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Pareto-Koopmans Efficiency and Network DEA

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ABSTRACT

Standard or black-box data envelopment analysis (DEA) evaluates the efficiency of the transformation of a DMU's exogenous inputs into its final outputs by ignoring what is going on in its divisions (sub-DMUs). To cope with this problem, network DEA (NDEA), which can provide adequate detail to management, has been developed and applied empirically. However, we show that some of the commonly used NDEA methods are inconsistent with the notion of Pareto-Koopmans efficiency. Since the original development of DEA, Pareto-Koopmans efficiency is a fundamental property used in DEA. From a Pareto-Koopmans efficiency perspective, therefore, we propose a two-phase NDEA approach that can provide information on both each DMU's overall (system) efficiency status and its divisions' efficiency scores. The proposed novel approach is developed based on the enhanced Russell graph model or equivalently the slacks-based model. We also propose several theorems and illustrate the proposed approach using two artificial numerical examples and a real-world data set.

Keywords: Data Envelopment Analysis, network DEA, dominance, divisional efficiency, network Russell efficiency, Pareto-Koopmans efficiency, sub-vector efficiency

1. Introduction

Standard or black-box data envelopment analysis (DEA) is a set of mathematical programming techniques for measuring the efficiency performance of decision making units (DMUs) that convert exogenous inputs into final outputs. In standard DEA, the internal production processes of DMUs are ignored and the exogenous inputs consumed and final outputs produced by the DMUs are the only consideration for efficiency evaluation. On the other hand, network DEA (NDEA) attempts to formulate the internal operations of the evaluated DMU and thus intermediate products (which are outputs coming from divisions (sub-processes) and inputs utilized by others) are explicitly taken into account. In other words, NDEA intends to open the black box so as to provide adequate detail to management and can provide detailed information on the efficiency of divisions (or sub-DMUs) at the assessed DMU as well as its efficiency status. NDEA can be thought of as a generalization of standard DEA.

DEA researchers developed various NDEA models for evaluating the efficiency of DMUs (Färe and Grosskopf (1996, 2000); Lewis and Sexton (2004); Prieto and Zofio (2007); Kao (2009, 2014); Tone and Tsutsui (2009); Lozano (2011); Du,Chen and Huo (2015)) and other researchers focused on the efficiency performance of DMUs which have internal series (e.g., two-stage or three-stage) structures (Sexton and Lewis (2003); Kao and Hwang (2008); Liang, Cook and Zhu (2008); Fukuyama and Weber (2010) ; Cook, Liang and Zhu (2010); Fukuyama, Weber and Xia (2015); Halkos, Tzeremes and Kourtzidis (2014); Akther, Fukuyama and Weber (2013)). A review of the NDEA models can be found in Kao (2014). The management of the DMU often would like to know the sources of inefficiency within it, but some of the existing network DEA methods do not fully provide information on the DMU's overall efficiency status that is consistent with Pareto-Koopmans efficiency with the full consideration of internal flows or intermediate products. Obviously, adoption of Pareto-Koopmans efficiency stems from the possibility principle or free disposal hull which is for example given in the A3 postulate of Cooper, Seiford, Tone and Zhu (2007). In NDEA Tone and Tsutsui (2009) used the possibility principle for only exogenous inputs and final outputs similar to standard DEA which doesn't have intermediate products.

In the DEA literature, there are two efficiency notions: weak efficiency and Pareto-Koopmans efficiency. The Farrell-Debreu measure is calculated based on the weakly efficient frontier and hence possibly existing nonzero slacks are ignored in its efficiency measurement. By contrast, the original model by Charnes, Cooper and Rhodes (CCR, 1978) is developed with the intension of incorporating the notion of full efficiency or Pareto-Koopmans efficiency. Charnes, Cooper and their associates have incorporated this efficiency notion with the use of the non-Archimedean infinitesimal. In a black-box setting, a DMU is Pareto-Koopmans efficient if and only if it is impossible to make an improvement in the utilization of any input or output without worsening some of the other inputs and/or outputs. Hence, Charnes, Cooper and their associates relate the CCR model to the notion of Pareto-Koopmans efficiency. See Charnes and Cooper (1984, 1985), Charnes, Cooper, Golany and Seiford (1985) and Cooper, Seiford, Tone and Zhu (2007) for detailed discussion of the difference between the two notions. The additive model, Russell models and slacks-based models are alternative methods to incorporate Pareto-Koopmans efficiency.

Another motivation for using Pareto-Koopmans efficiency is that we can provide a criterion to improve overall system efficiency in NDEA --- we will show how the notion can be utilized to obtain an efficient target based on this criterion.

The present study analyzes NDEA with respect to Pareto-Koopmans efficiency for the situation where intermediate products are not supplied or demanded outside the assessed system or DMU. Moreover, we assume that all intermediate products are desirable. The studies that are explicitly based on Pareto-Koopmans efficiency with respect to the evaluated DMU and its divisions, include Lewis and Sexton (2004) and Fukuyama and Mirdehghan (2012). In a two-stage problem where stage 1's outputs are the only inputs to stage 2, Lewis and Sexton (2004, p.1374) stated that a necessary condition for a DMU to be overall efficient is that each division is fully efficient, but efficiency in all sub-DMUs or divisions is not sufficient for overall efficiency of the DMU. Their definition of overall system efficiency is based on Pareto-Koopmans efficiency, which is the situation where it is not possible for a DMU to improve any exogenous input, final output or intermediate product without worsening some other exogenous inputs, final outputs or intermediate products. In a general NDEA setting where each division can have the three types of production variables, Fukuyama and Mirdehghan (2012) showed how to identify the overall system efficiency status of DMUs for the fixed link[1] formulation where each observed intermediate product is restricted between the intermediate output of one division and the intermediate input of another. For the free link formulation, however, a straightforward application of Fukuyama and Mirdehghan's method (2012) does not always identify the overall efficiency status of the assessed DMU which consists of divisions or sub-DMUs. Therefore, the purpose of the present study is not only to implement the notion of

[1] The term "fixed link" is used by Tone and Tsutsui (2009) to deal with the situation where the linking activities of a DMU are fixed and hence the intermediate products are discretionary (beyond the control of management). The present study develops an alternative general network framework that deals with Pareto-Koopmans efficiency. For a more discussion on Tone and Tsutsui's (2009) fixed link case, see Fukuyama and Mirdehghan (2012).

Pareto-Koopmans efficiency to the determination of a DMU's overall efficiency status but also to show how to gauge divisional efficiencies within the free link network framework[2]. For this purpose, we provide a novel necessary and sufficient condition for a DMU to be Pareto-Koopmans efficient in general NDEA. Our two-phase approach[3]is based on the enhance Russell graph model of Pastor et al. (1999). We adopt this model because: (i) the original NDEA contributions have been made based on slacks-based models (Tone and Tsutsui (2009, 2014); Kao (2014)); and (ii) the original efficiency measurement framework for dealing with positive slacks, or equivalently asymmetric scaling factors, is developed under input orientation (Färe and Lovell (1978)).

The organization of the paper is as follows. Section 2 provides two motivating examples as well as the basics and then develops a new network efficiency measurement framework. Section 3 makes comparisons with some other network models using two artificial numerical examples, and then shows the use of some existing approaches can lead to a non Pareto-Koopmans efficient solution. In section 4 we apply the proposed framework to the data of 27 Taiwanese banks as a real-life application. The last section concludes with several remarks. All the proofs of the Theorems are relegated to Appendix A.

2. Dominance, motivating examples and Network DEA

2.1Mathematical dominance and Pareto-Koopmans Efficiency

The basic definition of efficiency in multiple criteria decision making,

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[2]This research focuses on a framework in which inputs and outputs are not shared. See Castelli, Pesenti and Ukovich (2010) for a framework of shared inputs and outputs across divisions.

[3] In this paper the terms "model" and "measure" mean an efficiency measurement mathematical problem and its optimized objective function value, respectively. The two-phase "approach" is our proposed two-phase method, in which the Pareto-Koopmans efficiency statuses of DMUs are identified by solving two models in two phases.

particularly in DEA, is provided from a mathematical dominance (Pareto-Koopmans efficiency) perspective. However, Pareto-Koopmans efficiency is not necessarily utilized in NDEA. In actual fact, there are many NDEA studies, whose results are inconsistent with Pareto-Koopmans efficiency. That is, the evaluated DMU, rated as efficient in these NDEA models, can be dominated by another observed DMU, in which situation Pareto-Koopmans efficiency is violated.

In order to deal with this problem (the violation of Pareto-Koopmans efficiency) we suggest a two-phase approach based on three dominance notions --- our suggested definition of overall system efficiency in NDEA exactly corresponds to the case where the NR measure is one(i.e., phase-1 efficient) and all divisions are efficient.

We start with three mathematical dominance notions with respect to pair-wise comparisons: (i) full product vector dominance, (ii) sub-vector dominance, and (iii) dominance at the division level. DMU_a, consisting of divisions or sub-DMUs, is said to fully dominate DMU_b if the full product vector of DMU_a dominates the corresponding product vector of DMU_b, where the full vector comprises the total amounts of not only exogenous inputs and final outputs but also intermediate products. DMU_a is said to sub-vector dominate DMU_b if $\mathrm{DMU}_a's$ sub-vector dominates the corresponding sub-vector of DMU_b, where the sub-vector consists of only exogenous inputs and final outputs (without intermediate products). Therefore, the first definition of dominance considers all division's intermediate products, whereas the second does not. The third definition only deals with dominance at a division level of the evaluated DMU.

Here, we make pair-wise vector comparisons between different DMUs when internal flows exist. The three definitions are utilized to determine the Pareto-Koopmans efficiency status of the evaluated DMU. The three dominance notions are formally presented in the next section.

The adoption of vector dominance allows us to distinguish three kinds of efficiency: (a) Pareto-Koopmans efficiency, (b) sub-vector efficiency, and (c) divisional efficiency. Pareto-Koopmans efficiency at the evaluated DMU is determined by full-vector dominance. By contrast, the sub-vector efficiency status at the DMU is identified by sub-vector dominance. Sub-vector efficiency is often used to define efficiency in NDEA. Note that Pareto-Koopmans efficiency implies sub-vector efficiency but not the other way around. See Lewis and Sexton (2004, p.1374) and Castilli, Pesenti and Ukovich (2010, p.222) on this point.

