantitative Analysis* 20(4): 391-405.

Sobel, M. J. 2006. Discounting and Risk Neutrality. Case Western Reserve University, Technical Report Paper No. TM-724F. Available at SSRN: http://ssrn.com/abstract=883122 or http://dx.doi.org/10.2139/ssrn.883122

Van Mieghem, J. A. 2003. Capacity management, investment, and hedging: review and recent developments. *Manufacturing & Service Operations Management* 5(4) 269-302.

Wiggins, J., J. Blas. 2008. Food and consumer groups get hedge smart. *Financial Times*, August 17, 2008.

## Online Appendix

### Appendix A: Proofs

To prove Theorem 1, we need to introduce the following lemma (Lemma A.1).

**Lemma A.1:** *For* $0 \le k \le T-1$*, the following equation holds:*

$$CE_{\xi_{k+1}}\left[G_{r,k} + \tilde{R}_k(\boldsymbol{x}_k, \boldsymbol{y}_k) + V_{k+1}\left(\boldsymbol{x}^*_{k+1}, \boldsymbol{y}_{k+1}\right)\right] = V_k(\boldsymbol{x}_k, \boldsymbol{y}_k) \tag{15}$$

[*Proof*]: From equation (9), we know that $G_{r,k} = -J_k(\boldsymbol{x}_k, \boldsymbol{y}_k)$. Using the fact that $J_k(\boldsymbol{x}_k, \boldsymbol{y}_k)$ is the equivalent financial risk exposure independent of $\xi_{k+1}$, we have

$$\begin{aligned}
&CE_{\xi_{k+1}}\left[G_{r,k} + \tilde{R}_k(\boldsymbol{x}_k, \boldsymbol{y}_k) + V_{k+1}\left(\boldsymbol{x}^*_{k+1}, \boldsymbol{y}_{k+1}\right)\right] \\
&= CE_{\xi_{k+1}}\left[\tilde{R}_k(\boldsymbol{x}_k, \boldsymbol{y}_k) + V_{k+1}\left(\boldsymbol{x}^*_{k+1}, \boldsymbol{y}_{k+1}\right)\right] - J_k(\boldsymbol{x}_k, \boldsymbol{y}_k)
\end{aligned}$$

Then substituting expression (8) into the above equation, we get

$$CE_{\xi_{k+1}}\left[G_{r,k} + \tilde{R}_k(\boldsymbol{x}_k, \boldsymbol{y}_k) + V_{k+1}\left(\boldsymbol{x}^*_{k+1}, \boldsymbol{y}_{k+1}\right)\right] = V_k(\boldsymbol{x}_k, \boldsymbol{y}_k)$$

Note that due to the financial hedging with contingent claim $G_{r,k}$, the left-hand side of the above equation does not depend on the financial random factor $\omega_{k+1}$. □

**Proof of Theorem 1.** We can prove the optimality of $\boldsymbol{x}^*_k$ by induction.

First of all, we make the statement that for period $k$ ($0 \le k \le T$), the Bellman equation (12) holds, and the following equation is also true:

$$\begin{aligned}
&\max_{\boldsymbol{x}_k} E_k\left[u_\gamma\left(G_f + \sum_{j=0}^{k}\left(G_{r,j} + \tilde{R}_j\right) + V_{k+1}\left(\boldsymbol{x}^*_{k+1}, \boldsymbol{y}_{k+1}\right)\right)\right] \\
&= E_k\left[u_\gamma\left(G_f + \sum_{j=0}^{k-1}\left(G_{r,j} + \tilde{R}_j\right) + V_k(\boldsymbol{x}^*_k, \boldsymbol{y}_k)\right)\right]
\end{aligned} \tag{16}$$

By letting $V_{T+1} = 0$ and recalling the fact that $V_T = 0$, both sides of (16) are then reduced to (6), i.e., the objective function of JOFM; so (16) stands for $k = T$. Similarly, we know that (12) stands for $k = T$.

