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Abstract

In the usual data envelopment analysis (DEA) setting, as pioneered by Charnes et al. (1978), it is assumed that a set of decision making units (DMUs) is to be evaluated in terms of their relative efficiencies in converting a bundle of inputs into a bundle of outputs. The usual assumption in DEA is that each output is impacted by each and every member of the input set. One particular area of recent research is that relating to *partial* input to output impacts where the main issue addressed is that in many settings not all inputs impact all outputs. In that situation the authors view the DMU as consisting of a set of mutually exclusive subunits, with each subunit having its own unique bundle of inputs and outputs. Examined as well in this area, is the presence of multiple processes for generating sets of outputs. Missing from that earlier work is consideration of the presence of outputs in the form of by-products, giving rise to a *parent-offspring* phenomenon. One of the modelling complications there is that the parent assumes two different roles; as an input affecting the offspring, while at the same time being the dominant output. This gives rise to a model that we refer to as conditional two-stage. Another complication is that in the presence of multiple processes, by-products often arise out of only a subset of those processes. In the current paper we develop a DEA-type of methodology to handle partial input to output impacts in the presence of by-products.

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Keywords: DEA, multiple processes, by-products, dual role factors, partial impacts, conditional two-stage DEA

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1. Introduction

In the nearly 40 years since the development of the data envelopment analysis (DEA) methodology by Charnes et al. (1978), the original concepts have been applied to an enormous number of practical problem settings, and the model structure has been extended in many directions. Recent surveys include Cook et al (2009), Paradi et al. (2013), and Liu et al. (2013). Literally thousands of articles and books have been written on the subject.

In the usual DEA setting it is assumed that a set of decision making units (DMUs), such as a set of hospitals, is to be evaluated in terms of their relative efficiencies in converting a bundle of inputs into a bundle of outputs. More to the point, the usual assumption is that each output is impacted by each and every member of the input set. A related area of research has to do with non-homogeneous DMUs, specifically where some DMUs produce a collection of outputs that may differ partially from those produced by other DMUs. See Cook et al (2013). Connected to the non-homogeneity issue is the multiple component or multiproduct problem in DEA, introduced by Beasley (1995). There, he studied the problem of extracting both teaching and research efficiency scores in universities when both dedicated and shared resources were present. This joint determination of efficiencies was further explored by Molinero (1996) who re-examined the Beasley approach but from the envelopment side of the problem. Molinero specifically analyzed the dual variables and as well explored the theoretical justification for Beasley's approach. More recently Zu et al (2013) extend the works of Molinero and of Beasley through their RD-DEA methodology, allowing for multi-level characteristics and the associated indexes for those characteristics.

Connected to this multilevel area, and of relevance to the current paper, is that relating to partial input to output impacts, as presented in Imanirad et al. (2013). Those authors examine the problem of measuring the efficiencies of a set of 20 steel fabrication plants in the presence of four inputs and four outputs. The main issue addressed there was the fact that not all inputs impacted all outputs, much along the lines of Beasley (1995), Molinero (1996) and Zu et al (2013). Specifically, Imanirad et al (2013) viewed the DMU as consisting of a set of mutually exclusive subunits (or business units), wherein each subunit has its own unique bundle of inputs and outputs. In a recent paper by Li et al. (2015), a similar problem setting to that of Imanirad et el

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is examined, but where multiple processes are present for producing a given bundle of outputs.

Missing from the Li et al. (2015) paper, and from the earlier Imanirad et al. (2013) paper, is consideration of the presence of a 5^{th} output in the form of a by-product. This creates a form of *parent-offspring* phenomenon. One of the modelling complexities arising from this situation is that the parent assumes two different roles; as an input affecting the offspring, while at the same time being the dominant output. Another complication is that the by-product arises out of only a subset of the multiple processes. Other complexities are discussed below. The purpose of the current paper is to develop a DEA-based methodology for evaluating efficiency in the presence of multiple processes and by-products arising from those processes.

