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Late Season Low Inventory Assortment Effects in the Newsvendor Problem[1]

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Abstract: The assumption of the newsvendor being able to satisfy demand as long as on-hand inventory is positive does not hold for a non-homogenous product. Consumers who do not find a unit of the product which satisfies their secondary features preferences may not purchase the product even though the newsvendor has positive on-hand inventory. This is likely to occur late in the season as inventory level declines. We solve a newsvendor problem in which the probability of purchase by consumers is increasing in on-hand inventory for any inventory level below that which is needed to have a complete assortment. We identify the sufficient optimality condition for the order quantity. We show that, unlike the case of inventory-dependent demand models in the literature, the optimal order quantity may decrease due to the assortment effect. We investigate two types of pre-end of season discounts, immediate all-units and delayed, as ways to mitigate the late season assortment effect and show that in some cases, they can increase the newsvendor's profit and free up the shelf space for other products.

Keywords: Inventory control; Marketing

1 Introduction

The classical newsvendor problem is to find a perishable product's order quantity that maximizes the expected profit under probabilistic demand (Khouja 1999). If the newsvendor's inventory falls to zero then incoming demand goes unsatisfied. The assumption that the newsvendor can satisfy demand as long as on-hand inventory is positive does not hold for cases of heterogenous products such as fashion and food. For example, winter jackets that are ordered by a retailer in a complete assortment which includes six sizes, each in four colors, results in the complete assortment having 24 different size-color combinations. The retailer's order may include the same number of each size-color combination. For example, the retailer may order 240 jackets, i.e. 10 jackets in each color-size combination. Alternatively, the retailer may order a larger number of more popular sizes and smaller number of less popular sizes in the total 240 jackets order. In the early part of the selling season, there will be some units on-hand of every color-size combination. As sales continue and inventory

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level declines, it will be less likely to have a complete assortment, i.e. every color-size combination, in stock. When the inventory level reaches 23, it is impossible to have a complete assortment. Obviously, as some size-color combinations are no longer on-hand, an arriving consumer has a lower probability of finding a unit satisfying her/his secondary features preferences, such as color or exact size, and is less likely to purchase the product. We refer to this inability to fulfill consumers' secondary features preferences as the *assortment effect*. The assortment here refers to a product variations, e.g. a specific design winter jacket or real Christmas trees, and not within a product category, which is analyzed in multi-product newsvendor models with product substitution (Hübner and Kuhn 2012).

Examples of the assortment effect in the newsvendor are common. A real Christmas tree newsvendor is very unlikely to sell the last few trees as consumers do not even come in when the newsvendor has only a few trees left and and when they do come in, it is unlikely they will find a tree they like. This assortment effect is also observed for fresh produce, fresh meats, and clothing.

In this paper, we examine how the low inventory assortment effect influences the optimal order quantity and the expected profit of the newsvendor. We also examine the use of an immediate all-units pre-end of season discount, i.e. the discount is implemented as soon as inventory falls below the complete assortment level, due to high shelf space cost (Cachon 2001). We also examine the case in which the retailer follows optimal discounting, i.e. may delay the discount and sell some units at the regular price while inventory is below the complete assortment level before using a pre-end of season discount.

The remainder of this paper is organized as follows. In Section 2, we summarize the related literature. In Section 3, we quantify the impact of the assortment effect on sales and derive the optimal order quantity when the effect is present. In Section 4, we examine the use of an all-units pre-end of season discount offered as soon as the inventory level falls short of a complete assortment to free up the shelf space. In Section 5, we allow the discounting to be delayed, which may be optimal when shelf-space is not very costly, and identify the optimal inventory level at which to offer the pre-end of season discount. In Section 6, we present a numerical analysis and some insights. We present the conclusion and suggest directions for future research in Section 7.

2 Literature Review

Our model is related to models in four areas, inventory-dependent demand, assortment management, clearance pricing, and the newsvendor model. The role of inventory in influencing demand has been examined in both deterministic and stochastic inventory-dependent demand models. In these models, demand is assumed to be an increasing function of the inventory level. The increase in demand is attributed to the “advertising effect” in which a large shelf space and display quantity signal product popularity to consumers (Koschat 2008). The “advertising effect” depends on the quantity of displayed product, which may or may not be homogenous. For a comprehensive review of this literature the reader is referred to Urban (2006).

Inventory may also influence demand through the “selective effect” in which more units of a product which are not identical, e.g. fresh produce, real Christmas trees, fashion, provide consumers with more choices and induce them to purchase (Wang and Gerchak 2005). The “assortment effect” analyzed in this paper plays a similar role as the selective effect but in the direction of decreasing sales, i.e. if a high inventory level of a heterogenous product increases demand, then a low inventory level below complete assortment may decrease sales. When a product’s inventory level is low, some consumers whose reservation prices would have been met if they found the right product, i.e. one matching their secondary features preferences, will not purchase the product because the remaining selection lacks these features, e.g. color, size, freshness (Maddah and Bish 2007). In this case, sales will be lower than the actual demand in spite of the fact that the newsvendor does not stock out of the product.

Assortment can be measured in two ways, breadth and depth (Kok et al. 2009). Breadth is measured by the number of different product categories a retailer carries. Depth is measured by how many stock keeping units (SKUs) a retailer carries in each category. In this sense, depth includes both variants (SKUs) of the product, e.g. color sand size, a retailer carries and the inventory level for each SKU. Finding the optimal breadth of the assortment and its pricing over time has received considerable attention in the literature (Katsifou et al. 2014; Li 2007; Maddah et al. 2014).

The optimal depth and pricing of the assortment has received less attention than breadth. Caro and Gallien (2010) analyzed a problem in the fast-fashion industry where stores of a retail chain remove products from display whenever one of its key sizes stocks out and optimized the quantities to ship to stores to maximize profits. Caro and Gallien (2012) analyzed clearance pricing of a global retail chain in which some clearance inventory is moved between stores. Each product group, such as “woman blazers,” are broken into clusters and different pricing policies are applied to the clusters. Smith and Achabal (1998) examined clearance pricing and inventory policies for retail chains. They argued that while the sales rate is not affected by high inventory levels, the sales rate is decreased by low inventory levels, in particular for clothing when there is an incomplete selection of sizes and colors. They refer to an assortment of remaining inventory without complete selection of sizes and colors as “broken assortment.” Their model focused on the optimal price trajectory for the product. Abbott and Palekar (2008) defined the minimum presentation quantity as the minimum shelf space allocated for a product to generate sales corresponding to inherent demand and identified the optimal replenishment policy for a nonperishable product. Kalyanam et al. (2007) analyzed how the presence of every color and size, which they refer to as product attributes, affect sales. They found that substitution in case of shortages of some items is not common and was limited to a few colors. Furthermore, there was little substitution across sizes. This finding supports the conclusion that consumers who can’t find a unit with their preferred secondary features get less utility from purchasing one with different secondary features, e.g. color or size. Vakhutinsky et al. (2012) developed a markdown optimization (MDO) model in which demand is a function of price, seasonality, and inventory.

The inventory effect is captured using power of the ratio of on-hand inventory to a critical inventory level. They then obtained closed-form solution for price in continuous time and developed a solution algorithm to a dynamic programming formulation for the discrete time case.

There are some models in the literature which analyze pre-end of season (i.e. in-season) price adjustments for the newsvendor. Feng and Xiao (2000) identified the optimal times to switch between predetermined prices based on the time remaining in the season and inventory. They assumed demand follows a price-dependent poisson arrival process. Chung et.al. (2009) allowed an in-season price change which is implemented after the demand forecast is updated based on the realized demand up to the price change time. Forghani et al. (2012) assumed the demand rate is a random normal variable whose probability density function (pdf) is known and whose parameters depend on the selling price according to one of three functional forms. The authors allowed one in-season price adjustment. While our model also allows an in-season price adjustment, the purpose of the adjustment is to mitigate the assortment effect and the adjustment does not affect sales until the remaining inventory falls below the level needed for complete assortment, if it does. In this respect, our model can be used with any of the above models to make a second in-season price adjustment when or after the inventory level needed for complete assortment is reached.

The above late season assortment effect is also related to shelf space management. When the inventory level of a product falls below the complete assortment level, the newsvendor can keep the shelf space allocation to the product unchanged and try to sell the remaining units. However, when the shelf space cost is large (Cachon 2001), the newsvendor may decide to reallocate the shelf space of the product as soon as the complete assortment level is reached or at some level below it. The remaining units of the product are then grouped with other products whose remaining on-hand inventories are also less than their complete assortment levels in a discount area. For example, fashion retailers have racks displaying the "sale" or "clearance" sign where these products are compactly grouped. Some retailers have a minimum on-hand inventory for each product called “fixture fill” which is the quantity required for adequate presentation (Smith and Achabal 1998), which may be considered as the complete assortment level or a fraction of it.

