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Byeong Seok Ahn

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The analytic hierarchy process with interval preference statements

Byeong Seok Ahn[1]

College of Business and Economics, Chung-Ang University,, 221 Heukseok, Dongjak, Seoul 156-756, Republic of Korea

Abstract

In the analytic hierarchy process (AHP), interval judgments instead of precise ratios are widely accepted and can be practically used to solve decision-making problems when uncertainty exists because of scant information available or insufficient understanding of the problem. This paper presents a simple and effective method for finding the extreme points in a range of interval ratios (such as loose articulation, minimum number of interval ratios, and general interval ratios) and ultimately for establishing the dominance relations among alternatives using the identified extreme points. This is followed by an enumeration or simulation approach to manage situations in which the best or a full ranking of alternatives remains unidentified.

Keywords: Analytic hierarchy process (AHP); Interval ratio; Extreme point; Enumeration method; Simulation analysis

[1] *E-mail address:* bsahn@cau.ac.kr (B.S. Ahn)

1. Introduction

People frequently face the problem of choosing the best option from among several alternatives or of making a partial or full ranking of alternatives on the basis of multiple conflicting criteria. Such problems are termed multiple criteria decision-making (MCDM) problems and have received considerable attention in the decision science literature [15, 29, 30]. The analytic hierarchy process (AHP) provides a practical solution based on the *divide and conquer* principle, compared with other sophisticated MCDM methods. Since the introduction of AHP by Saaty [31], AHP has been successfully applied to a variety of real-world MCDM problems. See Vaidya and Kumar [36] for an extensive survey categorized by themes and areas of application.

The AHP decision process consists of three main parts: decomposition, measurement of preferences, and synthesis. In this paper, we focus on the latter two parts of the process. In measuring preferences, pairwise comparison judgments are mostly elicited as point estimates on a ratio scale from $1/9$ to 9. However, it is not uncommon for a decision-maker to be uncertain about his or her preferences, which can be attributed to two types of uncertainty: (a) uncertainty about the occurrence of events and (b) uncertainty about the range of judgments used to express preferences [32]. The first uncertainty is beyond the control of the decision-maker, whereas the second uncertainty is a consequence of the amount of information available to the decision-maker and his or her understanding of the problem [32]. Moreover, situations such as time pressure, lack of domain knowledge, limited attention, and information processing capabilities can heighten the uncertainty of the problem at hand [38]. In these circumstances, many people, when asked for a subjective judgment about the parameters of a problem, are reluctant to specify a unique number and would prefer to specify an interval within which the true judgment lies. To capture a decision-maker' uncertainty in making pairwise comparisons, many researchers have used interval ratio judgments to elicit the decision-maker's preferences instead of adhering to precise ratio judgments. However, the specific analysis of interval ratios differs depending on the assumptions of the interval ratios themselves and the aggregation methods: a fuzzy set approach to the interval ratio [10, 11, 20, 25], a distribution function approach for the weights of feasible region constructed by interval judgments [15], a simulation-based approach [8, 13, 32], and a goal programming approach [17].

Our focus in this paper is deriving priority vectors for three types of interval ratios (the loose articulation, minimum number of interval ratios, and general interval ratios) via extreme points. In the loose articulation, the decision-maker specifies that one factor is at least n times as important as another and so on. A consistent interval pairwise comparison matrix (PCM) can always be constructed if at least $(n-1)$ ratio bounds, the minimum number of interval ratios, are given when considering n factors. In the most general interval ratio case, a total of $n(n-1)/2$ ratio bounds are specified by the decision-maker. Judging from the range of preference formats, we deal with a variety of specifications compared with other established methods for using interval ratios. Furthermore, our proposed methods to derive the extreme points of interval ratios are distinct from previous ones. In the case of the loose articulation, cone theory and the inverse positivity property of a matrix provide the theoretical foundation for finding its extreme points. A dual programming technique provides another approach to obtain the same result. The analysis of the minimum number of interval ratios via the change of variables leads to their extreme points, which is easy to understand and apply. In the case of general interval ratios, we extract a minimum number of interval ratios, incorporate the remaining interval ratios into the extracted set one by one, and on each occasion modify the current extreme points until all remaining ones have been considered. This approach consistently results in the desired extreme points whereas Arbel's method [7] successively finds priority vectors by applying a technique based on a pivoting operation in the linear programming, but it often fails to produce all vertices. It should be noted that all of these methods to derive extreme points are valid when the set of interval ratios is not empty; otherwise, a fuzzy preference programming, a simulation-based approach, or a goal programming approach is more suitable for obtaining the priority vector.

We use the identified extreme points to establish dominance relations among the alternatives, unlike previous works that need to solve linear programs (LPs). We also offer an enumeration or simulation approach to manage situations in which the best or a full ranking of alternatives remains unidentified, as is often the case with a dominance criterion applied to interval PCMs.

The remainder of the paper is organized as follows. Section 2 describes the three types of interval ratios and methods to find their extreme points. In Section 3, we establish dominance relations among the alternatives by applying several decision-aiding approaches. A numerical example is illustrated in Section 4, followed by concluding remarks in Section 5.

2. Three types of interval ratios and their extreme points

A typical matrix of interval pairwise comparison judgments consists of $w_i/w_j \in [l_{ij}, u_{ij}]$, $i,j = 1,\cdots,n$ where l_{ij} and u_{ij} represent the lower and upper bounds, respectively. The range of bounds is assumed to be between $1/9$ and 9 inclusive, $u_{ij} = 1/l_{ji}$ and $l_{ij} = 1/u_{ji}$.

$$\begin{pmatrix} 1 & [l_{12},u_{12}] & \cdots & [l_{1n},u_{1n}] \\ [l_{21},u_{21}] & 1 & \cdots & [l_{2n},u_{2n}] \\ \vdots & \vdots & \vdots & \vdots \\ [l_{n1},u_{n1}] & [l_{n2},u_{n2}] & \cdots & 1 \end{pmatrix} \quad (1)$$

In the loose articulation, any $(n-1)$ judgments of the upper or lower triangle of the interval PCM are made while their upper bounds are limited to 9. Presumably, the decision-maker first selects the most important factor, for example, w_1, and then compares it with the other factors in a ratio scale, thus resulting in $w_1/w_2 \geq l_{12},\cdots,\ w_1/w_n \geq l_{1n}$. Another approach is to successively compare each factor with the others, resulting in $w_1/w_2 \geq l_{12}, w_2/w_3 \geq l_{23}$, $\cdots,\ w_{n-1}/w_n \geq l_{n-1\cdot n}$, which we adopt in this paper without loss of generality for further analysis (see also Form 2 below). The minimum number of interval ratios assumes that any $(n-1)$ interval ratios of (1) are known; the remaining interval ratios excluded in this specification can be obtained consistently with the given ratios. Finally, the general interval ratios require that every element of (1) be specified by ratio bounds, which is likely to cause information overload for the decision-maker and thus result in inconsistencies in the interval PCM. We summarize the three forms of uncertain judgments as follows:

Form 1 (LA: Loose Articulation): $w_j \geq l_j w_{j+1},\ 1 \leq j \leq n-1$

Form 2 (MR: Minimum Interval Ratios): $l_j w_{j+1} \leq w_j \leq u_j w_{j+1},\ 1 \leq j \leq n-1$ (2)

Form 3 (GR: General Interval Ratios): $l_{ij} w_j \leq w_i \leq u_{ij} w_j,\ 1 \leq i \leq n,\ i+1 \leq j \leq n$