Next, suppose that (12) and (16) holds for period $k+1$ onward ( $0 \le k \le T-1$ ), which implies that at period $k$ , the optimal operational decision $\boldsymbol{x}_k^*$ must satisfy the following optimization problem:

$$\max_{\boldsymbol{x}_k} E_k\left[u_\gamma\left(G_f + \sum_{j=0}^{k}\left(G_{r,j} + \tilde{R}_j\right) + V_{k+1}\left(\boldsymbol{x}_{k+1}^*, \boldsymbol{y}_{k+1}\right)\right)\right]$$

Now, it is straightforward to check the following decomposition:

$$\sum_{j=0}^{k}\left(G_{r,j} + \tilde{R}_j\right) = \sum_{j=0}^{k-1}\left(G_{r,j} + \tilde{R}_j\right) + G_{r,k} + \tilde{R}_k(\boldsymbol{x}_k, \boldsymbol{y}_k).$$

Using the above decomposition, we have

$$\begin{aligned}
&\max_{\boldsymbol{x}_k} E_k\left[u_\gamma\left(G_f + \sum_{j=0}^{k}\left(G_{r,j} + \tilde{R}_j\right) + V_{k+1}\left(\boldsymbol{x}_{k+1}^*, \boldsymbol{y}_{k+1}\right)\right)\right] \\
&= E_k\left[u_\gamma\left(G_f + \sum_{j=0}^{k-1}\left(G_{r,j} + \tilde{R}_j\right)\right.\right. \\
&\left.\left. + \max_{\boldsymbol{x}_k} CE_{\xi_{k+1}}\left[G_{r,k} + \tilde{R}_k(\boldsymbol{x}_k, \boldsymbol{y}_k) + V_{k+1}\left(\boldsymbol{x}_{k+1}^*, \boldsymbol{y}_{k+1}\right)\right]\right)\right] \\
&= E_k\left[u_\gamma\left(G_f + \sum_{j=0}^{k-1}\left(G_{r,j} + \tilde{R}_j\right) + \max_{\boldsymbol{x}_k} V_k(\boldsymbol{x}_k, \boldsymbol{y}_k)\right)\right]
\end{aligned}$$

The first equality follows from the $\Delta$-property (see Smith and Nau 1995) of the utility $u_\gamma(\cdot)$; the second equality follows from Lemma A.1. Therefore, the optimal operational decision $\boldsymbol{x}_k^*$ must maximizes the CE-based value for period $k$ :

$$\boldsymbol{x}_k^* = \arg\max_{\boldsymbol{x}_k} V_k(\boldsymbol{x}_k, \boldsymbol{y}_k) \tag{17}$$

It follows that the following equation holds for period $k$ :

$$\begin{aligned}
&\max_{\boldsymbol{x}_k} E_k\left[u_\gamma\left(G_f + \sum_{j=0}^{k}\left(G_{r,j} + \tilde{R}_j\right) + V_{k+1}\left(\boldsymbol{x}_{k+1}^*, \boldsymbol{y}_{k+1}\right)\right)\right] \\
&= E_k\left[u_\gamma\left(G_f + \sum_{j=0}^{k-1}\left(G_{r,j} + \tilde{R}_j\right) + V_k(\boldsymbol{x}_k^*, \boldsymbol{y}_k)\right)\right]
\end{aligned}$$

Thus, we have proved that the equation (16) must hold for period $k$. Besides, from (17), we know that the operational policy can be obtained by maximizing the CE-based value, i.e. the Bellman equation (12) stands for period $k$ also.