2.2 Motivating Examples

In this subsection, we provide two motivating examples from a model building point of view. Consider Figure 1 that consists of two DMUs, each of which has two divisions, and the DMUs employ one exogenous input, two intermediate products and one final output. The amounts of exogenous inputs and final outputs of the two DMUs are the same and the only difference between the two DMUs is the amount of intermediate products. Division 1 of DMU_2 produces a more amount of intermediate product than division 1 of DMU_1, even though the two DMUs consume the same amount of exogenous input and produce the same amount of final output. Based on the notion of dominance at the division level defined in the previous sub section, we conclude that Division 1 of DMU_2 and Division 2 of DMU_1 are efficient and Division 1 of DMU_1 and Division 2 of DMU_2 are inefficient. Now consider the two-stage constant-returns-to-scale network CCR (NCCR) model due to Kao and Hwang (2008). The NCCR model is a two-stage network model, in which the first division's outputs are the only inputs to the second stage. Clearly, the application of the NCCR model to Example 1 leads to the situation contradictory to Pareto-Koopmans efficiency because Divisions 1 and 2 are considered efficient for both DMUs.

<<Figure 1>>about here

Next, we show that the NCCR model does not find an efficiency score uniquely using Example 2 depicted in Figure 2. This result is of great importance because multiple solutions are inconsistent with Pareto-Koopmans efficiency. The efficiency of DMU_2 using the NCCR model is obtained by solving the following linear program:

$$
\begin{aligned}
Max \quad & 3u \\
s.t. \quad & 2v = 1 \\
& 2w_1 + w_2 - v \le 0 \qquad (1.1) \\
& 2u - 2w_1 - w_2 \le 0 \qquad (1.2) \\
& w_1 + 2w_2 - 2v \le 0 \qquad (1.3) \\
& 3u - w_1 - 2w_2 \le 0 \qquad (1.4) \\
& u, v, w_1, w_2 \ge \varepsilon
\end{aligned}
\tag{1}
$$

where ε is the non-Archimedean infinitesimal. If we evaluate DMU_1 and DMU_2, we see that DMU_1 is efficient and DMU_2 is inefficient with the efficiency score of 0.75. For DMU_1, Divisions 1 and 2 are efficient for all optimal solutions. However, we see that in evaluating inefficient DMU_2, the model produces multiple optimal solutions. One optimal solution is:

$$u = \frac{1}{4},\ v = \frac{1}{2},\ w_1 = \frac{1}{12},\ w_2 = \frac{1}{3}.$$

This optimal solution implies that Equation (1.3) is not binding and Equation (1.4) is binding, i.e., Division 1 of DMU_2 is inefficient and Division 2 of DMU_2 is efficient. In , there is another optimal solution:

$$\overline{u} = \frac{1}{4}, \overline{v} = \frac{1}{2}, \overline{w}_1 = \frac{1}{24}, \overline{w}_2 = \frac{5}{12}.$$

In this solution, Equation (1.3) and Equation (1.4) are not binding, i.e., Divisions

1 and 2 of DMU_2 are inefficient. However, Division 2 of DMU_2 should be efficient according to the definitions of dominance. Therefore, some optimal solutions from the NCCR model can be inconsistent with Pareto-Koopmans efficiency. Note that Division 2 of DMU_2 is identified as efficient or inefficient depending upon which optimal solution is considered. An important implication of Example 2 is that NCCR model (1) does not provide the Pareto-Koopmans efficiency status[4] for the case of multiple optima. The two-phase NR approach to be proposed in section 3 identifies the overall system efficiency status based on Pareto-Koopmans efficiency.

<<Figure 2>>about here

2.3Notation and Theoretical Results

This section starts with some definitions for the efficiency evaluation of DMUs and their divisions. Consider *J* DMUs consisting of *K* divisions. Let N^k, M^k and Q be the numbers of inputs and outputs of division *k* and intermediate products, respectively. Also we denote the link leading from division *k* to division *h* by (k,h) and the set of links by $L \subset \{1,...,K\} \times \{1,...,K\}$. Division *k* of DMU_j utilizes input vector $x_j^k \in \Re_+^{N^k}$ and intermediate product vector $z_j^{(h,k)} \in \Re_+^Q$, produced by division h $(h = 1,...,K)$, where $(h,k) \in L$, to produce output vector $y_j^k \in \Re_+^{M^k}$ and intermediate product vector $z_j^{(k,g)} \in \Re_+^Q$, consumed by division g $(g = 1,...,K)$, where $(k,g) \in L$. In

[4]We are not claiming that conventional network DEA models are ill-defined. Rather, we suggest using the two-phase NR approach when Pareto-Koopmans efficiency is relevant in efficiency evaluations.

other words, division k of DMU_j consumes $\left(x_j^k, \sum_{(h,k)\in L}^{h} z_j^{(h,k)}\right)$ to produce $\left(y_j^k, \sum_{(k,g)\in L}^{g} z_j^{(k,g)}\right)$.

Next we introduce a NDEA approach for evaluating the efficiency status of the evaluated DMU, and then prove the nonexistence of any observed DMU that dominates an overall efficient DMU. The NDEA technology consists of a finite collection of observations, and a DMU is expressed by a full vector of exogenous inputs, intermediate products and final outputs in NDEA. Considering these notations, we start with the following three definitions of mathematical dominance.

Definition 1 (fullvector dominance). DMU_j **fully** *dominates* DMU_o *if and only if*

$$\left(-x_j^g,\ y_j^g,\ -\sum_{(h,g)\in L}^{h} z_j^{(h,g)},\ \sum_{(g,h)\in L}^{h} z_j^{(g,h)}\right) \geq \left(-x_o^g,\ y_o^g,\ -\sum_{(h,g)\in L}^{h} z_o^{(h,g)},\ \sum_{(g,h)\in L}^{h} z_o^{(g,h)}\right) \qquad (2)$$

for each division $g = 1,...,K$, and

$$\left(-x_j^k,\ y_j^k,\ -\sum_{(h,k)\in L}^{h} z_j^{(h,k)},\ \sum_{(k,h)\in L}^{h} z_j^{(k,h)}\right) \neq \left(-x_o^k,\ y_o^k,\ -\sum_{(h,k)\in L}^{h} z_o^{(h,k)},\ \sum_{(k,h)\in L}^{h} z_o^{(k,h)}\right) \qquad (3)$$

for at least one division $k \in \{1,...,K\}$.

Definition 2 (sub-vector dominance). DMU_j **sub-vector** *dominates* DMU_o *if and only if* $\left(-x_j^1,...,\ -x_j^K,\ y_j^1,...,\ y_j^K\right) \underset{\neq}{\geq} \left(-x_o^1,...,-x_o^K,\ y_o^1,...,\ y_o^K\right)$.

Definition 1 is concerned with full-vector dominance, at the evaluated DMU_o, in the full space of exogenous inputs, final outputs and intermediate products. By contrast, Definition 2 is concerned with only the subspace of exogenous inputs and final outputs without the explicit consideration of the values of intermediate products. The following definition deals with divisional dominance.

Definition 3 (divisional dominance). *Division k of* DMU_j *dominates division k of* DMU_o *if and only if*

$$\left(-x_j^k,\ y_j^k,\ \ -\sum_{\substack{h\\(h,k)\in L}} z_j^{(h,k)},\ \sum_{\substack{h\\(k,h)\in L}} z_j^{(k,h)}\right) \gneq \left(-x_o^k,\ y_o^k,\ \ -\sum_{\substack{h\\(h,k)\in L}} z_o^{(h,k)},\ \sum_{\substack{h\\(k,h)\in L}} z_o^{(k,h)}\right)$$

An important characteristic of NDEA is that intermediate outputs of a division (say, Division k) should be greater than or equal to the sum of all intermediate inputs of other Divisions h where $(k,h)\in L$. In other words, it is reasonable that general NDEA models have the following conditions as additional constraints:

Fundamental network conditions

$$\underbrace{\sum_{j=1}^{J}\lambda_j^k \sum_{\substack{h\\(k,h)\in L}} z_{qj}^{(k,h)}}_{\text{intermediate output } q} - \underbrace{\sum_{\substack{h\\(k,h)\in L}}\sum_{j=1}^{J}\lambda_j^h z_{qj}^{(k,h)}}_{\text{intermediate input } q} \geq 0, \quad k=1,\ldots,K \ \text{ and } \ q=1,\ldots,Q \tag{4}$$

where $z_{qj}^{(k,h)}$ is a scalar. The first term on the left hand side of (4), $\sum_{j=1}^{J}\lambda_j^k\sum_{\substack{h\\(k,h)\in L}} z_{qj}^{(k,h)}$, represents the q^{th} intermediate output of Division k at the virtual DMU, and $\sum_{j=1}^{J}\lambda_j^h z_{qj}^{(k,h)}$ is the q^{th} intermediate input of Division h (produced by Division k) of the virtual DMU. The amount of the q^{th} intermediate product produced by Division k and consumed by some of the other Divisions h where $(h,k)\in L$ is the second term of , i.e., $\sum_{\substack{h\\(k,h)\in L}}\lambda_j^h z_{qj}^{(k,h)}$. The fundamental network conditions ensure that the amount produced of each intermediate product is no less than the consumed amount, under the assumption that there exist no supplies and demands of intermediate products from outside of the evaluated system. These constraints are obtained from the possibility principle (free disposal hull) of the production technologies on the intermediate products. Lozano (2011) implement the conditions in radial NDEA.

Now let us explain more in detail. We have the intermediate output q ($z_q^{(k,h)}$) of Division k of a DMU is less than the intermediate output of Division k of the projected point where $(k,h)\in L$, i.e., $\sum_{j=1}^{J}\lambda_j^k z_{qj}^{(k,h)} \geq z_q^{(k,h)}$. On the other hand, we have the intermediate input q ($z_q^{(k,h)}$) of Division h of a DMU is greater than the intermediate input quantify of Division h of the projected point, i.e., $\sum_{j=1}^{J}\lambda_j^h z_{qj}^{(k,h)} \leq z_q^{(k,h)}$, where $(k,h)\in L$. These relationships imply that

$$\sum_{\substack{h \\ (k,h)\in L}} \sum_{j=1}^{J}\lambda_j^k z_{qj}^{(k,h)} \geq \sum_{\substack{h \\ (k,h)\in L}} z_q^{(k,h)} \geq \sum_{\substack{h \\ (k,h)\in L}} \sum_{j=1}^{J}\lambda_j^h z_{qj}^{(k,h)}$$

The above relation concludes constraints .