Section 2 discusses in detail the issues surrounding the presence of subcontracting and by-products in a steel fabrication setting. The various complexities encountered there are elaborated. Section 3 develops a DEA-based parametric programming model for evaluating efficiency in the presence of multiple processes and by-products. Specifically, we accommodate the parent-offspring arrangement created in this manufacturing setting. In Section 4 we discuss the outcomes from the application of this model. Section 5 presents conclusions and offers insights into further research directions.

2. A Problem of Subcontracting and By-Products in Steel Fabrication

Consider the problem of measuring the relative efficiencies of a set of 20 steel fabrication plants where partial input to output impacts are present; this problem was first introduced in Imanirad et al. (2013). Figure 1 illustrates the partial impacts phenomenon.

Figure 1: Input to Output Impacts

Inputs	Sheet Steel	Flat Bar	Pipes/Cyl	Bearings
Labor	x	x	x	x
Shears	x	x		
Presses	x	x		
Lathes		x	x	x

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For example, while the manufacturing process for sheet steel requires only three of the four inputs, namely labor, shears and presses, flat bar products require, on the other hand, all four inputs.

Applied in the basic form of the Charnes et al. (1978) (CCR) model, the conventional DEA analysis may provide a distorted profile of the relative efficiency standing when partial input to output impacts are present. To counteract this phenomenon, Imanirad et al. (2013), as described above, proposed a non-homogenous DEA model to handle such partial input-to-output interactions. In brief, those authors developed a methodology that is based on the view that a DMU acts as a business unit consisting of K independent subunits, with each subunit k represented by a partial input and output bundle (I_k, R_k). From Figure 1 above, and referring to Imanirad et al. (2013), the (input, output) bundles for the three subunits are given by:

$$(I_1, R_1) = ((x_1, x_2, x_3), (y_1)),\ (I\ \ , R_2) = ((x_1, x_2, x_3, x_4), (y_2)),\ (I_3, R_3) = ((x_1, x_4), (y_3, y_4)) \quad (2.1)$$

The Imanirad et al. (2013) paper then introduced the idea of treating the efficiency of the DMU (the plant) as a weighted average of the efficiencies of its business subunits.

In a recent paper by Li et al. (2015), this problem was re-examined, and an additional feature was introduced, namely the use of multiple processes brought about by the presence of subcontracting. Specifically, the manufacture of one of the major product lines, cylindrical bearings, involves the use of specialized lathes which the plants have, but in limited supply. When insufficient lathe time is available, the plants resort to using a qualified subcontractor for the manufacture, in whole or part, of portions of this product. Thus, the product can be produced using different processes.

The two previous papers (Imanirad (2013) and Li et al. (2015)), do not address an important issue, namely the role played by by-products in this setting. In particular, the lathe (regular and specialized lathes) processes used in forming the bearings, generate steel shavings that can be recycled and therefore can be treated as an

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additional product. In the section to follow we investigate the issue of by-products and their appropriate modelling.

Before we proceed it is important to clarify the particular parent-offspring setting we are addressing in this paper. This is very important in that one might reasonably take the simplistic view that *if* there is a very direct and *fixed* relationship between the product and the associated offspring (e.g. a certain volume of the parent automatically generates a certain volume of the offspring), such as might be the case in certain chemical processes, then one could conceivably link the two products together as one, and then apply the standard DEA model. One might refer to the parent and offspring in this situation as co-products. In the present case, however, the precise amount of the by-product (steel shavings) is unknown, variable and dependent on the sizes of the raw steel that has to be ground down into bearings that come in various sizes and configurations. Given this, it is important to point out that certain complications arise relating to measuring the efficiency of the plant in regard to such recycled materials. As mentioned above, one complication is the necessity to account for the implied dual role on the part of the *parent* product (cylindrical bearings) in relation to its *offspring*, the by-product. Specifically, it is necessary to allow for the fact that product y_4 in its first role is an output from the manufacturing process, along with its offspring, but at the same time y_4, in its second role is a type of input that influences the amount of by-product being generated. Thus, in that latter role, the bearings are outputs in that they are generated by the applied inputs (lathes and labor), but at the same time the bearings generate the metal shavings as a recyclable offspring, hence are inputs. Viewed this way, we have a type of two stage process. That is, lathes and labor generate the bearings (stage 1), and the bearings generate the metal shavings (stage 2). This leads to what we shall call a *conditional two-stage DEA model*. There is a significant literature in the area of network DEA, and in particular, two stage DEA. See, for example, Kao et al. (2008), Yu et al. (2008), Li et al. (2012), Liang et al. (2011). Cook, Liang and Zhu (2010) provide an extensive review of the literature up to that time. A recent book by Cook and Zhu (2014) contains a broad literature on the subject. The earlier literature on multistage DEA, however, is not immediately applicable to the by-product problem addressed herein. While we use