2.1. Loose articulation (LA)

A loose articulation indicates that pairwise comparisons between the relevant elements at each level of the hierarchy are expressed in the form of weak inequalities [7]. This denotes a preference judgment such that factor i is at least l_{ij} times preferred to factor j, which produces the following set of constraints (3) about local priorities:

$$W_{LA} = \{w_j/w_{j+1} \geq l_j,\ 1 \leq j \leq n-1, w_n \geq 0,\ \sum_{j=1}^{n} w_j = 1\}^2 \quad (3)$$

where $l_j \in \left[\frac{1}{9}, 9\right]$. The set W_{LA} can be equivalently represented by matrix notation with the sum to unity constraint excluded,

$$\mathrm{Aw} \geq 0,\ \mathrm{w} \geq 0$$

where

$$\mathrm{A} = \begin{pmatrix} 1 & -l_1 & 0 & \cdots & 0 \\ 0 & 1 & -l_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -l_{n-1} \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \text{ and } \mathrm{w}^{\mathrm{T}} = (w_1, \cdots, w_n).$$

Notice that matrix A has special structure in its elements, and is a so-called class of Z-matrices whose off-diagonal entries are less than or equal to zero. Specifically, a Z-matrix satisfies $Z = (z_{ij}), z_{ij} \leq 0, i \neq j$. Furthermore, a Z-matrix is called an M-matrix, a class of *inverse-positive* matrices, when all the elements of its inverse are nonnegative. Of course, not every inverse-positive matrix is an M-matrix. The necessary and sufficient condition for a Z-matrix to be an M-matrix is that all of its principal minors are positive [16]. From these definitions, the $n \times n$ nonsingular matrix A belongs to a class of Z-matrix and further is an M-matrix because all of its principal minors are positive, judging from the upper diagonal matrix with positive diagonal elements. The extreme directions, and thereby the extreme points, can be easily identified based on the inverse-positive matrix. A closed convex cone C, defined by $C = \{\mathrm{w} \in R^n : \mathrm{Aw} \geq 0, \mathrm{w} \geq 0\}$, is a simplicial cone that has exactly n extremal rays because A is a nonsingular matrix of order n. It follows that $(AR_+^n)^* = (A^{-1})^T R_+^n$, based on the dual of C, defined by $C^* = \{\mathrm{y} \in R^n : \mathrm{s} \in C \to \mathrm{s} \cdot \mathrm{y} \geq 0\}$ where $\mathrm{s} \cdot \mathrm{y}$ denotes

2 For notational simplicity, we denote $l_j = l_{j(j+1)}$.

the inner product [9]. Therefore, a set of extremal vectors of C is composed of n column vectors of A^{-1}. To isolate the unique set of directions, each vector is normalized by its column sum, thereby yielding the extreme points.

According to the theory, we compute A^{-1} and then normalize it to obtain E, a matrix of the extreme points of W_{LA}, as in (4):

$$\mathrm{A}^{-1} = \begin{pmatrix} 1 & l_1 & l_1 l_2 & \cdots & l_1 \cdots l_{n-1} \\ 0 & 1 & l_2 & \cdots & l_2 \cdots l_{n-1} \\ 0 & 0 & 1 & \vdots & l_3 \cdots l_{n-1} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}, \ \mathrm{E} = \begin{pmatrix} 1 & \frac{l_1}{\beta_2} & \frac{l_1 l_2}{\beta_3} & \cdots & \frac{l_1 l_2 \cdots l_{n-1}}{\beta_n} \\ 0 & \frac{1}{\beta_2} & \frac{l_2}{\beta_3} & \cdots & \frac{l_2 l_3 \cdots l_{n-1}}{\beta_n} \\ 0 & 0 & \frac{1}{\beta_3} & \vdots & \frac{l_3 l_4 \cdots l_{n-1}}{\beta_n} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{\beta_n} \end{pmatrix} \qquad (4)$$

where $\beta_1 = 1$ and $\beta_j = 1 + l_{j-1}\beta_{j-1}$ for $2 \le j \le n$. See also the other approaches to find the extreme points of W_{LA} [6, 12, 22, 23, 27].

As an alternate method to determine the extreme points of W_{LA}, we present a dual programming approach that can be extended to other types of uncertain preference judgments beyond W_{LA}. To do so, we first formulate a primal LP (5) with a set of constraints W_{LA}:

minimize $\quad z_1 = c_1 w_1 + c_2 w_2 + \cdots + c_n w_n$

s.t.

$$w_j - l_j w_{j+1} \ge 0 \quad j = 1, \cdots, n \ \ (w_{n+1} = 0) \qquad (5)$$

$$\sum_{j=1}^{n} w_j = 1$$

where c_j, $j = 1, \cdots, n$ are arbitrary real numbers.

Then, we formulate a dual program (6) associated with (5) as follows:

maximize $z_2 = \mu_{n+1}$

s.t. $-l_{j-1}\mu_{j-1} + \mu_j + \mu_{n+1} \le c_j, \ j = 1, \cdots, n \ (l_0 = \mu_0 = 0) \qquad (6)$

$\mu_j \geq 0,\ j = 1, \cdots, n,\ \mu_{n+1}$: unrestricted in sign[3]

Theorem 1. *The optimal objective function value to (6) is obtained by*

$$\mu_{n+1}^* = min_{1 \leq j \leq n} \left(\frac{c_j + \sum_{i=1}^{j-1} \left(c_i \prod_{k=i}^{j-1} l_k \right)}{\beta_j} \right) \qquad (7)$$

where $\beta_1 = 1,\ \beta_j = 1 + l_{j-1}\beta_{j-1}$ *for* $2 \leq j \leq n$ *and* $\sum_{i=1}^{j-1} \left(c_i \prod_{k=i}^{j-1} l_k \right) = 0$ *for* $j = 1$.

Proof. It follows that $u_{n+1} \leq c_1$ from $u_1 \leq c_1 - u_{n+1}$ for $j = 1$ to satisfy the non-negative variable constraint $\mu_1 \geq 0$, which means that u_1 attains its maximum, $u_1^* = c_1 - u_{n+1}$ subject to $u_{n+1} \leq c_1 = \frac{c_1}{\beta_1}$. Similarly, it follows that $u_{n+1} \leq c_2 + l_1 u_1$ from $u_2 \leq c_2 + l_1 u_1 - u_{n+1}$ for $j = 2$ to satisfy $\mu_2 \geq 0$, yielding $u_2^* = c_2 + l_1 u_1^* - u_{n+1}$ subject to $u_{n+1} \leq \frac{c_2 + l_1 c_1}{1 + l_1} = \frac{c_2 + l_1 c_1}{\beta_2}$. Continuing in this manner, $u_{n+1} \leq \frac{c_n + l_{n-1} c_{n-1} + \cdots + l_1 \cdots l_{n-1} c_1}{\beta_n}$ for $j = n$; thus, the optimal objective function value reduces to a minimum of $\left\{ c_1, \frac{c_2 + l_1 c_1}{\beta_2}, \cdots, \frac{c_n + l_{n-1} c_{n-1} + \cdots + l_1 \cdots l_{n-1} c_1}{\beta_n} \right\}$.

Corollary 1. *The extreme points of* W_{LA} *reduce to* E *in (4).*

Proof. The dual optimal value μ_{n+1}^* can be equivalently represented as $\mu_{n+1}^* = \min[\mathrm{cE}]$ where $\mathrm{c} = (c_1, c_2, \cdots, c_n)$ and E in (4), which proves that E is the matrix of the extreme points of W_{LA} because $\mu_{n+1}^* = z_1^*$ at optimality, according to the primal-dual relationship.