In NDEA models having series structures, the intermediate output q of a stage (say stage k) must be greater than or equal to the intermediate inputs of the next stage (stage k+1). This set of constraints can be expressed as

$$\sum_{j=1}^{J}\lambda_j^k z_{qj}^{(k,k+1)} - \sum_{j=1}^{J}\lambda_j^{k+1} z_{qj}^{(k,k+1)} \geq 0, \quad q=1,\ldots,Q$$

Indeed, in the case of two-stage network problem, Chen, Cook and Zhu (2010) used the following constraints:

$$\sum_{j=1}^{J}\lambda_j^1 z_{qj}^{(1,2)} - \sum_{j=1}^{J}\lambda_j^2 z_{qj}^{(1,2)} \geq 0, \quad q=1,\ldots,Q$$

See Fukuyama and Weber (2010) for the use of this expression in the network directional slacks-based inefficiency measure.

Fundamental network constraints hold for any network structures in which the intermediate outputs are consumed and some of them can be retained (or wasted) within the DMU. As a result, Division k of the virtual DMU can be thought of as the entity that consumes $\left(\sum_{j=1}^{J}\lambda_j^k x_j^k, \sum_{j=1}^{J}\lambda_j^k \sum_{\substack{h \\ (h,k)\in L}} z_j^{(h,k)}\right)$ to produce $\left(\sum_{j=1}^{J}\lambda_j^k y_j^k, \sum_{j=1}^{J}\lambda_j^k \sum_{\substack{h \\ (k,h)\in L}} z_j^{(k,h)}\right)$.

In a nonparametric LP framework the positive slacks of inputs and outputs are likely to be present after the radial efficiency movement. In this case the efficiency scores obtained from these models overstate efficiency because they do not consider the non radial inefficiencies represented by the positive slacks. It is of great importance to account for non radial inefficiencies for the efficiency evaluation of DMUs. In standard DEA both the slacks-based model and the enhanced Russell graph model have coexisted while they are theoretically equivalent. The latter is a graph extension of Färe and Lovell's (1978) input-oriented Russell measure which is equivalent to the input-oriented slacks-based measure. In NDEA, the slacks-based model has predominantly been used and theoretically studied (see Tone and Tsutsui (2009, 2014); Fukuyama and Mirdehghan (2012)). In view of the historical developments of non radial efficiency measures/models, we would like to document NR efficiency results even if both formulations are equivalent --- if our results hold in NR, then they should hold for NSB, and vice versa.

Assuming that $y_{mj} > 0$ for all m and j, we introduce the following network Russell(NR) measure[5]:

[5]The measure E_o^{NR} adapts Pastor, Ruiz and Sirvent's (1999) enhanced Russell graph measure, which is equivalent to Tone's (2001) NSB measure. Note that the contribution of the present paper is to develop a two-phase NR approach, not the development of E_o^{NR} itself.

$$E_o^{NR} = \text{Min} \quad \frac{\sum_{g=1}^{K} w_g \frac{1}{N^g} \sum_{n=1}^{N^g} \hat{\theta}_n^g}{\sum_{g=1}^{K} w_g \frac{1}{M^g} \sum_{m=1}^{M^g} \hat{\phi}_m^g} \qquad (5a)$$

$$s.t. \quad \sum_{j=1}^{J} \hat{\lambda}_j^g x_{nj}^g \le \hat{\theta}_n^g x_{no}^g, \ \sum_{j=1}^{J} \hat{\lambda}_j^g y_{mj}^g \ge \hat{\phi}_m^g y_{mo}^g \quad \forall g, n, m \qquad (5b)$$

$$\sum_{j=1}^{J} \sum_{\substack{h \\ (g,h)\in L}} \hat{\lambda}_j^g z_{qj}^{(g,h)} - \sum_{j=1}^{J} \sum_{\substack{h \\ (g,h)\in L}} \hat{\lambda}_j^h z_{qj}^{(g,h)} \ge 0 \quad \forall g, q \qquad (5c)$$

$$\hat{\theta}_n^g \le 1, \hat{\phi}_m^g \ge 1, \hat{\lambda}_j^g \ge 0 \qquad \forall g, n, m, j \quad (5d)$$

where $w_g > 0$ and $\sum_{g=1}^{K} w_g = 1$. Here, $\hat{\theta}_n^g$, $\hat{\varphi}_m^g$, and $\hat{\lambda}_j^g$ are the variables. We refer to the optimal objective value of (5), denoted by E_o^{NR}, as the sub-vector NR measure because intermediate products do not appear in the objective function (5a). The positive and non-zero predetermined weights w_g $(g = 1,...,K)$ are the relative weights of division g which are determined with the consideration of its importance to managers of the DMU. It is clear that all constraints (5b) must be binding at the optimum.

We can transform the fractional model (5) to a linear programming model by using the Charnes-Cooper transformation.

Theorem 1: *The optimal objective value of Model* (5) *is equal to that of the following linear programming model:*

$$E_o^{NR} = \text{Min} \quad \sum_{g=1}^{K} w_g \frac{1}{N^g} \sum_{n=1}^{N^g} \theta_n^g \qquad (6a)$$

$$s.t. \quad \sum_{j=1}^{J} \lambda_j^g x_{nj}^g \le \theta_n^g x_{no}^g, \ \sum_{j=1}^{J} \lambda_j^g y_{mj}^g \ge \phi_m^g y_{mo}^g \quad \forall g, n, m \qquad (6b)$$

$$\sum_{j=1}^{J} \sum_{\substack{h \\ (g,h)\in L}} \lambda_j^g z_{qj}^{(g,h)} - \sum_{j=1}^{J} \sum_{\substack{h \\ (g,h)\in L}} \lambda_j^h z_{qj}^{(g,h)} \ge 0 \quad \forall g, q \qquad (6c) \qquad (6)$$

$$\sum_{g=1}^{K} w_g \frac{1}{M^g} \sum_{m=1}^{M^g} \phi_m^g = 1 \qquad (6d)$$

$$\theta_n^g \le t, \ \phi_m^g \ge t, \ \lambda_j^g \ge 0 \qquad \forall g, n, m, j \quad (6e)$$

where $\lambda_j^g, \theta_n^g, \varphi_m^g$ and t are the variables.

The total amount of intermediate output q to be produced by all relevant divisions of a virtual DMU, $\sum_{j=1}^{J}\sum_{(g,h)\in L}^{h} \lambda_j^g z_{qj}^{(g,h)}$, cannot be less than the total amount of intermediate output q produced actually by the divisions of the evaluated DMU, $\sum_{(g,h)\in L}^{h} z_{qo}^{(g,h)}$. Moreover, the total amount of intermediate input q to be supplied internally by all relevant divisions of a virtual DMU, $\sum_{j=1}^{J}\sum_{(g,h)\in L}^{h} \lambda_j^h z_{qj}^{(g,h)}$, cannot exceed the observed total amount of the intermediate input q supplied by the divisions of the evaluated DMU, $\sum_{(g,h)\in L}^{h} z_{qo}^{(g,h)}$. That is,

$$\begin{aligned} &\sum_{j=1}^{J}\sum_{\substack{h\\(g,h)\in L}} \lambda_j^g z_{qj}^{(g,h)} - \tilde{s}_q^{g+} = \sum_{\substack{h\\(g,h)\in L}} z_{qo}^{(g,h)}, \quad \tilde{s}_q^{g+} \geq 0 \ \ \forall q \quad (\text{intermediat output}) \ (7.1) \\ &\sum_{j=1}^{J}\sum_{\substack{h\\(g,h)\in L}} \lambda_j^h z_{qj}^{(g,h)} + \tilde{s}_q^{g-} = \sum_{\substack{h\\(g,h)\in L}} z_{qo}^{(g,h)}, \quad \tilde{s}_q^{g-} \geq 0 \ \ \forall q \quad (\text{intermediat input}) \ (7.2) \end{aligned} \tag{7}$$

The right hand side of Eq. (7.1) signifies the total amount of intermediate product q at DMU_o and is fixed as $\sum_{(g,h)\in L}^{h} z_{qo}^{(g,h)}$. For this fixed total amount, the slack of the constraint associated with intermediate product q for Division g is determined as $\tilde{s}_q^{g+} = \sum_{j=1}^{J}\sum_{(g,h)\in L}^{h} \lambda_j^g z_{qj}^{(g,h)} - \sum_{(g,h)\in L}^{h} z_{qo}^{(g,h)}$. Thus, Eq. (7.1) indicates the excesses in intermediate output q for Division g. This indicates that the linking activities are kept fixed. Eq. (7.2) can be interpreted similarly. Hence, we refer to Eq. (7) as fixed link conditions by following Tone and Tsutsui (2009, 2014).

However, in Tone and Tsutsui (2009), slacks are not included, i.e., Tone and Tsutsui's fixed link for intermediate products can be expressed as $\sum_{j=1}^{J}\sum_{\substack{h \\ (g,h)\in L}} \lambda_j^g z_{qj}^{(g,h)} = \sum_{\substack{h \\ (g,h)\in L}} z_{qo}^{(g,h)}$ in our terminology.

Note that Fukuyama and Mirdehghan (2012) include the fixed link conditions, but do not include the fundamental network conditions or equivalently (6c). Using (6) and , we define network-based Pareto-Koopmans efficiency as follows.

Definition 4: DMU_o *is sub-vector NR-efficient if and only if* $E_o^{NR} = 1$. *It is Pareto-Koopmans efficient if and only if it is sub-vector NR-efficient as well as the following relationships hold:*

$$\begin{aligned} \sum_{j=1}^{J} \sum_{\substack{h \\ (g,h)\in L}} \lambda_j^{g*} z_{qj}^{(g,h)} &= \sum_{\substack{h \\ (g,h)\in L}} z_{qo}^{(g,h)}, \ \forall g,q \\ \sum_{j=1}^{J} \sum_{\substack{h \\ (g,h)\in L}} \lambda_j^{h*} z_{qj}^{(g,h)} &= \sum_{\substack{h \\ (g,h)\in L}} z_{qo}^{(g,h)}, \ \forall g,q \end{aligned} \tag{8}$$

*where * indicates optimality in (6). We refer to the conditions as intermediate product binding conditions. It is sub-vector NR-inefficient if and only if* $E_o^{NR} < 1$.

