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Multi-output profit efficiency and directional distance functions ☆

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ABSTRACT

We extend a recently developed DEA methodology for cost efficiency analysis towards profit efficiency settings. This establishes a novel DEA toolkit for profit efficiency assessments in situations with multiple inputs and multiple outputs. A distinguishing feature of our methodology is that it assumes output-specific production technologies. In addition, the methodology accounts for the use of joint inputs, and explicitly includes information on the allocation of inputs to individual outputs. We also establish a dual relationship between our multi-output profit inefficiency measure and a technical inefficiency measure that takes the form of a multi-output directional distance function. Finally, we demonstrate the empirical usefulness of our methodology by an empirical application to a large service company.

1. Introduction

Production processes that generate multiple outputs are typically characterized by jointly used inputs, i.e. inputs that simultaneously benefit different outputs. These joint inputs give rise to economies of scope, which actually form a prime economic motivation for Decision Making Units (DMUs) to produce more than one output. In the current paper, we establish a methodology for multi-output profit efficiency evaluation that explicitly accounts for jointly used inputs. In particular, our methodology distinguishes between joint inputs and inputs that are allocated to specific outputs.

DEA analysis of multi-output production: The method that we develop fits within the popular Data Envelopment Analysis (DEA, after [8]) approach to productive efficiency measurement. This DEA approach is intrinsically nonparametric, which means that it does not require a parametric/functional specification of the (typically unknown) production technology. It “lets the data speak for themselves” by solely using technological information that is directly revealed by the observed production units. It then reconstructs the production possibility sets by (only) assuming standard production axioms (such as monotonicity and convexity).[1] A DMU's efficiency is measured as the distance of the corresponding input-output combination to the efficient frontier of this empirical production set. Typically, a DMU's efficiency can be computed by simple linear programming. Its nonparametric nature and its easy computation largely explain DEA's widespread use as an analytical research instrument and decision-support tool.

Recently, Cherchye et al. [10,9] introduced a novel DEA methodology to analyze cost efficiency in multi-output settings. The methodology assumes output-specific production technologies, accounts for joint inputs in the production process, and incorporates specific information on how inputs are allocated to individual outputs. As such they provide a formal modeling of the economies of scope that characterize the multi-output production process.[2] These authors have also shown that their cost efficiency measure evaluated at shadow prices is dually equivalent to a specific multi-output version of the [21–30] measure of (radial) input efficiency. This is an attractive feature, as DEA practitioners often use this Debreu–Farrell measure for evaluating the technical efficiency of a DMU's input use (when assuming a fixed output).

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[1] See [28,20,19,31,18] for extensive reviews of DEA. From an economic perspective, DEA itself is rooted in the structural approach to modeling efficient production behavior that was initiated by Afriat [1], Hanoch and Rothschild [34], Diewert and Parkan [23] and Varian [43]. Given the explicit economic motivation of our following analysis, our contribution also fits in this tradition of structural efficiency analysis.

[2] See also [42,40] for related work on modeling economies of scope in a DEA context.

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At this point, we remark that the methodology of [10,9] is closely related to several existing approaches in the DEA literature. Firstly, there is a clear connection with network DEA (see [26,29]). The literature on network DEA also makes use of what we define further as output-specific inputs. However, to the best of our knowledge, it abstracts from the possibility of jointly used inputs. These joint inputs play an important role in our approach because they define the interdependencies between the production processes associated with different outputs and, as a result, they characterize the economies of scope that underlie the observed production processes. Secondly, Salerian and Chan [41] and Despic et al. [22] present two alternative methods to model inputs that contribute to some outputs but not to others. As such, these models can actually be interpreted as special cases of our model with joint inputs, but without having output-specific inputs.

Summarizing, these alternative approaches have in common that they try to enhance the realism of the efficiency evaluation exercise by integrating information on the internal production structure. In a sense, we provide a unifying framework that integrates these existing approaches. This framework should be particularly attractive to empirical researchers who are familiar with standard DEA techniques and interested in the analysis of multi-output production characterized by joint inputs.

Profit efficiency analysis: The current paper extends this methodology for multi-output efficiency assessments to profit efficiency settings, which makes our paper fit in the extensive literature that studies profit efficiency and its extensions in a DEA context.[3] In many practical settings, profit efficiency is considered to be the best suited criterion for evaluating the performance of productive activities. In addition, by its very definition cost efficiency is a necessary condition for profit efficiency. Profit efficiency evaluations are generally more stringent than cost efficiency evaluations. As a result, they can signal additional sources of inefficiency and, thus, potential performance improvements. In this respect, as we will indicate, an appealing feature of our multi-output approach is that it also allows us to allocate a DMU's aggregate profit inefficiency to individual outputs. This helps to better identify specific output production processes where substantial profit efficiency gains are possible, which can usefully assist DMU managers to direct their performance improvement actions in an effective way (i.e. primarily towards outputs that are characterized by considerable inefficiency).

In developing our profit efficiency methodology, we also start from output-specific technologies and distinguish between joint inputs and output-specific inputs in the process of multi-output production. Next, we will show that our profit inefficiency measure under shadow prices has a dual representation as a directional distance function. We believe this is an interesting property, as directional distance functions have become increasingly popular as a technical inefficiency measure that simultaneously includes outputs produced and inputs used. Basically, this duality result extends the one of [7] towards our specific multi-output setting. A particular feature of our analysis here is that we explicitly account for output-specific technologies with jointly used inputs in establishing the duality relationship.

Outline: The rest of the paper is structured as follows. Section 2 introduces some necessary notation and terminology. Section 3 introduces our method for multi-output profit inefficiency measurement. Section 4 establishes the dual representation of our profit inefficiency measure as a directional distance function. Section 5 shows the practical usefulness of our method through an application to a large service company. Section 6 concludes.

[3] See, for example, [14,35,38,36,39,4] for recent contributions.

2. Preliminaries

The distinction between inputs and outputs becomes less relevant in profit efficiency analysis. Therefore, to simplify notation it will often be convenient to work with "netputs" in our following exposition. As we will explain, netput vectors simultaneously capture inputs used (as negative components) and outputs produced (as positive components). We will define this netput concept for our specific setting with joint and output-specific inputs. In turn, this will allow us to introduce our notion of output-specific technologies and, correspondingly, our particular concept of multi-output profit.

Netputs and multi-output technologies: We consider a production technology that uses N inputs to produce M outputs, which we represent by the vectors $\mathbf{X}=(x^1,\ldots,x^N)' \in \mathbb{R}^N_+$ and $\mathbf{Y}=(y^1,\ldots y^M)' \in \mathbb{R}^M_+$, respectively. Our method distinguishes between joint and output-specific inputs.

- *Output-specific* inputs are allocated to individual outputs m, i.e. they specifically benefit the production process of (only) the m-th output. In our formal analysis, we will use $\alpha_k^m \in [0,1]$ (with $\sum_{m=1}^M \alpha_k^m = 1$) to represent the fraction of the k-th output-specific input quantity that is allocated to output m.
- *Joint* inputs are not allocated to specific outputs but are simultaneously used in the production process of *all* the outputs. Clearly, these joint inputs generate interdependencies between the production processes of different outputs.[4]

In the following we will assume that the allocation parameters α_k^m are observed. We believe that in many instances this is not a strong assumption, since large firms nowadays often use cost systems that explicitly allocate inputs/costs to outputs (e.g. Activity Based Costing (ABC)) to support various strategic and operational decisions. These cost systems can be used to define the α_k^m (see, for example, our own empirical application in Section 5). Nevertheless, if this information is not available to the empirical analyst, we can make use of alternative approaches that are not based on observing this information, but try to reconstruct the decomposition (over outputs) of the output-specific inputs in DEA analysis itself.[5] Cherchye et al. [10] provide a discussion on how to integrate these techniques in the approach to multi-output efficiency analysis that we present here. These authors' discussion focused on a cost efficiency setting, but it readily extends to the profit efficiency setting that we consider in the current paper.