Our model assumes that the regular price is exogenous and therefore, unlike price-dependent demand newsvendor models (Jammernegg and Kischka 2013; Petruzzi and Dada 1999), sales depend only on the availability of a complete assortment. Our model contributes to the literature by examining the effect of a broken assortment on the optimal order quantity. While most models examine pricing when a broken assortment occurs, we identify the optimal order quantity in the presence of this assortment effect. We show that the newsvendor may respond to the assortment effect by increasing the order quantity. However, there are cases in which the newsvendor responds by decreasing the order quantity. We also analyze two policies the newsvendor can use in response to the assortment effect. In the first, the newsvendor discounts the product as soon as a broken assortment occurs, i.e. all-units discount. We find that if this discount is used, then the newsvendor is more likely to increase the order quantity. If all consumers are aware of the discount, then the discount may significantly increase the newsvendor's profit. We also identify the optimal discounting policy, i.e. the newsvendor does not discount the product as soon as a broken assortment occurs. We find that this policy of delaying the discount can result in a substantial increase in profit.

3 Optimal policy under the Assortment Effect

We assume that there are D consumers, where D is a random variable with pdf $h(d)$ and cumulative density function (cdf) $H(d)$. A list of notation used in the paper is shown in Table 1. We assume that consumer $i's$ reservation price R_i for a product satisfying her secondary features preferences is uniformly distributed on $[0,\ u]$. This results in the linear demand function commonly used in the literature (Huang et al. 2013). The reservation prices for a product lacking the preferred secondary features drop by γ and are therefore uniformly distributed on $[0-\gamma,\ u-\gamma]$. Since reservation prices can't be negative, this drop in utility can be treated as reservation prices that are uniformly distributed on $[0,\ u-\gamma]$ with demand of $D_1 = D\frac{u-\gamma}{u}$. Thus, the demand which can be lost due to lack of preferred secondary features is $D_2 = D\frac{\gamma}{u}$. In addition, we assume that the regular sale price is exogenously determined and is given by p. Using the notation in Table 1, we define $\rho = \frac{u-p}{u}$, $X = \rho D$, $v_e = p - \gamma$, $g = v_e - v$, $s_1 = s - 1$, $a = \frac{s_1}{s}$, $\lambda = u - p - \gamma$, and $\beta = \frac{\gamma}{u-p}$.

Table 1: List of notation

D	A random variable denoting the number of consumers
$h(d)$	The probability density function (pdf) of D
$H(d)$	The cumulative density function (cdf) of D
R_i	The reservation price of consumer i, a uniformly distributed random variable with a lower limit of 0
u	The upper limit of R_i
γ	Consumers' loss of utility due to the lack of product's preferred secondary features
s	The number of units below which the on-hand inventory does not have a complete assortment.
p	Price per unit
c	Cost per unit
v	Salvage value per unit in the post-season
Q	Order quantity, a decision variable
$f(x)$	The probability density function (pdf) of x
$F(x)$	The cumulative density function (cdf) of x

To illustrate the number of units below which the remaining on-hand inventory does not have a complete assortment s, consider the example of winter jackets. There are 24 color-size combinations and thus the minimum s is 24. However, it is unlikely that a 24 jackets of remaining inventory will each be of different color and size and thus s is larger than 24 units. As long as on-hand inventory is greater than or equal to s, an incoming consumer will find a product with her preferred secondary features and will buy the product if her reservation price is p or greater. When on-hand inventory reaches $s_1 = s - 1$ an incoming consumer has approximately an $a = \frac{s-1}{s} = \frac{s_1}{s}$ probability of finding the right product and making a purchase. If the right product is out-of-stock, the consumer's reservation price for the on-hand products, which lack her preferred secondary features, falls by γ. If $\gamma > R_i - p$ then the reservation price falls below p and the consumer will not purchase one of the remaining units of the product. If $\gamma \leq R_i - p$ then the reservation price is met and the consumer purchases one of the remaining units.

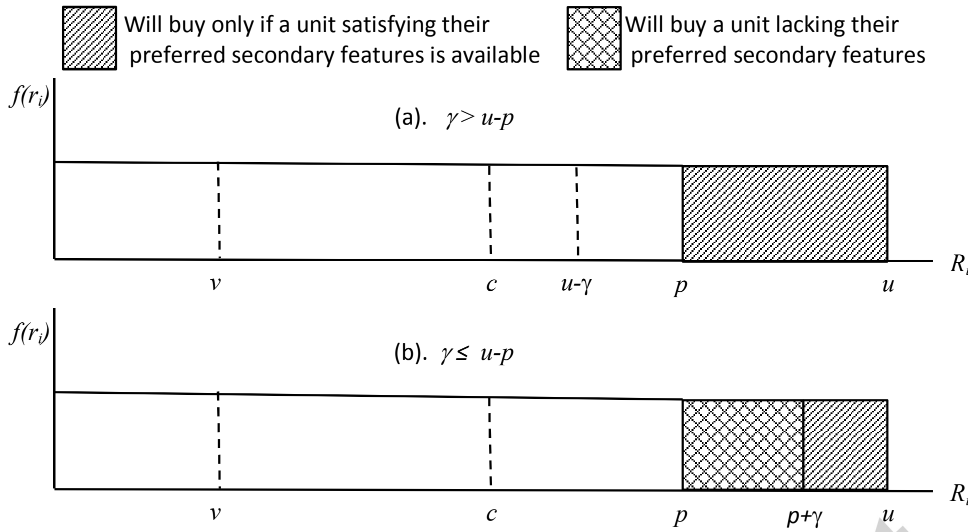
Finding the inventory level needed for a complete assortment, s, is an important input parameter which can be approximated from past data using product SKUs. Individual SKUs include information at the style-color-size level and sales patterns can be observed using them (Achabal et al. 2000). While the quantity ordered by the newsvendor may include disproportionately larger number of popular colors and sizes, the actual popularity of colors and sizes realized will likely be different. A product with m variations in color and size will need to have m different SKUs in stock to have a complete assortment (each individual SKU needs to have at least one unit in stock but can have more). Let T be the time index of the most recent period for which the data is available and n be the number of periods in a simple moving average. Let I_t, $t = T-n, T-(n-1), ...T$, be the minimum inventory level at which m different SKUs were available in I_t. The inventory level needed for complete assortment is approximated from the data as $s = \frac{\sum_{t=T-n}^{T} I_t}{n}$.

Under the above assumptions about consumers and their reservation prices, the demand for the product is

$$X = \frac{u-p}{u} D = \rho D \tag{1}$$

Since D is a random variable and ρ is a constant, X is a random variable with pdf $f(x)$ and cdf $F(x)$. There are two cases for the realized sales depending on consumers' loss of utility due to the lack of their preferred secondary features as shown in Figure 1. In the first case, as Figure 1a shows, $\gamma > u-p$ which implies that $p > u-\gamma$ and all consumers with $R_i > p$ who do not find the right product will not purchase a unit lacking their preferred secondary features for p. In the second case, as Figure 1b shows, $\gamma \leq u-p$, which implies that $p+\gamma \leq u$ and consumers with reservation prices $R_i \in [p+\gamma, u]$ will buy a unit lacking their preferred secondary features whereas consumers with $R_i \in [p, p+\gamma)$ will only purchase a unit satisfying their preferred secondary features.

Figure 1: Loss of utility and consumers' purchase decision features



For $\gamma > u - p$ sales before the end of the season at price p depend on Q and X as follows.

1. If $Q - s_1 \geq X$, then all incoming consumers will find a unit satisfying their preferred secondary features. In this case, X units will be sold at p and $Q - X$ will be salvaged at v in the post-selling season.

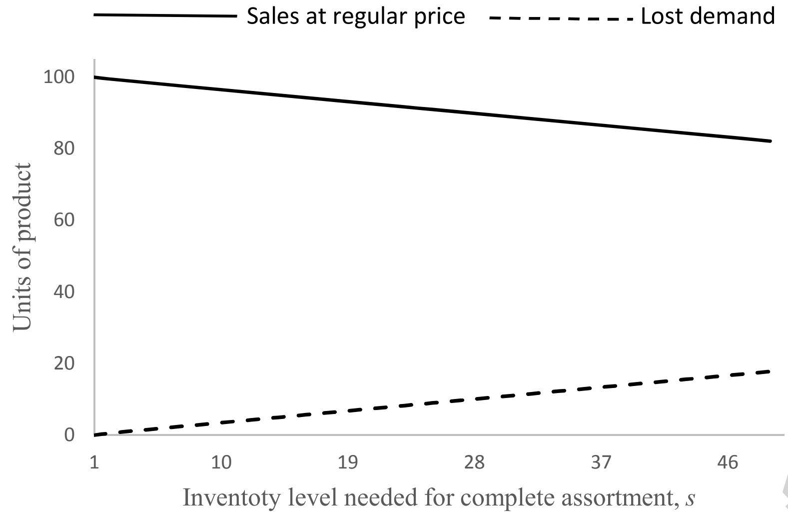
2. If $Q - s_1 < X$, then the first $Q - s_1$ consumers will find a unit satisfying their preferred secondary features and purchase it, some of the remaining $X - (Q - s_1)$ consumers will find a unit satisfying their preferred secondary features and purchase it. The remaining inventory will be salvaged at v. This outcome applies to both $Q - s_1 < X < Q$ and $X > Q$.