In Definition 4, we make a distinction between sub-vector NR efficiency and Pareto-Koopmans efficiency. The former notion of efficiency simply means the objective function value of (6a) is unity. In other words, $E_o^{NR} = 1$is not sufficient for Pareto-Koopmans efficiency. On the other hand, the latter requires not only sub-vector efficiency with the unity score of (6a),but also the intermediate product binding conditions (8), i.e., $\tilde{s}_q^{g-}$ and $\tilde{s}_q^{g+}$ defined in are zero in all possible optimal solutions of .It should be noted that this paper assumes that intermediate products are produced and supplied only within the assessed system or DMU, while some amounts of intermediate products can be retained within it. Therefore, Defintion 4 states as follows. The summation of intermediate inputs of the virtual DMU must equal to the summation of intermediate products of the assessed DMU and the summation of intermediate outputs of the virtual DMU must equal to the summation of the intermediate products of the assessed DMU in order to achieve Pareto-Koopmans efficiency.

Theorem 2: *Suppose that the DMU to be assessed is Pareto-Koopmans efficient. Then, there does not exist any observation which dominates this Pareto-Koopmans efficient DMU. Furthermore, there does not exist any observed DMU that sub-vector dominates a sub-vector NR-efficient DMU.*

The first and second parts of Theorem 2 correspond to Definition 1 (full-vector dominance) and Definition 2 (sub-vector dominance), respectively. The first part indicates that if DMU_o is Pareto-Koopmans efficient, then we cannot find any virtual DMU that performs better than DMU_o. Now let us consider the following linear programming problem:

$$
\begin{aligned}
\max \quad & \sum_{g=1}^{K}\sum_{q=1}^{Q}\left(\tilde{s}_q^{g-}+\tilde{s}_q^{g+}\right) \\
s.t. \quad & \sum_{j=1}^{J}\lambda_j^g x_{nj}^g = x_{no}^g,\ \ \sum_{j=1}^{J}\lambda_j^g y_{mj}^g = y_{mo}^g && \forall g,n,m \\
& \sum_{j=1}^{J}\sum_{\substack{h \\ (g,h)\in L}}\lambda_j^g z_{qj}^{(g,h)} - \sum_{j=1}^{J}\sum_{\substack{h \\ (g,h)\in L}}\lambda_j^h z_{qj}^{(g,h)} \ge 0 && \forall g,q \\
& \sum_{j=1}^{J}\sum_{\substack{h \\ (g,h)\in L}}\lambda_j^h z_{qj}^{(g,h)} + \tilde{s}_q^{g-} = \sum_{\substack{h \\ (g,h)\in L}} z_{qo}^{(g,h)} && \forall g,q \\
& \sum_{j=1}^{J}\sum_{\substack{h \\ (g,h)\in L}}\lambda_j^g z_{qj}^{(g,h)} - \tilde{s}_q^{g+} = \sum_{\substack{h \\ (g,h)\in L}} z_{qo}^{(g,h)} && \forall g,q \\
& \lambda_j^g \ge 0,\ \ \tilde{s}_q^{g-} \ge 0,\ \ \tilde{s}_q^{g+} \ge 0 && \forall g,j,q
\end{aligned} \tag{9}
$$

In , we incorporate the fundamental network constraints and the initial endowment constraints simultaneously. Using these constraints, we can establish the following theorem.

Theorem3: *Let* $E_o^{NR}=1$. *The optimal value of (9) is equal to zero if and only if, for all optimal solutions of (6), we have* $\sum_{j=1}^{J}\sum_{\substack{h\\(g,h)\in L}}\lambda_j^{g*}z_j^{(g,h)}=\sum_{\substack{h\\(g,h)\in L}}z_o^{(g,h)}$ *and* $\sum_{j=1}^{J}\sum_{\substack{h\\(g,h)\in L}}\lambda_j^{h*}z_j^{(g,h)}=\sum_{\substack{h\\(g,h)\in L}}z_o^{(g,h)}$ *for each g=1,…,K, where "*" indicates optimality in* (6).

Theorem 3 indicates that DMU_o is Pareto-Koopmans efficient if and only if $E_o^{NR}=1$ and the optimal value of equals zero. The significance of this theorem is that we can determine whether a DMU is Pareto-Koopmans efficient or not, by checking the value of E_o^{NR} and the optimal objective value of (9) even if (6) has multiple optimal solutions. The notion of Pareto-Koopmans efficiency is consistent with Lewis and Sexton's (2004) assertion as noted earlier.

The significance of Theorems2 and 3is that we can determine whether a DMU is Pareto-Koopmans efficient or not, by examining the objective value of and the optimal objective value of (9). Hence, we do not need to worry about the existence of multiple optimal solutions in our methodology, whereas the Pareto-Koopmans efficiency status cannot be identified using only an optimal solution in the NSB and NCCR Models.

In the NDEA literature, the efficiency scores of divisions are often obtained by decomposing the efficiency of the assessed DMU into divisional efficiencies (Kao (2009); Kao and Hwang (2008); Tone and Tsutsui (2009)). For example, the system efficiency in Kao and Hwang's (2008) two-stage NCCR model is decomposed into the two divisional efficiencies. That is, the divisional efficiency scores of stages 1 and 2 are obtained with respect to optimal multipliers by exploiting the relationship $E_o^{NCCR} = E_o^{NCCR,1} \times E_o^{NCCR,2}$, where E_o^{NCCR}, $E_o^{NCCR,1}$ and $E_o^{NCCR,2}$ are the efficiencies of DMU$_o$, Division 1 and Division 2, respectively.

Alternatively, one might obtain the efficiency of Division k as

$$\frac{\frac{1}{N^k}\sum_{n=1}^{N^k}\hat{\theta}_n^{k*}}{\frac{1}{M^k}\sum_{m=1}^{M^k}\hat{\phi}_m^{k*}} \qquad k = 1,\dots,K \tag{10}$$

where "*" indicates the optimality of or . Unfortunately, however, such a measure does not represent the divisional efficiency score that is consistent with the notion of dominance. Indeed, (10) does not even show whether the division is efficient or inefficient if (6) has multiple optimal solutions.

In addition to the problem of multiple optimal solutions, we provide another reason for not using Eq. . Suppose an efficient DMU consists of several divisions: all divisions except for one are efficient and the inefficient division's inefficiency comes from the inappropriate use of an intermediate input. If we use Eq. , which is the mean of only the scaling factors of exogenous inputs and final outputs of a division, then the score can be unity for all divisions because the optimal values for all scaling factors obtained from or in assessing the efficient DMU equal one. But one division is inefficient with respect to an intermediate input. Therefore, it implies a contradictory result. To obtain the efficiency of DMU$_o$'s Division k, we replace (5a) with

$$\frac{\frac{1}{N^k+Q}\left(\sum_{n=1}^{N^k}\hat{\theta}_n^{k*}+\sum_{q=1}^{Q}\hat{\bar{\theta}}_q^{k*}\right)}{\frac{1}{M^k+Q}\left(\sum_{m=1}^{M^k}\hat{\phi}_m^{k*}+\sum_{q=1}^{Q}\hat{\bar{\phi}}_q^{k*}\right)} \tag{11}$$

after appending the following constraints in :

$$\sum_{j=1}^{J}\sum_{\substack{h\\(h,k)\in L}}\lambda_j^k z_{qj}^{(h,k)} \le \hat{\bar{\theta}}_q^k \sum_{\substack{h\\(h,k)\in L}} z_{qo}^{(h,k)} \quad (\forall q), \qquad \sum_{j=1}^{J}\sum_{\substack{h\\(k,h)\in L}}\lambda_j^k z_{qj}^{(k,h)} \ge \hat{\bar{\phi}}_q^k \sum_{\substack{h\\(k,h)\in L}} z_{qo}^{(k,h)} \quad (\forall q)$$

$$\hat{\bar{\theta}}_q^k \le t \quad (\forall q,k), \quad \hat{\bar{\phi}}_q^k \ge t \quad (\forall q).$$

In order to find divisional efficiency scores uniquely we maximize over the constraints of along with the above additional constraints.

Using the Charnes-Cooper transformation, we suggest solving the following linear programming problem:

$$E_o^{NR,k} = \mathrm{Min}\frac{1}{N^k+Q}\left(\sum_{n=1}^{N^k}\theta_n^k+\sum_{q=1}^{Q}\bar{\theta}_q^k\right) \tag{12.1}$$

$$\begin{aligned}
s.t.\quad & \sum_{j=1}^{J}\lambda_j^k x_{nj}^k \le \theta_n^k x_{no}^k,\ \sum_{j=1}^{J}\lambda_j^k y_{mj}^k \ge \phi_m^k y_{mo}^k \quad \forall n,m && (12.2)\\
& \sum_{j=1}^{J}\sum_{\substack{h\\(h,k)\in L}}\lambda_j^k z_{qj}^{(h,k)} \le \bar{\theta}_q^k \sum_{\substack{h\\(h,k)\in L}} z_{qo}^{(h,k)} \quad \forall q && (12.3)\\
& \sum_{j=1}^{J}\sum_{\substack{h\\(k,h)\in L}}\lambda_j^k z_{qj}^{(k,h)} \ge \bar{\phi}_q^k \sum_{\substack{h\\(k,h)\in L}} z_{qo}^{(k,h)} \quad \forall q && (12.4)\\
& \sum_{j=1}^{J}\sum_{\substack{h\\(g,h)\in L}}\lambda_j^g z_{qj}^{(g,h)} - \sum_{j=1}^{J}\sum_{\substack{h\\(g,h)\in L}}\lambda_j^h z_{qj}^{(g,h)} \ge 0 \quad \forall g,q && (12.5)\\
& \frac{1}{M^k+Q}\left(\sum_{m=1}^{M^k}\phi_m^k+\sum_{q=1}^{Q}\bar{\phi}_q^k\right)=1, && (12.6)\\
& \lambda_j^g \ge 0, \quad \forall g,j && (12.7)\\
& \theta_n^k \le t,\ \bar{\theta}_q^k \le t, \quad \forall n,q && (12.8)\\
& \phi_m^k \ge t,\ \bar{\phi}_q^k \ge t, \quad \forall m,q && (12.9)
\end{aligned} \tag{12}$$

where $\lambda_j^g, \theta_n^k, \bar{\theta}_q^k, \varphi_m^k, \bar{\varphi}_q^k$ and t are the variables. The optimal value of (12), represented by $E_o^{NR,k}$, is the efficiency of division *k* of DMU_o. It is easy to see that if redundant constraints are deleted, then the number of constraints in (12.5) equals the number of indices $g=1,...,K,\ q=1,...,Q$ such that $(g,k)\in L$ or $(k,g)\in L$.