We will represent the allocation of inputs to outputs by means of a vector $\mathbf{A}^m \in \mathbb{R}^N_+$ for each output m, for which the entries are defined as (with $\alpha_k^m \in [0,1]$ and $\sum_{m=1}^M \alpha_k^m = 1$)

$$(\mathbf{A}^m)_k = \begin{cases} 1 & \text{if input } k \text{ is joint and used to produce output } m, \\ \alpha_k^m & \text{if input } k \text{ is output} - \text{specific and used to produce output } m, \\ 0 & \text{if input } k \text{is not used to produce output } m. \end{cases}$$

Then, each vector $\mathbf{A}^m$ defines the input vector $\mathbf{X}^m = \mathbf{A}^m \odot \mathbf{X}$, which thus contains the input quantities used in the production process of output m.[6]

As indicated above, we can often simplify our notation by working with netputs, which simultaneously stand for outputs and inputs. Specifically, we use $\mathbf{Z}=\begin{bmatrix}\mathbf{Y}\\ -\mathbf{X}\end{bmatrix} \in \mathbb{R}^{M+N}$ to denote the

[4] See [13] for the introduction of sub-joint inputs. These inputs play a similar role as joint inputs, but only for a *subset* of (instead of all) outputs. It is straightforward to include this third type of inputs in our methodology, but for the ease of the exposition we abstract from this in the current paper.

[5] See, for example, [17,16,3,37,44,24].

[6] The symbol $\odot$ stands for the Hadamard (or element-by-element) product.

aggregate netput vector. In a similar vein, $\mathbf{Z}^m = \begin{bmatrix} y^m \\ -\mathbf{X}^m \end{bmatrix} \in \mathbb{R}_+^{1+N}$ represents the netput vector that is specific to output m.

Our multi-output analysis will involve a specific representation of each output m's production technology. This technology defines the output-specific production possibility set

$$T^m = \{\mathbf{Z}^m \in \mathbb{R}^{1+N} | \mathbf{Z}^m \text{is technically feasible}\},$$

which contains all the combinations of output-specific and joint inputs (in $\mathbf{X}^m$) that can produce the output quantity y^m.

Prices and profits: To define profit, we use $\mathbf{P}_x = (p_x^1, \ldots, p_x^N)' \in \mathbb{R}_+^N$ for input prices and $\mathbf{P}_y = (p_y^1, \ldots, p_y^M) \in \mathbb{R}_+^M$ for output prices. Correspondingly, the netput price vector is given as $\mathbf{P} = \begin{bmatrix} \mathbf{P}_y \\ \mathbf{P}_x \end{bmatrix} \in \mathbb{R}_+^{M+N}$.

To incorporate our distinction between output-specific and joint inputs, we make use of output-specific input prices $\mathbf{P}_x^m \in \mathbb{R}_+^N$. First, for output-specific inputs, these prices coincide with the actual prices, i.e.

$$(\mathbf{P}_x^m)_k = (\mathbf{P}_x)_k \quad \text{for } k \text{ an output} - \text{specific input.}$$

Next, following Cherchye et al. [10,9] we make use of output-specific prices $(\mathbf{P}_x^m)_k$ for every joint input k. Essentially, these prices $(\mathbf{P}_x^m)_k$ capture the fractions of the aggregate input price $(\mathbf{P}_x)_k$ that are allocated to individual outputs m. Efficient production requires the output-specific prices $(\mathbf{P}_x^m)_k$ to add up to the aggregate DMU-level prices, i.e. they must satisfy

$$\sum_{m=1}^{M} (\mathbf{P}_x^m)_k = (\mathbf{P}_x)_k \quad \text{for } k \text{ a joint input.}$$

As explained in detail by Cherchye et al. [12], these output-specific prices have a similar interpretation as Lindahl prices for public goods. Specifically, Pareto efficient provision of public goods equally requires these Lindahl prices to sum up to the aggregate prices.

Taken together, the output-specific price vector $\mathbf{P}^m$ for each output m is given as

$$\mathbf{P}^m = \begin{bmatrix} p_y^m \\ \mathbf{P}_x^m \end{bmatrix}.$$

Correspondingly, for every output m we can define the output-specific profit

$$\pi^m = \mathbf{P}^{m'} \mathbf{Z}^m.$$

In turn, by summing these output-specific profit, we obtain the aggregate profit[7]

$$\pi = \sum_{m=1}^{M} \pi^m = \sum_{m=1}^{M} \mathbf{P}^{m'} \mathbf{Z}^m = \mathbf{P}'\mathbf{Z}.$$

The last equality also shows that summing the profit levels associated with individual netputs $\mathbf{Z}^m$ yields, by construction, the DMU's profit level defined in terms of the aggregate netput $\mathbf{Z}$. Given this, we will work with the sum profit $\sum_{m=1}^{M} \mathbf{P}^{m'} \mathbf{Z}^m$ in what follows, without explicitly considering $\mathbf{P}'\mathbf{Z}$.

3. Multi-output profit efficiency

In practice, the true production technology is typically unknown. Therefore, in empirical efficiency evaluations, we need to reconstruct the production possibilities from a set of T observed DMUs. In what follows, we assume a setting in which we observe, for each DMU t, the netput vectors $\mathbf{Z}_t^1, \ldots, \mathbf{Z}_t^M$, which contain the joint and output-specific inputs, as well as the resulting outputs.

In the current section, we will additionally assume that the empirical analyst also observes the associated netput price vector $\mathbf{P}_t = \begin{bmatrix} \mathbf{P}_{y,t} \\ \mathbf{P}_{x,t} \end{bmatrix} \in \mathbb{R}_+^{M+N}$. At this point, two remarks are in order. First, the assumption of observed prices is often restrictive in empirical settings. In the next section, we will show how we can relax this assumption by using shadow prices. Second, throughout we will assume that we do not have any information about the output-specific prices for the joint inputs, which typically holds true in practical applications (including our own application in Section 5). However, it is worth to indicate that, if extra information on output-specific prices were available, it would actually be fairly easy to integrate this information in our profit efficiency analysis.

Taken together, we assume that we observe a data set

$$S = \{(\mathbf{Z}_t^1, \ldots, \mathbf{Z}_t^M, \mathbf{P}_t) | t = 1, \ldots, T\}.$$

Empirical efficiency criterion: Following a nonparametric approach, we reconstruct the production possibilities while avoiding (non-verifiable) parametric assumptions regarding the DMUs' technologies. In our profit efficiency analysis, we (only) use the following minimalistic prior regarding the production possibility sets.

Axiom T1 (observability means feasibility): Observing the netputs $\mathbf{Z}_t^1, \ldots, \mathbf{Z}_t^M$ implies for all $m = 1, \ldots, M$ that $\mathbf{Z}^m \in T^m$.

This axiom has a very natural interpretation. Basically, it says that what we observe is certainly feasible. Or, if we observe the netput vector $\mathbf{Z}_t^m = \begin{bmatrix} y_t^m \\ -\mathbf{X}_t^m \end{bmatrix}$, then we conclude that the input $\mathbf{X}_t^m$ can effectively produce the output y_t^m.[8]

Adopting our above notation of the previous section, we let $\mathbf{P}_t^m = \begin{bmatrix} p_{y,t}^m \\ \mathbf{P}_{x,t}^m \end{bmatrix}$ represent the m-th output-specific prices for DMU t, with the subvector $\mathbf{P}_{x,t}^m$ containing the prices for the output-specific and joint inputs as characterized in Section 2. Then, building on Axiom T1, we obtain our empirical condition for profit efficient production behavior.

