Author's Accepted Manuscript

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PII: S0305-0483(15)00197-8
DOI: http://dx.doi.org/10.1016/j.omega.2015.09.004
Reference: OME1594

To appear in: *Omega*

Received date: 15 November 2014
Revised date: 17 September 2015
Accepted date: 18 September 2015

Cite this article as: Rolf Färe, Hirofumi Fukuyama, Shawna Grosskopf an Valentin Zelenyuk, Cost Decompositions and the Efficient Subset, *Omega* http://dx.doi.org/10.1016/j.omega.2015.09.004

Cost Decompositions and the Efficient Subset

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September 17, 2015

Abstract: This paper develops two cost decompositions based on the multiplicative Russell and additive slack-based (in)efficiency measurement frameworks. While the multiplicative cost decomposition is a straightforward extension of the standard cost decomposition, the decomposition we develop in this paper incorporates slacks directly so that efficiency is measured relative to the efficient subset. To show the applicability of our novel approach, we provide an illustration using a data set used in the literature.

Keywords: cost efficiency; efficient subset; Russell measure; directional distance function; slacks; data envelopment analysis (DEA)

JEL Classification: D24

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1. Introduction

The measurement of input technical efficiency relative to the efficient subset of an input set goes back to Färe (1975) who proposed minimizing inputs one at a time, i.e., nonradially. Later, Färe and Lovell (1978) proposed what they called the Russell measure (also referred to by others as the Färe-Lovell measure) which was also nonradial but summed over the individual input inefficiency components. These measures eliminated all technical inefficiencies including those due to 'slacks' as opposed to the radial Farrell (1957) measure of input technical efficiency which uses the isoquant rather than the efficient subset as the reference for technical efficiency. Thus, when the efficient subset differs from the isoquant, radial measures of technical efficiency such as the Farrell measure and nonradial measures may differ. Furthermore this may affect not only technical efficiency but allocative efficiency as well, resulting in different decompositions of the overall (e.g., cost or revenue) efficiency. This discrepancy is the motivation for considering nonradial measures as part of a decomposition of the overall Farrell measure of cost or revenue efficiency. The first such result was obtained by Färe, Grosskopf and Zelenyuk (2007). Their decomposition comes from introducing a multiplicative version of the Russell measure; and here we expand on their result. In this paper we will focus on the cost efficiency or input orientation, but similar decompositions can be developed for revenue efficiency as well.

The introduction of the directional distance functions[1], by Chambers, Chung and Färe (1998), is another alternative nonradial (additive) way of estimating technical (in)efficiency. In fact, the directional input distance function may be turned into a slack-based additive efficiency measure[2]. We can identify two classes of nonradial slack-based technical efficiency measures, a multiplicative and an additive measure, both with an indication property such that the multiplicative (additive) measure equals one

[1] The directional distance functions are production counterparts of Luenberger's shortage function which is based on the utility function and the consumption possibility set. See Luenberger (1995). The directional input distance function is a production counterpart of Luenberger's (1992) benefit function developed in a consumer context.

[2] See Färe and Grosskopf (2010).

(zero) if and only if the input vector belongs to the efficient subset. The efficient subset is particularly important in efficiency measurement because input vectors in the efficient subset cannot be reduced without decreasing at least one input and/or increasing at least one output. On the other hand, if the input vector is in the isoquant but not in the efficient subset, then it is possible to reduce at least one input given a fixed level of outputs. Whereas the Farrell measure and the directional input distance function are constructed relative to an isoquant, the slack-based measures are constructed relative to the efficient subset. Consequently, it is of great interest to develop efficiency analysis based on a slack-based efficiency measurement framework.

In this paper we introduce both an additive and a multiplicative slack-based approach for the decomposition of the Farrell cost efficiency measure. The rest of the paper unfolds as follows. While Section 2 describes the basics, Section 3 introduces a multiplicative cost decomposition and its corresponding allocative efficiency measure based on the multiplicative Russell measure developed by Färe, Grosskopf and Zelenyuk (2007). Section 4 introduces a new cost decomposition based on the additive Russell measure. Section 5 extends the additive approach into data envelopment analysis (DEA) and provides an empirical illustration using a real-life data set documented in Banker and Maindiratta (1988). The last section gives a brief summary.

2. Background and Methodology

In this section, we outline the theoretical background for our paper. Then we develop the methodology for it by building on the inequality of Mahler (1939). Let $x \in \Re_{+}^{N}$ be an input vector and $y \in \Re_{+}^{M}$ be an output vector. The input requirement sets are defined as

$$L(y) = \{x : x \text{ can produce } y\}, \quad y \in \Re_{+}^{M} \tag{1}$$

and are our representation of the technology, which is assumed to be a nonempty, closed, strongly disposable set satisfying the boundedness of $\{y : x \in L(y)\}$ as well as no free lunch and convexity of $L(y)$. For the details of these regularity conditions, see for example Färe and Primont (1995). Technology can equivalently be expressed as the production possibility set $T = \{(x, y) : x \in L(y)\}$, i.e., $x \in L(y) \Leftrightarrow (x, y) \in T$. The following two subsets of (1) are important for the paper. The isoquant for $y \in \mathbb{R}^M_+$ is defined as

$$IsoqL(y) = \{x : x \in L(y) \text{ and if } \lambda < 1 \text{ then } \lambda x \notin L(y)\}$$

and the efficient subset is

$$EffL(y) = \{x : x \in L(y) \text{ and if } x' \leq x \text{ then } x' \notin L(y)\}^3.$$

Clearly, $EffL(y) \subseteq IsoqL(y)$. Shephard's (1953) input distance function is defined as

$$D_i\left(y, x\right) = \sup\ \{\lambda : x / \lambda \in L(y)\}$$

which characterizes the production technology (1). Now assume that input prices $w \in \Re^N_{++}$ are given, then the cost function is

$$C\left(y, w\right) = \min_x \{w \cdot x : x \in L(y)\}$$

where $w \cdot x$ is the inner product, i.e., $w \cdot x = \sum_{n=1}^N w_n x_n$.