Let $\lfloor X \rceil$ denote the closest integer to X, Proposition 1 identifies the quantity which will be sold at price p when $X > Q - s_1$.

Proposition 1. *If $\gamma > u - p$ and $X > Q - s_1$ then the quantity which will be sold at a price of p is $Q - s_1 + \sum_{i=1}^{\lfloor X \rceil - Q + s_1} a^i \approx Q - s_1 a^{s_1 - Q + X}$.*

Proposition 1 is illustrated in Figure 2. As can be seen, for $X > Q - s_1$, as the inventory level needed for a complete assortment increases, sales at the regular price p decrease and the amount of lost demand increases due to consumers not finding a unit of the product satisfying their secondary features preferences.

Figure 2: Sales and lost demand as a function of inventory needed for a complete assortment $Q = 100,\ x = 100$



We assume that the optimal order quantity satisfies $Q - s_1 > 0$, which implies that the newsvendor orders at least a complete assortment. The revenue of the newsvendor is

$$R = \begin{cases} Xp + v(Q - X) & If\ 0 < X \leq Q - s_1 \\ (Q - s_1 a^{s_1 - Q + X})p + v s_1 a^{s_1 - Q + X} & If\ X > Q - s_1 \end{cases}$$

and the expected profit is

$$\begin{aligned} Z_1 = &\int_0^{Q-s_1} [Xp + v(Q - X)]\, f(x)\, dx + \int_{Q-s_1}^{\infty} p \left(Q - s_1 a^{s_1 - Q + X}\right)\, f(x)\, dx \\ &+ \int_{Q-s_1}^{\infty} v s_1 a^{s_1 - Q + x} f(x)\, dx - cQ. \end{aligned} \tag{2}$$

Let $\ln(\cdot)$ denote the natural logarithm. In Proposition 2, we show that the profit function in Equation (2) is concave and identify the sufficient optimality condition for the order quantity.

Proposition 2. *If $\gamma > u - p$, then the expected profit given by Equation (2) is concave with the optimal order quantity given by the unique solution to*

$$p - (p - v)F(Q - s_1) + (p - v)s_1 \ln(a) \int_{Q-s_1}^{\infty} a^{s_1 - Q + x} f(x)\, dx - c = 0 \tag{3}$$

As we show in the numerical analysis section, the optimal order quantity given by the solution to Equation (3) may be larger or smaller than the classic (i.e. without the assortment effect) newsvendor's optimal order quantity.

For $\gamma \le u-p$, if $Q-s_1<X$, then every unit of incoming demand after the first $Q-s_1$ units can be thought of as having two parts. The first part is given by $\left(1-\frac{\gamma}{u-p}\right)$, which comes from a consumer whose drop in utility is small enough so that she/he will buy a unit of the remaining inventory with or without their preferred secondary features. The second part is given by $\beta=\frac{\gamma}{u-p}$, which comes from a consumer whose drop in utility is large enough such that she/he will only buy a unit satisfying her/his preferred secondary features. Thus, if there are $X-(Q-s_1)$ units of demand remaining after the first $Q-s_1$ units of demand are satisfied, then the potential sales from the portion of the $X-(Q-s_1)$ units from consumers whose reservation prices are on $[p+\gamma,u]$ is $(1-\beta)(X+s_1-Q)$. The potential sales from the portion of $X-(Q-s_1)$ units from consumers whose reservation prices are on $[p,p+\gamma)$, is $\frac{\left(1-s^{-(s_1+x-Q)\beta}\, s_1^{(s_1+x-Q)\beta}\right)((u-p)s_1-\lambda)}{u-p}$, which is derived in the Appendix. Thus, the total potential sales from the additional $X-(Q-s_1)$ units of demand is

$$q=(1-\beta)(X+s_1-Q)+\frac{(\lambda-s_1(u-p))\left(s_1^{\beta(s_1-Q+x)}\, s^{-\beta(s_1-Q+X)}-1\right)}{u-p} \tag{4}$$

Solving $q=s_1$ yields the demand level at which all Q units will be sold for p per unit as

$$\check{x}=A_1+\frac{\lambda}{(u-p)(\beta-1)}-\frac{W\left(\frac{a^{\frac{\beta\lambda}{(u-p)(\beta-1)}}\beta(s_1(p-u)+\lambda)\ln(a)}{(u-p)(\beta-1)}\right)}{\beta\ln(a)}+Q=k+Q, \tag{5}$$

where the $W(y)$ function gives the solution for y in $z=ye^y$. Thus, there are two possibilities when $X\ge Q-s_1$:

1. $Q-s_1\le X\le\check{x}$ and some of the remaining s_1 units will be sold at p and the rest will be salvaged at v, and

2. $X>\check{x}$ and all the remaining s_1 units will be sold at p.

The expected profit is

$$\begin{aligned}Z_1=&\int_0^{Q-s_1}[xp+v(Q-x)]\,f(x)\,dx+\int_{Q-s_1}^{\check{x}}p\,[qp+v(Q-q)]\,f(x)\,dx\\&+\int_{\check{x}}^{\infty}Qp\,f(x)\,dx-cQ\end{aligned} \tag{6}$$

From Equation (5), $\check{x}$ is a linear function of Q and similar to Proposition 2, if $\gamma \leq u - p$, then the expected profit given by Equation (6) is concave and the optimal order quantity is given by the unique solution to

$$\begin{aligned} &- F(k+Q) + \int_{Q-s_1}^{k+Q} \left(\beta - \frac{\beta(\ln(a) - \ln(s_1))(s_1 p - s_1 u + \lambda) s_1^{\beta(s_1 - Q + x)} s^{\beta(Q - x - s_1)}}{p-u} \right) f(x)\, dx \\ &+ \frac{f(k+Q) s^{-\beta(s_1+k)} \left(s^{\beta(s_1+k)}((1-\beta)(s_1+k)(p-u)+\lambda) - s_1^{\beta(s_1+k)}(s_1 p - s_1 u + \lambda) \right)}{p-u} + \frac{p-c}{p-v} = 0 \end{aligned} \tag{7}$$

4 An All-units Pre-end of Season Discount

In this section, we analyze the use of an all-units pre-end of season discount to $v_e = p - \gamma$ per unit, where the subscript e denotes an early all-units discount, implemented as soon as on-hand inventory level falls below the level needed for a complete assortment and there is some demand outstanding. Thus, the newsvendor does not wait until the end of the season to discount the product to a unit price of $v < v_e$. Since the product has higher value to consumers before the selling season's end, the assumption that $v_e > v$ is quite reasonable. The discount induces consumers whose reservation prices are met at p but are unable to find a unit of the product satisfying their preferred secondary features to purchase a unit which does not satisfy their secondary features preferences. The discount is offered as soon as the inventory level reaches s_1 because the shelf space is expensive (Cachon 2001) and the newsvendor wants to use it for other products for which a complete assortment is on-hand. We analyze the case of optimal discount timing in which the discount may be offered at inventory level below s_1 or not at all in the next section.

The pre-end of season discount can also possibly lead to increased sales because consumers whose reservation prices are between v_e and p will have their reservation prices met if they find a unit which satisfies their secondary features preferences. We first begin with a worst-case scenario where only consumers whose reservation prices are above p visit the newsvendor.

In this case, three outcomes are possible:

1. If $X \leq Q - s_1$, then X units will be sold at p and $Q - X$ units will be salvaged at v.

2. If $Q - s_1 < X \leq Q$, then $Q - s_1$ units will be sold at p, $X - (Q - s_1)$ units will be sold at v_e, and $Q - X$ will be salvaged at v.

3. If $X > Q$, then $Q - s_1$ units will be sold at p and s_1 units will be sold at v_e.

The resulting revenue of the newsvendor is

$$R_e = \begin{cases} Xp + v(Q - X) & If\ X \leq Q - s_1 \\ (Q - s_1)p + (X - Q + s_1)v_e + (Q - X)v & If\quad Q - s_1 < X \leq Q \\ (Q - s_1)p + s_1 v_e & If\ X > Q \end{cases}$$

Define $g = v_e - v$, the expected profit can be written as

$$\begin{aligned} Z_e = & \int_0^{Q-s_1} [xp + v(Q - x)] f(x)\, dx + \int_{Q-s_1}^{Q} p[Q - s_1 - g(Q - x) + v_e s_1] f(x)\, dx \\ & + \int_Q^{\infty} (pQ - \gamma s_1) f(x)\, dx - cQ. \end{aligned} \tag{8}$$

Proposition 3 identifies the sufficient optimality condition for the order quantity.