2.2. Minimum interval ratios (MR)

Suppose that without loss of generality, the $(n - 1)$ interval ratios are successively given as

[3] A dual variable μ_{n+1} corresponds to the sum to unity constraint in the primal.

follows:

$$W_{MR} = \{l_j w_{j+1} \leq w_j \leq u_j w_{j+1},\ 1 \leq j \leq n-1, w_n \geq 0,\ \sum_{j=1}^{n} w_j = 1\} \qquad (8)$$

where l_j and u_j are a decision-maker's preferences taken from the $1/9 - 9$ comparison scale. This type of information is often used to determine the ranking of factors that one should infer when the decision-maker uses interval judgments rather than point estimates [7, 8, 15, 19, 21, 37].

The other missing ratios in the interval PCM can be induced by using the given interval ratios. For instance, a ratio bound $l_{ik} \leq \frac{w_i}{w_k} \leq u_{ik}$ for $i < k,\ k \neq i+1,\ 1 \leq i, k \leq n$ can be inferred by multiplying the given ratio bounds sequentially, thus resulting in:

$$\prod_{j=i}^{k-1} l_j \leq \prod_{j=i}^{k-1} \frac{w_j}{w_{j+1}} \leq \prod_{j=i}^{k-1} u_j. \qquad (9)$$

Theorem 2. *Given* $(n-1)$ *ratio bounds,*

(a) the total number of extreme points is 2^{n-1} *and*

(b) all extreme points are determined by solving 2^{n-1} *systems of linear equations:*

$$\frac{w_j}{w_{j+1}} = l_j \ (or\ u_j),\ 1 \leq j \leq n-1,\ \sum_{j=1}^{n} w_j = 1. \qquad (10)$$

Proof. See Ahn and Park [5].

Remark. It is noteworthy that Theorem 2 is valid when the ratio bounds induced by (9) belong to $[1/9, 9]$; otherwise, we rely on the procedure in Section 2.3.

To illustrate, consider a set of ratio bounds adopted from Arbel [7]:

$$W_{MR} = \left\{1 \leq \frac{w_1}{w_2} \leq 2, 2 \leq \frac{w_2}{w_3} \leq 3, w_1 + w_2 + w_3 = 1\right\}.$$

Multiplying the two given ratio bounds yields a ratio bound of w_1/w_3 such as $\{2 \le w_1/w_3 \le 6\}$, which belongs to $[1/9, 9]$ and thus completes a consistent interval PCM, as shown in (11):

$$A = \begin{pmatrix} 1 & [1,2] & [2,6] \\ [\frac{1}{2}, 1] & 1 & [2,3] \\ [\frac{1}{6}, \frac{1}{2}] & [\frac{1}{3}, \frac{1}{2}] & 1 \end{pmatrix} \quad (11)$$

Introducing q_i such that $q_1 = \frac{w_1}{w_2}$, $q_2 = \frac{w_2}{w_3}$, and $q_3 = \frac{w_3}{w_1}$, leads to a set Q wherein, in particular, the product of q_is appears differently from the sum to unity constraint in W_{MR}:

$$Q = \{q: 1 \le q_1 \le 2, 2 \le q_2 \le 3, \tfrac{1}{6} \le q_3 \le \tfrac{1}{2}, q_1 \cdot q_2 \cdot q_3 = 1\}$$[4].

Furthermore, we obtain the equivalent sets R and S by letting $r_i = \ln q_i$, $i = 1,2,3$, and successively $s_1 = r_1$, $s_2 = r_2 - \ln 2$, and $s_3 = r_3 + \ln 6$:

$$R = \{r: 0 \le r_1 \le \ln 2, \ln 2 \le r_2 \le \ln 3, -\ln 6 \le r_3 \le -\ln 2, r_1 + r_2 + r_3 = 0\}$$

$$S = \left\{s: 0 \le s_1 \le \ln 2, 0 \le s_2 \le \ln \tfrac{3}{2}, 0 \le s_3 \le \ln 3, s_1 + s_2 + s_3 = \ln 3\right\}.$$

The change of variables such that $t_i = s_i / \ln 3$, $i = 1, 2, 3$ yields set T:

$$T = \{t: 0 \le t_1 \le a, 0 \le t_2 \le b, 0 \le t_3 \le c, t_1 + t_2 + t_3 = 1\}$$

where $a = \frac{\ln 2}{\ln 3} < 1$, $b = \frac{\ln 3/2}{\ln 3} < 1$, and $c = 1$ Then selecting *at least* two end points of t_i summing to one gives the extreme points of T as follows:

$$(a, 0, 1-a),\ (a, b, 0),\ (0, b, 1-b), \text{ and } (0, 0, 1) \quad (12)$$

For each extreme point in terms of t_i, make the change of variables backward:

$$t \to s \to r \to q \to w.$$

Specifically, for $t = (a, 0, 1-a)$, multiplying each element by $\ln 3$ gives

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[4] It is possible to directly determine a set of extreme vectors $\left\{\left(1, 2, \frac{1}{2}\right), \left(1, 3, \frac{1}{3}\right), \left(2, 2, \frac{1}{4}\right), \left(2, 3, \frac{1}{6}\right)\right\}$ from the set Q to find the extreme points of W_{MR}. Note that a feasible vector, for instance, $\left(1, \frac{5}{2}, \frac{2}{5}\right)$ in Q, is not considered an extreme vector.

$s = (\ln 2, 0, \ln\frac{3}{2})$, adding $(0, \ln 2, -\ln 6)$ to s gives $r = (\ln 2, \ln 2, \ln\frac{1}{4})$, applying $q_i = e^{r_i}$ to r gives $q = (2, 2, \frac{1}{4})$, and finally solving the following system of equations gives an extreme point $\left(\frac{4}{7}, \frac{2}{7}, \frac{1}{7}\right)$:

$$w_1 = 2w_2,\ w_2 = 2w_3,\ w_1 + w_2 + w_3 = 1.$$

Similar computations for the other extreme vectors in (12) result in corresponding extreme points in terms of w_i such as $\left(\frac{6}{10}, \frac{3}{10}, \frac{1}{10}\right)$, $\left(\frac{3}{7}, \frac{3}{7}, \frac{1}{7}\right)$, and $\left(\frac{2}{5}, \frac{2}{5}, \frac{1}{5}\right)$.

Theorem 2 serves as a means to find the extreme points of W_{MR} as the solutions of the systems of linear equations derived by combining the lower or upper bounds of $\frac{w_j}{w_{j+1}}$ in W_{MR} as many as 2^{n-1} times. These findings are equivalent to those by Arbel [7].

2.3. General interval ratios (GR)

Consider a set of interval ratios W_{GR} consisting of $n(n-1)/2$ interval ratios, and divide them into two subsets W_{MR} as in (8) and W_{OT}, the others that are not included in W_{MR}, $W_{GR} = W_{MR} \cap W_{OT}$:

$$W_{GR} = \{l_{ij}w_j \le w_i \le u_{ij}w_j, 1 \le i \le n, i+1 \le j \le n, \sum_{j=1}^{n} w_j = 1\}.$$

We summarize a procedure for finding the extreme points of W_{GR} in four steps:

Step 1: Determine the extreme points of W_{MR} by Theorem 2 in Section 2.2;

Step 2: Select a ratio bound in W_{OT} and incorporate it into W_{MR};

Step 3: Manipulate one of three cases, which occur depending on the selected ratio bound:

(a) some modifications of the current extreme points if it is binding,

(b) no modifications of the current extreme points if it is redundant,

(c) inconsistency if it is infeasible;

Step 4: If case (c) occurs, then stop[5]; otherwise repeat Step 2 until W_{OT} is empty.