From the Mahler (1939) inequality, we have

$$\frac{C(y, w)}{w \cdot x} \leqq \frac{1}{D_i\,(y, x)}$$

where $C(y, w) / w \cdot x$ is referred to as the cost efficiency measure and $1 / D_i\,(y, x)$ is called the technical (Farrell) input oriented efficiency measure. An allocative efficiency measure, call it $AE(y, x, w)$, is then defined as the multiplicative residual required to close the inequality, so that

$$\frac{C(y, w)}{w \cdot x} = \frac{1}{D_i\,(y, x)} \times AE(y, x, w) \tag{2}$$

which is sometimes referred to as the Farrell decomposition of cost efficiency.

Throughout the paper, we use the above approach:

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[3] "$\leq$" means "$\leqq$ but $\neq$".

i) Start with the cost inequality.

ii) Derive a technical efficiency measure (input oriented).

iii) Complete the decomposition by introducing an allocative efficiency measure (input oriented).

3. The Multiplicative Approach[4]

The Russell measure (RM) has the indication property that it yields unity (or 100%) if and only if the input vector belongs to the efficient subset $EffL(y)$. Here we follow Färe, Grosskopf and Zelenyuk (2007) and define the multiplicative Russell measure as

$$RM^{mult}\left(y,x\right)=\min_{\lambda_1,\ldots,\lambda_N}\left\{\left(\prod_{n=1}^{N}\lambda_n\right)^{1/N}:\begin{array}{l}(\lambda_1 x_1,\ldots,\lambda_N x_N)\in L(y),\\ 0<\lambda_n\leqq 1,\ n=1,\ldots,N\end{array}\right\}. \tag{3}$$

This definition differs from the additive (original) Russell measure (Färe and Lovell (1978)), which we denote as $RM^{add}\left(y,x\right)$, whose objective function was additive, i.e.,

$$\frac{\sum_{n=1}^{N}\lambda_n}{N}.$$

For the rest of the paper, we assume that *x* is strictly positive, i.e., $x_n>0,\, n=1,\ldots,N$.

We note that

$$RM^{mult}\left(y,x\right)=1 \quad \text{if and only if} \quad x\in EffL(y)$$

and that

$$(\lambda_1^* x_1,\ldots,\lambda_N^* x_N)\in L(y)$$

where $\lambda_n^*,\ n=1,\ldots,N$, are the optimizers of (3). Together with the cost inequality we get

[4] This section follows and expands on Färe, Grosskopf and Zelenyuk (2007).

$$C(y,w) \le w_1\lambda_1^* x_1 + \ldots + w_N\lambda_N^* x_N$$
$$= RM^{mult}(y,x)\left(\frac{\lambda_1^* w_1 x_1}{\left(\prod_{n=1}^{N}\lambda_n^*\right)^{1/N}} + \ldots + \frac{\lambda_N^* w_N x_N}{\left(\prod_{n=1}^{N}\lambda_n^*\right)^{1/N}} \right)$$

and multiplying both sides with $1/w\cdot x$ yields

$$\frac{C(y,w)}{w\cdot x} \leq RM^{mult}(y,x)\left(\delta_1^* s_1 + \ldots + \delta_N^* s_N\right) \tag{4}$$

where

$$\delta_i^* = \frac{\lambda_i^*}{\left(\prod_{n=1}^{N}\lambda_n^*\right)^{1/N}}, \qquad i = 1,\ldots,N \tag{5}$$

and $s_n = \dfrac{w_n x_n}{w\cdot x}$, ($n$ = 1,…, N) are the factor shares and $\sum_{n=1}^{N} s_n = 1$. The expression $RM^{mult}(y,x)\left(\delta_1^* s_1 + \ldots + \delta_N^* s_N\right)$ shows a technical efficiency component in comparison with the standard cost efficiency measure, $C(y,w)/wx$, because the new Equation (4) is equivalent to $C(y,w) \le w_1\lambda_1^* x_1 + \ldots + w_N\lambda_N^* x_N$. Incidentally, note that:

$$RM^{mult}(y,x) = 1 \quad \text{if and only if} \quad \lambda_1^* = \ldots = \lambda_N^* = 1. \tag{6}$$

Furthermore, note that by closing the inequality with allocative efficiency, we have a cost decomposition relative to the efficient subset $EffL(y)$. Three comments are of particular consideration here:

i) $\delta_n^* = \dfrac{\lambda_n^*}{\left(\prod_{n=1}^{N}\lambda_n^*\right)^{1/N}}$ is the n-th input efficiency relative to the multiplicative Russell measure for $n = 1,\ldots,N$.

ii) If $\lambda_n^* = \lambda^*\ \forall n = 1,\ldots,N$, then the Farrell decomposition is obtained.

iii) The share s_n is the weight reflecting relative importance of n-th input in total cost.

Using (4), we propose the following decomposition:

$$\frac{C(y,w)}{w \cdot x} = RM^{mult}(y,x) \times \left(\delta_1^* s_1 + ... + \delta_N^* s_N\right) \times AE^{RM}(y,x,w)$$

where $AE^{RM}(y,x,w)$ is the (residual) allocative efficiency based on the multiplicative Russell measure.

Before closing this section, several remarks are in order[5]. First, $RM^{mult}(y,x)$ is a nonlinear program and hence it is relatively more difficult to solve. But $RM^{mult}(y,x)$ can be also viewed as a geometric programming problem. The objective function of the multiplicative Russell model is a monomial of positive variables $\lambda_1,...,\lambda_N$ and the constraints are shown in linear inequality form. Therefore, $RM^{mult}(y,x)$ is a relatively simple geometric programming problem once the efficient subset (input efficient frontier) is constructed. Interior-point algorithms for geometric programming have been developed and the algorithms can solve large-scale geometric programs[6].

Second, the basic properties of input-oriented efficiency measures proposed by Färe and Lovell (1978) are: (i) input-indication, (ii) strong input-monotonicity, and (iii) input-homogeneity. Since we have $RM^{mult}(y,x)=1$ if and only if $\lambda_1^* = ... = \lambda_N^* = 1$ as (6) states, the input-indication property holds. Since $\left(\prod_{n=1}^{N} \lambda_n^*\right)^{1/N}$ is increasing in $(\lambda_1,...,\lambda_J) > 0$, $RM^{mult}(y,x)$ satisfies strong input-monotonicity. Note that the strong input-monotonicity property of the additive Russell measure was proved by Färe and Lovell (1978). Regarding input-homogeneity, $RM^{mult}(y,x)$ does not satisfy this property but it does satisfy its weaker version called sub-input-homogeneity. This

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[5] We add these remarks to incorporate the referees' constructive comments and suggestions.