Definition 1 (*Profit efficiency*). Let $S = \{(\mathbf{Z}_t^1, \ldots, \mathbf{Z}_t^M, \mathbf{P}_t) | t = 1, \ldots, T\}$ be a data set. Then, DMU t is profit efficient if there exist, for all outputs m, output-specific price vectors $\mathbf{P}_t^m = \begin{bmatrix} p_{y,t}^m \\ \mathbf{P}_{x,t}^m \end{bmatrix} \in \mathbb{R}_+^{1+N}$, such that

(i) $(\mathbf{P}_{x,t}^m)_k = (\mathbf{P}_{x,t})_k$ for output-specific inputs k,
(ii) $\sum_{m=1}^{M} (\mathbf{P}_{x,t}^m)_k = (\mathbf{P}_{x,t})_k$ for joint inputs k,
(iii) $\mathbf{P}_t^{m'} \mathbf{Z}_t^m \geq \mathbf{P}_t^{m'} \mathbf{Z}_s^m$ for all observations $s = 1, \ldots, T$.

In words, this definition states that DMU t is profit efficient if, for the input and output prices that apply to t (captured by $\mathbf{P}_t^m$ for every output m), there does not exist another observed DMU s (with netput vector $\mathbf{Z}_s^m$) that attains a larger profit. As such, given our multi-output setting, we have a separate profit efficiency criterion for each different output m.

While we do observe the aggregate prices $\mathbf{P}_t$, we typically do not observe the output-specific prices $\mathbf{P}_t^m$ because of jointly used inputs (i.e., for a joint input k, we do not observe the price fraction $(\mathbf{P}_{x,t}^m)_k$ that is borne by m). Therefore, the criterion in Definition 1 (only) requires that there exists *at least one* possible specification of these prices that makes the observed behavior of DMU t consistent with profit efficiency. As soon as such a specification exists, we conclude that profit efficient behavior cannot be

[7] To obtain the last equality we use that $\sum_{m=1}^{M} \mathbf{P}^{m'} \mathbf{Z}^m = \sum_{m=1}^{M} (p_y^m y^m) - \sum_{m=1}^{M} (\mathbf{P}_x^{m'} \mathbf{X}^m) = \mathbf{P}_y' \mathbf{Y} - \mathbf{P}_x' \mathbf{X} = \mathbf{P}'\mathbf{Z}$.

[8] Essentially, this axiom excludes measurement errors. Importantly, however, it is fairly easy to extend our methodology to account for measurement problems. For compactness, we will not discuss this question here, but refer to [10] for a detailed treatment.

rejected given the information that is available (contained in the data set S).

Measuring profit efficiency: In practice, if a DMU t does not meet the profit efficiency criterion in Definition 1, we quantify the degree of profit inefficiency as the extent to which actual profit deviates from maximum profit. In what follows, we will introduce a method to measure profit inefficiency in our multi-output framework. In doing so, we will adapt the "directional" profit efficiency framework of [7] to our particular setting.

As a first step, we define, for each output m, the profit function

$$\pi_t^m(\mathbf{P}_t^m) = \max_{s \in \{1,\dots,T\}} \left(\mathbf{P}_t^{m'} \mathbf{Z}_s^m\right),$$

which gives the maximum attainable profit over the observed set S for the prices $\mathbf{P}_t^m$ that apply to DMU t. Correspondingly, when summing over all outputs m, we obtain the aggregate profit function

$$\pi_t(\mathbf{P}_t^1,\dots,\mathbf{P}_t^M) = \sum_{m=1}^{M} \pi_t^m(\mathbf{P}_t^m).$$

In the sequel, we will focus on the profit inefficiency measure[9]

$$PE_t^C(\mathbf{P}_t^1,\dots,\mathbf{P}_t^M) = \frac{\sum_{m=1}^M \pi_t^m(\mathbf{P}_t^m) - \sum_{m=1}^M \left(\mathbf{P}_t^{m'} \mathbf{Z}_t^m\right)}{\sum_{m=1}^M \mathbf{P}_t^{m'} \mathbf{g}_{\mathbf{Z}^m}},$$

which we see as a natural translation of Chambers et al's "directional" profit inefficiency measure to our specific multi-output setting. In this definition, each $\mathbf{g}_{\mathbf{Z}^m}$ represents the directional distance vector for the output m.[10] Equivalently, we can also express it as $\mathbf{g}_{\mathbf{Z}^m} = \begin{bmatrix} g_{y^m} \\ \mathbf{g}_{\mathbf{X}^m} \end{bmatrix}$, with $g_{y^m} \in \mathbb{R}$ and $\mathbf{g}_{\mathbf{X}^m} \in \mathbb{R}^N$ defining the output and input directions, respectively. In practice, these directional vectors are chosen by the empirical analyst prior to the actual efficiency evaluation (see our own application in Section 5 for example specifications of $\mathbf{g}_{\mathbf{Z}^m}$). Clearly, $PE_t^C(\mathbf{P}_t^1,\dots,\mathbf{P}_t^M) = 0$ reveals profit efficiency, while higher values PE_t^C indicate a greater degree of profit inefficient behavior.

In general, the value of the measure $PE_t^C(\mathbf{P}_t^1,\dots,\mathbf{P}_t^M)$ will depend on the output-specific input prices that are used to evaluate the joint inputs (and contained in $(\mathbf{P}_t^1,\dots,\mathbf{P}_t^M)$). As indicated above, these prices are typically not known by the empirical analyst. In what follows, we will choose prices that minimize the value of the profit inefficiency measure PE_t^C for DMU t under evaluation, i.e. we solve

$$PE_t^C = \min_{\mathbf{P}_t^1,\dots,\mathbf{P}_t^M \in \mathbb{R}_+^{1+N}} PE_t^C(\mathbf{P}_t^1,\dots,\mathbf{P}_t^M),$$

where each output-specific price vector $\mathbf{P}_t^m$ is subject to the conditions outlined in Definition 1. Intuitively, by minimizing the profit inefficiency, we actually choose "most favorable" prices $\mathbf{P}_t^m$ for DMU t under evaluation. In other words, we evaluate DMU t in the best possible light, which gives this DMU the benefit of the doubt in the absence of true price information. Attractively, this falls in line with usual DEA efficiency analysis, which typically can be given a similar benefit-of-the-doubt interpretation.[11]

We conclude that DMU t meets our empirical profit efficiency criterion in Definition 1 if and only if $PE_t^C = 0$. In that case, there effectively does exist a specification of the prices $\mathbf{P}_t^m$ that makes the observed production behavior profit maximizing over the data set S. By contrast, profit inefficiency occurs if $PE_t^C > 0$, with higher values revealing a greater degree of profit inefficiency.

As a final remark, we note that the measure PE_t^C can be computed by means of linear programming. The associated program has a structure that is formally analogous to the one of (**LP-1**) that we present below.[12] Given this direct similarity, and for the sake of compactness, we do not report it here.

4. Shadow prices and duality

In the previous section, we have assumed that the empirical analyst knows the netput price vector $\mathbf{P}_t$ for every DMU t. In practical applications, however, reliable price information is often not available. In such a case, we can conduct efficiency analysis with endogenously defined shadow prices. In what follows, we will apply this shadow pricing idea to the multi-output profit efficiency framework set out above. Next, we will show that the resulting profit inefficiency measure (under shadow prices) has a dually equivalent representation as a multi-output directional distance function, which establishes a multi-output version of the original duality result in [7].