To illustrate, suppose the following interval PCM [8]:

$$\mathrm{A} = \begin{pmatrix} 1 & [2,5] & [2,4] & [1,3] \\ \left[\frac{1}{5},\frac{1}{2}\right] & 1 & [1,3] & [1,2] \\ \left[\frac{1}{4},\frac{1}{2}\right] & \left[\frac{1}{3},1\right] & 1 & \left[\frac{1}{2},1\right] \\ \left[\frac{1}{3},1\right] & \left[\frac{1}{2},1\right] & [1,2] & 1 \end{pmatrix}$$

To begin, partition the set of interval ratios into two subsets W_{MR} and W_{OT} where

$$W_{MR} = \left\{2 \le \frac{w_1}{w_2} \le 5, 1 \le \frac{w_2}{w_3} \le 3, \frac{1}{2} \le \frac{w_3}{w_4} \le 1, \sum_{j=1}^{4} w_j = 1\right\}$$

$$W_{OT} = \left\{2 \le \frac{w_1}{w_3} \le 4, 1 \le \frac{w_1}{w_4} \le 3, 1 \le \frac{w_2}{w_4} \le 2\right\}.$$

The interval PCM proves to be consistent because each element in W_{OT} belongs to its counterpart induced by multiplying the appropriate elements in W_{MR}, such as $2 \le \frac{w_1}{w_3} = \frac{w_1}{w_2} \times \frac{w_2}{w_3} \le 15$, $1 \le \frac{w_1}{w_4} \le 15$, and $\frac{1}{2} \le \frac{w_2}{w_4} \le 3$. Therefore, the next step is finding the extreme points of W_{MR} by solving eight systems of linear equations based on Theorem 2. The resulting extreme points are: $\mathrm{v}_1 = \left(\frac{1}{3},\frac{1}{6},\frac{1}{6},\frac{1}{3}\right)$, $\mathrm{v}_2 = \left(\frac{5}{9},\frac{1}{9},\frac{1}{9},\frac{2}{9}\right)$, $\mathrm{v}_3 = \left(\frac{1}{2},\frac{1}{4},\frac{1}{12},\frac{1}{6}\right)$, $\mathrm{v}_4 = \left(\frac{5}{7},\frac{1}{7},\frac{1}{21},\frac{2}{21}\right)$, $\mathrm{v}_5 = \left(\frac{2}{5},\frac{1}{5},\frac{1}{5},\frac{1}{5}\right)$, $\mathrm{v}_6 = \left(\frac{5}{8},\frac{1}{8},\frac{1}{8},\frac{1}{8}\right)$, $\mathrm{v}_7 = \left(\frac{6}{11},\frac{3}{11},\frac{1}{11},\frac{1}{11}\right)$, and $\mathrm{v}_8 = \left(\frac{3}{4},\frac{3}{20},\frac{1}{20},\frac{1}{20}\right)$.

Then select any ratio bound in W_{OT}, for instance $C_1 = \left\{2 \le \frac{w_1}{w_3} \le 4\right\}$, and incorporate it into W_{MR}, which divides the current set of extreme points into two subsets; $M_1 = \{\mathrm{v}_1, \mathrm{v}_5\}$ satisfying C_1, and $\overline{M}_1 = \{\mathrm{v}_2, \mathrm{v}_3, \mathrm{v}_4, \mathrm{v}_6, \mathrm{v}_7, \mathrm{v}_8\}$ not satisfying C_1. To see whether C_1 intersects with the edge connecting, for example, a pair of $\mathrm{v}_1 \in M_1$ and $\mathrm{v}_3 \in \overline{M}_1$, construct a line segment (i.e., a convex combination) between v_1 and v_3 in (13) and check whether

[5] To deal with an inconsistent interval PCM, identify the constraint where the inconsistency first occurs while finding the extreme points and provide it to the decision-maker to revise. Then proceed to find the extreme points that characterize the revised uncertain statements.

feasible λs are found by solving equations (14a) and (14b):

$$\begin{pmatrix} \frac{1}{3} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{3} \end{pmatrix} + \begin{pmatrix} \frac{1}{2} - \frac{1}{3} \\ \frac{1}{4} - \frac{1}{6} \\ \frac{1}{12} - \frac{1}{6} \\ \frac{1}{6} - \frac{1}{3} \end{pmatrix} \lambda = \begin{pmatrix} \frac{1}{3} + \frac{1}{6}\lambda \\ \frac{1}{6} + \frac{1}{12}\lambda \\ \frac{1}{6} - \frac{1}{12}\lambda \\ \frac{1}{3} - \frac{1}{6}\lambda \end{pmatrix},\ 0 \le \lambda \le 1. \qquad (13)$$

$$\frac{w_1}{w_3} = \frac{\left(\frac{1}{3}+\frac{1}{6}\lambda\right)}{\left(\frac{1}{6}-\frac{1}{12}\lambda\right)} = 2 \text{ for } \frac{w_1}{w_3} \ge 2 \qquad (14a)$$

$$\frac{w_1}{w_3} = \frac{\left(\frac{1}{3}+\frac{1}{6}\lambda\right)}{\left(\frac{1}{6}-\frac{1}{12}\lambda\right)} = 4 \text{ for } \frac{w_1}{w_3} \le 4. \qquad (14b)$$

The solutions to (14a) and (14b) are $\lambda = 0$ and $\lambda = \frac{2}{3}$, respectively, which give the current extreme point $v_1 = \left(\frac{1}{3}, \frac{1}{6}, \frac{1}{6}, \frac{1}{3}\right)$ and a candidate extreme point $\left(\frac{4}{9}, \frac{2}{9}, \frac{1}{9}, \frac{2}{9}\right)$, respectively.[6] A further check to see whether the candidate extreme point is generated from invalid line segments in spite of a feasible λ proves it to be legitimate, because it gives eight basic variables when substituted for the eight constraints (six constraints in W_{MR} plus two constraints in C_1) used to find the candidate extreme point. Repeat this procedure until every line segment connecting the pairs of M_1 and $\overline{M}_1$ has been examined. Table 1 lists the modified extreme points resulting from the incorporation of C_1 into W_{MR}.

Table 1

Modified extreme points resulting from the incorporation of C_1 into W_{MR}.

Vertex #1	Vertex #2	Vertex #3	Vertex #4	Vertex #5	Vertex #6
$\frac{1}{3}$	$\frac{2}{5}$	$\frac{1}{2}$	$\frac{4}{9}$	$\frac{4}{7}$	$\frac{1}{2}$
$\frac{1}{6}$	$\frac{1}{5}$	$\frac{1}{8}$	$\frac{2}{9}$	$\frac{1}{7}$	$\frac{1}{4}$
$\frac{1}{6}$	$\frac{1}{5}$	$\frac{1}{8}$	$\frac{1}{9}$	$\frac{1}{7}$	$\frac{1}{8}$
$\frac{1}{3}$	$\frac{1}{5}$	$\frac{1}{4}$	$\frac{2}{9}$	$\frac{1}{7}$	$\frac{1}{8}$

[6] A pair of (v_1, v_2) produces $\lambda = 0$ for $\frac{w_1}{w_3} \ge 2$ and no feasible λ for $\frac{w_1}{w_3} \le 4$.

Likewise, select $C_2 = \left\{1 \leq \frac{w_1}{w_4} \leq 3\right\}$ and $C_3 = \left\{1 \leq \frac{w_2}{w_4} \leq 2\right\}$ sequentially in W_{OT} and compute modified extreme points to obtain the final six vertexes shown in Table 2.

Table 2

The final extreme points of W_{GR}.

Vertex #1	Vertex #2	Vertex #3	Vertex #4	Vertex #5	Vertex #6	Min	Max
0.4	0.444	0.5	0.462	0.48	0.522	0.4	0.522
0.2	0.222	0.167	0.231	0.24	0.174	0.167	0.24
0.2	0.111	0.167	0.154	0.12	0.130	0.111	0.2
0.2	0.222	0.167	0.154	0.16	0.174	0.154	0.222

This manual process of obtaining a complete set of extreme points becomes computationally intensive as the number of factors being compared increases, although the results of Theorem 2 significantly reduce the burden of computation. To enhance accuracy and speed, we developed an Excel VBA (Visual Basic for Applications) program to find the extreme points of the general interval ratios and used it to solve the illustrative example in Section 4.