[6] Boyd and Vandenberghe (2004) reported that a standard algorithm based on the interior-point method can solve a complicated geometric program with 10000 constraints and 1000 variables within a minute on a desktop computer. However, such a computational issue is beyond the scope of this paper.

proof[7] is similar to that of the sub-input-homogeneity property of Färe and Lovell's (1978) additive Russell measure.

Third, $RM^{mult}(y,x)$ is units invariant. This property can also be proved by adapting the proof for the slack-based directional technology distance function (see Färe, Fukuyama, Grosskopf and Zelenyuk 2015). For the sake of completeness, we include such a proof in the Appendix A1.

Fourth, regarding the issue of multiple optimal solutions, the multiplicative Russell measure may have alternative optimal solutions. However, all known DEA-based linear programming efficiency measures also face the problem of possible non-unique optimal solutions. See for example Fukuyama et al. (2014) for the discussion of multiple optimal solutions in the standard constant and variable returns to scale DEA models.

Finally, there is an alternative multiplicative DEA model in the literature. The model, called the multiplicative DEA model, was proposed by Charnes, Cooper, Seiford and Stutz (1982). The multiplicative DEA model can be linearized by taking the natural logarithms. However, the linear form of the multiplicative DEA model needs to be constructed based on the nonstandard log-linear production possibility set. In contrast, the multiplicative Russell measure (3) is constructed relative to the standard production possibility set T and hence it cannot be linearized.

4. The Additive Approach

Our additive approach builds upon the concept of the slack-based directional input distance function by Färe and Grosskopf (2010). The slack-based directional input distance function can be thought of as a generalization of the directional input distance function for a directional input vector of ones, $1_N = (1,...,1) \in \mathbb{R}^N_+$. To see this, let us first introduce a directional vector $g = \left(g_1,...,g_N\right) \in \mathbb{R}^N_+$.

Relative to g, the directional input distance function is defined as

[7] See Russell (1985) and Dmitruk and Koshevoy (1991). Therefore, these basic properties of $RM^{mult}(y,x)$ can easily be proved.

$$\vec{D}_i\left(y,x;g\right)=\sup_{\beta}\left\{\beta:\quad (x_1-\beta\cdot g_1,...,x_N-\beta\cdot g_N)\in L(y),\ \ \beta\geqq 0\right\}.$$

Now we take $g=1_N$, so that the directional input distance function is written as

$$\vec{D}_i\left(y,x;1_N\right)=\sup_{\beta}\left\{\beta:\quad (x_1-\beta\cdot 1,...,x_N-\beta\cdot 1)\in L(y),\ \ \beta\geqq 0\right\} \tag{7}$$

Allowing for asymmetric scaling in (7), the slack-based directional input distance function[8] is defined as

$$S\vec{D}_i\,(y,x;1_N)=\max_{\beta_1,...,\beta_N}\left\{\beta_1+...+\beta_N:(x_1-\beta_1\cdot 1,...,x_N-\beta_N\cdot 1)\in L(y),\quad \beta_n\geqq 0\ \forall n\right\} \tag{8}$$

where $\beta_1^*,...,\beta_N^*$ are the optimizers in the linear programming problem (8). Assuming $g=1_N$ indicates that each component is endowed with a unit of measurement. For example, if x_1 is labor hours, the first component is one unit of labor hours. This makes our indicator independent of the unit of measurement[9].

Here we have assumed that $x_n>0,\ n=1,...,N$ for illustrative purposes of this section; however, we need not assume the positivity of the inputs to define $S\vec{D}_i\,(y,x;1_N)$ because zeros do not cause feasibility problems. Färe and Grosskopf (2010) proved that

$$S\vec{D}_i\,(y,x;1_N)=0 \quad \text{if and only if} \quad x\in EffL(y)\,.$$

Since $(x_1-\beta_1^*\cdot 1,...,x_N-\beta_N^*\cdot 1)\in L(y)$, the cost inequality

$$w\cdot x\geqq C(y,w) \quad \text{for all } \ x\in L(y)$$

yields

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[8] Note that when the multiplication based on $g=1_N=(1,...,1)$ is used as in (8), we leave "1" to emphasize that the directional vector is (1,…,1) and that each "1" refers to the particular unit of measurement of that particular input.

[9] See Färe, Fukuyama, Grosskopf and Zelenyuk (2015) for the units invariance property of the slack-based directional technology distance function.

$$w_1(x_1 - \beta_1^* \cdot 1) + ... + w_N(x_N - \beta_N^* \cdot 1) \geqq C(y,w)$$

implying that

$$w \cdot x - C(y,w) \geqq w_1 \beta_1^* \cdot 1 + ... + w_N \beta_N^* \cdot 1 .$$

Now consider the situation where $x \notin EffL(y)$. Multiplying and dividing by $S\vec{D}_i(y,x;1_N) = \sum_{n=1}^{N} \beta_n^*$ yields

$$w \cdot x - C(y,w) \geqq S\vec{D}_i(y,x;1_N)\left(\frac{w_1\beta_1^*}{\sum_{n=1}^N \beta_n^*} + ... + \frac{w_N\beta_N^*}{\sum_{n=1}^N \beta_n^*}\right). \quad (9)$$

Also, dividing both sides of (9) by the value of the directional vector

$$w \cdot g = \sum_{n=1}^N w_n \cdot 1 \quad \left(g = 1_N\right),$$

we have

$$\frac{w \cdot x - C(y,w)}{\sum_{n=1}^N w_n \cdot 1} \geqq S\vec{D}_i(y,x;1_N)\left(\frac{\alpha_1\beta_1^*}{\sum_{n=1}^N \beta_n^*} + ... + \frac{\alpha_N\beta_N^*}{\sum_{n=1}^N \beta_n^*}\right) \quad (10)$$

where $\alpha_n = \dfrac{w_n}{\sum_{n=1}^N w_n \cdot 1}$ $\left(n = 1,...,N\right)$ are the price shares and hence $\sum_{n=1}^N \alpha_n = 1$.