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the name "conditional two-stage", we actually solve the efficiency problem as a single stage problem. This is explained below.

A second issue is the need to ensure that the values (weights) assigned to the bearings and by product are independent of the processes that generate them. This means that while the sub-bundles (processes) can be viewed as mutually exclusive business units, they need to be considered as a "group" in modelling efficiency, rather than being evaluated independently. A third issue relating to the parent and offspring products is the need to impose constraints relating to the relative proportions of these products coming from any given process. As will be shown below, the specification of these constraints must be done in an indirect rather than direct manner.

In the following section we develop a methodology for evaluating efficiency in the presence of by-products and multiple processes.

3. Measuring Efficiency with Multiple Processes and By-Products

Consider the case of 20 manufacturing plants that produce 4 products as shown in Figure 2, namely sheet steel, flat bar, pipes and cylinders, and cylindrical bearings, which are denoted as y_1, y_2, y_3, y_4 respectively. Labor, shears, presses, lathes and subcontract dollars make up the set of inputs denoted as x_1, x_2, x_3, x_4, x_5, respectively.

Figure 2: Input to Output Impacts with Subcontracting

Outputs

Inputs	Sheet Steel	Flat Bar	Pipes/Cyl	Cy Bearings
Labor	X	X	X	X
Shears	X	X		
Presses	X	X		
Lathes		X	X	X
Subcontract				X

Assume that the bearings y_4 are produced using 3 processes, namely:

Process 1: Bearings produced under this process are made using only in-house resources (labor and lathes);

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Process 2: The set of bearings manufactured under this process are partially completed in-house, while the finishing operation is done externally (subcontracted) on specialized lathes;

Process 3: Bearings manufactured under this process are completed in their entirety by way of a subcontractor.

In notational terms, the total production of cylindrical bearings y_4 is comprised of three lots, namely $\{y_4^q\} = \{y_4^1, y_4^2, y_4^3\}$, with y_4^q denoting the numbers of bearings manufactured under processes $q = 1,2,3$. Note that $y_4 = y_4^1 + y_4^2 + y_4^3$. In the particular case examined, the third subunit, (I_3, R_3) from Imanirad et al (2013), can then be viewed as two different subunits, namely $(I_3, R_3) = \{(x_1, x_4), (y_3)\}$, and $(I_4, R_4) = \{(x_1, x_4, x_5), (y_4)\}$, with the latter consisting of three mutually exclusive parts that we call *sub-bundles*:

$$(I_4^1, R_4^1) = \{(x_1, x_4), (y_4^1)\}, (I_4^2, R_4^2) = \{(x_1, x_5), (y_4^2)\}, (I_4^3, R_4^3) = \{(x_5), (y_4^3)\}. \qquad (3.1)$$

That is, the portion y_4^1 of the cylindrical bearings y_4 is made using inputs (x_1, x_4), y_4^2 is manufactured using (x_1, x_5) and y_4^3 is made using subcontracting resources x_5.

Because the use of subcontract resources is a frequent and "as needed" occurrence, details as to the exact values of the three portions $y_4^q, q = 1,2,3$ are not available, but can be specified only within known ranges $c_q \le y_4^q \le d_q$. The issue is how to derive the efficiency of each of the subunits, sub-bundles, and then the overall efficiency of the DMU.