Proposition 3. *If $F(x)$ is strictly increasing, then the expected profit in Equation (8) is concave with the optimal order quantity given by the unique solution to*

$$p - c - \gamma\ F(Q - s_1) - g\ F(Q) = 0 \tag{9}$$

In the numerical analysis section we show that it is suboptimal to use an all-units quantity discount for a product when no consumers with reservation prices on $[v_e, p]$ visit the newsvendor. However, because of shelf space scarcity and the possibility of having other products with higher contribution to profits on hand, freeing up the shelf space with an all-units discount may be optimal.

In the above analysis, we assumed that consumers with reservation prices between v_e and p are not aware of the discount and do not visit the newsvendor. In a complete information scenario, consumers whose reservation prices are between v_e and p are aware of the discount and will visit the newsvendor. If these consumers find a unit which satisfies their secondary feature preferences, they will purchase it. The additional consumers with $R_i \in [v_e, p]$ cause a change when $Q - s_1 < X \leq Q$. In this case, $Q - s_1$ units will be sold at p. Assuming that consumers with reservation prices above p arrive first since they are more interested in the product (higher reservation prices), $X - (Q - s_1)$ units will be sold at v_e. For the remaining $Q - (Q - s_1) - [X - (Q - s_1)] = Q - X$ units, some will be sold at v_e to consumers whose reservation prices are between v_e and p who find a unit which satisfies their secondary features preferences and the rest will be salvaged at v. This is a conservative assumption since it decreases the probability of consumers with reservation prices on $[v_e, p]$ finding a unit satisfying their preferred secondary features compared to mixed arrivals of the two consumers groups. We derive the quantity sold to consumers whose reservation prices are between v_e and p in Proposition 4.

Proposition 4. *If $Q - s_1 < X \leq Q$ then the quantity sold for a price of $v_e = p - \gamma$ is $(Q - X) \sum_{i=1}^{\lfloor \beta X \rceil} \frac{s_1^{i-1}}{s^i} \approx (Q - X) \left(1 - s^{-\beta X} s_1^{\beta X}\right)$.*

Using Z_e from Equation (8), the expected profit can be written as

$$Z_{e1} = Z_e + \int_{Q-s_1}^{Q} g(s^{-\beta x} s_1^{\beta x} - 1)(Q - x) f(x)\, dx. \tag{10}$$

The last term in Equation (10) is the additional revenue from consumers whose reservation prices are on $[v_e, p]$ and pay v_e for the product. The expected profit Z_{e1} in Equation (10) is not concave over the entire range of $Q \geq 0$. In Proposition 5, we show that Z_{e1} is unimodal and identify the sufficient optimality condition for the order quantity.

Proposition 5. *The expected profit given by Equation (10) is unimodal with the optimal order quantity given by the unique solution to*

$$\begin{aligned} &\int_{Q-s_1}^{Q} g s^{-\beta x} s_1^{\beta x} f(x)\, dx - s_1 g f(Q - s_1)(s_1^{\beta(Q-s_1)} s^{-\beta(Q-s_1)} - 1) \\ &- 2\gamma\, F(Q - s_1) - c + F(Q)(\gamma - g) + p = 0 \end{aligned} \tag{11}$$

In the numerical analysis section, we show that for some range of γ, if consumers whose reservation prices are $R_i \in [v_e, p]$ visit the newsvendor, then using an early discount will substantially increase the optimal expected profit above the no-early discount case.

5 Optimal pre-end of Season Discount

In this section, we analyze the case in which the newsvendor follows an optimal pre-end of season discounting when on-hand inventory level falls below the complete assortment level and there is some demand outstanding. The newsvendor may discount the product when the on-hand inventory is less than or equal to s_1 or not at all. In other words, the newsvendor may not discount the product to v_e as soon as inventory level reaches s_1 since the probability of a consumer whose reservation price is between p and u finding a unit which satisfies her preferred secondary features is high. As the inventory level decreases, this probability decreases and at some remaining inventory level, while there is some demand outstanding, it may become optimal to offer a pre-end of season discount and sell the product for v_e per unit. If $\gamma > u - p$, then there are three possible cases under this policy:

1. If $X \leq Q - s_1$, then X units will be sold at p and $Q - X$ units will be sold at v.

2. If $Q - s_1 < X \leq Q$, then $Q - s_1$ units will be sold at p. For the remaining s_1 units, the newsvendor has two choices. In the first choice, no discount to v_e is used and the quantity sold for p is $\sum_{i=1}^{X-(Q-s_1)} a^i = s_1 - a^{s-Q+X}$. The revenue from the remaining s_1 units can be written as

$$R_1 = (v-p)s_1^{s-Q+x} s^{Q-x-s_1} + s_1(p-v) + Qv. \tag{12}$$

In the second choice the newsvendor discounts the price to v_e beginning with the x_o^{th} consumer, $x_o < x$, resulting in a revenue of

$$R_2 = (v-p)s_1^{s-Q+x_o} s^{Q-x_o-s_1} + s_1(p-v) + Qv + (v+v_e)(x-x_o). \tag{13}$$

3. If $Q < X$, then in addition to the two above choices for $X \leq Q - s_1$, the newsvendor

can sell some of the remaining s_1 units at p and the remaing at v_e by manipulating x_o. For no units to be sold at v requires that the realized sales after inventory level reaches s_1 is equal to or greater than s_1 which can be written as $s_1 < (s_1 - a^{s-Q+x_o} s) + (X - x_o)$. This condition can be restated as $X < x_c$, where $x_c = a^{s_1+x_o-Q} s_1 + x_o$. If $X < x_c$, then the revenue is given by

$$R = p\left(s_1 - a^{s-Q+x_o} s\right) + (X - x_o)v_e. \tag{14}$$

We can prove Lemma 1 which identifies the optimal discounting policy for the case in which the additional revenue is given by Equation (13).

Lemma 1. *For $\gamma > u - p$, the amount of realized demand at which the retailer should discount the product to v_e per unit is*

$$x_o = Q + \delta_1 - s \tag{15}$$

where $\delta_1 = \frac{\ln\left(\frac{g}{s(p-v)\ln(a)}\right)}{\ln(a)}$.

Solving for when x_o in Equation (15) is equal to Q, i.e. $x_o = Q$, yields

$$v_o = v - sa^s \ln(a)(p - v). \tag{16}$$

If $v_e > v_o$, then $x_o < Q < x_c$ and the revenue is

$$R_{e2} = \begin{cases} Xp + v(Q - X) & If\ X \leq Q - s_1 \\ pQ - s_1(p - v)a^{s_1-Q+X} & If\ Q - s_1 < X \leq x_o \\ pQ - s_1(p - v)a^{s_1-Q+x_o} + g(X - x_o) & If\ x_o < X \leq x_c \\ pQ - a^{s_1-Q+x_o} s_1\gamma & If\ X > x_c \end{cases}$$

and the expected profit is

$$\begin{aligned} Z_{e2} = {} & \int_0^{Q-s_1} [xp + v(Q - x)] f(x)\, dx + \int_{Q-s_1}^{x_o} [pQ - s_1(p - v)a^{s_1-Q+x}] f(x)\, dx + \int_{x_o}^{Q} [pQ - a^{s_1-Q+x_o} s_1(p - v) \\ & + g(x - x_o)] f(x)\, dx + \int_Q^{x_c} [pQ - a^{s_1-Q+x_o} s_1(p - v) + g(x - x_o)] f(x)\, dx + \int_{x_c}^{\infty} (pQ - a^{s_1-Q+x_o} s_1\gamma) f(x)\, dx \end{aligned} \tag{17}$$

Similarly, if $v_e < v_o$, then $Q < x_o < x_c$ and the expected profit is

$$
\begin{aligned}
Z_{e2} = & \int_0^{Q-s_1} [xp + v(Q-x)] f(x)\, dx + \int_{Q-s_1}^{Q} [pQ - s_1(p-v)a^{s_1-Q+x}] f(x)\, dx + \int_Q^{x_o} [pQ - s_1(p-v)a^{s_1-Q+x}] f(x)\, dx \\
& + \int_{x_o}^{x_c} [pQ - a^{s_1-Q+x_o} s_1(p-v) + g(x-x_o)] f(x)\, dx + \int_{x_c}^{\infty} [pQ - a^{s_1-Q+x_o} s_1 \gamma] f(x)\, dx
\end{aligned}
\tag{18}
$$

Proposition 6 identifies the sufficient optimality condition.