For a technically inefficient firm, our new Mahler-type inequalities (9) and (10) can be closed by including a residual allocative inefficiency similar to (2), but additive. Specifically, a decomposition based on the slack-based directional input distance function, which provides a technical inefficiency score, is obtained by

$$\frac{w \cdot x - C(y,w)}{\sum_{n=1}^N w_n \cdot 1} = S\vec{D}_i(y,x;1_N) \times \left(\frac{\alpha_1\beta_1^*}{\sum_{n=1}^N \beta_n^*} + ... + \frac{\alpha_N\beta_N^*}{\sum_{n=1}^N \beta_n^*}\right) + AIneff\left(y,x,w;1_N\right) \quad (11)$$

where $AIneff(y,x,w;1_N)$ is the allocative inefficiency[10].

The slack-based directional input distance function $S\vec{D}_i(y,x;1_N) = \sum_{n=1}^N \beta_n^*$

[10] $AIneff\left(y,x,w;1\right)$ is defined as the residual of normalized cost inefficiency and the slack-based directional distance function and hence it is not a directional distance function.

represents the total (input) technical inefficiency and $\frac{S\vec{D}_i(y, x; 1_N)}{N}$ is the average (input) technical inefficiency.

We also introduce the normalized versions of the slack-based efficiency measures. Specifically, the normalized slack-based directional total technical inefficiency term is given by $S\vec{D}_i(y, x; 1_N) \times \left(\frac{\alpha_1 \beta_1^*}{\sum_{n=1}^N \beta_n^*} + ... + \frac{\alpha_N \beta_N^*}{\sum_{n=1}^N \beta_n^*} \right)$, which can be directly compared to cost inefficiency $w \cdot x - C(y, w)$ normalized by $\sum_{n=1}^N w_n \cdot 1$. In Equation (10), $\alpha_n = \frac{w_n}{\sum_{n=1}^N w_n \cdot 1}$ is the weight of the n-th input price. The inequality (10) means that we can directly compare $S\vec{D}_i(y, x; 1_N)$ and $wx - C(y, w)$ after normalizing them with the use of $\sum_{n=1}^N w_n \cdot 1$ and $\left(\frac{\alpha_1 \beta_1^*}{\sum_{n=1}^N \beta_n^*} + ... + \frac{\alpha_N \beta_N^*}{\sum_{n=1}^N \beta_n^*} \right)$. These adjusted terms in Equation (7) are used to properly define allocative inefficiency $AIneff\,(y, x, w; 1_N)$.

For a technically efficient firm, $\left(w \cdot x - C(y, w)\right) / \sum_{n=1}^N w_n \cdot 1 \geq 0$ is used and the corresponding allocative inefficiency is defined to close this inequality gap. Note that

i) $\frac{\beta_n^*}{\sum_{n=1}^N \beta_n^*}$ is the *n*-th input efficiency.

ii) if $\beta_n^* = \beta_n$ for all *n*, then $S\vec{D}_i\,(y, x; 1_N) = N \cdot \vec{D}_i\left(y, x; 1_N\right)$ and hence

$$\frac{w \cdot x - C(y, w)}{\sum_{n=1}^N w_n \cdot 1} \geqq \vec{D}_i\,(y, x; 1_N)$$

where $\vec{D}_i\left(y, x; 1_N\right)$ is the directional input distance function given in (7).

Finally, we show how the slack-based directional input distance function $S\vec{D}_i\,(y, x; g)$

is related to the additive Russell measure $RM^{add}(y,x)$ developed by Färe and Lovell (1978), where

$$RM^{add}(y,x) = \min_{\lambda_1,\dots,\lambda_N} \left\{ \lambda_1 + \dots + \lambda_N : \left(\lambda_1 x_1, \dots, \lambda_N x_N \right) \in L(y),\ 0 < \lambda_n \leqq 1 \quad \forall n \right\}.$$

Suppose $g = x$, then we have

$$\begin{aligned} \vec{SD}_i\left(y,x;x\right) &= \max_{\beta_1,\dots,\beta_N} \left\{ \beta_1 + \dots + \beta_N : (x_1 - \beta x_1, \dots, x_N - \beta_N x_N) \in L(y),\ 1 > \beta_n \geqq 0\ \forall n \right\} \\ &= -\min_{\beta_1,\dots,\beta_N} \left\{ (-\beta_1) + \dots + (-\beta_N) : \left((1-\beta_1) x_1, \dots, (1-\beta_N) x_N \right) \in L(y),\ 1 > \beta_n \geqq 0\ \forall n \right\} \\ &= N - \min_{\lambda_1,\dots,\lambda_N} \left\{ \lambda_1 + \dots + \lambda_N : \left(\lambda_1 x_1, \dots, \lambda_N x_N \right) \in L(y),\ 0 < \lambda_n \leqq 1\ \forall n \right\}, \quad \lambda_n = 1 - \beta_n \\ &= N\left(1 - RM^{add}(y,x)\right). \end{aligned} \tag{12}$$

Note that $\beta_n \geqq 0$ in (8) is replaced by $1 > \beta_n \geqq 0$ on the right-hand side of the first equation of (12). This result shows that, since $RM^{add}(y,x)$ and $1 - RM^{add}(y,x)$ are respectively efficiency and inefficiency measures, $\vec{SD}_i\left(y,x;x\right)/N$ is an inefficiency measure.

In the production economics literature, there is a measure called the (input-oriented) Zieschang measure (Zieschang 1984). The Zieschang measure first projects an inefficient firm radially onto the isoquant and then tries to minimize the resultant input vector to reach the efficient subset for a given output level. In contrast, the slack-based directional input distance function directly searches for a point on the efficient subset[11]. Therefore, the projection points are likely to differ between the Zieschang measure and the slack-based directional input distance function and hence the decomposition based on the Zieschang measure differs from ours.

The slack-based directional input distance function has the properties[12] of

[11] A similar comparison can be made for the multiplicative Russell measure.

[12] $\vec{SD}_i\,(y,x;1_N)$ is units invariant. This property can be proved easily by adapting the proof for the slack-based directional technology distance function from Färe, Fukuyama, Grosskopf and Zelenyuk (2015), who cited the present paper as a CEPA working paper.

input-indication, strong input-monotonicity and sub input-homogeneity. However, the Zieschang measure fails to satisfy strong input-monotonicity (even weak input-monotonicity fails). See Russell (1988) on this point. In contrast, the slack-based directional input distance function satisfies strong input-monotonicity.