Consider now the above processes for producing bearings y_4, where a *by-product* (steel shavings) denoted by z_4, is generated by two of the processes q=1 and 2 involving in house components. As mentioned in the previous section, in these two processes we shall refer to the bearings as the *parent* and the by-product as the *offspring*, adopting terminology from materials requirements planning (MRP). We point out that while recyclable steel may be generated under process 3 as well, it is not handed back to the contractor (the plant), hence the plant cannot claim to benefit

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from it. In the presence of the by- product, the sub-bundles corresponding to processes 1, 2 and 3 are denoted here by:

$$(I_4^1, R_4^1) = \{(x_1, x_4), (y_4^1, z_4^1)\}, (I_4^2, R_4^2) = \{(x_1, x_5), (y_4^2, z_4^2)\}, (I_4^3, R_4^3) = \{(x_5), (y_4^3)\} \quad (3.1)$$

We point out that under the *mixed* process (I_4^2, R_4^2), we have broadly classified all in-house resources used as "labor". This classification used by the plants is rather general and possibly misleading, in that part of the in-house portion of the manufacturing operation on bearings involves machine work that does create some amount of by-product that we denote as z_4^2.

The model we develop below is tailored to the specific application herein, involving steel fabrication. One might argue that it would be more appropriate to develop a more *general* model that could be applied in any given parent/offspring situation. It is important to emphasize at this point, however, that many parent/offspring relations in manufacturing settings can be very complex, can take many different forms, and will depend completely on the particular process involved. As discussed earlier, in chemical processes such relations are generally well defined; in creating a given set of main products in petroleum production, for instance, these products will *collectively* generate different types of waste materials during the refining process. Some of these materials may be recyclable, hence having positive value; other waste materials have no positive value, and in fact can give rise to significant disposal costs.

In the steel fabrication setting described herein, one can conceive of various scenarios. While we look at a very specific process involving the generation of "waste" steel shavings from the production of bearings, another possible scenario might be one where a number of products (e.g. bearings and pipes and cylinders) lead as well to the generation of waste steel. Moreover, the two by-products generated (call them z_4, z_5) may or may not be distinguishable from each other. If indistinguishable, then we have the case of a single by-product resulting from the production of two or more parent products which would need to be modelled in a manner somewhat similar

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to that below. If, however, the by-products are distinguishable, then they need to be considered as two separate entities, with different modelling considerations.

In summary, because the parent/by-product interaction can take many different forms, it is difficult to conceive of a model structure that would encompass those various forms. This being the case, we have developed a model structure below that caters to a particular parent/by-product situation.

The Model

To address efficiency measurement in the presence of by-products, we introduce the following "splitting" variables:

α_{ik}: the proportion of input x_i consumed by subunit k

α_{i4}^q: the proportion of input x_i consumed by sub-bundle q in subunit 4

β_4^q: the proportion of product y_4 made using process q, where $q=1,2,3$

γ_4^q : the proportion of by-product z_4 made under process q, where $q=1,2$

We assume that these proportions are known only within limits:

$$a_{ik1} \le \alpha_{ik} \le a_{ik2}, \ \ k=1,2,3$$

$$a_{i41} \le \alpha_{i4} \le a_{i42} \qquad (3.2)$$

$$b_{41}^q \le \beta_4^q \le b_{42}^q$$

$$g_{41}^q \le \gamma_4^q \le g_{42}^q$$

The basic principle to be used here is to view the DMU as consisting of a set of separate business units, with the overall efficiency of the DMU being a convex combination of the efficiencies of those business units. We proceed in three steps:

Step 1: Derive a split of inputs and outputs across the respective subunits and sub-bundles;

Step 2: Using the resulting inputs and outputs arising from step 1, derive efficiency scores for the subunits and sub-bundles;

Step 3: Combine, by way of a convex combination, the subunit and sub-bundle efficiency scores to arrive at the aggregate efficiency of the DMU.