Proposition 6. *The expected profit given by Equations (17) and (18) are concave and the optimal order quantities are given by the unique solutions to*

$$
\begin{aligned}
& p - c - (p-v)F(Q-s_1) + g\left(F(Q+\delta_1 - s) - F\left(\frac{s\left(a^{\delta_1} s_1 + a(Q+\delta_1 - s)\right)}{s_1} \right) \right) \\
& + (p-v) \int_{Q-s_1}^{Q+\delta_1-s} a^{s_1-Q+x} s_1 \ln(a) f(x)\, dx = 0,
\end{aligned}
\tag{19}
$$

if $v_e > v_o$, *or*

$$
\begin{aligned}
& p - c - (p-v)F(Q-s_1) + g\left[F(Q+\delta_1 - s) - F\left(a^{\delta_1 - 1} s_1 - s + Q + \delta_1 \right) \right] \\
& + (p-v) \int_{Q-s_1}^{Q+\delta_1-s} a^{s_1-Q+x} s_1 \ln(a) f(x)\, dx = 0,
\end{aligned}
\tag{20}
$$

if $v_e \leq v_o$, *respectively.*

If $\gamma \leq u - p$, then the following changes occur. For $Q - s_1 < X \leq Q$, discounting the product to v_e results in $(1-\beta)(x_o - Q + s_1)$ units of sales at p to consumers who will buy a unit lacking their preferred secondary features. In addition, the demand from consumers who will only buy a unit satisfying their preferred secondary features is $s_1 + \beta - 1 - (s_1 + \beta - 1)s^{Q-x-s_1} s_1^{s_1-Q+x}$. As a result, the condition of case 3 (for $\gamma > u - p$) changes to $Q \leq X \leq \check{x}$, where $\check{x}$ is given by Equation (5). Also, if $X > \check{x}$ then all Q units will be sold for p per unit. Define the function

$$
h(x_e) = -(s_1 + \beta - 1)s^{Q-x_e-s_1} s_1^{s_1-Q+x_e} - s_1\beta + 2s_1 + \beta + \beta Q - Q - \beta x_e + x_e - 1. \tag{21}
$$

which is the total sales at price of p after the first $q_1 = Q - s_1$ of demand are satisfied if the discount to v_e is implemented at the x_e^{th} unit. There are two threshold values of x (see proof of Lemma 1 for similar analysis) given by:

$$
\dot{x} = \frac{\ln\left(\frac{\beta(u-p)(2p-u-v)}{\ln(a)(p-v)(s_1(p-u)+\lambda)} \right)}{\ln(a)} + s + Q, \tag{22}
$$

and

$$\ddot{x} = \frac{a^{-Q}\left((s_1-1)(p-u)a^{s_1+\dot{x}} - \gamma a^{s+\dot{x}} + a^{Q}(-(\lambda(-s_1+Q+1)+\gamma\dot{x}))\right)}{p-u}, \tag{23}$$

which are needed to identify the newsvendor's optimal discounting policy for maximizing the revenue for the remaining s_1 units as follows. We examine one case is which the loss of utility is large. Let γ_c be the value of γ which solves $\dot{x} = \ddot{x}$, then if $\gamma > \gamma_c$, $\dot{x} < \ddot{x} < \check{x}$ and the newsvendor will follow one of four policies depending on the realized demand.

1. If $Q - s_1 < x \leq \dot{x}$, then do not to discount to v_e, but discount to v.

2. If $\ddot{x} < x \leq \check{x}$, then discount to v_e and then to v.

3. If $\dot{x} < x \leq \ddot{x}$, then discount only to v_e.

4. If $x > \check{x}$ then do not discount to v_e or v.

Thus, with the case of $X < Q - s_1$, the expected revenue is made up of five parts

$$Z_a = \int_0^{Q-s_1} [px + v(Q-x)]f(x)\,dx, \tag{24}$$

$$Z_b = \int_{Q-s_1}^{\dot{x}} [-\beta(p-v)(s_1 - Q + x) + h(x)(p-v) + px + Qv - vx]f(x)\,dx, \tag{25}$$

$$Z_c = \int_{\dot{x}}^{\ddot{x}} [-\beta(p-v)(s_1 - Q + \dot{x}) + h(\dot{x})(p-v) + p\dot{x} + Qv - vx + v_e x - v_e\dot{x}]f(x)\,dx, \tag{26}$$

$$Z_d = \int_{\ddot{x}}^{\check{x}} [-\beta(p-v_e)(s_1 - Q + \ddot{x}) + h(\ddot{x})(p-v_e) + p\ddot{x} + Qv_e - v_e\ddot{x}]f(x)\,dx, \tag{27}$$

$$Z_e = \int_{\check{x}}^{\infty} p\,Qf(x)\;dx. \tag{28}$$

Resulting in a total profit of

$$Z = Z_a + Z_b + Z_c + Z_d + Z_e - cQ \tag{29}$$

While a necessary optimality condition is obtainable, it is quite complicated. We therefore use a nonlinear optimization algorithm utilizing Newton's method to find a solution. We show in the numerical analysis section that using optimal discounting has the potential of substantially increasing profits when the loss of utility from the lack of preferred secondary features is large.

6 Numerical Analysis

Consider the following base example. Demand is normally distributed with mean and standard deviation of 700 units and $\sigma = 52.5$ units, respectively. Consumers reservation prices are uniformly distributed on $[0, \$140]$, i.e. $u = \$140$. The selling price, the salvage value, and cost per unit of the product are $p = \$100$, $v = \$25$ and $c = \$70$, respectively. Thus, demand at $p = \$100$ is normally distributed with a mean of $\mu = (\frac{140-100}{140})700 = 200$ units and standard deviation of $\sigma = 15$ units. Consumers who are unable to find a unit satisfying their secondary features preferences suffer a $\gamma = \$34$ loss in utility, thus the pre-end of season discount price if it is used is $v_e = p - \gamma = 100 - 34 = \$66$. The quantity on-hand below which the product has less than complete assortment is $s = 70$ units.

Ignoring the assortment effect results in an optimal classic newsvendor order quantity of $Q^* = F^{-1}(\frac{100-70}{100-25}) = 196.2$ units and if there was no assortment effect the optimal expected profit would be \$5565.36. If the assortment effect is present with $s = 70$ units but the newsvendor still orders 196.2 units, the actual profit will be \$4537.72, which is 18.5% below the \$5565.36 expected by the newsvendor. If the newsvendor takes the assortment effect into consideration, then the optimal order quantity decreases to $Q^* = 176.3$ units and the optimal expected profit is $Z^* = \$4617.74$, a 1.73% increase over ignoring the assortment effect. If the newsvendor uses a pre-end of season discount to $v_e = \$66$ when on-hand inventory reaches 69 units and consumers whose reservation prices are between \$66 and \$100 do not visit the newsvendor (because they are not aware of the discount), then the optimal order quantity increases to $Q^* = 209.268$ units and the optimal expected profit is $Z^* = \$3451.35$ and the newsvendor is considerably worse-off with the discount unless the shelf space is very expensive. If all consumers are aware of the discount, which brings in consumers whose

$R_i \in [66, 100]$, then the optimal order quantity is $Q^* = 246.5$ units and the optimal expected profit is $Z^* = \$4674.98$, which is 0.97% increase from the adjusted newsvendor optimal profit and which gives the newsvendor the use of the shelf space sooner.

Figure 3a shows that, contrary to inventory-dependent demand models in the literature, the assortment effect may lead to a decrease in the optimal order quantity when the post-season salvage value, v, is small. When v is small, the newsvendor wants to avoid overstocking. The assortment effect reduces the effective demand, i.e. for a realized demand of X units, less than X units are sold at p even though inventory is available, which increases excess inventory. Therefore, the newsvendor responds by decreasing the order quantity. As v increases, both the optimal order quantities of the classic newsvendor and the adjusted newsvendor increase. With the assortment effect, the payoff from increasing the order quantity is higher than in the classic newsvendor because there is a probability that the product will be sold for p instead of salvaging it at v. As a result the optimal order quantity with the assortment effect increases at a larger rate and at $v = \$42.1$ it exceeds the classic newsvendor's optimal order quantity. Figure 3b shows that the newsvendor is better off taking the assortment effect into account for any v and particularly when v is large or small. This is because for v in the middle range, as Figure 3a shows, the optimal order quantities of the classic and the adjusted newvendors are close.