3. Interval AHP based on extreme points

The previous section dealt with methods to find the extreme points of a range of interval ratios that evaluate pairwise comparisons between relevant elements at each level of a hierarchy. In this section, we introduce several decision-aiding approaches, all of which actively use the identified extreme points in the context of AHP. To do this, we define terminology as follows:

- $A = \{x, y, z, \cdots\}$: a finite discrete set of alternatives
- $I = \{1, 2, \cdots, n\}$: a finite discrete set of criteria of multiple layers of hierarchy
- $D(i) \subset I$: a set of criteria structured immediately under a criterion i, that is, the set of direct successors of a criterion i

- T: a set of twig level criteria, $T = \{i \in I | D(i) = \emptyset\}$
- L_h: a set of the *h*th leveled criteria, $h = 0, 1, \cdots, m$, with L_0 as the topmost level (goal) and $L_h \subseteq I,\ h \neq 0$
- $N_{D(i)}$: the number of extreme points of the interval PCM for criteria $D(i)$
- $\mu_j^k(x)$: the *k*th entry of alternative x in a set of extreme points of the interval PCM for alternatives with respect to a criterion $j \in T$
- ρ_j^k: the *k*th entry of criterion j in a set of extreme points of the interval PCM for criteria $j \in D(i),\ k = 1, \cdots, N_{D(i)}$

Salo and Hämäläinen [33] presented an efficient algorithm for synthesizing interval judgments into dominance relations on the alternatives under a hierarchical tree. This algorithm passes the pairwise comparison values to an immediately upper node and then finally to the topmost goal by solving a series of LPs. Ahn et al. [3] extended their work to accommodate different types of incomplete preferences. Here, unlike the previous approaches that require solving LPs, Theorem 3 enables us to establish dominance relationships among the alternatives using only extreme points. Of course, this is made possible by knowing all the extreme points that characterize various types of interval PCMs.

Theorem 3. *For $j \in T$, compute the values of pairwise comparison*

$$\pi_j(x, y) = min_k[\mu_j^k(x) - \mu_j^k(y)].$$

For level L_h, $h = 0, \cdots, m-1$ and for each $j \notin T$, calculate the values of pairwise comparison

$$\pi_j(x, y) = min_{1 \le m \le N_{D(j)}} \sum_{k \in D(j)} \pi_k(x, y) \rho_k^m.$$

Then alternative x *is at least preferred to* y, $x \succcurlyeq_p y$ *if and only if* $\pi_0(x,y) \geq 0$.

Proof. It follows that

$$\min_{V_j}[v_j(x) - v_j(y)] = \min_k[\mu_j^k(x) - \mu_j^k(y)],\ j \in T \text{ and}$$

$$\min_{W_{D(j)}} \sum_{k \in D(j)} \pi_k(x,y)\, w_k = \min_{1 \leq m \leq N_{D(j)}} \sum_{k \in D(j)} \pi_k(x,y) \rho_k^m$$

where V_j is a set of constraints (i.e., incomplete statements) for the jth attribute values of alternatives $v_j(\cdot) \in V_j$, $j \in T$ and $W_{D(j)}$ is a set of constraints on the weights $w_k \in W_{D(j)}$ $k \in D(j)$ [3, 33].

The purpose of decision-making is generally to find the best alternative or a rank-ordering of alternatives. The pairwise dominance criterion applied to interval PCMs often fails to accomplish this goal while identifying some non-dominated alternatives. One way to cope with this problem is to ask the decision-maker to state more specific preferences; thereby, reducing the feasible region formed by the interval PCMs. Though such an interactive approach makes sense and occasionally yields the desired result, it can be difficult to acquire more specific information from the decision-maker or it could end with a deadlock in which s/he is unwilling to provide more restrictive values [3]. Some methods, including maximax, maximin, and minimax regret,[7] have been advanced to deal with this situation [4, 27, 34]. The dominance measuring method, based on the outranking concept, computes the dominating and dominated measures of each alternative by combining pairwise dominance values appropriately. Their difference is then considered the net dominance value that one alternative has over all the other alternatives. The net dominance values are thus a measure of

[7] The classical decision rules in incomplete decision-making problems use the lower and upper bounds of each alternative such that $LB_j(x) = \min_k[\mu_j^k(x)]$ and $UB_j(x) = \max_k[\mu_j^k(x)]$ for $j \in T$, and $LB_h(x) = \min_{1 \leq m \leq N_{D(j)}} \sum_{k \in D(j)} LB_k(x)\rho_k^m$ and $UB_h(x) = \max_{1 \leq m \leq N_{D(j)}} \sum_{k \in D(j)} UB_k(x)\rho_k^m$ for each $j \notin T$.

the preference strength for alternatives in the sense that a larger net value is better [4]. This idea has been extended to more elaborate dominance measuring methods [1, 14, 18, 24]. Finally, we introduce an enumeration method that considers all possible combinations of the extreme points to rank alternatives based on how many combinations support each one. Note that the range for each alternative's global priority weight resulting from this enumeration method is equivalent to the bounds of absolute dominance. When we denote the global priority weights of alternatives x and y by $Agg(x) \in [LB(x), UB(x)]$ and $Agg(y) \in [LB(y), UB(y)]$, respectively, then the degree of preference of x over y, defined by (15), shows how much one alternative is preferred to the other considering the overlapping portion of two intervals [35, 40]:

$$\mathrm{d}(Agg(x) > Agg(y)) = \frac{\max(UB(x)-LB(y),0)-\max(LB(x)-UB(y),0)}{UB(x)-LB(x)+UB(y)-LB(y)} \quad (15)$$

Obviously, $d(Agg(x) > Agg(y)) + d(Agg(y) > Agg(x)) = 1$ and $d(Agg(x) > Agg(y)) = d(Agg(y) > Agg(x)) = 0.5$ when $Agg(x) = Agg(y)$. Furthermore, we can collect other statistics as a basis to evaluate alternatives [13]: the number of times alternative *x* obtains rank *r* and the number of times alternative *x* scores better than alternative *y*.

The total number of combinations of the extreme points depends on the problem size, characterized by the number of layers of the hierarchy, the number of criteria in each layer, and the number of alternatives for the lowest level criteria. Obviously, the computations required to obtain the frequencies of each alternative over the others increase in a multiplicative fashion as the problem size increases. Therefore we rely on a simulation approach that considers only some combinations of the extreme points when the problem is too large to adopt an enumeration approach. A hybrid approach takes into account all combinations of extreme points with respect to criteria while randomly selecting extreme points of alternatives for the lowest level criteria. This is based on the idea that the local priority vectors of criteria are, in general, more influential than those of alternatives in computing the global priority vector, as exemplified in the illustrative example.

4. An illustrative example

We illustrate the proposed method with an international supplier selection problem in which the purchasing department of a company considers three competing suppliers (x, y, z) that are evaluated by the hierarchy of criteria shown in Figure 1 [2, 26, 28, 39].