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Step 1: Splitting Inputs and Outputs across Subunits and Sub-bundles

As described above, the intention is to view the aggregate efficiency of the DMU as a convex combination of the efficiency scores of the subunits and sub-bundles. That being the case, we begin by looking at the efficiency ratio for a subunit. Given the definition of the input to output bundles (I_k, R_k) for the subunits (see (2.1)), we would define the ratio for (I_k, R_k) as

$$e_k = u_k y_{kj_0} / \sum_{i \in I_k} v_i \alpha_{ik} x_{ij_0} \qquad (3.3)$$

In the specific case under investigation, we note that there is only one product y_k in each of the output bundles R_k; the more general case of multiple products is not included here, but is straightforward.

For the three processes q=1, 2 and 3, the (sub-bundle) efficiency ratios take a different form. Specifically, we argue that with the introduction of the by-product z_4, or offspring, the primary or parent product, y_4 (bearings) takes on a *dual* role. On the one hand, y_4 is an output along with the by-product, but at the same time it acts as a type of input that influences the amount of by-product produced.

The idea of dual role factors was first introduced by Beasley (1995) in connection with an application of DEA in ranking universities. There, the factor "research funds" was viewed both as an input affecting the amount of research carried out, and at the same time was an output that enhanced the reputation of the institution. Beasley's treatment of such a factor was to place it in both the numerator and denominator of the efficiency ratio. That is, letting w denote the dual role factor, Beasley suggested defining the efficiency ratio for the DMU as $(\sum u_r y_{rj} + u_{R+1} w)/(\sum v_i x_{ij} + v_{I+1} w)$. Unfortunately, as pointed out by Cook et al. (2006), this approach is flawed in that all DMUs become efficient simply by setting all input and output multipliers to zero except for those associated with the dual role factor. Cook et al. (2006) suggested an alternate approach that would see the factor treated as both an output, and as a *nondiscretionary* input, meaning that in that latter role, it is placed in the numerator with a negative sign, as per

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$(\sum u_r y_{rj} + u_{R+1}w - v_{I+1}w)/(\sum v_i x_{ij})$. Viewed this way, at the optimum only one of the two multipliers of *w* (namely u_{R+1} or v_{I+1}) will be positive, and will signal whether the dominant role of the factor is that of an input or an output.

Given the discussion above, and referring to (3.3), it can be argued that the appropriate representation of the sub-bundle *q* efficiency ratio (for *q*=1, 2) would be given by:

$$e_4^q = (u_4^1 \beta_4^q y_{4j_o} - u_4^2 \beta_4^q y_{4j_o} + u_5 \gamma_4^q z_{4j_o}) / \sum_{i \in I_4^q} v_i \alpha_{i4}^q x_{ij_o} \quad (3.4a)$$

For *q*=3, we argue that an appropriate representation for the efficiency ratio is:

$$e_4^3 = (u_4^1 \beta_4^3 y_{4j_o} + u_4^2 \beta_4^3 y_{4j_o}) / \sum_{i \in I_4^3} v_i \alpha_{i4}^3 x_{ij_o} \quad (3.4b)$$

Note that in (3.4b) we utilize the sum, rather than the difference of the two numerator terms involving the parent product. Some explanation is in order. The status of a dual role variable (y_{4j}) as an *input*, is logical only when there is an output (the offspring z_{4j}) to support. In the case of the third process there is no by-product going back to the plant from the subcontractor, meaning that if the parent product is deemed to have input status, then there would be *no output*. Hence, in this situation, the only status that can realistically be held by the parent y_{4j}, is that of an *output*. As will be shown below, at the optimum either u_4^1 or u_4^2 will be zero, meaning that the apparent "double counting" implied by the sum of two terms in the numerator of (3.4b), will not actually materialize.