Shadow prices: If we do not observe the true prices that apply to each DMU t, the relevant data set becomes

$$\widehat{S} = \{(\mathbf{Z}_t^1,\dots,\mathbf{Z}_t^M) | t = 1,\dots,T\}.$$

When only $\widehat{S}$ (instead of S) is given, we are forced to use a weakened version of the efficiency criterion in Definition 1. Specifically, we can (only) check whether there exists at least one feasible "shadow" price specification that supports profit efficiency of the evaluated DMU t.

Definition 2 (*Shadow profit efficiency*). Let $\widehat{S} = \{(\mathbf{Z}_t^1,\dots,\mathbf{Z}_t^M)|t = 1,\dots,T\}$ be a data set. Then, DMU t is shadow profit efficient if there exist, for each output m, non-zero output-specific shadow price vectors $\widehat{\mathbf{P}}_t^m = \begin{bmatrix} \widehat{p}_{y,t}^m \\ \mathbf{P}_{x,t} \end{bmatrix} \in \mathbb{R}_+^{1+N}$ such that $\widehat{\mathbf{P}}_t^{m'} \mathbf{Z}_t^m \ge \widehat{\mathbf{P}}_t^{m'} \mathbf{Z}_s^m$ for all observations $s = 1,\dots,T$.

In this case, we can choose the shadow price vector $\widehat{\mathbf{P}}_t^m$ freely (except from the non-zero and non-negativity constraints). Implicitly, the shadow prices $(\widehat{\mathbf{P}}_{x,t}^m)_k$ for the joint inputs k define the (aggregate) DMU prices $(\widehat{\mathbf{P}}_{x,t})_k = \sum_{m=1}^M (\widehat{\mathbf{P}}_{x,t})_k$. Next, we note that shadow prices for the output-specific inputs can be different for different outputs, i.e. for output-specific inputs k we can have $(\widehat{\mathbf{P}}_{x,t}^m)_k \neq (\widehat{\mathbf{P}}_{x,t}^{m'})_k$ when $m \neq m'$.[13]

Following our reasoning of the previous section, we can evaluate our shadow profit efficiency criterion by the following efficiency measure, which endogenizes the (shadow) price

[9] We assume that the denominator $(\sum_{m=1}^M \mathbf{P}_t^{m'} \mathbf{g}_{\mathbf{Z}^m})$ is positive. For the shadow profit inefficiency measure $\widehat{PE}_t^C$ that we introduce below, this is guaranteed by the normalization constraint $\sum_{m=1}^M \widehat{\mathbf{P}}_t^{m'} \mathbf{g}_{\mathbf{Z}^m} = 1$.

[10] In DEA applications, the directional vectors are often DMU-specific, i.e. we have $\mathbf{g}_{\mathbf{Z}^m} = \mathbf{g}_{\mathbf{Z}_t^m}$. It is common in the literature to drop the subscript t for compactness.

[11] See, for example, [15] for a detailed discussion of the benefit-of-the-doubt interpretation of common DEA models.

[12] The only difference between the linear program for PE_t^C and the program (**LP-1**) for $\widehat{PE}_t^C$ involves the inclusion of the price information contained in the data set S (whereas (**LP-1**) applies to shadow pricing). This price information is easily included in the form of linear constraints, which obviously does not interfere with the linear programming nature of (**LP-1**).

[13] In principle, of course, one can impose the constraint that $(\widehat{\mathbf{P}}_{x,t}^m)_k = (\widehat{\mathbf{P}}_{x,t}^{m'})_k$ for some output-specific input k (and $m \neq m'$), which obtains a stronger efficiency criterion. We refer to [11] for an exploration of such a stronger criterion in a multi-output cost efficiency setting that is formally close to the profit efficiency setting that we consider here.

selection in the efficiency evaluation process:

$$\widehat{PE}_t^C = \min_{\widehat{\mathbf{P}}_t^1,\ldots,\widehat{\mathbf{P}}_t^M \in \mathbb{R}_+^{1+N}} \frac{\sum_{m=1}^M \pi_t^m(\widehat{\mathbf{P}}_t^m) - \sum_{m=1}^M \left(\widehat{\mathbf{P}}_t^{m'} \mathbf{Z}_t^m\right)}{\sum_{m=1}^M \widehat{\mathbf{P}}_t^{m'} \mathbf{g}_{\mathbf{Z}^m}}.$$

Similar to before, $\widehat{PE}_t^C$ selects the most favorable netput price vectors $\widehat{\mathbf{P}}_t^m$ to evaluate DMU t's shadow profit efficiency, which effectively applies the benefit-the-doubt pricing in the absence of full price information. It is easy to verify that DMU t satisfies the shadow profit efficiency criterion in Definition 2 if and only if $\widehat{PE}_t^C = 0$, which reveals that there exists at least one possible specification of the shadow price vectors $\widehat{\mathbf{P}}_t^m$ under which DMU t is profit maximizing over the data set $\widehat{S}$.

To operationalize the measure $\widehat{PE}_t^C$, we need to normalize the denominator. In what follows, we will use

$$\sum_{m=1}^M \widehat{\mathbf{P}}_t^{m'} \mathbf{g}_{\mathbf{Z}^m} = 1.$$

Then, we can formulate our (shadow) profit inefficiency measure $\widehat{PE}_t^C$ as solving the linear program (**LP-1**)

$$\widehat{PE}_t^C = \min_{\substack{\widehat{\pi}_t^1,\ldots,\widehat{\pi}_t^M \in \mathbb{R} \\ \widehat{\mathbf{P}}_t^1,\ldots,\widehat{\mathbf{P}}_t^M \in \mathbb{R}_+^{1+N}}} \sum_{m=1}^M \widehat{\pi}_t^m - \sum_{m=1}^M \left(\widehat{\mathbf{P}}_t^{m'} \mathbf{Z}_t^m\right)$$

s.t.

$$\forall m \in \{1,\ldots,M\} : \widehat{\pi}_t^m \geq \widehat{\mathbf{P}}_t^{m'} \mathbf{Z}_s^m \quad \text{for all } s \in \{1,\ldots,T\},$$

$$\sum_{m=1}^M \widehat{\mathbf{P}}_t^{m'} \mathbf{g}_{\mathbf{Z}^m} = 1,$$

where each $\widehat{\pi}_t^m$ represents $\pi_t^m(\widehat{\mathbf{P}}_t^m)$, i.e. the maximum attainable profit (over the data set $\widehat{S}$) in the production of output m given the output-specific prices $\widehat{\mathbf{P}}_t^m$ that apply to the evaluated DMU t.

As a final note, apart from the "aggregate" profit inefficiency measure $\widehat{PE}_t^C$, we can also define profit inefficiency measures $\widehat{PE}_t^{C,m}$ that are specific to individual outputs m. In particular, let $\widehat{\pi}_t^{m*}$ and $\widehat{\mathbf{P}}_t^{m*}$ solve the above linear problem. Then, we can use

$$\widehat{PE}_t^{C,m} = \frac{\widehat{\pi}_t^{m*} - \widehat{\mathbf{P}}_t^{m*'} \mathbf{Z}_t^m}{\widehat{\mathbf{P}}_t^{m*'} \mathbf{g}_{\mathbf{Z}^m}}$$

Clearly, for $\widehat{PE}_t^C = 0$ we will have $\widehat{PE}_t^{C,m} = 0$ for all m. However, if $\widehat{PE}_t^C > 0$, the measures $\widehat{PE}_t^{C,m}$ allow us to allocate DMU t's profit inefficiency to specific outputs. We will illustrate this feature in our empirical application in Section 5.