5. DEA Implementation

To further develop our analysis provided in Section 4, we utilize non-parametric frontier models which have been popularized as data envelopment analysis (DEA) by Charnes, Cooper and Rhodes (1978), Färe, Grosskopf and Lovell (1985) and many others.[13] Let $\left\{(x_{1j},...,x_{Nj},y_{1j},...,y_{Mj}): j=1,2,\ldots,J\right\}$ be observations consisting of J decision making units or firms. We assume that inputs and outputs are nonnegative but each input and output consists of at least one positive quantity. The reference technology of the conceptual input requirement set, which allows for constant returns to scale (CRS) is constructed as

$$L^{CRS}(y)=\left\{x\in\mathbb{R}_{+}^{N}:\sum_{j=1}^{J}x_{nj}z_j \leqq x_n\ \forall n,\ \sum_{j=1}^{J}y_{mj}z_j \geqq y_m\ \forall m,\ z\geq 0\right\},\ y\in\mathbb{R}_{+}^{M} \tag{13}$$

where z is a J-dimensional vector of intensity variables. The reference technology under the variable returns to scale (VRS) environment is constructed as

$$L^{VRS}(y)=\left\{x\in\mathbb{R}_{+}^{N}:\sum_{j=1}^{J}x_{nj}z_j \leqq x_n\ \forall n,\ \sum_{j=1}^{J}y_{mj}z_j \geqq y_m\ \forall m,\ \sum_{j=1}^{J}z_j=1,\ z\geq 0\right\},\ y\in\mathbb{R}_{+}^{M}\ . \tag{14}$$

The convexity restriction $\sum_{j=1}^{J}z_j=1$ in (14) allows for variable returns to scale. Using the two basic DEA-based representations (13) and (14), we develop two slack-based directional input distance functions:

$$S\vec{D}_i^{CRS}(y,x;1_N)=\max_{\beta_1,...,\beta_N}\left\{\sum_{n=1}^{N}\beta_n:(x_1-\beta_1\cdot 1,...,x_N-\beta_N\cdot 1)\in L^{CRS}(y),\ \ \beta_n \geqq 0\ \forall n\right\}, \tag{15}$$

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[13] See Asmild and Pastor (2010), Atici and Podinovski (2015), Cook, Tone and Zhu (2014) and Liu, Lu, Lu, and Lin (2013) to mention just a few recent works in this journal.

$$S\vec{D}_i^{VRS}(y,x;1_N)=\max_{\beta_1,\dots,\beta_N}\left\{\sum_{n=1}^{N}\beta_n:(x_1-\beta_1\cdot 1,\dots,x_N-\beta_N\cdot 1)\in L^{VRS}(y),\quad \beta_n \geqq 0\ \forall n\right\}. \tag{16}$$

Since $L^{VRS}(y)\subseteq L^{CRS}(y)$, the following must hold:

$$S\vec{D}_i^{VRS}(y,x;1_N)\leqq S\vec{D}_i^{CRS}(y,x;1_N). \tag{17}$$

The discrepancy between $S\vec{D}_i^{CRS}(y,x;1_N)$ and $S\vec{D}_i^{VRS}(y,x;1_N)$ can be written as

$$SSIneff(y,x;1_N)=S\vec{D}_i^{CRS}(y,x;1_N)-S\vec{D}_i^{VRS}(y,x;1_N) \tag{18}$$

which we refer to as a slack-based scale inefficiency measure. $SSIneff(y,x)$ can be thought of as the (in)efficiency associated with firm size being not at the CRS level. When $SSIneff(y,x;1_N)=0$ the firm operates at the constant returns to scale level, which is consistent with the socially optimal scale of utilizing resources. When $SSIneff(y,x;1)>0$, a gain in production can obtained if the firm adjusts to produce at the optimal scale.

Using (18) and the cost efficiency decomposition (11), we obtain a DEA-based cost decomposition[14] as

$$\begin{aligned}\frac{w\cdot x-C^{CRS}(y,w)}{\sum_{n=1}^{N}w_n\cdot 1}&=\left(SSIneff(y,x;1_N)+S\vec{D}_i^{VRS}(y,x;1_N)\right)\\&\quad\times\left(\frac{\alpha_1\beta_1^{CRS*}}{S\vec{D}_i^{CRS}(y,x;1_N)}+\dots+\frac{\alpha_N\beta_N^{CRS*}}{S\vec{D}_i^{CRS}(y,x;1_N)}\right)+AIneff^{CRS}(y,x,w;1_N),\end{aligned} \tag{19}$$

where $C^{CRS}(y,w)=\min_x\left\{w\cdot x: x\in L^{CRS}(y)\right\}$ is the cost function and $AIneff^{CRS}(y,x,w;1_N)$ the allocative inefficiency, both developed assuming CRS technology, via (13). Note also that $\beta_n^{CRS*},\ n=1,\dots,N,$ are the solutions to (15).

[14] Note that it is possible to define allocative inefficiency as a multiplicative term by dividing the left-hand side of (19) with $\left(SSIneff(y,x;1_N)+S\vec{D}_i^{VRS}(y,x;1_N)\right)\left(\sum_{n=1}^{N}\frac{\alpha_n\beta_n^{CRS*}}{S\vec{D}_i^{CRS}(y,x;1_N)}\right)$ from the right-hand side.

6. An Illustration

In this section, we apply our newly developed additive approach to the data set given in Banker and Maindiratta (1988), which consists of 20 observations. The data were obtained from a division of a large decentralized U.S. manufacturing firm observed over 20 quarters. The data set consists of one output (y_1) and labor (x_1), material (x_2) and capital (x_3). The data and descriptive statistics are given in Table 1. See Banker and Maindiratta (1988) for the details of the data. Our slack-based estimates of cost efficiency and its components are reported in Table 2. The average estimates of $SSIneff\,(y,x;1_N)$ and $S\vec{D}_i^{VRS}(y,x;1_N)$ are 4239.39 and 4791.84, respectively. This shows that slack-based scale inefficiency is less severe than slack-based directional inefficiency on the average. The major source of slack-based technical CRS-inefficiency ($S\vec{D}_i^{CRS}(y,x;1_N)$) varies substantially: slack-based scale inefficiency ($SSIneff\,(y,x;1_N)$) is larger than the slack-based VRS-directional inefficiency $S\vec{D}_i^{VRS}(y,x;1_N)$ for nine DMUs and the reverse is true for eight DMUs. DMUs 8, 12 and 17 are CRS-efficient. The normalized slack-based directional technical inefficiency with CRS technology, which we denote as $NS\vec{D}_i^{CRS}(y,x;1_N)$,

$$\begin{aligned} NS\vec{D}_i^{CRS}(y,x;1_N) &= S\vec{D}_i^{CRS}(y,x;1_N) \times \left(\sum_{n=1}^{N} \frac{\alpha_n \beta_n^{CRS*}}{S\vec{D}_i^{CRS}(y,x;1_N)} \right) \\ &= \left(SSIneff\,(y,x;1_N) + S\vec{D}_i^{VRS}(y,x;1_N) \right) \times \left(\sum_{n=1}^{N} \frac{\alpha_n \beta_n^{CRS*}}{S\vec{D}_i^{CRS}(y,x;1_N)} \right) \end{aligned} \tag{20}$$

is, on average, greater than $AIneff^{\,CRS}(y,x,w;1_N)$; only three observations were allocatively inefficient relative to CRS technology.