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We now wish to derive the overall efficiency of each DMU, taking into consideration input-to-output impacts operating in the presence of multiple processes. Here, we view the efficiency of a DMU as a weighted average (using weights W_k, W_4^q) of the efficiency scores of the K=3 subunits and the 3 sub-bundles. Specifically, using the notation in (3.3), (3.4a) and (3.4b), the aggregate efficiency model takes the form:

$$e_{agg} = \sum_{k=1}^{3} W_k e_k + \sum_{q=1}^{3} W_4^q e_4^q \quad (3.5)$$

In the earlier papers by Imanirad et al. (2013) and Li et al. (2015), it is argued that it is appropriate to choose the weight to be assigned to a given subunit or sub-bundle to be

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the proportion of total weighted inputs consumed by that subunit or sub-bundle. As discussed in the earlier reference literature, this can be reasonably justified from an accounting standpoint.

Specifically, for k=1,2,3 and q=1,2,3 we define the weights as:

$$W_k = \sum_{i \in I_k} v_i \alpha_{ik} x_{ij_0} / [\sum_{k=1}^{3} \sum_{i \in I_k} v_i \alpha_{ik} x_{ij_0} + \sum_{q=1}^{3} \sum_{i \in I_4^q} v_i \alpha_{i4}^q x_{ij_o}] \quad (3.6a)$$

$$W_4^q = \sum_{i \in I_4^q} v_i \alpha_{i4}^q x_{ij_o} / [\sum_{k=1}^{3} \sum_{i \in I_k} v_i \alpha_{ik} x_{ij_0} + \sum_{q=1}^{3} \sum_{i \in I_4^q} v_i \alpha_{i4}^q x_{ij_o}] \quad (3.6b)$$

Note that a different set of weights W_k, W_4^q is derived for each DMU j_o; for notational convenience we have not shown the j_o index here.

We point out that some reasonable lower bounds should be imposed on the W_k, W_4^q, to prevent some of these weights from approaching zero. Since the denominators in (3.6a) and (3.6b) are set to unity, as discussed below, the bounds need only be applied to the numerators, specifically

$$\sum_{i \in I_k} v_i \alpha_{ik} x_{ij_0} \geq f_k, \; k = 1,2,3, \text{ and } \sum_{q=1}^{3} \sum_{i \in I_4^q} v_i \alpha_{i4}^q x_{ij_o} \geq f_4 \quad (3.6c)$$

In is emphasized that we focus here only on the overall score (and weight) for the three processes together, but obviously we could restrict the weight for each of those processes individually.

Given the format of the weights, the aggregate efficiency score becomes:

$$e_{agg} = \frac{\sum_{k=1}^{3} u_k y_{kj_0} + \sum_{q=1}^{2} (u_4^1 \beta_4^q y_{4j_o} - u_4^2 \beta_4^q y_{4j_o} + u_5 \gamma_4^q z_{4j_o}) + (u_4^1 \beta_4^3 y_{4j_o} + u_4^2 \beta_4^3 y_{4j_o})}{\sum_{k=1}^{3} \sum_{i \in I_k} v_i \alpha_{ik} x_{ij_0} + \sum_{q=1}^{3} \sum_{i \in I_4^q} v_i \alpha_{i4}^q x_{ij_o}} \quad (3.7)$$

Clearly (3.7) is nonlinear, which can partially be dealt with by observing that

$$\sum_{k=1}^{3} \alpha_{ik} + \sum_{q=1}^{3} \alpha_{ik}^q = 1 \; \forall i, \quad \sum_{q=1}^{3} \beta_4^q = 1, \quad \sum_{q=1}^{2} \gamma_4^q = 1, \quad (3.8)$$

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since with (3.8), expression (3.7) becomes:

$$\frac{\sum_{k=1}^{3} u_k y_{kj_0} + u_4^1 y_{4j_o} - u_4^2[1-2\beta_4^3] y_{4j_o} + u_5 z_{4j_o}}{\sum_{i=1}^{5} v_i x_{ij_o}} \tag{3.9}$$

The term $u_4^2[1-2\beta_4^3]y_{4j_o}$ accounts for the fact that process 3 cannot have the parent product designated as an input. Note that if none of the parent product is produced under process 3 (wherein $\beta_4^3 = 0$), then that product is allowed to assume full input status. Since this term is nonlinear, we treat β_4^3 as a ***parameter*** in the optimization model below.