Remark: If division *k* does not have intermediate inputs, then *Q* and $\sum_{q=1}^{Q}\bar{\theta}_q^k$ in the objective (12.1) along with constraints (12.3) and $\bar{\theta}_q^k \le t$ of (12.8) should be omitted. Moreover, if division *k* does not have any intermediate outputs, then *Q* and $\sum_{q=1}^{Q}\bar{\phi}_q^k$ in (12.6) along with constraints (12.4) and $\bar{\varphi}_q^k \ge t$ of (12.9) should be deleted.

Now we are ready to formally define divisional efficiency with respect to $E_o^{NR,k}$ in .

Definition 5(**Divisional Efficiency**). *Division* $k\in\{1,...,K\}$ *of* DMU_o *is divisionally efficient if and only if* $E_o^{NR,k}=1$. *It is divisionally inefficient if and only if* $E_o^{NR,k}<1$.

Considering Definition 5, we obtain the following theorem associated with divisional efficiency.

Theorem 4: *If division k of* DMU_o *is divisionally efficient according to* $E_o^{NR,k}$, *then there does not exist any observed DMU whose division k dominates division k of* DMU_o.

The measure $E_o^{NR,k}$ represents divisional efficiency. We refer to our proposed two-phase framework consisting of and as the two-phase NR approach. We name this method "two-phase" because in the first phase we obtain sub-vector NR efficiency by or , and in the second phase we maximize slacks of intermediate products using model if the sub-vector NR efficiency is one.

3. Comparison with NDEA methods: Two artificial numerical examples

In this section, we compare the two-phase NR approach with Liang, Cook and Zhu's (2008) centralized authority method as well as two standard network methods: one is the network slacks-based (NSB) model (Tone and Tsutsui (2009, 2014) and the other is the network CCR (NCCR) model (Kao (2009); Kao and Hwang (2008, 2010)). To start with, we define Tone and Tsutsui's (2014) fixed link model as follows:

$$
\begin{aligned}
NSB_o = \quad & \text{Min} \frac{\sum_{g=1}^{K} w^g \left(1 - \frac{1}{N^g + Q}\left(\sum_{n=1}^{N^k} \frac{s_n^{g-}}{x_{no}^g} + \sum_{q=1}^{Q} \frac{\tilde{s}_q^{g-}}{z_{qo}^g}\right)\right)}{\sum_{g=1}^{K} w^g \left(1 + \frac{1}{M^g + Q}\left(\sum_{m=1}^{M^k} \frac{s_m^{g+}}{y_{mo}^g} + \sum_{q=1}^{Q} \frac{\tilde{s}_q^{g+}}{z_{qo}^g}\right)\right)} \\
s.t. \quad & \sum_{j=1}^{J} \lambda_j^g x_{nj}^g = x_{no}^g - s_n^{g-}, \ \sum_{j=1}^{J} \lambda_j^g y_{mj}^g = y_{mo}^g + s_m^{g+} \qquad \forall g, n, m \\
& \sum_{j=1}^{J} \lambda_j^g z_{qj}^{(g,h)} = z_{qo}^{(g,h)} + \tilde{s}_q^{g+}, \ \sum_{j=1}^{J} \lambda_j^h z_{qj}^{(g,h)} = z_{qo}^{(g,h)} - \tilde{s}_q^{g-} \qquad \forall g, h, q \\
& s_n^{g-} \geq 0, \ s_m^{g+} \geq 0, \tilde{s}_q^{g-} \geq 0, \ \tilde{s}_q^{g+} \geq 0, \lambda_j^g \geq 0 \qquad \forall g, n, m, q, j
\end{aligned}
\tag{13}
$$

Note that the NR model is different from Tone and Tsutsui's (2014) fixed link and free link NSB formulations with respect to intermediate products. Whereas Tone and Tsutsui (2014) utilize $\sum_{j=1}^{J} \lambda_j^g z_{qj}^{(g,h)} \geq z_{qo}^{(g,h)}, \ \sum_{j=1}^{J} \lambda_j^h z_{qj}^{(g,h)} \leq z_{qo}^{(g,h)} \quad (\forall g, h, q)$ for the fixed link case and $\sum_{j=1}^{J} \lambda_j^g z_{qj}^{(g,h)} - \sum_{j=1}^{J} \lambda_j^h z_{qj}^{(g,h)} \geq 0 \quad (\forall g, h, q)$ for the free link case, the present paper uses $\sum_{j=1}^{J} \sum_{\substack{h \\ (g,h) \in L}} \lambda_j^g z_{qj}^{(g,h)} - \sum_{j=1}^{J} \sum_{\substack{h \\ (g,h) \in L}} \lambda_j^h z_{qj}^{(g,h)} \geq 0 \quad (\forall g, q)$ as shown in (6c) of NR model (6).Tone and Tsutsui (2014) developed a dynamic-network DEA model. We can obtain if we confine ourselves to the static situation.

The NSB of Division *g* (NSB divisional efficiency) is calculated as

$$NSB_o^{g*} = \frac{1 - \frac{1}{N^g + Q}\left(\sum_{n=1}^{N^k} \frac{s_n^{g-*}}{x_{no}^g} + \sum_{q=1}^{Q} \frac{\tilde{s}_n^{g-*}}{z_{qo}^g}\right)}{1 + \frac{1}{M^g + Q}\left(\sum_{m=1}^{M^k} \frac{s_m^{g+*}}{y_{mo}^g} + \sum_{q=1}^{Q} \frac{\tilde{s}_q^{g+*}}{z_{qo}^g}\right)} \tag{14}$$

where the star "*" shows an optimal solution of . That is, the NSB measure for division *g* is calculated based on the average efficiencies associated with not only the exogenous inputs and final outputs but also intermediate products. Hence, the NSB measure for the evaluated DMU is constructed in the full product space and hence is a full-space vector measure.

Now return to Example 1, whose data are given in Figure 1. Table 1 shows the results of the NCCR, Liang et al. (2008), NSB fixed link and free link, and two-phase NR models where an efficiency score of unity indicates efficiency. We see that DMU_1, DMU_2 and the two divisions are efficient according to the NCCR, Liang et al.'s models as well as the NSB free link formulation. The result of NSB fixed link method based on and shows that DMU_1 is inefficient because the efficiency of its Division 1 is 0.8, but DMU_2 is rated as efficient with the two divisions being efficient. By contrast, the two-phase NR approach shows that the first divisional efficiency score of DMU_1 is 0.8 and the second divisional efficiency score of DMU_2 is 0.83333. Moreover, the second division of DMU_1 and the first division of DMU_2 are efficient[6]. Note that the efficiencies of the two DMUs equal one and so the two DMUs are sub-vector efficient,

[6]Note that the constraints, corresponding to intermediate products, used in Tone and Tsutsui (2009, 2014) are different from that constraints used in the network Russell model(6). Therefore the results based on Tone and Tsutsui (2009) may be different from the results of the proposed approach.

while the maximization of slacks (Model) is nonzero and hence these DMUs are Pareto-Koopmans inefficient. We see that the results obtained from the two-phase NR approach correctly identify Pareto-Koopmans efficiency as is expected.

<<Table 1>>about here

We can easily see that our two-phase NR procedure produces a unique score for the efficiency of the assessed DMU and its divisions and so it can solve the problem of Example 2. On the contrary, NCCR cannot find the efficiency scores of the divisions uniquely. The results of NCCR, NSB and NR of Example 2, whose data are given in Figure 2, are listed in Table 2. Although the divisional efficiencies are uniquely determined from NSB for this example, it is not the case in general, i.e., the NSB is not capable of finding divisional efficiencies uniquely similar to NCCR.

<<Table 2>>about here

In the next section, we examine how the two-phase NR approach can be used to achieve Pareto-Koopmans efficiency using Taiwanese banking data. Here we also provide estimates based on Tone and Tsutsui's (2014) NSB model.

4. Real-life Application: Taiwanese banks

In this section, we apply the two network methods (NSB and two-phase NR) discussed in section 3 to the data set used in Kao and Hwang (2010). Their data set consists of 27 banks in Taiwan. Following Wang, Gopal and Ziont's (1997) study, Kao and Hwang (2010) specified the problem of evaluating the impact of information technology (IT) on bank performance into two stages in series. The framework identifies Division 1 as an IT-related activity and deposits as the intermediate output from IT. Division 2 is a profit generating process, in which a bank utilizes the deposits as funds to provide loans to customers and invest in securities. That is, Division 1uses three inputs to produce one intermediate product and this intermediate product is consumed by Division 2 to produce two final outputs. The three inputs are IT budget, fixed assets and the number of employees; and the intermediate product is represented by the dollar value of deposits; and the two final outputs are the earned profit and the percentage of loans recovered. The data are given in Appendix A1.

Since there is a substantial difference in the bank size in the sample, we analyze the sample under the assumption of variable returns to scale. Our proposed approach can be used to provide a Pareto-Koopmans efficient target for banks in the sample.