<<Table 2>>about here

In contrast, in the Farrell case under CRS, all but one observation were allocatively inefficient (Table 3). Furthermore, allocative efficiency components are relatively more important for both CRS and VRS technologies (see also the legend of Table 3). This is what we would expect: the additive, nonradial approach will eliminate slacks in input usage in addition to the proportional 'overuse' identified in the radial Farrell approach to measuring technical efficiency. In the Farrell decompositions this slack would then be attributed to allocative inefficiency rather than technical inefficiency. On the other hand both approaches agree that only one observation is cost efficient under CRS—observation 13.

<<Table 3>>about here

Finally, we examine scale efficiencies. Table 4 compares three scale efficiency measures. The first scale efficiency measure is obtained by transforming our scale inefficiency measure into a normalized slack-based scale efficiency measure,

$$NSSE(y,x;1_N) = 1 - \frac{SSIneff(y,x;1_N)}{w \cdot x}\left(\sum_{n=1}^{N} \frac{w_n \beta_n^{CRS*}}{\sum_{n=1}^{N} \beta_n^{CRS*}}\right) \tag{21}$$

to compare it with the two popular scale efficiency measures: the Farrell and the cost-based scale efficiency measures. See Appendix A2 for the derivation of (21). Note that cost-based scale efficiency $CSE(y,w)$ is defined as

$$CSE(y,w) = \frac{C^{CRS}(y,w)}{C^{VRS}(y,w)},$$

which can be shown also as the ratio of the cost efficiency measure w.r.t. CRS technology to the cost efficiency measure w.r.t. VRS technology, due to the relationship between the cost function and cost efficiency (see Färe and Grosskopf (1985) and Zelenyuk (2014) for more details and the relationship between the two measures).

According to Table 4, the three methods (Farrell, slack-based scale efficiency and cost-based scale efficiency) give similar values for our data example.

<<Table 4>>about here

Note that a useful practical advantage of the nonradial approach illustrated here is the additional information provided by the individual input scores, providing guidance to the firm as to how to reduce cost, but also providing the basis for individual factor productivities.

7. Summary and Conclusions

We have proposed new cost inefficiency decompositions (multiplicative and additive forms) for the multiplicative Russell and the slack-based (in)efficiency measures which gauge (in)efficiencies relative to the efficient subset, rather than the isoquant. Then, focusing on the additive form, we illustrate the additive cost inefficiency decomposition in a DEA framework. As expected, the slack-based decomposition typically results in a greater share of cost inefficiency due to technical rather than allocative inefficiency, which is illustrated in our empirical example.

The decomposition results based on the slack-based directional input distance function can be extended to a slack-based directional technology distance function, which is a dual to the profit function[15]. However, it is important to note that cost inefficiency and profit inefficiency have different economic interpretations and hence separate treatments are needed. In fact, the use of an unconstrained profit function is very restrictive in some real-life situations because profit maximization requires cost minimization and revenue maximization simultaneously. A firm, operating in a stagnant economic situation, may try to minimize costs rather than to maximize profits. In this case, the projection points suggested by the profit maximization framework are less practical because attaining the profit maximizing objective requires much effort if the outputs are required to be increased in the stagnant economic situation. Furthermore, cost (in)efficiency can be decomposed into allocative (in)efficiency and a scale (in)efficiency component (as is illustrated in our empirical example). In contrast, scale inefficiency is not a component of long-run profit inefficiency because profit-maximizing projection points will not be on the increasing returns to scale portion of the frontier.

Moreover, it is interesting to note that our decomposition method can be extended to Briec's (2000) slack-based directional distance function and Fukuyama and Weber's (2010) and Akther et al.'s (2013) two-stage network directional distance models.

[15] This can be done by expanding on Färe, Fukuyama, Grosskopf and Zelenyuk (2015).

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Table 1: Banker and Maindiratta Data

No.	y_1	x_1	x_2	x_3	w_1	w_2	w_3
1	94593	30722	38054	8184.00	1	1	1
2	95921	28365	35795	8119.00	1	1	1
3	76852	25445	31814	8079.00	1	1	1
4	94141	30648	41743.4	8667.62	1.04686	1.017	1.047
5	102132	33279	41364.8	8881.57	1.04688	1.017	1.047
6	100341	30828.1	40124.9	8744.03	1.04684	1.017	1.047
7	81755	27360.1	32910.5	8109.84	1.04689	1.017	1.047
8	95154	31544.9	39720.8	9659.96	0.09571	1.046	1.094
9	91393	33485.8	40893.9	8889.40	1.09593	1.046	1.094
10	90752	30725.8	39137.7	8808.96	1.09553	1.046	1.094
11	75033	27881.6	32143.4	8442.41	1.09628	1.046	1.094
12	85681	30042.7	28737.6	81113.79	1.15742	1.109	1.16
13	87399	24799.7	32198.4	6962.07	1.15682	1.109	1.16
14	80469	27676.7	38023.4	6887.93	1.15556	1.109	1.16
15	65009	25173.9	30527.5	7051.72	1.15397	1.109	1.16
16	86443	28634.7	43111.2	8520.29	1.22135	1.124	1.232
17	94454	28289.2	46075.6	7384.74	1.22015	1.124	1.232
18	84361	26157.7	39393.2	7344.97	1.21861	1.124	1.232
19	76176	23490	33694	7351.46	1.21992	1.124	1.232
20	75775	23078.5	31686.8	7311.08	1.27937	1.194	1.318
mean	86692	28381.4	36857.5	11725.69	1.06771	1.069	1.123
std dev	9736	3039.2	4848.6	16350.50	0.24402	0.057	0.093
max	102132	33485.8	46075.6	81113.79	1.27937	1.194	1.318
min	65009	23078.5	28737.6	6887.93	0.09571	1	1