Since ultimately it is the aggregate efficiency score that we wish to maximize, it is appropriate that we use such an aggregate model to derive the split of the inputs and outputs. The aggregate efficiency model is given by (3.10). We note that in (3.10) we have added additional notation, namely M_i, denoting the set of all subunits and sub-bundles that have input x_i as a member. For example, $M_4 = \{k = 2,3; q = 1\}$; that is input x_4 is a member of subunits *k*= 2 and 3 and sub-bundle *q*=1.

We point out that the first three constraints in (3.10) restrict the subunit and sub-bundle efficiency ratios to not exceed1. This insures that the model for the second step derivation of these scores will be feasible. Clearly, we should also impose a similar restriction on the ratio representing the aggregate score for each DMU *j*. However, in the presence of the subunit and sub-bundle restrictions, the constraints on the aggregate scores will be redundant, and can therefore be omitted.

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$$e_{agg} = \max \frac{\sum_{r=1}^{3} u_k y_{kj_0} + u_4^1 y_{4j_o} - u_4^2[1-2\beta_4^3] y_{4j_o} + u_5 z_{4j_o}}{\sum_{i=1}^{5} v_i x_{ij_o}}$$

subject to

$$u_k y_{kj} / \sum_{i \in I_k} v_i \alpha_{ik} x_{ij} \le 1, \qquad k = 1,2,3, \forall j$$

$$\sum_{i \in I_k} v_i \alpha_{ik} x_{ij_0} \ge f_k, \; k = 1,2,3$$

$$\sum_{q=1}^{3} \sum_{i \in I_4^q} v_i \alpha_{i4}^q x_{ij_o} \ge f_4$$

$$(u_4^1 \beta_4^q y_{4j} - u_4^2 \beta_4^q y_{4j} + u_5 \gamma_4^q z_{4j}) / \sum_{i \in I_4^q} v_i \alpha_{i4}^q x_{ij} \le 1, \quad q = 1,2, \quad \forall j$$

$$(u_4^1 \beta_4^3 y_{4j} - u_4^2[1-2\beta_4^3] y_{4j_o}) / \sum_{i \in I_4^3} v_i \alpha_{i4}^3 x_{ij} \le 1, \quad \forall j \tag{3.10}$$

$$\sum_{k \in M_i} \alpha_{ik} + \sum_{q \in M_i} \alpha_{i4}^q = 1, \quad \forall i$$

$$\sum_{q=1}^{3} \beta_4^q = 1$$

$$\sum_{q=1}^{2} \gamma_4^q = 1$$

$$a_{ik1} \le \alpha_{ik} \le a_{ik2}, \forall i, k = 1,2,3$$

$$a_{i41}^q \le \alpha_{i4}^q \le a_{i42}^q, \forall q$$

$$b_{41}^q \le \beta_4^q \le b_{42}^q, \forall q$$

$$g_{41}^q \le \gamma_4^q \le g_{42}^q, q = 1,2$$

$$u_k, u_4^1, u_4^2, u_5, v_i, \alpha_{ik}, \beta_4^q, \gamma_4^q \ge 0$$

Following the usual procedure for transforming the fractional programming problem to linear form, we let $t = 1/\sum_i v_i x_{ij_o}$ and let $\upsilon_i = tv_i, \mu_4^1 = tu_4^1, \mu_4^2 = tu_4^2, \mu_5 = tu_5$

Using the following transformation of variables

$$\delta_{ik} = \upsilon_i \alpha_{ik}, \delta_{i4}^q = \upsilon_i \alpha_{i4}^q, \xi_4^{1q} = \mu_4^1 \beta_4^q, \xi_4^{2q} = \mu_4^2 \beta_4^q, \rho_4^q = \mu_5 \gamma_4^q,$$

it is observed that

$$\sum_{k=1}^{3} \delta_{ik} + \sum_{q=1}^{3} \delta_{i4}^q = \upsilon_i, \sum_{q=1}^{3} \xi_4^{1q} = \mu_4^1, \sum_{q=1}^{3} \xi_4^{2q} = \mu_4^2, \sum_{q=1}^{2} \rho_4^q = \mu_5$$