Dual representation: Interestingly, our shadow profit inefficiency measure has a dual representation as a multi-output version of the directional distance function introduced by Chambers et al. [7]. We believe this is an appealing property as directional distance functions are frequently used in DEA technical efficiency evaluations that simultaneously account for inputs used and outputs produced.

This directional distance function representation appears from the dual version of our linear program (**LP-1**). Specifically, let λ_s^m represent the dual variables for the first constraint (for each output m and DMU s) and β the dual variable for the second constraint of that program. Then, the dual can be written as (**LP-2**)

$$\widehat{PE}_t^C = \max_{\lambda_s^1,\ldots,\lambda_s^M,\beta \in \mathbb{R}_+} \beta$$

$$\forall m \in \{1,\ldots,M\} : \sum_{s=1}^T \lambda_s^m \mathbf{Z}_s^m \geq \mathbf{Z}_t^m + \beta \mathbf{g}_{\mathbf{Z}^m},$$

$$\forall m \in \{1,\ldots,M\} : \sum_{s=1}^T \lambda_s^m = 1.$$

To interpret $\widehat{PE}_t^C$ as a multi-output version of the directional function, we first note that in the case of a single output m, Chambers et al's original version of the general directional distance function is defined as

$$\overrightarrow{D}(\mathbf{Z}_t^m; \mathbf{g}_{\mathbf{Z}^m}) = \max\{\beta \mid (\mathbf{Z}_t^m + \beta \mathbf{g}_{\mathbf{Z}^m}) \in T^m\}.$$

As a natural extension towards our framework with output-specific technologies, we can define the multi-output version of this distance function as

$$\overrightarrow{D}(\mathbf{Z}_t^1,\ldots,\mathbf{Z}_t^M; \mathbf{g}_{\mathbf{Z}^1},\ldots,\mathbf{g}_{\mathbf{Z}^M}) = \max\{\beta \mid \forall m \in \{1,\ldots,M\} : (\mathbf{Z}_t^m + \beta \mathbf{g}_{\mathbf{Z}^m}) \in T^m\}.$$

Then, it is easy to see that we obtain

$$\widehat{PE}_t^C = \overrightarrow{D}(\mathbf{Z}_t^1,\ldots,\mathbf{Z}_t^M; \mathbf{g}_{\mathbf{Z}^1},\ldots,\mathbf{g}_{\mathbf{Z}^M}),$$

if we define the production possibility set of output m as

$$T^m = \{\mathbf{Z}^m \mid \mathbf{Z}^m \leq \sum_{s=1}^T \lambda_s^m \mathbf{Z}_s^m, \sum_{s=1}^T \lambda_s^m = 1,\ \lambda_s^m \geq 0\},$$

i.e. the convex monotone hull of the observed netput vectors $\mathbf{Z}_s^m$. Actually, this convex monotone hull of observed netput vectors is often used as an (empirical) production possibility set in practical DEA analysis. Banker et al. [2] first proposed this technology specification in the DEA literature.[14] A distinguishing feature of our framework is that it uses this specification to construct a production possibility set for each different output m. This follows naturally from our particular set-up, which explicitly considers output-specific production technologies (while accounting for interdependencies through joint inputs).

Summarizing, we conclude that our shadow profit inefficiency measure can also be represented as a multi-output technical inefficiency measure. In particular, it can be characterized as a multi-output directional distance function defined for output-specific technologies that are convex and monotone. A specific feature of this characterization is that it accounts for joint input use in the process of multi-output production.

Example. To illustrate the directional distance function representation of our shadow profit inefficiency measure, we make use of a fictitious example with three DMUs A, B and C that produce two outputs y^1 and y^2 by using one input x. The fact that we consider

Table 1
Data and input allocation. for the three DMUs.

DMU	y^1	y^2	x	$\alpha^1 x$	$\alpha^2 x$
DMU A	3	5	5	2	3
DMU B	5	1	5	4	1
DMU C	4	2	13	8	5

[14] Banker et al. [2] show that we obtain the convex monotone hull as a DEA-type technology approximation if we add the technology assumptions convexity and monotonicity to our Axiom 1. In words, monotonicity implies that the outputs and inputs are freely (or strongly) disposable; i.e. producing less outputs cannot lead to use more inputs and using more inputs never reduces the outputs. It also implies that marginal rates of substitution/transformation (between inputs, outputs, and inputs and outputs) are nowhere negative or, in other words, there is no congestion. Next, convexity says that convex combinations of feasible netput vectors are themselves also technically feasible. This implies that marginal rates of substitution/transformation (between inputs, output and inputs and outputs) are nowhere increasing. The fact that the dual representation of our shadow profit inefficiency measure implies a production set that is convex and monotone follows from the result that these technology properties are essentially "irrelevant" for profit efficiency analysis (i.e. imposing the properties will not interfere with the profit efficiency results). See, for example, [43] for a detailed discussion of this last point.

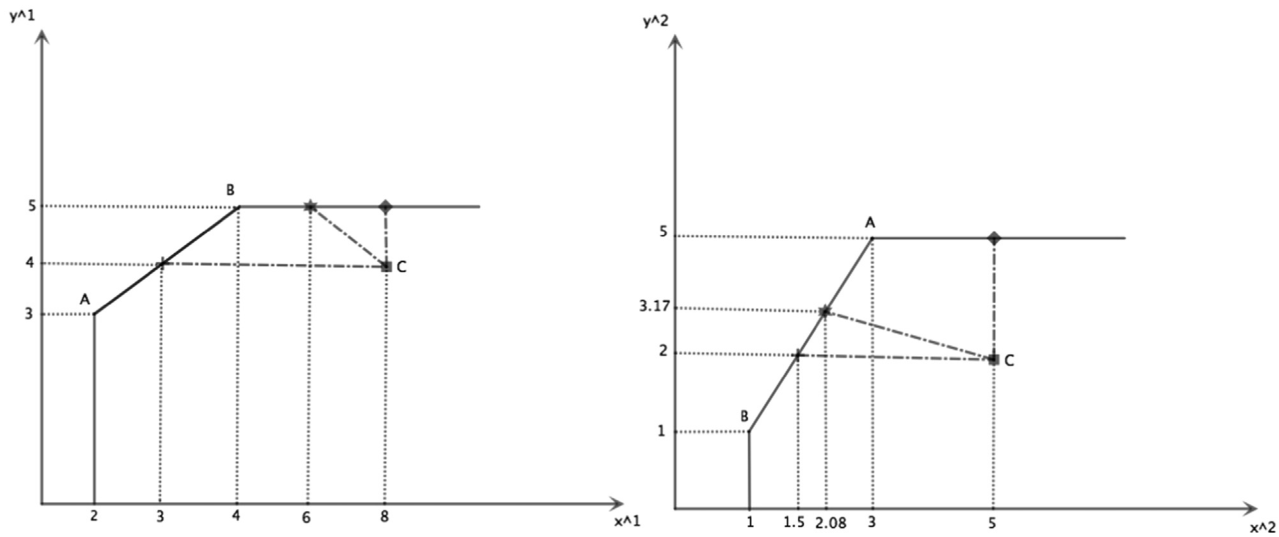

Fig. 1. Production process – 2 outputs/1 allocated input.

Table 2
Efficiency scores for the three DMUs.