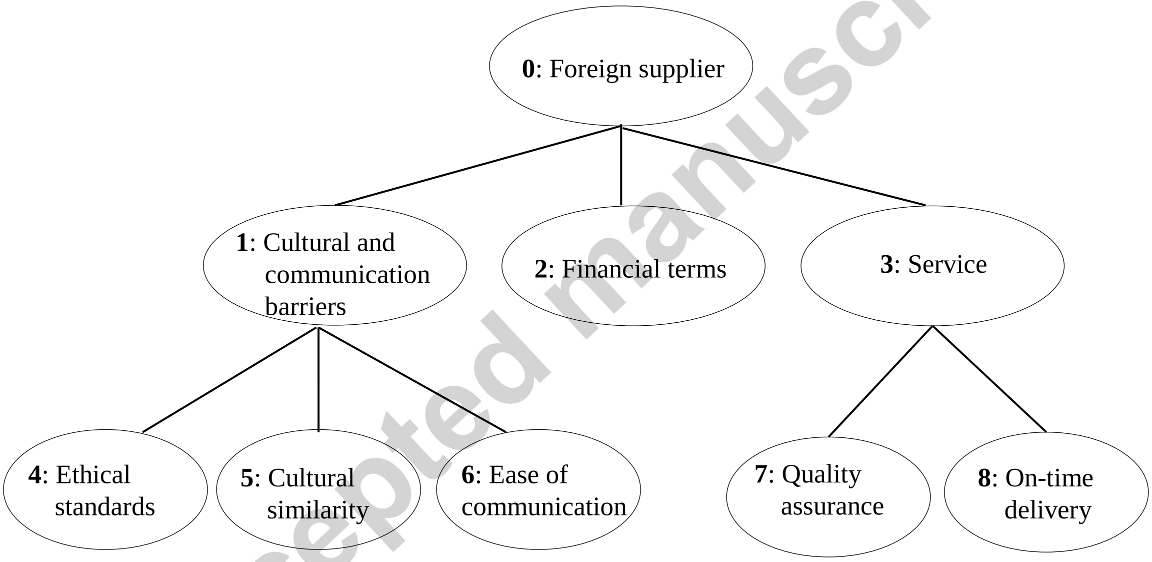

Fig. 1. An example of a hierarchy of criteria.

Suppose that the purchasing department makes approximate ratio comparisons on the hierarchy of criteria, as shown in Table 3. Specifically, the interval ratios between the criteria succeeding to *foreign supplier* can be equivalently written as a set of constraints $W_{D(0)}$:

$$W_{D(0)} = \left\{\frac{1}{4}w_2 \le w_1 \le \frac{1}{2}w_2, 2w_3 \le w_2 \le 3w_3, w_1 + w_2 + w_3 = 1, w_1, w_2, w_3 \ge 0\right\}.$$

Similarly, the interval ratios between the criteria succeeding to *cultural and communication barriers*, and *service* are denoted by $W_{D(1)}$ and $W_{D(3)}$ respectively:

$$W_{D(1)} = \left\{w_5 \leq w_4 \leq 2w_5, w_4 \leq \frac{1}{3}w_6,\ w_5 \leq \frac{1}{2}w_6, w_4 + w_5 + w_6 = 1, w_4, w_5, w_6 \geq 0\right\}$$

$$W_{D(3)} = \{w_7 = w_8, w_7 + w_8 = 1, w_7, w_8 \geq 0\}.$$

Table 3

Pairwise comparisons between criteria.

	Foreign supplier (Form 2)				Cultural and communication barriers (Forms 1 & 3)				Service (Form 3)	
	1	2	3		4	5	6		7	8
1		$\left[\frac{1}{4}, \frac{1}{2}\right]$		4		$[1, 2]$	$\left[-, \frac{1}{3}\right]$	7		$[1, 1]$
2			$[2, 3]$	5			$\left[-, \frac{1}{2}\right]$	8		
3				6						

Furthermore, suppose that for each lowest level criterion $T = \{2, 4, 5, 6, 7, 8\}$, the purchasing department's interval ratios between alternatives are given in Table 4. Taking the criterion *ethical standards* as an example, the interval ratios between alternatives can be equivalently written by a set of constraints V_4:

$$V_4 = \{v_4(y) \leq v_4(x) \leq 2v_4(y), 3v_4(z) \leq v_4(y) \leq 4v_4(z), 5v_4(z) \leq v_4(x) \leq 6v_4(z), v_4(x) + v_4(y) + v_4(z) = 1, v_4(x), v_4(y), v_4(z) \geq 0\}$$

Table 4

Pairwise comparisons between alternatives with respect to criteria.

	4: Ethical standards (Form 3)				**5**: Cultural similarity (Form 1)				**6**: Ease of communication (Form 3)		
	x	y	z		x	y	z		x	y	z
x		$[1, 2]$	$[5, 6]$	x		$[1, -]$		x		$[1, 2]$	$[1, 1]$
y			$[3, 4]$	y			$[1, -]$	y			$\left[\frac{1}{4}, \frac{1}{2}\right]$
z			$-$	z			$-$	z			

2: Financial terms (Forms 1 & 3)	x	y	z	**7**: Quality assurance (Forms 1 & 3)	x	y	z	**8**: On-time delivery (Form 3)	x	y	z
x		$[1,2]$	$[3,-]$	x		$[1,2]$	$\left[-,\frac{1}{4}\right]$	x		$[2,4]$	$[1,2]$
y			$[2,3]$	y			$\left[-,\frac{1}{3}\right]$	y			$\left[\frac{1}{4},\frac{1}{3}\right]$
z				z				z			

To select one of the three competing suppliers (x, y, z), we first determine the local priority vectors in terms of the extreme points from the interval PCMs in Tables 3 and 4 (Tables 5 and 6).

Table 5

Extreme points of interval ratios between criteria.

	#1	#2	#3	#4		#1	#2	#3		#1
1	$\frac{3}{19}$	$\frac{3}{11}$	$\frac{1}{4}$	$\frac{1}{7}$	4	0	$\frac{1}{5}$	$\frac{2}{9}$	7	$\frac{1}{2}$
2	$\frac{12}{19}$	$\frac{6}{11}$	$\frac{2}{4}$	$\frac{4}{7}$	5	0	$\frac{1}{5}$	$\frac{1}{9}$	8	$\frac{1}{2}$
3	$\frac{4}{19}$	$\frac{2}{11}$	$\frac{1}{4}$	$\frac{2}{7}$	6	1	$\frac{3}{5}$	$\frac{6}{9}$		

Table 6

Extreme points of interval ratios between alternatives with respect to criteria.

4: Ethical standards	#1	#2	#3	#4	**5**: Cultural similarity	#1	#2	#3	**6**: Ease of communication	#1
x	$\frac{5}{9}$	$\frac{5}{10}$	$\frac{6}{10}$	$\frac{6}{11}$	x	1	$\frac{1}{2}$	$\frac{1}{3}$	x	$\frac{4}{10}$
y	$\frac{3}{9}$	$\frac{4}{10}$	$\frac{3}{10}$	$\frac{4}{11}$	y	0	$\frac{1}{2}$	$\frac{1}{3}$	y	$\frac{2}{10}$
z	$\frac{1}{9}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{11}$	z	0	0	$\frac{1}{3}$	z	$\frac{4}{10}$
2: Financial terms	#1	#2	#3	#4	**7**: Quality assurance	#1	#2	#3	**8**: On-time delivery	#1	#2	#3
x	$\frac{4}{7}$	$\frac{6}{10}$	$\frac{3}{6}$	$\frac{3}{7}$	x	0	$\frac{1}{6}$	$\frac{2}{11}$	x	$\frac{4}{9}$	$\frac{3}{7}$	$\frac{4}{8}$
y	$\frac{2}{7}$	$\frac{3}{10}$	$\frac{2}{6}$	$\frac{3}{7}$	y	0	$\frac{1}{6}$	$\frac{1}{11}$	y	$\frac{1}{9}$	$\frac{1}{7}$	$\frac{1}{8}$
z	$\frac{1}{7}$	$\frac{1}{10}$	$\frac{1}{6}$	$\frac{1}{7}$	z	1	$\frac{4}{6}$	$\frac{8}{11}$	z	$\frac{4}{9}$	$\frac{3}{7}$	$\frac{3}{8}$

The next step is to establish the dominance between a pair of alternatives x and y (see Table 7 for other pairs of alternatives). According to Theorem 3, the pairwise comparison values from the lowest level L_2 are passed to an immediately upper level L_1 and finally to the topmost level L_0, $\pi_0(x,y)$, as shown below.