We now turn to prove that the hedging portfolio $G_h$ defined by (9), (10) and (11) is optimal. Given the optimal operational policy $\left\{\boldsymbol{x}_k^*\right\}_{k=0}^{T-1}$, let the corresponding replicating portfolio be $\left\{G_{r,k}\right\}_{k=0}^{T-1}$. We have:

$$\max_{G_h,\left\{\boldsymbol{x}_k^*\right\}_{k=0}^{T-1}} E_0\left[u_\gamma\left(G_h + \sum_{k=0}^{T-1}\tilde{R}_k\right)\right] = \max_{G_f}\left\{\max_{\left\{\boldsymbol{x}_k^*\right\}_{k=0}^{T-1}} E_0\left[u_\gamma\left(G_f + \sum_{k=0}^{T-1}\left(G_{r,k} + \tilde{R}_k\right)\right)\right]\right\}$$
$$= \max_{G_f} E_0\left[u_\gamma\left(G_f + V_0^*\right)\right]$$

The first equality follows directly from (11), while the second equality is derived by applying (16) iteratively from the final period $T-1$ to the initial period 0. □

**Proof of Theorem 2.** From Harrison and Kreps (1979), we know that if the original financial market characterized by the price vector $\boldsymbol{X}_t$ is complete with respect to $\{\mathcal{F}_t\}_{0\le t\le T}$, so is the financial market characterized by the discounted price vector $\boldsymbol{X}_t' = \beta^t\boldsymbol{X}_t$. Then the validity of the dynamic program (13) follows directly from Theorem 1. □

**Proof of Proposition 1.** Consider an arbitrary function $W(\boldsymbol{y};\eta)$ which is concave in $\boldsymbol{y}$. For any $\boldsymbol{y}_1$, $\boldsymbol{y}_2$, and $\phi\in[0,1]$, let $\boldsymbol{y}_\phi = \phi\boldsymbol{y}_1 + (1-\phi)\boldsymbol{y}_2$. We have:

$$E_\eta\left[\exp\left(-\gamma\cdot W(\boldsymbol{y}_\phi;\eta)\right)\right] \le E_\eta\left[\exp\left(-\gamma\cdot W(\boldsymbol{y}_1;\eta)\right)^\phi \exp\left(-\gamma\cdot W(\boldsymbol{y}_2;\eta)\right)^{1-\phi}\right]$$
$$\le E_\eta\left[\exp\left(-\gamma\cdot W(\boldsymbol{y}_1;\eta)\right)\right]^\phi E_\eta\left[\exp\left(-\gamma\cdot W(\boldsymbol{y}_2;\eta)\right)\right]^{1-\phi}$$

The first inequality follows from the concavity of $W(\boldsymbol{y};\eta)$, while the second inequality follows from the Hölder inequality. From (7), we can directly verify the following inequality:

$$CE_\eta\left[W(\boldsymbol{y}_\phi;\eta)\right] \ge \phi CE_\eta\left[W(\boldsymbol{y}_1;\eta)\right] + (1-\phi)CE_\eta\left[W(\boldsymbol{y}_2;\eta)\right]$$

It then follow that the value function $V(\boldsymbol{y}) = CE_\eta\left[W(\boldsymbol{y};\eta)\right]$ is concave. □

**Proof of Theorem 3.** The first part of this theorem is proved by induction. It is clear that $V_T^* = 0$ is concave. Then, suppose that at period $k+1$ ($0 \le k \le T-1$), the value function $V_{k+1}^*(\cdot)$ is concave also. To finish the induction, we need to prove that $V_k^*(\boldsymbol{y}_k)$ is also concave. Note firstly that concavity is a C3 property (closed convex cone property, see Smith and McCardle 2001). From Proposition 2 and Proposition 3 in Smith and McCardle (2001), we know that both the expectation operator $E_k^Q\left[\cdot\right]$ and the maximization operator $\max_{\boldsymbol{x}_k}(\cdot)$ preserve the concavity of value functions. In addition, from Proposition 1 in this paper, we know that concavity is also preserved by the CE operator $CE_{\xi_{k+1}}\left[\cdot\right]$. Then, from (13), we can deduce that $V_k^*(\boldsymbol{y}_k)$ is concave if the composite function $V_{k+1}^*\left(\boldsymbol{y}_{k+1}(\boldsymbol{x}_k, \boldsymbol{y}_k)\right)$ is concave with respect to $\boldsymbol{x}_k$ and $\boldsymbol{y}_k$. This is evident since the condition (A1) requires the transition function $\boldsymbol{y}_{k+1}(\boldsymbol{x}_k, \boldsymbol{y}_k)$ to be affine.