DMU	$\widehat{PE}^{C,E}$	$\widehat{PE}^{C,I}$	$\widehat{PE}^{C,O}$
DMU A	0	0	0
DMU B	0	0	0
DMU C	0.250	0.625	0.250

only a single input makes it possible to graphically represent the efficiency analysis. We assume that input x is output-specific and we denote the allocation to output 1 by $\alpha^1 x$ and to output 2 by $\alpha^2 x$.[15] Table 1 presents the relevant output and input numbers.

As explained above, our shadow profit inefficiency measure can be represented as a multi-output directional distance function defined with respect to output-specific production sets that are constructed as convex monotone hulls of the netput vectors given in Table 1. These possibility sets are given in Fig. 1, which also represents the three DMUs under evaluation. For both outputs, the production frontiers are fully defined by DMUs *A* and *B*. For output 1, DMU B is efficient because there are no other DMUs producing more output, and DMU A is efficient because it uses less input than the other two DMUs. A similar reasoning holds for output 2. In this case, DMU A produces the most output and DMU B uses the least input.

DMU C is clearly not on any frontier, which implies that it is inefficient. As explained above, measuring the degree of inefficiency requires the specification of a direction vector. For the current example, we consider the same three specifications of direction vectors as in our following empirical application, i.e. $\mathbf{g}_{\mathbf{Z}^m} = (0, \mathbf{X}^m)$, $\mathbf{g}_{\mathbf{Z}^m} = (y^m, 0)$ and $\mathbf{g}_{\mathbf{Zm}} = (y^m, \mathbf{X}^m)$. We denote the associated efficiency measures as $\widehat{PE}^{C,I}$, $\widehat{PE}^{C,O}$ and $\widehat{PE}^{C,E}$. As we will explain in more detail in Section 5, these efficiency measures correspond to quantifying profit inefficiency in terms proportional input reduction (for $\mathbf{g}_{\mathbf{Zm}} = (0, \mathbf{X}^m)$), proportional output expansion (for $\mathbf{g}_{\mathbf{Zm}} = (y^m, 0)$) and simultaneous input reduction and output expansion (for $\mathbf{g}_{\mathbf{Zm}} = (y^m, \mathbf{X}^m)$), respectively.

The efficiency results for the three measures are presented in Table 2. Let us consider in more detail the results for the measure $\widehat{PE}^{C,I}$, which quantifies the degree of inefficiency in terms of proportional input reduction. DMUs *A* and *B* both have an inefficiency score equal to zero, since they are situated on the production frontiers for both outputs. To calculate the inefficiency score of DMU C, we first need to consider both outputs individually. From the left panel of Fig. 1 we learn that, for output 1, a convex combination of DMUs *A* and *B* (with $\lambda_A^1 = \lambda_B^1 = 1/2$) defines a potential input reduction of 62.5% (from 8 to 3 input units). Similarly, from the right panel of Fig. 1, we find that a convex combination of DMUs *A* and *B* (with $\lambda_A^2 = 0.25$ and $\lambda_B^2 = 0.75$) reveals a potential input reduction of 70% (from 5 to 1.5 input units). Using our above definition, the multi-output distance function equals $\min\{62.5\%, 70\%\} = 62.5\%$, which corresponds to the maximum equi-proportionate input reduction when simultaneously accounting for the production possibility constraints that apply to the two individual outputs.[16]

A directly analogous interpretation holds for the other two directions of measurement, i.e. $\widehat{PE}^{C,O}$ (output expansion) and $\widehat{PE}^{C,E}$ (simultaneous input and output improvement). For compactness, we will not discuss this in detail. One notable observation from Table 2 is that the particular choice of direction vector impacts the efficiency results. This will also hold for our empirical application in Section 5.

5. Empirical application

We illustrate the empirical usefulness of our (shadow) profit efficiency method by an application to the input and output data that were also analyzed by Cherchye et al. [10] in their original study. These authors focused on multi-output cost efficiency. As such, we complement this first study by assessing the profit efficiency of the same DMUs. In the following, we will first discuss the specificities of our data. Subsequently, we present the findings of our empirical analysis.

[15] To facilitate our discussion, we here consider a situation with a single, output-specific input. However, analogous examples with joint inputs are easy to construct.

[16] As an additional note, we can verify that, for the same data set, the DEA model of Banker et al. [2] obtains a maximum input reduction for DMU C equal to 61.5% (from 13 to 5 units of the aggregate input x), which is below the reduction of 62.5% that we identify through our methodology. Just like our DEA model, the original model of Banker, Charnes and Cooper also uses a convex monotone hull representation of the production technology. However, a basic difference is that it does not consider output-specific production possibility sets. This illustrates that our use of output-specific possibility sets enhances the "dicriminatory power" (i.e. potential to detect inefficient production behavior) of the DEA evaluation. See also [9] for a similar argument in a cost efficiency setting.

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After showing our results for the DMUs' aggregate profit inefficiency, we also consider output-specific profit inefficiencies.

The data: Our data set contains input and output information for 290 offices (DMUs) of a large European service company. Each DMU uses 7 inputs, i.e. three types of labor (x^1, x^2 and x^3), three types of transport (x^4, x^5 and x^6) and other overhead cost (x^7), for the production of 7 outputs. Thus, we have $N=7$ and $M=7$. All 7 inputs are expenditures, so prices times quantities. In a sense, this effectively accounts for input quality differences across DMUs. Specifically, higher quality inputs typically have higher prices, which in turn lead to higher expenditure levels (for given quantities). See, for example, [5,6,32,39] on the relevance of taking input quality into account in DEA assessments.

The service company uses an "activity-based costing system", which allows us to allocate the first 6 inputs to the 7 individual outputs. That is, adopting the terminology of Section 2, the three types of labor and transport are output-specific. The other overhead cost is modeled as a joint input, which simultaneously benefits the production of all 7 outputs. We refer to [10] for more detailed information on the input and output data that we use.[17]

Because we have no data on input and output prices, we conduct a shadow profit efficiency analysis by using the methodology that we presented in Section 4. That is, we evaluate each DMU's profit efficiency by using "most favorable" input and output prices. In particular, we consider two exercises. Our first exercise does not impose any restriction on the possible prices (except from non-negativity). Obviously, this computes profit efficiency of DMUs in the best possible light. However, since no guidance is given in the shadow price selection process, it may well use shadow prices that are very far from the prices really faced by the offices under evaluation. Therefore, our second exercise makes use of price restrictions that better guarantee "realistic" shadow prices. These restrictions have been defined in consultation with the management of the service company, and were also used by Cherchye et al. [10] in their original study.

Multi-output profit efficiency: To compute the shadow profit inefficiency measure $\widehat{PE}^C$, we first need to specify the directional vector $\mathbf{g}_{\mathbf{Z}^m}$ for each output m. To demonstrate the versatility of our approach, we will consider three different directional vectors. These three directional vectors are the most popular ones in applied DEA analysis and, as we will explain below, imply alternative interpretations of the observed degree of profit inefficiency.[18] The first two directional vectors are:

- $\mathbf{g}_{\mathbf{Z}^m}=(y^m,\mathbf{0})$ for each output m, which measures profit inefficiency in terms of proportional output increase, and
- $\mathbf{g}_{\mathbf{Z}^m}=(0,\mathbf{X}^m)$ for each output m, which measures profit inefficiency in terms of proportional input reduction.

An attractive feature of these directional vectors is that they imply shadow profit inefficiency measures that have a dual representation in terms of the input and output oriented [21–30] technical efficiency measures, respectively.[19]

The third specification of the directional vector is:

- $\mathbf{g}_{\mathbf{Z}^m}=(y^m,\mathbf{X}^m)$ for each output m, which measures profit inefficiency in terms of equiproportional output increase and input reduction.