For level L_2:

$$\pi_4(x,y) = \min_{1\le k\le 4}\left(\mu_4^k(x)-\mu_4^k(y)\right) = \min\left\{\frac{5}{9}-\frac{3}{9},\frac{5}{10}-\frac{4}{10},\frac{6}{10}-\frac{3}{10},\frac{6}{11}-\frac{4}{11}\right\} = \frac{1}{10},$$

$$\pi_5(x,y) = 0,\ \pi_6(x,y) = \frac{2}{10}$$

For level L_1:

$$\pi_1(x,y) = \min_{1\le k\le 3}\left(\pi_4(x,y)\rho_4^k + \pi_5(x,y)\rho_5^k + \pi_6(x,y)\rho_6^k\right)$$

$$= \min\ \left(\frac{1}{10},0,\frac{2}{10}\right)\begin{pmatrix} 0 & \frac{1}{5} & \frac{2}{9} \\ 0 & \frac{1}{5} & \frac{1}{9} \\ 1 & \frac{3}{5} & \frac{6}{9} \end{pmatrix} = \frac{7}{50}$$

$$\pi_2(x,y) = 0,\ \pi_3(x,y) = \frac{1}{7}$$

For level L_0:

$$\pi_0(x,y) = \min_{1\le k\le 4}\left(\pi_1(x,y)\rho_1^k + \pi_2(x,y)\rho_2^k + \pi_3(x,y)\rho_3^k\right)$$

$$= \min\ \left(\frac{7}{50},0,\frac{1}{7}\right)\begin{pmatrix} \frac{3}{19} & \frac{3}{11} & \frac{1}{4} & \frac{1}{7} \\ \frac{12}{19} & \frac{6}{11} & \frac{2}{4} & \frac{4}{7} \\ \frac{4}{19} & \frac{2}{11} & \frac{1}{4} & \frac{2}{7} \end{pmatrix} = \frac{347}{6650}$$

Table 7

Pairwise dominance values for each criterion[8].

Criteria	Pair					
	$\pi_j(x,y)$	$\pi_j(y,x)$	$\pi_j(x,z)$	$\pi_j(z,x)$	$\pi_j(y,z)$	$\pi_j(z,y)$
0	0.052	-0.31	0.018	-0.324	-0.145	-0.108
1	0.14	-0.38	0	-0.3	-0.2	-0.04
2	0	-0.3	0.259	-0.5	0.143	-0.286
3	0.143	-0.233	-0.5	0.188	-0.667	0.375
4	0.1	-0.3	0.4	-0.5	0.2	-0.3
5	0	-1	0	-1	0	-0.5
6	0.2	-0.2	0	0	-0.2	0.2
7	0	-0.091	-1	0.5	-1	0.5
8	0.286	-0.375	0	-0.125	-0.333	0.25

Thus, alternative x dominates both y and z, based on $\pi_0(x,y),\ \pi_0(x,z) > 0$, but the dominance relation between y and z is undetermined:

$$x \to y,\ x \to z.$$

If the proposed method relies on interactions with the decision-maker to obtain a complete ranking and s/he is willing to modify preference statements of the alternatives for "Financial terms" as in V_2', we can obtain a complete ranking of $x \to z \to y$, based on $\pi_0(x,y) = 0.196$, $\pi_0(x,z) = 0.000$, and $\pi_0(z,y) = 0.057$.

$$V_2' = \{v_2(y) \le v_2(x) \le 2v_2(y), \tfrac{1}{2}v_2(z) \le v_2(y) \le v_2(z), 2v_2(z) \le v_2(x),$$
$$v_2(x) + v_2(y) + v_2(z) = 1, v_2(x), v_2(y), v_2(z) \ge 0\}$$

However, in general, much effort is required to reach such a conclusion, not a single trial like this.

When pairwise dominance fails to provide a full ranking, an alternative approach is to perform a simulation analysis. Note that our simulation analysis is quite different from

[8] The absolute bounds of the alternatives are computed as $[LB_0, UB_0] = [0.363, 0.542], [0.202, 0.351], [0.214, 0.364]$ for alternatives x, y, z. See Appendix A for details.

previously reported ones, which use random observations generated from each interval ratio and then the eigenvector method to derive the local priority vector in the crisp PCMs.[9] In contrast, our simulation analysis chooses the extreme points randomly and perform the AHP aggregations using the chosen extreme points, which requires a much smaller number of simulation runs. Enumerating all possible combinations of extreme points to see how many combinations support each alternative requires the consideration of 5184 different combinations for this small example. It is believed that the analysis produces a reliable outcome with a much smaller number of simulation runs than required by the enumeration method.

We designed a simulation study as follows.

1. In this example, the number of extreme points for criteria is small (12 different combinations); thus, we considered all of them in the analysis.
2. For each priority vector of criteria, we randomly selected 10 different combinations of priority vectors for alternatives from Table 7, thus yielding a total of 120 simulation runs.[10]
3. For the local priority vectors for criteria and alternatives chosen, we conducted the AHP aggregation to obtain a global priority vector for alternatives.

The simulation analysis produced the statistical results shown in Table 8. Obviously, alternative x outperformed the other two alternatives in terms of the ranges of the global priority vectors. Similarly, as was observed in the absolute bounds and pairwise dominance values, alternatives y and z overlap each other. If the purpose of the decision is to

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[9] The computational experiments comparing the proposed approach and simulation-based approaches [32] are described in Appendix B.

[10] We also performed the simulation analysis for 5% and 10% of all combinations of extreme points to see what proportion of extreme points needs to be sampled in this example. At most 10% of all combinations of extreme points was sufficient to infer the superiority of alternatives. See Table 8 for details.

determine a full ranking of alternatives, we can record the frequency that one alternative is preferred over the other, thus reaching the conclusion that z is more frequently preferred to y than y to z in 120 simulation runs:

$$88 \text{ counts } (z \succ \text{y}) > 32 \text{ counts } (\text{y} \succ z).$$

If the frequencies between y and z were close to each other, unlike this example, it might be better to consider the difference between the sum of the global priority weights as a measure of the preference strength by computing the sum of global priority weights of z for the instances of $z \succ y$ and that of y for the instances of $y \succ z$.

Table 8

Statistical analysis.

	Enumeration			120 runs (2.3%)			264 runs (5%)			516 runs (10%)		
	x	*y*	*z*	*x*	*y*	*z*	*x*	*y*	*z*	*x*	*y*	*z*
min.	.3633	.2017	.2136	.3655	.2122	.2244	.3633	.2067	.2206	.3655	.2077	.2136
max.	.5420	.3506	.3639	.5184	.3458	.3520	.5264	.3471	.3619	.5420	.3487	.3639
avg.				.4522	.2624	.2854	.4500	.2627	.2874	.4558	.2600	.2843
std.				.0410	.0355	.0292	.0426	.0352	.0302	.0414	.0325	.0293
$z \succ$ y					.27*	.73		.30	.70		.29	.71

*: $1 - \%(z \succ \text{y})$

5. Concluding remarks

In this paper, we presented an AHP-based decision-making method for uncertain judgments that include a range of interval ratios: loose articulation, minimum number of interval ratios, and general interval ratios. Our approach is distinct from previous studies in several ways. First, we deal with interval ratios in a unified framework in which the extreme points of the loose articulation can be obtained by a formula or dual programming approach, and those for a minimum number of interval ratios and the general interval ratios can be obtained by a simple procedure.