We now turn to the second part of the theorem. Using conditions (A2) and (A3), the Bellman equation (13) can be rewritten as:

$$V_k^*(\boldsymbol{y}_k) = \max_{\boldsymbol{x}_k} E_k^Q\left[CE_{\xi_{k+1}}\left[V_{k+1}^*\left(\tilde{A}_k\boldsymbol{x}_k + \tilde{c}_k\right) + \beta^k \tilde{J}_k(\boldsymbol{x}_k)\right]\right] + E_k^Q\left[\beta^k \tilde{L}_k(\boldsymbol{y}_k)\right]$$

Then, it is straightforward to see the validity of (14). □

**Proof of Corollary 1.** From (2) and (3), conditions (A2) and (A3) can be satisfied if we let:

$$\tilde{J}_k(x_k) = r\min(x_k, \tilde{D}_k) - h\left|x_k - \tilde{D}_k\right|^+ - q\left|\tilde{D}_k - x_k\right|^+ - S_k x_k$$

and $\tilde{L}_k(y_k) = S_k y_k$. Applying Theorem 3, we know that the value function $V_k^*(\boldsymbol{y}_k)$ is concave and a base-stock policy is optimal. □

**Appendix B: The Expected-Value-Based approach**

Under the complete market assumption (i.e., $\left\{\xi_k\right\}_{k=0}^T$ do not come into play), the uncertain operating profit $\Pi(\alpha)$ can always be replicated by a dynamic hedging portfolio. Thus, we can use the risk-neutral valuation technique to "value" the operating profit, simply by calculating the expected value of the random payoff using the risk-neutral probability measure $Q$. As a result, an EV-based approach can be established to solve the JOFM in this situation. Following Birge (2000), we can write the operational problem in the form of Bellman equation as follows (Let $V_T^* = 0$):

$$V_k^*(\boldsymbol{y}_k) = \max_{\boldsymbol{x}_k} E_k^Q \left[ V_{k+1}^* \left( \boldsymbol{y}_{k+1}(\boldsymbol{x}_k, \boldsymbol{y}_k) \right) + \beta^k \tilde{R}_k(\boldsymbol{x}_k, \boldsymbol{y}_k) \right] \tag{18}$$

**Theorem A.1**. *Suppose that the complete market assumption holds. Then, the optimal operational policy* $\alpha^* = \{\boldsymbol{x}_k^*\}_{k=0}^{T-1}$ *can be obtained directly by solving the dynamic program (18). That is, the EV-based approach applies.*

**Proof.** As $\{\xi_k\}_{k=0}^{T}$ do not come into play, the validity of Theorem A.1 follows immediately from Theorem 2. □

When compared with Bellman equation (13), Bellman equation (18) does not involve the nonlinear operator $CE_{\xi_{k+1}}[\cdot]$. As a result, it would be easier to derive some structural properties for the value function of (18) and then characterize its optimal solution, by using techniques summarized in Smith and McCardle (2001). However, as noted above, the EV-based approach is derived under the complete market assumption, which would be too restrictive in many cases.

In some recent papers (e.g., Birge 2000, Goel and Gutierrez 2011), the EV-based approach is extended to incorporate nonfinancial random factors. Generally, the associated Bellman equation can be formulated as follows:

$$\hat{V}_k^*(\boldsymbol{y}_k) = \max_{\boldsymbol{x}_k} E_k^Q \left[ E_{\xi_{k+1}} \left[ V_{k+1}^* \left( \boldsymbol{y}_{k+1}(\boldsymbol{x}_k, \boldsymbol{y}_k) \right) + \beta^k \tilde{R}_k(\boldsymbol{x}_k, \boldsymbol{y}_k) \right] \right] \tag{19}$$

The difference between Bellman equations (13) and (19) is that the nonlinear CE operator $CE_{\xi_{k+1}}[\cdot]$ in equation (13) is replaced by the linear expectation operator $E_{\xi_{k+1}}[\cdot]$ in equation (19).