Figure 3: Optimal order quantity and expected profit vs. post-season salvage value

$\mu = 200,\ \sigma = 15,\ \gamma = 34,\ s = 70,\ p = 100,\ c = 70$

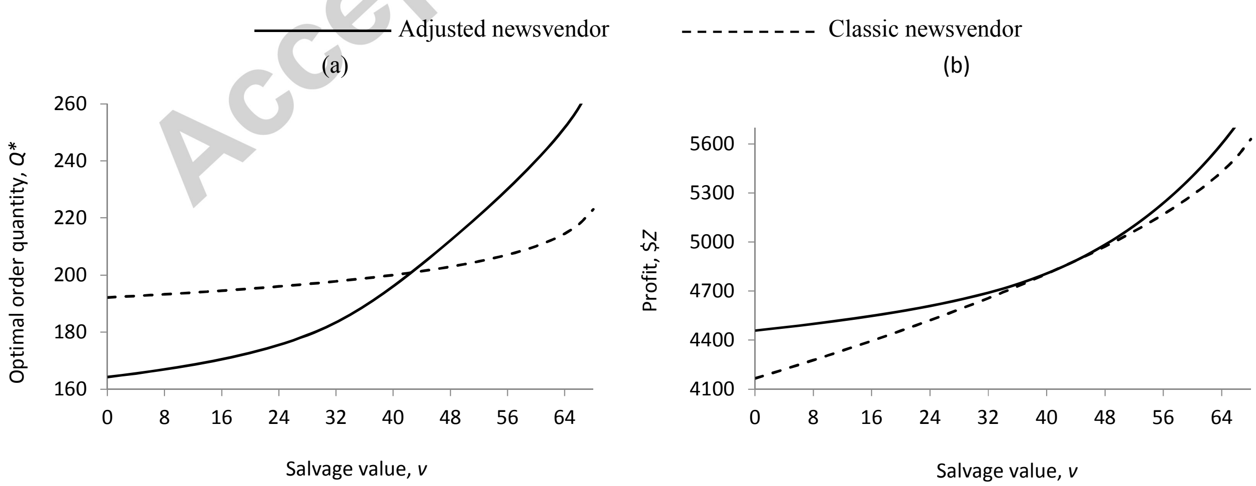


Figure 4a shows that the optimal order quantity with the assortment effect (adjusted newsvendor) is increasing in the standard deviation at low standard deviation whereas the classic newsvendor order quantity is decreasing. The decrease in the classic newsvendor order quantity is because $\frac{p-c}{p-v} < 0.5$ which pushes the order quantity further to the left of the mean as the standard deviation increases. Figure 4b shows that taking the assortment effect into account instead of ordering the classic newsvendor's order quantity leads to a larger increase in the expected profit at small demand standard deviation. As the standard deviation increases, the variability in demand leads to smaller area under the pdf for which $X > Q - s_1$ and diminished assortment effect.

Figure 4: Optimal order quantity and expected profit vs. standard deviation of demand

$\mu = 200,\ \gamma = 34,\ s = 70,\ p = 100,\ c = 70,\ v = 25$

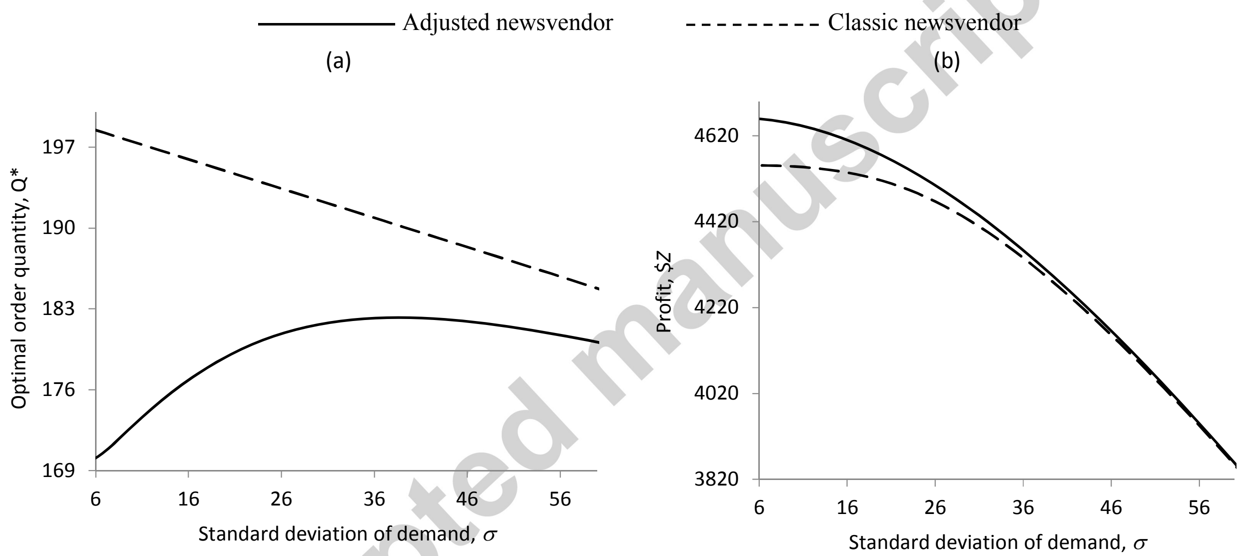


Figure 5a shows that the optimal order quantity is decreasing in the inventory level needed for a complete assortment, s, while the classic newsvendor's optimal order quantity is independent of s. As s increases, the newsvendor orders a smaller quantity in order to increase the probability of having $X > Q - s_1$ and to increase the probability of selling the remaining s_1 units to the large number of consumers after the first $Q - s_1$ consumers are satisfied. The impact of s on profit is shown in Figure 5b where the optimal expected profit under the assortment effect is decreasing in s. Again as s increases, the assortment effect has a larger negative impact on selling inventory beyond demand of $Q - s_1$ and the profit decreases. At large values of s, ignoring the assortment effect can have a significant negative impact on profit.

Figure 5: Optimal order quantity and expected profit vs. inventory level needed for a complete assortment

$\mu = 200,\ \sigma = 15,\ \gamma = 34,\ p = 100,\ c = 70,\ v = 25$

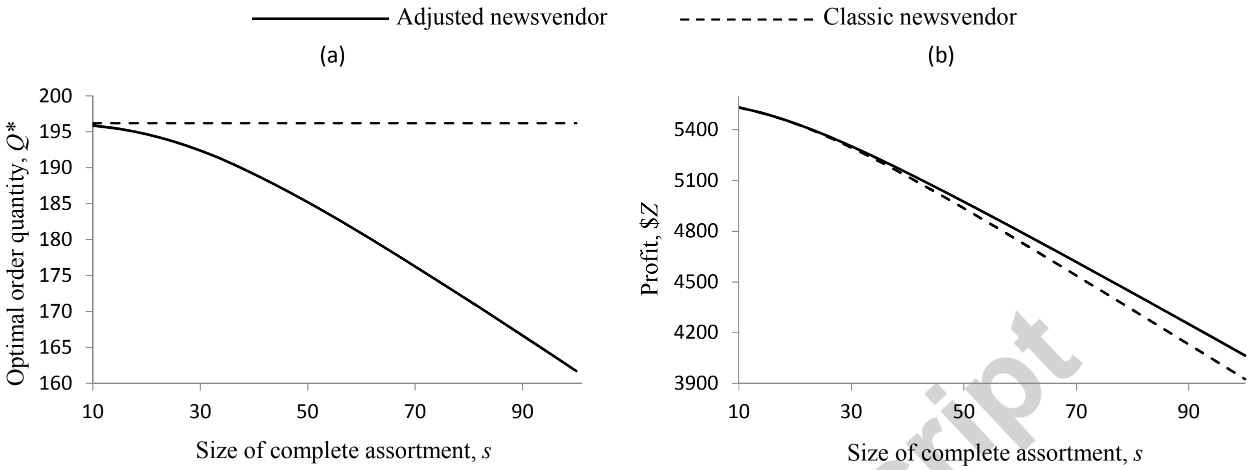


Figure 6a shows the optimal order quantities when there is a pre-end of season discount price $v_e > v$ at which consumers whose reservation prices are met will purchase a unit without their preferred secondary features. The figure shows two cases. In the first case, consumers whose reservation prices are between v_e and p do not visit the newsvendor and in the second case they do. When they do, additional demand is generated from those consumers who find a unit satisfying their secondary features preferences. In both cases, Figure 6a shows that the optimal order quantity is decreasing in v_e (i.e. increasing in γ). This is because discounting the product to v_e when on-hand inventory reaches s_1 at low v_e makes the discounting unprofitable comparing to selling it for p and the newsvendor increases Q as v_e decreases to increase the probability of selling more units at p. At high v_e, if $X > Q - s_1$ then selling some of the s_1 units for v_e is profitable and the newsvendor decreases Q as v_e increases in order to avoid salvaging inventory for v. Figure 6b shows that implementing a pre-end of season discount is suboptimal when v_e is large, a $v_e >$ \$67.12 for the case in which additional consumers with $R_i \in [67.12, 100]$ visit the newsvendor. For $v_e <$ \$67.12, implementing a pre-end of season discount can be very profitable when the discount attracts consumers whose reservation prices are $R_i \in [67.12, 100]$. For example, if a 35% discount ($\gamma = 35$) compensates consumers for the loss of utility resulting from purchasing a product which lacks their preferred secondary features, then implementing a pre-end of season discount results in a 2.6% increase in profit from the adjusted newsvendor.