Table 2: Estimation Results

firm	β_1^{CRS*}	β_2^{CRS*}	β_3^{CRS*}	$NCIneff(y,w,x)$	$SSIneff(y,x;1_N)$	$\vec{SD}_i^{VRS}(y,x;1_N)$	$\sum_{n=1}^{N}\frac{\alpha_n\beta_n^{CRS*}}{\vec{SD}_i^{CRS}(y,x;1_N)}$	$AIneff^{CRS}(y,x,w;1_N)$
1	3880.98	3205.28	648.87	2578.38	1757.79	5977.34	0.3333333	0
2	1147.16	457.04	478.08	694.09	2082.28	0	0.3333333	0
3	3638.04	3501.19	1957.09	3032.11	6009.24	3087.08	0.3333333	0
4	3936.14	7061.20	1168.49	4026.30	1647.35	10518.48	0.3309510	0
5	4298.77	3738.66	745.89	2919.90	8783.32	0	0.3324370	0
6	2356.07	3158.58	751.02	2078.22	6265.67	0	0.3316842	0
7	4161.90	2791.39	1597.36	2850.74	3215.71	5334.94	0.3333942	0
8	4544.70	4665.40	2080.14	3891.82	1894.87	9395.37	0.3007188	2373.21
9	7552.79	7224.08	1609.17	5437.13	975.90	15410.15	0.3318143	0
10	4974.68	5704.03	1579.80	4062.59	819.27	11439.23	0.3314096	0
11	6590.79	4500.72	2465.40	4520.70	7165.87	6391.03	0.3334609	0
12	0	0	0	1456.68	0	0	0	1456.68
13	0	0	0	0	0	0	0	0
14	4843.41	8378.06	477.89	4509.33	13699.36	0	0.3291635	0
15	6727.42	6577.73	1873.20	5033.89	15178.36	0	0.3316493	0
16	4106.27	11265.00	1634.37	5504.24	544.69	16460.95	0.3236715	0
17	0	0	0	4004.32	0	0	0	4004.32
18	2220.04	8314.02	624.90	3586.46	1730.92	9428.04	0.3213973	0
19	1874.86	5630.23	1283.40	2851.49	6394.39	2394.09	0.3244575	0
20	1577.14	3770.76	1274.96	2162.92	6622.86	0	0.3265839	0
mean	3421.56	4497.17	1112.50	3260.07	4239.39	4791.84	0.2791396	391.71
s.d.	2232.67	3095.01	737.20	1493.75	4449.78	5563.77	0.1205369	1044.28
max	7552.79	11265.00	2465.40	5504.24	15178.36	16460.95	0.3334609	4004.32
min	0	0	0	0	0	0	0	0

Table 3: Farrell Decompositions

firm	$FCE^{CRS}(y,x,w)$	$F^{CRS}(y,x)$	$FAE^{CRS}(y,x,w)$	$FCE^{VRS}(y,x,w)$	$F^{VRS}(y,x)$	$FAE^{VRS}(y,x,w)$
1	0.89949	0.92045	0.97723	0.92233	0.96183	0.95894
2	0.97119	0.97945	0.99157	1	1	1
3	0.86078	0.88063	0.97746	0.95275	0.97288	0.97932
4	0.85020	0.87157	0.97548	0.87070	0.90288	0.96436
5	0.89463	0.91567	0.97702	1	1	1
6	0.92138	0.92357	0.99763	1	1	1
7	0.87439	0.89560	0.97632	0.92130	0.93867	0.98150
8	0.84219	0.87505	0.96244	0.87961	0.89195	0.98616
9	0.80275	0.82194	0.97666	0.81462	0.84131	0.96827
10	0.84396	0.84974	0.99319	0.85450	0.85724	0.99680
11	0.80074	0.83110	0.96348	0.90492	0.94098	0.96169
12	0.93437	1	0.93437	0.94892	1	0.94892
13	1	1	1	1	1	1
14	0.81200	0.92098	0.88167	0.86631	1	0.86631
15	0.75760	0.76956	0.98446	0.98830	1	0.98830
16	0.79036	0.85660	0.92268	0.79714	0.86113	0.92569
17	0.84990	1	0.84990	0.87162	1	0.87162
18	0.84953	0.91513	0.92832	0.87325	0.94796	0.92119
19	0.86510	0.92018	0.94013	0.96388	0.98978	0.97383
20	0.89350	0.93166	0.95903	1	1	1
mean	0.86570	0.90394	0.95845	0.92151	0.95533	0.96464
s.d.	0.06071	0.06215	0.03911	0.06596	0.05539	0.04034
max	1	1	1	1	1	1
min	0.75760	0.76956	0.84990	0.79714	0.84131	0.86631

Legend:

	CRS-efficiency	VRS-efficiency
Farrell cost:	$FCE^{CRS}(y,x,w) = C^{CRS}(y,w)/w \cdot x$	$FCE^{VRS}(y,x,w) = C^{VRS}(y,w)/w \cdot x$
Farrell technical:	$F^{CRS}(y,x) = \min\left\{\lambda > 0 : \lambda x \in L^{CRS}(y)\right\}$	$F^{VRS}(y,x) = \min\left\{\lambda > 0 : \lambda x \in L^{VRS}(y)\right\}$
Farrell allocative:	$FAE^{CRS}(y,x,w) = FCE^{CRS}(y,x,w)/F^{CRS}(y,x)$	$FAE^{VRS}(y,x,w) = FCE^{VRS}(y,x,w)/F^{VRS}(y,x)$
Decomposition:	$FCE^{CRS}(y,x,w) = F^{CRS}(y,x) \times FAE^{CRS}(y,x,w)$	$FCE^{VRS}(y,x,w) = F^{VRS}(y,x) \times FAE^{VRS}(y,x,w)$