We opt for using this additional specification because it simultaneously considers output and input improvements in the DMUs' efficiency assessment. Basically, in its dual form, the resulting profit inefficiency measure combines the Debreu–Farrell input and output efficiency measures in a single metric.

Table 3 summarizes the results of our first profit efficiency analysis, in which we do not impose any restrictions on the possible shadow prices (except from non-negativity). $\widehat{PE}^{C,E}$ refers to profit inefficiency with equiproportionate output increase and input reduction, $\widehat{PE}^{C,I}$ to profit inefficiency with input reduction only, and $\widehat{PE}^{C,O}$ to profit inefficiency with output increase only. We provide summary statistics on the distribution of the inefficiency measures, as well as information on the number of efficient DMUs (in absolute and relative terms).[20]

Interestingly, we observe quite some variation in profit inefficiency across the DMUs in our sample, for all three directional vectors under consideration. For the measure $\widehat{PE}^{C,E}$ the mean profit inefficiency amounts to 15.9%. On average, the offices should equiproportionally reduce inputs and expand outputs by 15.9% to attain shadow profit efficiency. Next, for the measure $\widehat{PE}^{C,I}$, we find a mean inefficiency of 27.7%. Thus, if output is kept fixed, we need an average input reduction of 27.7% to achieve efficiency. Finally, the mean value of $\widehat{PE}^{C,O}$ equals 31.1%, which signals that profit efficiency requires an average output expansion of 31.1% when inputs are fixed at their given level.

We also observe that the numbers of efficient DMUs are not the same for our three specifications of the directional vectors. This may seem surprising at first sight, since our three efficiency measures are based on the same (shadow) profit efficiency criterion. The different results in Table 3 pertain to the different directional vectors underlying $\widehat{PE}^{C,E}$, $\widehat{PE}^{C,I}$ and $\widehat{PE}^{C,O}$. In particular, it is well-known in the DEA literature that different directions of efficiency measurement can yield alternative (in)efficiency classifications. For example, it is well possible that a DMU may not be able to increase its outputs without affecting its inputs (i.e. efficient in terms of $\widehat{PE}^{C,O}$), while it can decrease its inputs for the given outputs (i.e. inefficient in terms of $\widehat{PE}^{C,I}$). See, for instance, [45,33] for related discussions.[21]

The profit efficiency scores reported in Table 3 do not impose any restriction on possible shadow prices. As indicated above, this implies that the profit efficiency results are based on (shadow) prices that may be very far from the prices really faced by the DMUs. We anticipate this concern in our second efficiency measurement exercise, which uses price restrictions provided by the management of the service company under evaluation.

[17] Cherchye et al. [10] also explain that confidentiality and strict non-disclosure agreements prohibit us from providing more details on the nature and operations of the service company under study.

[18] Our three directional vectors are DMU specific, which implies that we cannot aggregate the efficiency scores to a regional or firm level; see [27] for more discussion. This is in line with the objective of the management of the service company under study, who wanted to benchmark the individual DMUs. As should be clear from our discussion above, alternative choices, such as a common directional vector for all DMUs, are also feasible.

[19] In particular, $\mathbf{g}_{\mathbf{Z}^m}=(y^m,\mathbf{0})$ obtains (a multi-output version of) the Debreu–Farrell output measure (DF_O) minus one as the outcome of our program (**LP-2**), i.e. $\widehat{PE}^C=DF_O-1$ (where $DF_O\geq 1$ and $DF_O=1$ indicates efficiency). Similarly, $\mathbf{g}_{\mathbf{Z}^m}=(0,\mathbf{X}^m)$ obtains one minus (a multi-output version of) the Debreu–Farrell input measure (DF_I) as the outcome of (**LP-2**), i.e. $\widehat{PE}^C=1-DF_I$ (where $DF_I\leq 1$ and $DF_I=1$ indicates efficiency). See, for example, [7,25] for a detailed discussion on the relations between directional distance functions and Debreu–Farrell efficiency measures (including dual representations). Using our results in Section 4, we can extend these authors' arguments to our particular multi-output setting.

[20] It can be verified that, for a given DMU, the value of $\widehat{PE}^{C,E}$ can never exceed the values of $\widehat{PE}^{C,O}$ and $\widehat{PE}^{C,I}$, by the very construction of these measures. This definitional property also appears from the results in Table 3.

[21] This also relates to the so-called "slack problem" that received considerable attention in the DEA literature. In terms of the shadow price representation of DEA measures which we adhere to here, this slack problem corresponds (dually) to the possibility of zero shadow prices, which prevails in the case without price restrictions (as for the efficiency results in Table 3).

Table 3
Multi-output profit efficiencies without price restrictions.

Statistics	$\widehat{PE}^{C,E}$	$\widehat{PE}^{C,I}$	$\widehat{PE}^{C,O}$
Min	0	0	0
Mean	0.159	0.277	0.370
Median	0.149	0.286	0.311
Max	0.647	1	1
St. dev.	0.132	0.198	0.326
# Efficient	33	27	33
% Efficient	11.38	9.31	11.38

Table 6
Share of the total production per output (sample average).

Outputs	Share of the total production (%)
Output 1	90.78
Output 2	0
Output 3	6.84
Output 4	0.32
Output 5	0.04
Output 6	0.91
Output 7	1.09
Total	100

Table 4
Multi-output profit efficiencies with price restrictions.

Statistics	$\widehat{PE}^{C,E}$	$\widehat{PE}^{C,I}$	$\widehat{PE}^{C,O}$
Min	0	0	0
Mean	0.175	0.322	0.460
Median	0.167	0.322	0.452
Max	0.647	1	1
St. dev.	0.136	0.195	0.362
# Efficient	0	0	0
% Efficient	0	0	0

Table 5
Output-specific profit efficiencies.

Statistics	$\widehat{PE}^{C,E}$	$\widehat{PE}^{C,1}$	$\widehat{PE}^{C,2}$	$\widehat{PE}^{C,3}$	$\widehat{PE}^{C,4}$	$\widehat{PE}^{C,5}$	$\widehat{PE}^{C,6}$	$\widehat{PE}^{C,7}$
Min	0	0	0	0	0	0	0	0
Mean	0.175	0.126	0.953	0.615	0.543	0.392	0.514	0.442
Median	0.167	0.096	1	0.689	0.638	0.390	0.622	0.489
Max	0.647	0.792	1	1	1	0.929	0.990	0.995
St. dev.	0.136	0.131	0.191	0.337	0.270	0.282	0.266	0.249
# Efficient	0	82	7	27	23	45	27	28
% Efficient	0	28.28	30.43	9.31	7.93	15.52	9.31	9.66

As an introductory note to our results for this second exercise, we remark that it may actually be that no DMU is found to be profit efficient at the aggregate level. As discussed in Section 4 (at the end of the paragraph "Shadow prices"), the overall profit inefficiency measure $\widehat{PE}^{C}$ can be seen as the aggregate of output-specific profit inefficiency measures $\widehat{PE}^{C,m}$. From this perspective, aggregate profit efficiency (i.e. $\widehat{PE}^{C}=0$) requires profit efficiency for each individual output m (i.e. $\widehat{PE}^{C,m}=0$ for all m), with the output-specific efficiencies evaluated for output-specific production possibility sets. Thus, if no individual DMU is profit efficient for all outputs simultaneously, then we effectively obtain that no DMU is overall profit efficient. This will turn out to be the case for our profit efficiency analysis with price restrictions.[22]

Table 4 provides a first summary of our profit efficiency results computed under price restrictions. We consider the same three directional vectors as before. By construction, the inefficiency scores in this table reveal greater (shadow) profit inefficiency than the ones in Table 3, because we limit the feasible ranges of shadow prices. But, in general, the efficiency patterns we observe are roughly similar to the ones in Table 3. However, one notable and important difference is that we no longer have any DMU that is labeled efficient at the aggregate level. As explained above, this indicates that there are no DMUs that simultaneously produce all 7 outputs efficiently when restricting the possible shadow prices.