Figure 6: Optimal order quantity and expected profit vs. pre-end of season discount price

$\mu = 200,\ \sigma = 15,\ s = 70, p = 100,\ c = 70,\ v = 25$

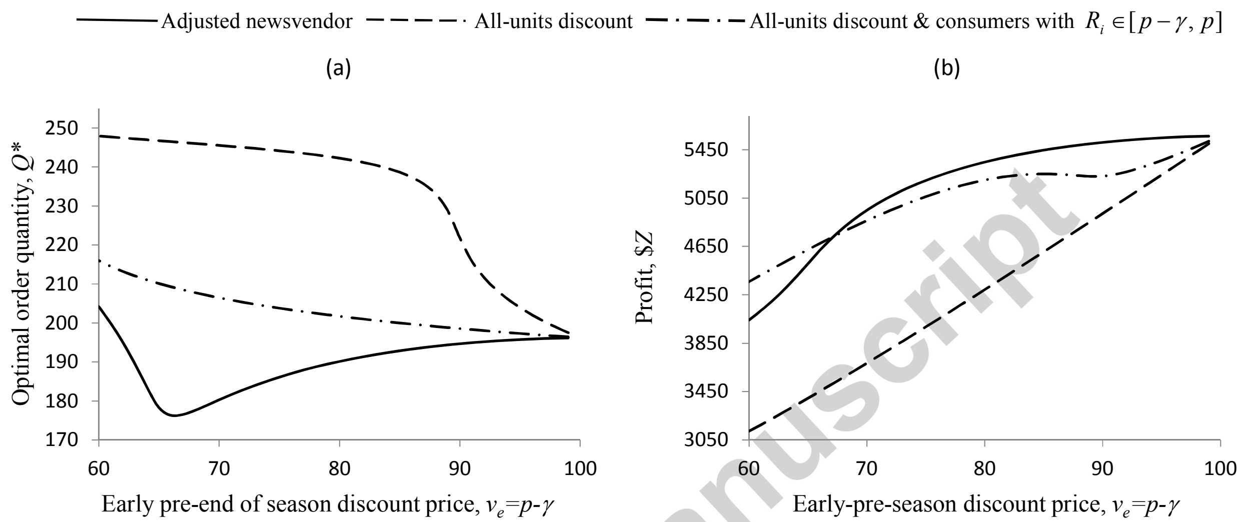
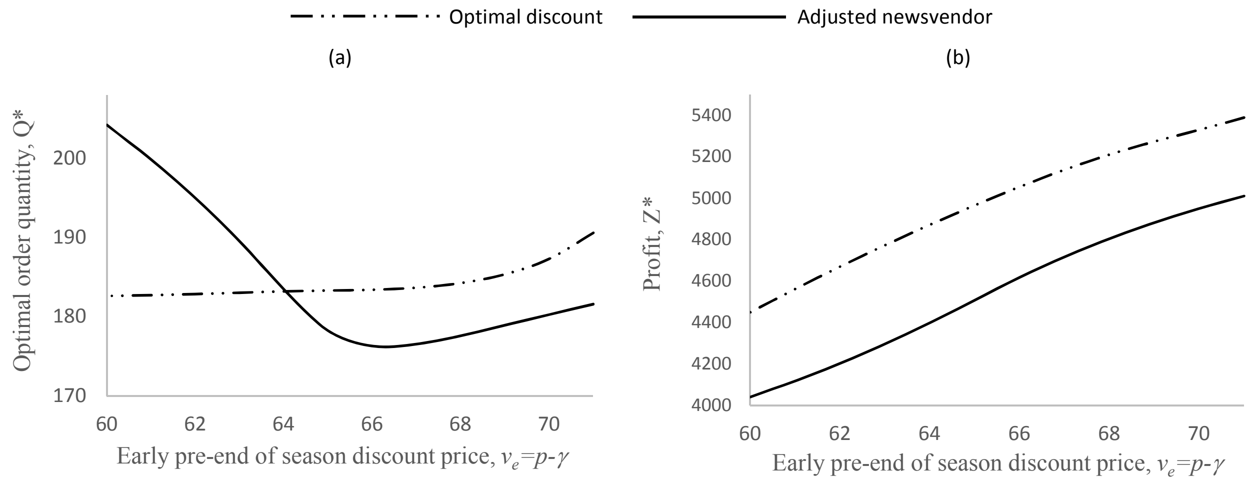


Figure 7 shows the optimal order quantity and optimal profit from using optimal pre-end of season discounting and the adjusted newsvendor. As Figure 7a shows, while the adjusted newsvendor's order quantity is decreasing then increasing in v_e, the order quantity with optimal discounting is monotonically increasing in v_e. Figure 7b shows that optimal discounting can result in substantial increase in profit from the adjusted newsvendor in which no discounting to v_e is used. This advantage of optimal discounting decreases as v_e increases (γ decreases).

Thus far, our analysis has focused on demand as a continuous random variable, which is reasonable when the demand and the complete assortment size are relatively large. For small demand and assortment size, the approximation provided by the continuous case may lack the needed accuracy because the probabilities of consumers finding units satisfying their preferred secondary features are based on whole numbers while our analysis uses a

Figure 7: Optimal discounting vs. all-units discount

$\mu = 200,\ \sigma = 15,\ s = 70, p = 100,\ c = 70,\ v = 25$



continuous approximation. Thus, for small discrete demand and small complete assortment size, a marginal analysis approach may provide better solutions.

7 Conclusion

We developed a model in which the newsvendor sells a non-homogenous product due to variation in secondary features such as color, size, condition, etc. A consumer whose reservation price for a product satisfying her/his preferred secondary features is met buys the product if it is available. If the product is lacking the preferred secondary features, consumers' utility from the product decreases. Thus, when on-hand inventory falls below the level needed for a complete assortment, i.e. all variations in secondary features are no longer available, there is a deceasing probability of a consumer finding the right product, which we refer to as the assortment effect.

We find that for low post-season salvage value, the assortment effect may lead to a decrease in the optimal order quantity from the classic newsvendor. For large post-season salvage value, the optimal order quantity is larger than the optimal order quantity of the classic newsvendor. We also find that the assortment effect has a larger influence on the optimal order quantity when the demand has a small standard deviation and the inventory level needed for a complete assortment is large. Furthermore, the assortment effect has a large influence on the optimal expected profit when demand has a small standard deviation, the inventory level needed for a complete assortment is large, and the post-season salvage value is small.

We examined the use of a pre-end of season discount as soon as inventory level drops below the level needed for a complete assortment to compensate consumers whose reservation prices are met but are unable to find a unit which satisfies their preferred secondary features. We found that this discount leads to an increase in the optimal order quantity and, if the loss of utility from lack of secondary features is small, then the discount can lead to a substantial increase in profit when all consumers are aware of the discount. We also examined the use of an optimal discount in which the newsvendor may discount the product at an inventory level below the level needed for complete assortment or not at all. We found that this can be substantially more profitable than the all-units discount.

Our model has some limitations which provide some avenues for future research. We used some simplifying assumption about the order of consumer arrivals, other arrival orders can be examined. We limited our focus to the retailer and did not address the problem in a supply chain context which would include a manufacturer's response. We also did not consider the possibility for selling inventory when its level reaches the complete assortment level to off-price retailers who buy brand and designer products from regular retailers.

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Appendix

Proof. **Proposition 1**.

For on-hand inventory level greater than s_1, the inventory contains complete assortment. At $q = s_1$, an incoming consumer has a $Pr_1 = \frac{s_1}{s}$ probability of finding the right product and purchasing it resulting in an expected sale of $1 \times Pr_1 = Pr_1$ units and a remaining inventory of $q_2 = s_1 - Pr_1 = \frac{s_1^2}{s}$ units. For the next incoming consumer, the probability of finding the right product is $Pr_2 = \frac{q_2}{s} = (\frac{s_1}{s})^2$ and the remaining inventory is $q_3 = q_2 - (\frac{s_1}{s})^2 = \frac{s_1^3}{s^2}$ units. For the next incoming consumer, the probability of finding the right product is $Pr_3 = \frac{q_3}{s} = (\frac{s_1}{s})^3$. This results in a total demand after inventory level reaches s_1 of $\sum_{i=1}^{X-Q+s_1} (\frac{s_1}{s})^i = s_1 \left(1 - \left(\frac{s_1}{s}\right)^{X-Q+s_1}\right)$. Adding the prior sales of $Q - s_1$ gives a total sales of $Q - s_1 + s_1 \left(1 - \left(\frac{s_1}{s}\right)^{X-Q+s_1}\right) = Q - s(\frac{s_1}{s})^{s_1 - Q + X}$ for price p per unit. □

Proof. **Proposition 2**.
The second derivative of Z with respect to Q is

$$\frac{d^2 Z}{dQ^2} = -(p-v)s_1(\ln(a))^2 \int_{Q-s_1}^{\infty} a^{s_1 - Q + x} f(x)\,dx - (p-v)f(Q-s_1)(1 - s_1 \ln(a)) \tag{30}$$

Since $s > 1$, $s_1 > 0$ and the first term in Equation (30) is negative. Also, since $a = \frac{s_1}{s} < 1$, $\ln(a) < 0$. Thus $-s_1 \ln(a) > 0$ and $(1 - s_1 \ln(a)) > 0$. Thus $-(p-v)f(Q-s_1)(1-s_1 \ln(a)) < 0$. Therefore, $\frac{d^2 Z}{dQ^2} < 0$, Z is concave, and the solution to $\frac{dZ}{dQ} = 0$ given by Equation (3) is a sufficient optimality condition. □