As aforementioned, it would be easier to analyze a Bellman equation with just the linear expectation operator.

## Appendix C: A discussion on creating independent random variables by transformation

This appendix provides a technical discussion on how to construct a new pair of independent random variables from two originally correlated random variables. In our context, this can be interpreted as identifying a new nonfinancial random factor that is independent of the financial random factor. It typically requires finding an appropriate transformation of a random variable. Unfortunately, as far as we know, there is no guaranteed method that can find such a transformation in general cases. Therefore, we present three different methods that can help identifying the required transformation in many cases. We believe that this is already sufficient to demonstrate that the independence assumption between the financial and nonfinancial random factors is not very restrictive (see Remark 1 in section 2.1). For simplicity, we assume that all functions in this appendix are sufficiently smooth, i.e., at least have continuous second-order derivatives.

Let us formally introduce the concept of independent random variables. Consider two random variables $\tilde{x}$ and $\tilde{y}$ that have a joint probability density function $f(x,y)$. Then, we have the following definition of their independence.

**Definition B.1.** The two random variables $\tilde{x}$ and $\tilde{y}$ are said to be independent if and only if the joint density function $f(x,y)$ has the following *separating property*:

$$f(x,y) = g_1(x)g_2(y)$$

where $g_1(x)$ and $g_2(y)$ are the marginal density functions of $\tilde{x}$ and $\tilde{y}$, respectively.

If the above separating property does not hold, then the two random variables are said to be dependent or correlated. In this case, we may apply some transformations to create a new pair of random variables from $\tilde{x}$ and $\tilde{y}$.

First, if $\tilde{x}$ and $\tilde{y}$ are (jointly) normally distributed, the required transformation is easy to obtain. Indeed, one can define a new random variable $\tilde{p}$ through a linear combination of $\tilde{x}$ and $\tilde{y}$:

$$\tilde{p} = \tilde{x} - \frac{\operatorname{cov}(\tilde{x}, \tilde{y})}{\sigma_y^2} \tilde{y} \tag{20}$$

where $\operatorname{cov}(\tilde{x}, \tilde{y})$ is the covariance between $\tilde{x}$ and $\tilde{y}$, while $\sigma_y$ the standard deviation of $\tilde{y}$. By doing so, we obtain a new pair of independent random variables $(\tilde{p}, \tilde{y})$.

Second, suppose that $\tilde{x}$ and $\tilde{y}$ do not follow the normal distribution, but can still be represented as functions of other normally distributed random variables; that is, $\tilde{x} = a(\tilde{x}_1)$ and $\tilde{y} = b(\tilde{y}_1)$, where $\tilde{x}_1$ and $\tilde{y}_1$ are normally distributed (e.g., lognormal distributions). Then, the formula (20) can be applied on $\tilde{x}_1$ and $\tilde{y}_1$ to create a new random variable $\tilde{p}$ through a linear combination of $\tilde{x}_1$ and $\tilde{y}_1$:

$$\tilde{p} = \tilde{x}_1 - \frac{\operatorname{cov}(\tilde{x}_1, \tilde{y}_1)}{\sigma_{y_1}^2} \tilde{y}_1$$

Because $\tilde{p}$ is independent of $\tilde{y}_1$, it is also independent of $\tilde{y} = b(\tilde{y}_1)$.