Output-specific multi-output profit efficiency: As indicated in Section 4, we can allocate a DMU's aggregate profit inefficiency to individual outputs by computing output-specific profit inefficiencies. We believe that this provides useful management input as it helps to better identify specific output production processes where substantial profit efficiency gains are possible. By using this information, DMU managers can direct their performance improvement actions in a more effective way.

We will illustrate this practice for the (aggregate) profit inefficiency measure $\widehat{PE}^{C,E}$ with price restrictions (see also Table 4). Table 5 summarizes our results.[23] In that table, each $\widehat{PE}^{C,m}$ gives the profit inefficiency specific to output m ($m=1,\dots,7$). These measures have the same interpretation as the aggregate measure $\widehat{PE}^{C,E}$, but now the equiproportionate input reduction and output expansion specifically applies to the m-th output production process.

Table 5 reveals substantial heterogeneity across the 7 outputs.[24] For example, we find that the production processes of outputs 1 and 2 appear to be most efficient, in terms of both the average inefficiency (only 16.7% and 9.6%) and the number of efficient DMUs (28.28% and 31.03%). By contrast, the offices are, on average, less efficient in the production of outputs 3, 4, 6 and 7 and the biggest number of inefficient DMUs is found for output 5. It is useful to relate these observations to the production shares of the 7 outputs that are given in Table 6. Interestingly, output 1 has by far the greatest average share, whereas the production share of output 2 is virtually zero.

Apart from revealing interesting efficiency patterns at the level of the full sample of DMUs (see Table 5), our output-efficiency scores also provide useful information for individual DMUs. We illustrate this by means of Table 7, which shows the inefficiency results for two selected DMUs. In parenthesis, we also report the weights of the output-specific profit efficiency scores to the aggregate profit efficiency score. These weights sum to one by construction.

DMU 40 is close to efficient in terms of our aggregate efficiency score, whereas DMU 161 exhibits considerable inefficiency in terms of its aggregate score. The output-specific efficiency scores give a more

[22] As a related remark, if no price restrictions (except from non-negativity) are imposed (as in our first exercise, with results in Table 3), it can well be that the aggregate efficiency score is fully determined by a single output. In this case, all other outputs get an implicit weight of zero in the calculation of the overall profit efficiency. By implication, if there are no price restrictions, we always have at least one DMU that is profit efficient at the aggregate level (as there is at least one DMU that is profit efficient for each single output). We can exclude that a single output fully defines a DMU's aggregate efficiency score by imposing restrictions that exclude zero shadow prices (as in our second exercise, with results in Table 4).

[23] The results for the measures $\widehat{PE}^{C,I}$ and $\widehat{PE}^{C,O}$ are reported in Tables 8 and 9. The interpretation of these tables is directly analogous to the one of Table 5.

[24] To compute the percentages of efficient DMUs that are reported in Table 5, we only take into account DMUs that produce non-zero output quantities, i.e. all 290 DMUs for all the outputs except from output 2 (22 DMUs) and output 3 (285 DMUs).

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Table 7
Output-specific profit efficiencies for selected DMUs.

DMU	$\widehat{PE}^{C}$	$\widehat{PE}^{C,1}$	$\widehat{PE}^{C,2}$	$\widehat{PE}^{C,3}$	$\widehat{PE}^{C,4}$	$\widehat{PE}^{C,5}$	$\widehat{PE}^{C,6}$	$\widehat{PE}^{C,7}$
DMU 40	0.00367	0	1	0.345	0.601	0	0.581	0
		(0.132)	(0.00172)	(0.00256)	(0.000429)	(0.534)	(0.00138)	(0.328)
DMU 161	0.386	0.312	1	0.505	0.750	0.293	0.694	0.622
		(0.808)	(0.0517)	(0.0699)	(0.0113)	(0.00361)	(0.0369)	(0.0186)

Table 8
Output-specific profit inefficiencies for the input reduction direction.

Statistics	$\widehat{PE}^{C}$	$\widehat{PE}^{C,1}$	$\widehat{PE}^{C,2}$	$\widehat{PE}^{C,3}$	$\widehat{PE}^{C,4}$	$\widehat{PE}^{C,5}$	$\widehat{PE}^{C,6}$	$\widehat{PE}^{C,7}$
Min	0	0	0	0	0	0	0	0
Mean	0.322	0.215	0.960	0.710	0.663	0.526	0.639	0.587
Median	0.322	0.193	1	0.826	0.784	0.610	0.777	0.675
Max	1	0.868	1	1	1	1	1	1
# Efficient	0	77	7	28	24	41	25	27
% Efficient	0	26.55	40.90	9.66	8.28	14.14	8.62	9.31

Table 9
Output-specific profit inefficiencies for the output expansion direction

Statistics	$\widehat{PE}^{C}$	$\widehat{PE}^{C,1}$	$\widehat{PE}^{C,2}$	$\widehat{PE}^{C,3}$	$\widehat{PE}^{C,4}$	$\widehat{PE}^{C,5}$	$\widehat{PE}^{C,6}$	$\widehat{PE}^{C,7}$
Min	0	0	0	0	0	0	0	0
Mean	0.460	0.363	0.961	0.830	0.837	0.709	0.840	0.799
Median	0.452	0.263	1	1	1	1	1	1
Max	1	1	1	1	1	1	1	1
# Efficient	0	71	7	29	23	33	24	24
% Efficient	0	24.48	40.90	10.00	7.93	11.38	8.28	8.28

balanced picture of the efficiency performance of these two DMUs. For example, we find that the high efficiency of DMU 40 is particularly due to its efficient production of the outputs 1, 5 and 7. However, there is substantial potential to realize efficiency gains in the production of the outputs 2, 4 and 6 (and also, but to a far lesser extent, in the production of output 3). As for DMU 161, we find that the high level of aggregate inefficiency is caused by inefficient production of all the outputs. But, again, we observe substantial heterogeneity across outputs (with output-specific inefficiencies ranging from 29.3% to 100%).

We believe these two examples clearly show the usefulness of our output-specific efficiency measures to direct the attention of DMU managers towards individual outputs that are characterized by profit inefficiency. Importantly, this holds not only for DMUs with low efficiency (like DMU 161) but also for DMUs of which the aggregate performance is close to efficient (like DMU 40).

6. Conclusion

We presented a novel DEA toolkit for profit efficiency analysis in the context of multi-output production. A distinguishing feature of our methodology is that it assumes output-specific production technologies. In addition, the methodology accounts for the use of joint inputs, and explicitly includes information on the allocation of inputs to specific outputs.

We have specified a multi-output profit inefficiency measure when prices are observed, as well as a shadow profit inefficiency measure that can be used if prices are unknown. Our framework also allows us to define output-specific profit inefficiency measures, which allocate a DMU's aggregate profit inefficiency to individual outputs. Finally, we established a dual relationship between our multi-output profit inefficiency measure and a technical inefficiency measure that takes the form of a multi-output directional distance function.

We illustrated our methodology by an empirical application to a large European service company. This demonstrated the practical usefulness of our measure for (shadow) profit inefficiency at the aggregate DMU level. Next, we showed that our output-specific profit inefficiency measures provide useful management input. They can identify individual outputs that are characterized by substantial inefficiency, so that performance improvement actions can be directed primarily towards these outputs.

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