To do so let us denote the differences $H_o^1(\mathrm{DEP})$ and $H_o^2(\mathrm{DEP})$ between the deposits produced and consumed by Division 1 and Division 2, respectively, as

$$\begin{aligned} H_o^1(\mathrm{DEP}) &= \sum\nolimits_{j=1}^{27} \mathrm{DEP}_j \lambda_j^{1\&} - \sum\nolimits_{j=1}^{27} \mathrm{DEP}_j \lambda_j^{2\&} \\ H_o^2(\mathrm{DEP}) &= \sum\nolimits_{j=1}^{27} \mathrm{DEP}_j \lambda_j^{1\&\&} - \sum\nolimits_{j=1}^{27} \mathrm{DEP}_j \lambda_j^{2\&\&} \end{aligned} \quad (15)$$

where $\mathrm{DEP}_j = z_j^{(1,2)}$ is Bank *j*'s observed value of deposits. Note that & and && indicate optimality of associated to Divisions 1 and 2, respectively. The indexes $H_o^1(DEP)$ and $H_o^2(\mathrm{DEP})$ represent the retained amounts of deposits corresponding to Divisions 1 and 2, respectively. Similarly, using the optimal solution of (6), $H_o(DEP)$ is defined as $H_o(\mathrm{DEP}) = \sum\nolimits_{j=1}^{27} \mathrm{DEP}_j \lambda_j^{1*} - \sum\nolimits_{j=1}^{27} \mathrm{DEP}_j \lambda_j^{2*}$ where '*' indicates the optimality of (6).

Table 2 reports the efficiency scores based on the two methods where $w_1=w_2=0.5$. All efficiency measures calculated by the NSB and NR models are different for all inefficient banks and divisions, although the efficiency status of DMUs and their divisions are the same in the two methods. According to the NSB model, banks 7, 9, 18, 20, and 27 are efficient because $E_j^{NR} = E_j^{NR,1} = E_j^{NR,2} = 1$ $(j = 7, 9, 18, 20, 27)$ and

$H_j^1 = H_j^2 = 0 \ (j = 7,\ 9,\ 18,\ 20,\ 27)$.

<<Table 3>>about here

Now consider banks 4, 13, 17, 21, 22 and 26 whose second division is inefficient. The last three columns in Table 3 provide the amounts of slacks associated with the constraints of the intermediate product for these inefficient banks based on models and . The H_o^2 values for banks 4, 17 and 22 are positive and those for banks 13, 21and 26 are zero. If Bank 4 decreases deposits by 4 billion dollars, the bank and its two divisions become efficient. Note that this bank's Division 1 was efficient and it still remains efficient, though we have decreased Division 1's output (deposit). Hence Bank 4 will become overall system efficient by decreasing 4 billion dollars of Bank 4's intermediate product. Similar results can be obtained for Bank 17 and Bank 22 if they decrease deposits by 2 and 7 billion dollars, respectively.

For Banks 13, 21 and 26 whose wastes of deposits represented by H_o^2 are zero, we cannot make these banks efficient only with changes of the amounts of deposits. In order to obtain their Pareto-Koopmans efficient targets, they need to change exogenous inputs and/or final outputs from a managerial perspective.

The same result is obtained for Banks 3 and 16, for which their second divisions are efficient and their first divisions are inefficient. That is, we cannot improve these banks and their divisions to make them efficient by increasing their intermediate inputs (deposits), because the values of H_o^1 are zero for Banks 3 and 16.

5. Conclusions

Mathematical dominance is the fundamental property asserted by Charnes and Cooper (1984, 1985) and Charnes, Cooper, Golany and Seiford (1985) in standard DEA models.

However, the dominance notions (criteria or rules) are not fully utilized to evaluate the efficiency of DMUs and divisions in NDEA. By incorporating the three vector-based notions of dominance, we suggested a novel two-phase network Russell (NR) approach for evaluating the overall efficiency performances of DMUs with network internal structures. We also presented and proved several theorems to show the validity of the two-phase NR approach. Clearly, the two-phase NR approach is also applicable to DMUs having series or parallel structures as well as other NDEA models. For the purpose of illustration, we employed Kao and Hwang's (2010) data on Taiwanese banks as well as two artificial numerical examples. We demonstrated that, in the free link case, the two-phase NR approach can always identify the Pareto-Koopmans efficiency status which is not identified by standard NDEA models, though Fukuyama and Mirdehghan (2012) had dealt with the efficiency status identification for the fixed link.

Before concluding, several remarks are in order. First, the two-phase NR approach is particularly relevant when a DMU manager would like to improve the overall operations of the system (or organization) further, even when exogenous inputs and final outputs are optimally allocated. In fact, Lewis and Sexton (2004, on page 1394) asserted that providing this kind of information is of great importance for further improving operations when a DMU is sub-vector efficient.

Second, Pareto-Koopmans efficiency implies that all of its divisions are fully efficient, but the condition that a DMU is sub-vector efficient is only necessary for Pareto-Koopmans efficiency. As was shown in our examples in section2, it is possible that all observed DMUs are overall system inefficient (see also Lewis and Sexton (2004); Castelli, Pesenti and Ukovich (2010, p.222)). Third, by opening a black box inherent to a DMU, we can identify the sources of inefficiency at the DMU by using the two-phase NR approach.

Fourth, the two-phase NR approach is capable of providing divisional efficiency scores even under multiple optima. Chen, Cook, Kao and Zhu (2013) stated that while the NSB envelopment form is incapable of providing the divisional efficiency scores, the NCCR multiplier form can provide both overall and divisional efficiency scores. However, as shown in the present paper, the NCCR model is inconsistent with Pareto-Koopmans efficiency. While the two-phase NR approach does not give the overall efficiency score[7] of the assessed DMU, it not only provides divisional efficiency scores but also determines its Pareto-Koopmans efficiency status.

Fifth, it is straightforward to endogenize the weights associated with the importance of divisions. However, such a NR setting will be nonlinear. To avoid nonlinearity, the present paper only discussed the two-phase NR approach under the assumption that the weights be known a priori.

The sixth remark is about our treatment of intermediate products in the modeling. This paper focused on the case where all intermediate products are not leakage variables, i.e., they are produced, consumed and possibly retained in the system. Interesting future extensions include: relaxing this assumption, developing a NDEA approach from a dominance perspective and sensitivity analysis on the intermediate products to transform an inefficient DMU to an overall efficient DMU.

Seventh, it should be noted that finding the reference set and the benchmarks of DMUs is one of the most important issues in the standard DEA which some researchers (such as Jahanshahloo, Shirzadi and Mirdehghan (2008); Krivonozhko, Førsund and Lychev (2012); Mehdiloozad, Mirdehghan, Sahoo and Roshdi (2015)) have focused on.

[7]As stated in this paper, other models such as NCCR do not provide the overall efficiency score which is consistent with Pareto-Koopmans efficiency.

Finding the benchmarks of DMUs is another important subject in network DEA which we need more studying.

Finally, we conclude this section with the following remark. We focused on the situation where the notions of dominance (particularly complete dominance involving intermediate products) are applicable. However, it should be noted that there can be other situations where dominance related to intermediate products is not the rule or norm. The models dealing with such situations are those of Tone and Tsutsui (2009, 2014) and Kao and Hwang (2010). Liang, Cook and Zhu's (2008) approach based on cooperative and non-cooperative games also deals with such a situation.

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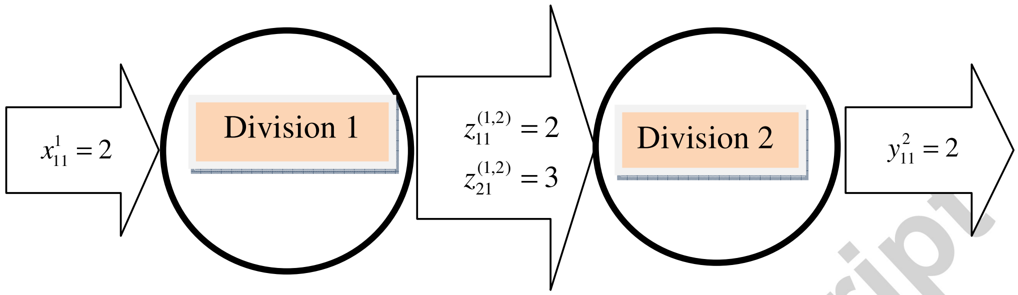
Figure 1:Example 1.

DMU 1



DMU 2

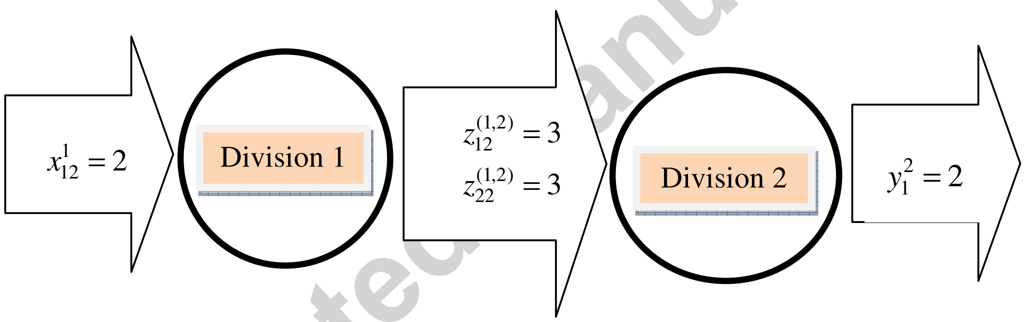
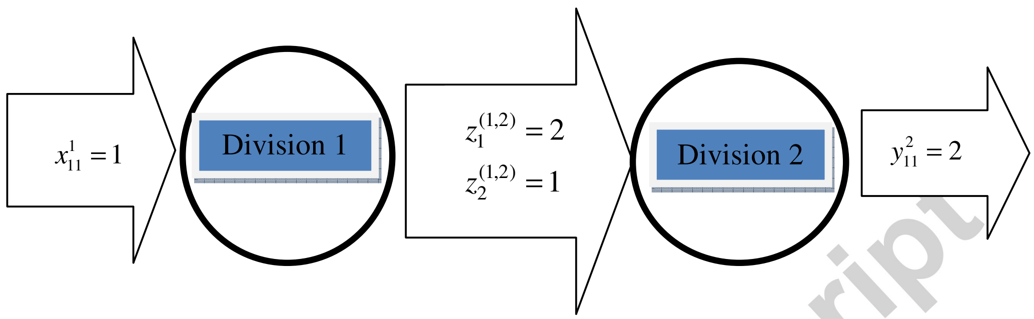