Derivation of additional sales for $\gamma \leq u - p$.
Sales to the remaining $X - (Q - s_1)$ consumers after the first $(Q - s_1)$ units of inventory are sold depend on the order of arrivals of consumers. We follow an optimistic scenario for the newsvendor in order to demonstrate the assortment effect on the newsvendor even under favorable conditions. In this scenario, consumers whose drop in utility is sufficiently large and will not buy a unit lacking their preferred secondary features have a priority in the choice of the remaining units. The total demand from consumers whose reservation prices are on $[p, p+\gamma)$ is $\frac{\gamma}{u-p}[X - (Q - s_1)]$. The first of these consumers can select from the remaining inventory of $k_1 = s_1 - (1 - \frac{\gamma}{u-p})$, where the term $(1 - \frac{\gamma}{u-p}) = (1 - \beta)$ is used to account for partial reduction in inventory from consumers with a small drop in utility. Thus, the probability of the first consumer finding a unit satisfying his/her preferred secondary features is $Pr_1 = \frac{k_1 - (1-\beta)}{s}$. The remaining inventory becomes $k_2 = s_1 - Pr_1$ and the probability of the second consumer finding a unit satisfying her/his preferred secondary features is $Pr_2 = \frac{k_2 - (1-\beta)}{s} = \frac{s_1(\lambda - (u-p)s_1)}{s^2(u-p)}$. The remaining inventory is $k_3 = s_1 - Pr_1 - Pr_2$. Continuing this process for all $\frac{\gamma}{u-p}[X - (Q - s_1)]$ consumers results in a total sales of $\sum_{i=1}^{\lfloor \beta(X+s_1-Q) \rceil} \frac{s_1^{i-1}(\lambda + s_1(u-p))}{s^i(u-p)} \approx \frac{\left(1 - s^{-(s_1+x-Q)\beta} s_1^{(s_1+x-Q)\beta}\right)((u-p)s_1 - \lambda)}{u-p}$. Similar analysis can be used to derive the additional sales after inventory reaches s_1 for other orders of arrivals of consumers.

Proof. **Proposition 3**.
The second derivative of Z_e with respect to Q is

$$\frac{d^2 Z_e}{dQ^2} = -\gamma\, f(Q - s_1) - g\, f(Q) < 0. \tag{31}$$

Thus, Z_e is concave. Solving $\frac{dZ_e}{dQ} = 0$ yields Equation (9). □

Proof. **Proposition 4.**
When on-hand inventory level reaches s_1, the newsvendor implements a discount to v_e and $X - (Q - s_1)$ consumers purchase the product resulting is a $Q - X > 0$ remaining on-hand inventory. By the assumption on the reservation prices being uniformly distributed on $[0, u]$, the additional number of consumers whose reservation prices for a unit satisfying their secondary features preferences are met is $\frac{\gamma}{u}D$. Substituting for D using Equation (1) gives $\frac{\gamma}{u}\frac{Xu}{u-p} = \frac{X\gamma}{u-p}$ as the additional demand due to the price discount. Since the first consumer has $Q - X < s_1$ units to chose from, her probability of finding the right product and purchasing it is $Pr_1 = \frac{Q-X}{s_1}$ and a remaining inventory becomes $Q - X - Pr_1 = \frac{s_1(Q-X)}{s}$. For the next incoming consumer, the probability of finding the right product is $Pr_2 = \frac{\frac{s_1(Q-X)}{s}}{s} = \frac{s_1(Q-X)}{s^2}$. Continuing with the same process, the probability of the third consumer finding the right product is $Pr_3 = \frac{(s_1)^2(Q-X)}{s^3}$ and so on. This results in a total demand after inventory level reaches $Q - X$ of $(Q-X)s^{-\frac{\gamma X}{u-p}}\left(s^{\frac{\gamma X}{u-p}} - s_1^{\frac{\gamma X}{u-p}}\right) = (Q-X)(1 - s_1^{\frac{\gamma X}{u-p}} s^{\frac{\gamma X}{p-u}})$. □

Proof. **Proposition 5.**
From Proposition 2, Z_{e1} is concave in Q. For a given value of X, Proposition 4 shows that the additional demand from consumers whose reservation prices are between v_e and p is $(Q-X)(1 - s_1^{\frac{\gamma X}{u-p}} s^{\frac{\gamma X}{p-u}})$. These units will be sold for v_e per unit instead of v resulting in an additional revenue of $R_A = g(Q-X)(1 - s_1^{\frac{\gamma X}{u-p}} s^{\frac{\gamma X}{p-u}})$. Taking the first derivative of R_A with respect to Q gives

$$\frac{dR_A}{dQ} = \left(1 - \frac{s_1^{\frac{\gamma X}{u-p}}}{s^{\frac{\gamma X}{u-p}}}\right) g.$$

Since $s > s_1$ and $g > 0$ $\frac{dR_A}{dQ}$ is a positive constant. Thus, R_A increases linearly with Q. Since Z_e is concave, it decreases at an increasing rate after the Q^* identified in Proposition 3. At Q^*, $\frac{dZ_e}{dQ} = 0$ and $\frac{dR_A}{dQ}$ is a positive constant and thus $\frac{dZ_e}{dQ} < \frac{dR_A}{dQ}$. Therefore, Z_{e1} increases for a range of Q above Q^* until $\frac{dZ_e}{dQ} = \frac{dR_A}{dQ}$ where a new optimal, denoted $Q_1^* > Q^*$ is reached and beyond which Z_{e1} decreases and thus Z_{e1} is unimodal. Therefore, the solution to $\frac{dZ_{e1}}{dQ} = 0$ given by Equation (11) is a sufficient optimality condition. □

Proof. **Lemma 1.**
If $X > Q - s_1$ then the newsvendor must determine when to discount the product to v_e. Define y as the realized part of demand at which the newsvendor should discount the product to v_e. $y = Q - s_1$ implies discounting as soon as inventory level falls below the level needed for complete assortment level. The other extreme is to not discount at all and to lower the price from p to v at the end of the season. When inventory level i.e. demand, reaches $Q - s_1$,

the newsvendor, knowing the remaining unrealized demand, maximizes additional revenue from the remaining on-hand inventory which is given by

$$R_A = (X-y)v_e + p\sum_{i=1}^{y-Q+s_1} a^i + v\left(Q-(X-y)-\sum_{i=1}^{y-Q+s_1} a^i\right)$$

$$= Qv - \frac{(p-v)\left(a^{1+s_1 y-Q}-a\right)}{1-a} + g(y-X).$$

Taking the second derivative of R_A with respect to y yields $\frac{d^2R_A}{dy^2} = -(p-v)a^{s-Q+y}s\ln^2(a) < 0$. Thus, R_A is concave in y. Setting $\frac{dR_A}{dy} = -g - a^{s-Q+y}s(p-v)\ln(a) = 0$ yields the optimal value of y, denoted by x_o, and given by Equation (15). $\square$

Proof. **Proposition 6**.
We provide proof for the case of $v_e > v_o$, the case $v_e < v_o$ can be similarly proved. Taking the second derivative of Z_{e2} with respect to Q and simplifying gives

$$\begin{aligned}\frac{d^2Z_{e2}}{dQ^2} &= -agf\left(a^{-1+\delta_1}s_1 - s + Q + \delta_1\right) - a(p-v)\int_{Q-s_1}^{Q+\delta_1-s} a^{s_1-Q+X}s_1pf(x)(\ln(a))^2\,dx \\ &\quad - a(p-v)f(Q-s_1)(1+s_1\ln(a))\end{aligned} \tag{32}$$

Since $-a(p-v)\int_{s_1+Q}^{Q+\delta_1-s} a^{s_1-Q+x}s_1f(x)(\ln(a))^2\,dx < 0$, if

$$-agf\left(a^{\delta_1-1}s_1 + Q + \delta_1 - s\right) - a(p-v)f(Q-s_1)(1+s_1\ln(a)) < 0 \tag{33}$$

then $\frac{d^2Z_{e2}}{dQ^2} < 0$. For $v_e > v_o$, $\delta_1 > 0$ and since $Q > s$, $-agf\left(a^{\delta_1-1}s_1 + Q + \delta_1 - s\right) < 0$. The sign of the remaining term $\Omega = -a(p-v)f(Q-s_1)(1+s_1p\ln(a))$ is determined by $(1+s_1p\ln(a))$ whose derivative with respect to s is $\frac{1}{s}+\ln(a) < 0$ and is therefore decreasing in s. Since $\lim_{s\to\infty}(1+s_1\ln(a)) = 1$ and $\lim_{s\to 0}(1+s_1\ln(a))) = 0$. Thus,$(1+s_1p\ln(a)) > 0$ for $s > 1$ and $\Omega < 0$. This establishes that $\frac{d^2Z_{e2}}{dQ^2} < 0$ and Z_{e2} is concave. Thus, the unique solution to $\frac{dZ_{e2}}{dQ} = 0$ given by (20) is optimal. $\square$