Besides, one may also employ some established empirical/statistical results to identify the independent financial and nonfinancial random factors. Because this method is rather straightforward, we just illustrate it by a simple example: Kouvelis et al. (2012) have discussed the correlation between volatile commodity price and demand, which are modeled as lognormal random variables. Their formulation essentially implies a log-linear equation that links price and demand:

$$\log(D_{k+1}) = \rho \log(S_{k+1}) + \varepsilon_{k+1} + Const$$

where $\varepsilon_{k+1}$ is a random variable independent of $S_{k+1}$. Thus, one can use the new pair $S_{k+1}$ and $\varepsilon_{k+1}$ as the respective financial and nonfinancial random factors when applying the CE-based approach.

Finally, we introduce a general method on how to create a new pair of independent random variables. This method is quite theoretical, and involves solving a difficult partial differential equation (PDE). Thus, it may only be regarded as the "last resort" if all the above methods fail to work.

Let $\tilde{p}$ be a new random variable such that we can express $\tilde{x}$ as $x = g(p, y)$. Then, the joint density function for the new pair of random variables $(\tilde{p}, \tilde{y})$ is $f(g(p,y),y)g_p(p,y)$. By definition, if $\tilde{p}$ and $\tilde{y}$ are independent of each other, then the new joint density function must have the separating property, that is, there are two density functions $m_1(p)$ and $m_2(y)$ such that $f(g(p,y),y)g_p(p,y) = m_1(p)m_2(y)$ , or $\log[f(g(p,y),y)g_p(p,y)] = \log[m_1(p)] + \log[m_2(y)]$ (To avoid triviality, we proceed over the support of the joint density function to avoid potential zero points of the probability density function). Thus,

$$\frac{\partial^2}{\partial p \partial y}\log[f(g(p,y),y)g_p(p,y)] = 0 \qquad (21)$$

Calculating the derivatives and rearranging the terms, we get a PDE for $g(p, y)$:

$$ff_{xy}g_p + 2ff_xg_{py} + (ff_{xx} - f_x^{\,2})g_pg_y - f_xf_yg_p^{\,2} + ff_xg_pg_{py} = 0$$

We can obtain the desired transformation by solving this PDE for $g(p, y)$. Once $g(p, y)$ is obtained, from (21) there exist two functions $a(p)$ and $b(y)$ such that the joint density function of $(\tilde{p}, \tilde{y})$ takes the following separating form: $f(g(p,y),y)g_p(p,y) = a(p)b(y)$. Thus, we have

$$\iint_{R^2} f(g(p,y),y)g_p(p,y)dydp = \int_R a(p)dp \cdot \int_R b(y)dy = \iint_{R^2} f(x,y)dxdy = 1$$

It follows that $m_1(p) = a(p)\Big/\int_R a(z)dz$ and $m_2(p) = b(y)\Big/\int_R b(z)dz$ are the respective marginal density functions for $\tilde{p}$ and $\tilde{y}$, and that $(\tilde{p}, \tilde{y})$ is a pair of independent random variables.

Generally, we can solve the PDE using a variety of numerical methods (e.g., the finite element method, see Renardy and Rogers 2004), providing that the solution exists. So we would like to provide a discussion on the existence of solutions for PDEs. According to the Cauchy–Kovalevskaya theorem (Renardy and Rogers 2004), a sufficient condition for the existence of the solution is that the coefficients of the PDE are (locally) analytic functions. In our case, this amounts to requiring that the probability density function is analytic in its support. Instances of such density functions include many commonly-used density functions (e.g., normal/lognormal density) in continuous space. Still, it is worth noting that in rare cases, a PDE where the coefficients are not analytic may not have conventional smooth solutions, and one may need to explore possible “weak solutions” for the equation (Renardy and Rogers 2004). However, mathematical methods on these “weak solutions” of PDEs, which may involve complex generalized functions and complicated functional space analysis, are well beyond our current scope. For more discussions on the solutions to PDEs, see Renardy and Rogers (2004) and references therein.

**Highlights**:

- A new approach to quantify the effects offinancial hedging on operational policies.
- Theapproachalleviatessome of the difficultiesarising from market incompleteness.
- A certainty equivalent operator is employedto formulate the Bellman equation.
- The base-stock policy can remain optimal under specific conditions.