Figure 2:Example 2

DMU 1



DMU 2

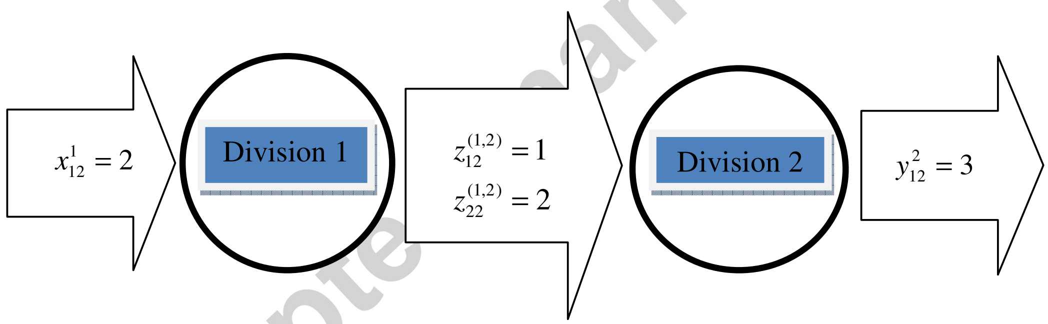


Figure 3: Two-Stage Intermediation Process for Bank j

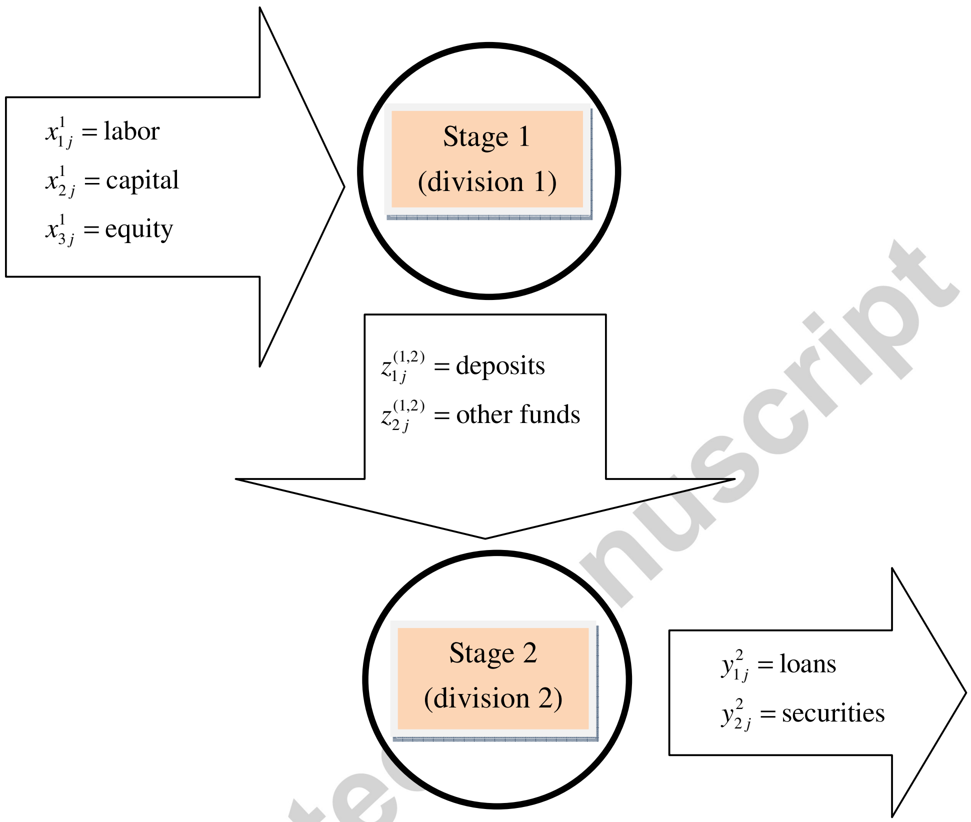


Table 1:Efficiencies of DMU and Divisions for Example 1: Constant Returns to Scale

	NCCR model			Central authority KH model			NSBfixed link model			NSBfree link model			NR model		
	Kao and Hwang (2010)			Liang et al. (2008)			Tone and Tsutsui (2009)			Tone and Tsutsui (2009)			Present study		
DMU	E_o^{NCCR}	$E_o^{\text{NCCR,1}}$	$E_o^{\text{NCCR,2}}$	E_o^{NR}	$E_o^{\text{Central,1}}$	$E_o^{\text{Central,2}}$	$E_{o,fixed}^{\text{NSB}}$	$E_{o,fixed}^{\text{NSB,1}}$	$E_{o,fixed}^{\text{NSB,2}}$	$E_{o,free}^{\text{NSB}}$	$E_{o,free}^{\text{NSB,1}}$	$E_{o,free}^{\text{NSB,2}}$	E_o^{NR}	$E_o^{\text{NR,1}}$	$E_o^{\text{NR,2}}$
1	1	1	1	1	1	1	0.8333	0.8	1	1	1	1	1	0.8	1
2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0.8333

Note:Tone and Tsutsui's (2009) NSB free link model and the NR model only differs in constraints associated with intermediate products.

Table 2: Efficiencies of DMUs and Divisions for Example 2: Constant Returns to Scale

	NCCR model			NSBfree link model			NR model		
	Kao and Hwang (2010)			Tone and Tsutsui (2009)			Present study		
DMU	E_o^{NCCR}	$E_o^{NCCR,1}$	$E_o^{NCCR,2}$	$E_{o,free}^{NSB}$	$E_{o,free}^{NSB,1}$	$E_{o,free}^{NSB,2}$	E_o^{NR}	$E_o^{NR,1}$	$E_o^{NR,2}$
1	1	1	1	1	1	1	1	1	5/6
2	3/4	7/8, 3/4*	1, 6/7*	4/7	2/5	1	3/4	2/5	1

*The divisional efficiencies of $DMU_2$are not obtained uniquely for the NCCR model due to the existence ofmultiple optimal solutions.The NCCR scores are calculated using the two optimal solutions which are given in subsection 2.2.

Table 3: VRS Results of 27 Banks in Taiwan.

Bank	Free link case of NSB model (Tone and Tsutsui (2009))			Two-phase NR approach (present study)			Two-phase NR approach (difference between optimal deposits in Division 1 and Division 2)		
	$E_{o,free}^{\mathrm{NSB}}$	$E_{o,free}^{NSB,1}$	$E_{o,free}^{\mathrm{NSB},2}$	E_o^{NR}	$E_o^{NR,1}$	$E_o^{NR,2}$	H_o (DEP)	H_o^1(DEP)	H_o^2(DEP)
1	0.6809	0.7753	0.9319	0.5733	0.5506	0.8458	0	0	0
2	0.7257	0.7707	0.9984	0.5485	0.5075	0.9960	0	0	0
3	0.6036	0.6794	1	0.4334	0.3588	1	0	0	0
4	0.8567	1	0.8568	0.8421	1	0.7136	0	0	4.5408
5	0.6778	0.8117	0.8983	0.6684	0.6235	0.7546	0	4.9571	3.0033
6	0.5519	0.7130	0.8389	0.3091	0.4259	0.6779	0	0	0
7	1	1	1	1	1	1	0	0	0
8	0.6280	0.7508	0.9348	0.5768	0.5016	0.8304	0	0	4.9775
9	1	1	1	1	1	1	0	0	0
10	0.5251	0.6864	0.9106	0.4076	0.3729	0.7680	0	0	0
11	0.5218	0.6906	0.9156	0.4033	0.3697	0.7680	0	0	0
12	0.6634	0.8585	0.8227	0.6344	0.7169	0.5947	0	3.0865	5.9421
13	0.9248	1	0.9251	0.6283	1	0.8503	0	0	0
14	0.5160	0.6733	0.8803	0.2851	0.3466	0.7295	0	0	0
15	0.6176	0.6979	0.9415	0.3575	0.3898	0.8740	0	0	1.6567
16	0.7629	0.7994	1	0.5996	0.5644	1	0	0	0
17	0.6665	1	0.7499	0.4998	1	0.4822	0	0	2.3673
18	1	1	1	1	1	1	0	0	0
19	0.6892	0.8620	0.8272	0.5516	0.7241	0.6544	0	0	2.0531
20	1	1	1	1	1	1	0	0	0
21	0.9395	1	0.9428	0.8893	1	0.8856	0	0	0
22	0.8200	1	0.8201	0.8082	1	0.6402	0	2.4280	3.6369
23	0.8305	0.9621	0.8691	0.8317	09242.	0.7328	0	3.3654	1.8266
24	0.9308	0.9350	0.9958	0.8498	0.8700	0.9915	0	0	0
25	0.6202	0.8409	0.8061	0.5460	0.6640	0.5728	0	0	0
26	0.7558	1	0.7565	0.5668	1	0.5130	0	0	0
27	1	1	1	1	1	1	0	0	0

Notes: (i) Banks 4, 13, 17, 21, 22 and 26 are Pareto-Koopmans inefficient but their first divisions are efficient; (ii) Banks 3 and 16 are Pareto-Koopmans inefficient but their second divisions are efficient; (iii) DMUs 7, 9, 18, 20 and 27 are Pareto-Koopmans efficient.

Appendix A1: Proofs of Theorems

Proof of Theorem 1: Letting $t^{-1}=\sum_{g=1}^{K} w_g \frac{1}{M^g}\sum_{m=1}^{M^g}\hat{\phi}_m^g$ in (5), we convert the variables into new variables as

$$\phi_m^g = t\hat{\phi}_m^g \ \forall m,g, \quad \theta_n^g = t\hat{\theta}_n^g \ \forall n,g, \quad \lambda_j^g = t\hat{\lambda}_j^g \ \forall j,g$$

This conversion yields a desired result. ■

Proof of Theorem 2: Let DMU_o be overall system efficient, then $E_o = 1$, i.e., $\theta_n^{g*}=1$ and $\phi_m^{g*}=1$ for *g=1,…,K*, $n=1,...,N^g$ and $m=1,...,M^g$. By contradiction, suppose that DMU_p completely dominates DMU_o where *o* and *p* are in $\{1,...,J\}$. It implies $x_p \le x_o$ and $y_p \ge y_o$. Consequently, the vector $\left(\hat{\lambda}^1,...,\hat{\lambda}^K,\hat{\theta}^1,...,\hat{\theta}^K,\hat{\phi}^1,...,\hat{\phi}^K,\hat{t}\right)$ is an optimal solution to Model (6) (in evaluating DMU$_o$) where $\hat{\lambda}_p^g=1$ and $\hat{\lambda}_j^g=0$ for $j\neq p$ as well as $\hat{t}=1$, $\hat{\theta}_n^g=1$ and $\hat{\phi}_m^g=1$ for *g=1,…,K*, $n=1,...,N^g$, and $m=1,...,M^g$. Since all constraints (6b) are binding in optimality, we have $x_p^g = x_o^g$, $y_p^g = y_o^g$. Also, DMU_o is overall system efficient. It follows from Definition 4 that we have $\sum_{\substack{h\\(g,h)\in L}} z_p^{(g,h)} = \sum_{\substack{h\\(g,h)\in L}} z_o^{(g,h)}$, *g=1,…,K*, for the above optimal solution of (6).