We also introduced two dominance criteria geared to actively use the extreme points to prioritize competing alternatives without solving many LP problems. After applying the dominance criteria, we can use an enumeration or a simulation approach to manage situations in which the best or a full ranking of alternatives remains unidentified.

Obviously the efficient and successful derivation of extreme points from different formats of interval ratios is a critical point for the development of the proposed method; therefore, the underlying assumption is the existence of consistent interval PCMs. To partially handle inconsistency, we attempt to identify which interval ratios cause inconsistency while solving a system of equations to obtain a set of modified extreme points. Therefore, systematic diagnosis of inconsistency and recommendation of a proper (upper or lower) bound for resolving inconsistency, which can be accomplished more effectively by developing a DSS (decision support system) to better communicate with the decision-maker, is a direction for future study.

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Appendix A

The data in Table A.1 denote the absolute bounds of the alternatives at each criteria level according to the absolute dominance criterion, which leads to the conclusion that x dominates y on the basis of $LB_0(x) > UB_0(y)$. The absolute bounds fail to establish x over z by a narrow margin, even though the pairwise dominance value in Table 7 establishes x over z.

Table A.1

Absolute bounds of alternatives at each criteria level.

Criteria	Alternative					
	x		y		z	
	LB_j	UB_j	LB_j	UB_j	LB_j	UB_j
0	0.363	0.542	0.202	0.351	0.214	0.364
1	0.4	0.56	0.18	0.3	0.258	0.4
2	0.429	0.6	0.286	0.429	0.1	0.167
3	0.214	0.341	0.056	0.155	0.521	0.722
4	0.5	0.6	0.3	0.4	0.091	0.111
5	0.333	1	0	0.5	0	0.333
6	0.4	0.4	0.2	0.2	0.4	0.4
7	0	0.182	0	0.167	0.667	1
8	0.429	0.5	0.111	0.143	0.375	0.444

Appendix B

We derive a global priority vector for prioritizing alternatives in the AHP problem which is characterized by three factors: the number of layers in the hierarchy, the number of criteria in each layer, and the number of alternatives. Further, various types of interval ratios in addition to these factors need to be considered to compare the proposed approach (extreme point approach) with the well-known simulation-based approach [32]. Nevertheless, to obtain a partial answer to how similar their priority vectors are, we create two interval PCMs (B.1) and (B.2), each of which has a different dimension and level of uncertainty by varying α. As a preliminary study, we set $\alpha = 0$ in (B.1) to make it a crisp consistent PCM and find that the extreme point method results in the same priority vector $(w_1, w_2, w_3) = (0.6, 0.3, 0.1)$ as the eigenvector method.

$$\begin{pmatrix} 1 & [2, 2+\alpha] & \\ & 1 & [3, 3+\alpha] \\ & & 1 \end{pmatrix} \quad \text{(B.1)}$$

$$\begin{pmatrix} 1 & [2, 4+\alpha] & [3, 5+\alpha] & [3, 5+\alpha] \\ & 1 & [\frac{1}{2}, 1+\alpha] & [\frac{1}{2}, 1+\alpha] \\ & & 1 & [\frac{1}{3}, 1+\alpha] \\ & & & 1 \end{pmatrix} \quad \text{(B.2)}$$

The interval PCM (B.3) is perfectly consistent and the ranges of local priority vectors resulting from two comparing methods are somewhat different as shown in Table B.1 due to their dissimilar theoretical backgrounds.

$$\begin{pmatrix} 1 & [2, 4] & [1, 4] & [\frac{1}{3}, 4] \\ & 1 & [\frac{1}{2}, 1] & [\frac{1}{6}, 1] \\ & & 1 & [\frac{1}{3}, 1] \\ & & & 1 \end{pmatrix} \quad \text{(B.3)}$$

Table B.1

A range of priority weights from extreme point and simulation analysis.

	Extreme point		Simulation analysis*	
	min	max	min	max
w_1	0.182	0.571	0.227	0.560
w_2	0.077	0.200	0.080	0.200
w_3	0.111	0.286	0.111	0.265
w_4	0.143	0.545	0.150	0.455

* The local priority vectors of IR > 0.1 are excluded.

For (B.1) and (B.2), we calculate two summary statistics[11]; *sum of average weights differences* and *degree of conformity* for α varying from 1 to 6. First, the priority vectors from the two methods show greater discrepancy as the uncertainty (α) increases in terms of the sum of average weights differences, which is computed by

$$\sum_i \left| w_{iA}^E(\alpha) - w_{iA}^S(\alpha) \right|$$

where $w_{iA}^E(\alpha)$ and $w_{iA}^S(\alpha)$ are the *i*th average weight from the extreme point method and simulation-based analysis respectively for a given α.

Second, the degree of conformity[12], represented by the following formula, measures the degree of overlap between the two intervals:

$$d\left(w_i^E(\alpha) > w_i^S(\alpha)\right) = \frac{max\left(w_{iU}^E(\alpha) - w_{iL}^S(\alpha), 0\right) - max\left(w_{iL}^E(\alpha) - w_{iU}^S(\alpha), 0\right)}{\left(w_{iU}^E(\alpha) - w_{iL}^E(\alpha)\right) + \left(w_{iU}^S(\alpha) - w_{iL}^S(\alpha)\right)}$$

where $w_{iL}^{E(S)}(\alpha)$ and $w_{iU}^{E(S)}(\alpha)$ are the lower and upper bounds of the *i*th weight from the extreme point method (simulation-based analysis) for a given α. Figure B.1 depicts the trend of degree of conformity from two methods applied to the interval PCM (B.1) over α. It shows that the interval of each weight overlaps less and less as α increases, noting that

[11] The raw data for deriving summary statistics are not included in the paper.

[12] We also call it the degree of preference in Section 3.

identical intervals correspond to the degree of conformity of 0.5.

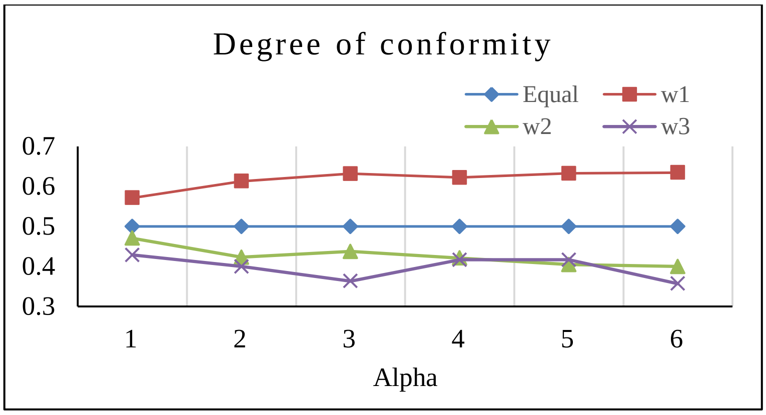

Fig. B.1 The trend of degree of conformity over α.

In summary, the local priority vectors from the two methods show much difference in terms of the measures used. Further, if we extend this analysis to the global priority vectors, there is no apparent reason that we will have comparable consequences for the two methods as the global priority vectors are determined by the aforementioned various factors.

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Highlights

- Interval ratio judgments are classified into three categories in the AHP.
- They include loose articulation, minimum of ratio bounds, and general type.
- We develop efficient methods to find their vertexes.
- The vertexes are utilized to rank the alternatives via dominance or simulation analysis.
- The extreme point-based simulation analysis is advocated if dominance criterion fails.