Table 4: Scale Efficiency

firm	$NSSE(y,x;1_N)$	$SE(y,x)$	$CSE(y,w)$
1	0.97716	0.95698	0.97524
2	0.97119	0.97945	0.97119
3	0.90803	0.90518	0.90347
4	0.97972	0.96533	0.97646
5	0.89463	0.91567	0.89463
6	0.92138	0.92357	0.92138
7	0.95276	0.95412	0.94909
8	0.97689	0.98106	0.95746
9	0.98825	0.97697	0.98543
10	0.98957	0.99125	0.98766
11	0.89468	0.88323	0.88487
12	1	1	0.98467
13	1	1	1.00000
14	0.81200	0.92098	0.93730
15	0.75760	0.76956	0.76657
16	0.99329	0.99474	0.99150
17	1	1	0.97509
18	0.97666	0.96537	0.97284
19	0.90185	0.92969	0.89752
20	0.89350	0.93166	0.89350
mean	0.93946	0.94724	0.94129
s.d.	0.06648	0.05460	0.05602
max	1	1	1
min	0.75760	0.76956	0.76657

Appendix A1: Units invariance of $RM^{mult}(y,x)$

Proof: To show that $RM^{mult}(y,x)$ is independent of the units of measurement, let Ω_x and Ω_y be arbitrary strictly positive diagonal matrices that transform the units of measurement of, respectively, inputs and outputs (x and y), into

$\tilde{x} = \Omega_x x = (x_1u_1,....,x_Nu_N)'$ and $\tilde{y} = \Omega_y y = (y_1v_1,....,y_Nv_N)'$, and let $\tilde{T}$ be the same technology as T but expressed in the new units of measurement, i.e.,

$$\tilde{T} = \{(\tilde{x},\tilde{y}) : \tilde{x} = \Omega_x x \text{ can produce } \tilde{y} = \Omega_y y\} \tag{22}$$

then it should also be the case that

$$(x,y) \in T \quad \Leftrightarrow \quad (\tilde{x},\tilde{y}) \in \tilde{T}$$

and therefore (for x > 0) we have

$$\begin{aligned}
RM^{mult}(\tilde{y},\tilde{x}) &= \min_{\lambda_1,...,\lambda_N} \left\{ \left(\prod_{n=1}^{N} \lambda_n \right)^{1/N} : \begin{array}{l} (\lambda_1 \tilde{x}_1,...,\lambda_N \tilde{x}_N, \tilde{y}_1,...,\tilde{y}_M) \in \tilde{T}, \\ 0 < \lambda_n \leqq 1, \ n = 1,...,N \end{array} \right\} \\
&= \min_{\lambda_1,...,\lambda_N} \left\{ \left(\prod_{n=1}^{N} \lambda_n \right)^{1/N} : \begin{array}{l} (\lambda_1 x_1 u_1,...,\lambda_N x_N u_N, y_1 v_1,..., y_M v_M) \in \tilde{T}, \\ 0 < \lambda_n \leqq 1, \ n = 1,...,N \end{array} \right\} \\
&= \min_{\lambda_1,...,\lambda_N} \left\{ \left(\prod_{n=1}^{N} \lambda_n \right)^{1/N} : \begin{array}{l} (\lambda_1 x_1,...,\lambda_N x_N, y_1,..., y_M) \in T, \\ 0 < \lambda_n \leqq 1, \ n = 1,...,N \end{array} \right\} \\
&\qquad \text{due to (22)} \\
&= RM^{mult}(y,x)
\end{aligned}$$

Thus, for any scalar-type transformation of units of measurement given by $\tilde{x} = \Omega_x x$ and $\tilde{y} = \Omega_y y$, we have $RM^{mult}(\tilde{y},\tilde{x}) = RM^{mult}(y,x)$.

Appendix A2: $1 \geqq \frac{SSIneff(y,x;1_N)}{w \cdot x}\left(\sum_{n=1}^{N} \frac{w_n \beta_n^{CRS*}}{\sum_{n=1}^{N} \beta_n^{CRS*}}\right) \geqq 0$.

We show this result as follows. Since the equality relationship in (19) implies that

$$\frac{w \cdot x - C^{CRS}(y,w)}{\sum_{n=1}^{N} w_n \cdot 1} \geqq \left(SSIneff(y,x;1_N) + S\vec{D}_i^{VRS}(y,x;1_N)\right) \times \left(\sum_{n=1}^{N}\left(\alpha_n \times \frac{\beta_n^{CRS*}}{\sum_{n=1}^{N} \beta_n^{*}}\right)\right),$$

we have

$$\frac{1 - C^{CRS}(y,w)/w \cdot x}{\sum_{n=1}^{N} w_n \cdot 1} \geqq \left(\frac{SSIneff(y,x;1_N)/w \cdot x}{\sum_{n=1}^{N} w_n \cdot 1} + \frac{S\vec{D}_i^{VRS}(y,x;1_N)/w \cdot x}{\sum_{n=1}^{N} w_n \cdot 1}\right) \times \left(\sum_{n=1}^{N} \frac{w_n \beta_n^{CRS*}}{\sum_{n=1}^{N} \beta_n^{CRS*}}\right).$$

Therefore, we have

$$1 - \frac{C^{CRS}(y,w)}{w \cdot x} \geqq \left(\frac{SSIneff(y,x;1_N)}{w \cdot x} + \frac{S\vec{D}_i^{VRS}(y,x;1_N)}{w \cdot x}\right) \times \left(\sum_{n=1}^{N} \frac{w_n \beta_n^{CRS*}}{\sum_{n=1}^{N} \beta_n^{CRS*}}\right) \quad \Leftrightarrow$$

$$1 \geqq \frac{C^{CRS}(y,w)}{w \cdot x} + \left(\frac{SSIneff(y,x;1_N)}{w \cdot x} + \frac{S\vec{D}_i^{VRS}(y,x;1_N)}{w \cdot x}\right) \times \left(\sum_{n=1}^{N} \frac{w_n \beta_n^{CRS*}}{\sum_{n=1}^{N} \beta_n^{CRS*}}\right)$$

Since $SSIneff(y,x;1_N) \geqq 0$, $S\vec{D}_i^{VRS}(y,x;1_N) \geqq 0$, $\beta_n^{CRS*} \geqq 0$, $w_n > 0$, we have

$$1 \geqq \frac{SSIneff(y,x;1_N)}{w \cdot x}\left(\sum_{n=1}^{N} \frac{w_n \beta_n^{CRS*}}{\sum_{n=1}^{N} \beta_n^{CRS*}}\right) \geqq 0.$$