Moreover, we have

$$\sum_{g=1}^{K}\sum_{\substack{h\\(g,h)\in L}} z_{qp}^{(g,h)} = \sum_{g=1}^{K}\sum_{\substack{h\\(g,h)\in L}} z_{qo}^{(g,h)} \qquad \text{for all } q\in\{1,...,Q\}. \tag{A}$$

Since DMU_p completely dominates DMU_o, we have $\sum_{\substack{h\\(h,g)\in L}} z_{qp}^{(h,g)} \le \sum_{\substack{h\\(h,g)\in L}} z_{qo}^{(h,g)}$ for all indices $q\in\{1,...,Q\}$ and $g\in\{1,...,K\}$. Since $x_p^g = x_o^g$, $y_p^g = y_o^g$ and

$\sum_{\substack{h \\ (g,h)\in L}} z_p^{(g,h)} = \sum_{\substack{h \\ (g,h)\in L}} z_o^{(g,h)}$ for at least a pair of indices q and g, we have a constraint with strict inequality, i.e., there exist indices $q \in \{1,...,Q\}$ and $g \in \{1,...,K\}$ such that $\sum_{\substack{h \\ (h,g)\in L}} z_{qp}^{(h,g)} < \sum_{\substack{h \\ (h,g)\in L}} z_{qo}^{(h,g)}$. It implies that

$$\sum_{g=1}^{K} \sum_{\substack{h \\ (h,g)\in L}} z_{qp}^{(h,g)} < \sum_{g=1}^{K} \sum_{\substack{h \\ (h,g)\in L}} z_{qo}^{(h,g)} \quad . \tag{B}$$

On the other hand, we have

$$\sum_{g=1}^{K} \sum_{\substack{h \\ (h,g)\in L}} z_{qp}^{(h,g)} = \sum_{g=1}^{K} \sum_{\substack{h \\ (g,h)\in L}} z_{qp}^{(g,h)} \quad \text{and} \quad \sum_{g=1}^{K} \sum_{\substack{h \\ (h,g)\in L}} z_{qo}^{(h,g)} = \sum_{g=1}^{K} \sum_{\substack{h \\ (g,h)\in L}} z_{qo}^{(g,h)} \tag{C}$$

which are the sums of all q^{th} intermediate products linked among all divisions of DMU_p and DMU_o, respectively.

From (A), (B) and (C), we have a desired contradiction. Therefore, there do not exist any DMUs that completely dominate DMU_o which is overall system efficient. The proof of the second part is similar and hence we omit it. ■

Proof of Theorem 3: We can prove Theorem 3 by considering Models (7) and (9).

Proof of Theorem 4: Let Division k of DMU_o be divisionally efficient. By contradiction, suppose that Division k of DMU_p $\left(p \in \{1,...,J\},\ p \neq o\right)$ dominates Division k of DMU_o, i.e.,

$$\left(-x_p^k, y_p^k, \ -\sum_{\substack{h \\ (h,k)\in L}} z_p^{(h,k)}, \ \sum_{\substack{h \\ (k,h)\in L}} z_p^{(k,h)}\right) \gneq \left(-x_o^k, y_o^k, \ -\sum_{\substack{h \\ (h,k)\in L}} z_o^{(h,k)}, \ \sum_{\substack{h \\ (k,h)\in L}} z_o^{(k,h)}\right).$$

Let $\frac{1}{\hat{t}} = \frac{1}{M^k + Q}\left(\sum_{m=1}^{M^k}\frac{y_{mp}^k}{y_{mo}^k} + \sum_{q=1}^{Q}\frac{\sum_{\substack{h \\ (k,h)\in L}} z_{qp}^{(k,h)}}{\sum_{\substack{h \\ (k,h)\in L}} z_{qo}^{(k,h)}}\right)$. From the previous relations, it is clear that $\hat{t} \le 1$, $x_{np}^k \le x_{no}^k$ and $\sum_{\substack{h \\ (h,k)\in L}} z_{qp}^{(h,k)} \le \sum_{\substack{h \\ (h,k)\in L}} z_{qo}^{(h,k)}$ for $n=1,\ldots,N^k$ and $q=1,\ldots,Q$. If we set $\hat{\lambda}^g = \left(\hat{\lambda}_1^g,\ldots,\hat{\lambda}_J^g\right)$ where $\hat{\lambda}_p^g = \hat{t}$ and $\hat{\lambda}_j^g = 0$ for $j \ne p$, $g=1,\ldots,K$ as well as $\hat{\theta}_n^k = \frac{\hat{t}x_{np}^k}{x_{no}^k}$,

$\hat{\varphi}_m^k = \frac{\hat{t}y_{mp}^k}{y_{mo}^k}$, $\hat{\hat{\theta}}_q^k = \hat{t}\frac{\sum_{\substack{h \\ (h,k)\in L}} z_{qp}^{(h,k)}}{\sum_{\substack{h \\ (h,k)\in L}} z_{qo}^{(h,k)}}$ and $\hat{\hat{\varphi}}_q^k = \hat{t}\frac{\sum_{\substack{h \\ (k,h)\in L}} z_{qp}^{(k,h)}}{\sum_{\substack{h \\ (k,h)\in L}} z_{qo}^{(k,h)}}$, then vector

$\left(\hat{\lambda}^1,\ldots,\hat{\lambda}^K, \hat{\theta}_1^k,\ldots,\hat{\theta}_{N^k}^k, \hat{\varphi}_1^k,\ldots,\hat{\varphi}_{M^k}^k, \hat{\hat{\theta}}_1^k,\ldots,\hat{\hat{\theta}}_Q^k, \hat{\hat{\varphi}}_1^k,\ldots,\hat{\hat{\varphi}}_Q^k, \hat{t}\right)$ is a feasible solution of (12).

Here we consider two cases for scalar $\hat{t}$:

Case 1. $\hat{t} < 1$; In this case, we have $\hat{\theta}_n^k < 1$ and $\hat{\hat{\theta}}_q^k < 1$. Hence, the objective value of (12) is less than 1, i.e., $E_o^k < 1$. This is a contradiction to the supposition that Division k of DMU_o is divisionally efficient.

Case 2. $\hat{t} = 1$; In this case we have $y_{mp}^k = y_{mo}^k$ and $\sum_{\substack{h \\ (k,h)\in L}} z_{qp}^{(k,h)} = \sum_{\substack{h \\ (k,h)\in L}} z_{qo}^{(k,h)}$ for $m=1,\ldots,M^k$ and $q=1,\ldots,Q$. Division k of DMU_p dominates division k of DMU_o. According to definition 3, it follows that there exist at least one index $n \in \{1,\ldots,N^k\}$ or $q \in \{1,\ldots,Q\}$ such that $\frac{x_{np}^k}{x_{no}^k} < 1$ or $\frac{\sum_{\substack{h \\ (h,k)\in L}} z_{qp}^{(h,k)}}{\sum_{\substack{h \\ (h,k)\in L}} z_{qo}^{(h,k)}} < 1$. It implies that at least one of $\hat{\theta}_n^k$ or $\hat{\hat{\theta}}_q^k$ is less than one, i.e., the objective value of (12) for this feasible solution is less than one and it is a contradiction to the supposition that Division k of DMU_o is efficient. ■

Appendix A2: Data of 27 Banks in Taiwan

Bank	Fixed assets ($billions) x_1	IT budget ($billions) x_2	#of employees (thousand) $x_3\mathbf{x_3}$	Deposits ($billions) zz	Profit ($billions) y_1	Fraction of loans recovered y_2
1	0.150	0.713	13.3	14.478	0.232	0.986
2	0.170	1.071	16.9	19.502	0.340	0.986
3	0.235	1.224	24.0	20.952	0.363	0.986
4	0.211	0.363	15.6	13.902	0.211	0.982
5	0.133	0.409	18.485	15.206	0.237	0.984
6	0.497	5.846	56.42	81.186	1.103	0.955
7	0.060	0.918	56.42	81.186	1.103	0.986
8	0.071	1.235	12.0	11.441	0.199	0.985
9	1.500	18.120	89.51	124.072	1.858	0.972
10	0.120	1.821	19.8	17.425	0.274	0.983
11	0.120	1.915	19.8	17.425	0.274	0.983
12	0.050	0.874	13.1	14.342	0.177	0.985
13	0.370	6.918	12.5	32.491	0.648	0.945
14	0.440	4.432	41.9	47.653	0.639	0.976
15	0.431	4.504	41.1	52.630	0.741	0.981
16	0.110	1.241	14.4	17.493	0.243	0.988
17	0.053	0.450	7.6	9.512	0.067	0.980
18	0.345	5.892	15.5	42.469	1.002	0.948
19	0.128	0.973	12.6	18.987	0.243	0.985
20	0.055	0.444	5.9	7.546	0.153	0.987
21	0.057	0.508	5.7	7.595	0.123	0.987
22	0.098	0.370	14.1	16.906	0.233	0.981
23	0.104	0.395	14.6	17.264	0.263	0.983
24	0.206	2.680	19.6	36.430	0.601	0.982
25	0.067	0.781	10.5	11.581	0.120	0.987
26	0.100	0.872	12.1	22.207	0.248	0.972
27	0.0106	1.757	12.7	20.670	0.253	0.988

Pareto-Koopmans Efficiency and Network DEA

Highlights

1. Network DEA models generalize standard DEA models.
2. We propose a two-phase network DEA approach from a Pareto-Koopmans perspective.
3. The approach can provide information on overall efficiency status and divisional efficiency.