Treating β_4^3 as a parameter, the *parametric linear* version of problem (3.10) now becomes (3.11). More to the point, for each DMU j_o we solve (3.11) by scanning

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the allowable range for β_4^3, namely $b_{41}^3 \le \beta_4^3 \le b_{42}^3$, in increments of say of 0.01. From the solution of (3.11) for each chosen value of β_4^3, we choose that value that yields the highest resulting score.

$$e_{agg} = \max \sum_{k=1}^{3} \mu_k y_{rj_0} + \mu_4^1 y_{4j_o} - \mu_4^2 [1 - 2\beta_4^3] y_{4j_o} + \mu_5 z_{4j_o}$$

subject to

$$\sum_{i=1}^{5} \upsilon_i x_{ij_o} = 1$$

$$\mu_k y_{kj} - \sum_{i \in I_k} \delta_{ik} x_{ij} \le 0, \qquad k = 1,2,3, \quad \forall j$$

$$\sum_{i \in I_k} \delta_{ik} x_{ij_o} \ge f_k, \, k = 1,2,3$$

$$\sum_{q=1}^{3} \sum_{i \in I_4^q} \delta_{i4}^q x_{ij_o} \ge f_4$$

$$(\xi_4^{1q} y_{4j} - \xi_4^{2q} y_{4j} + \rho_4^q z_{4j}) - \sum_{i \in I_4^q} \delta_{i4}^q x_{ij} \le 0, \quad q = 1,2, \quad \forall j$$

$$(\xi_4^{13} y_{4j} + 2\xi_4^{23} y_{4j} - \mu_4^2 y_{4j}) - \sum_{i \in I_4^3} \delta_{i4}^3 x_{ij} \le 0, \qquad \forall j \qquad (3.11)$$

$$\sum_{k \in M_i} \delta_{ik} + \sum_{q \in M_i} \delta_{i4}^q = \upsilon_i, \, \forall i$$

$$\sum_{q=1}^{3} \xi_4^{1q} = \mu_4^1$$

$$\sum_{q=1}^{3} \xi_4^{2q} = \mu_4^2$$

$$\xi_4^{23} - \mu^2 \beta_4^3 = 0$$

$$\sum_{q=1}^{2} \rho_4^q = \mu_5$$

$$\mu_4^1 \le My$$

$$\mu_4^2 \le M(1-y)$$

$$a_{ik}\upsilon_i \le \delta_{ik} \le \bar{a}_{ik}\upsilon_i, \, i \in I_k, \, k = 1,2,3,$$

$$a_{14}^q \upsilon_i \le \delta_{i4}^q \le \bar{a}_{i4}^q \upsilon_i, \, i \in I_4^q, \, q = 1,2,3$$

$$b_4^q \mu_4^1 \le \xi_4^{1q} \le \bar{b}_4^q \mu_4^1, \, \forall q$$

$$b_4^q \mu_4^2 \le \xi_4^{2q} \le \bar{b}_4^q \mu_4^2, \, \forall q$$

$$g_4^q \mu_5 \le \rho_4^q \le \bar{g}_4^q \mu_5, \, q = 1,2$$

$$\mu_k, \mu_4^1, \mu_4^2, \mu_5, v_i, \delta_{ik}, \delta_{ik}^q, \xi_4^{1q}, \xi_4^{2q}, \rho_4^q \ge 0, \, y \text{ binary}$$

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Note that we have included the constraint $\xi_4^{23} - \mu^2 \beta_4^3 = 0$ in model (3.11) to reflect the fact that ξ_4^{23} is a function of the parameter β_4^3.

To insure that the dual role variable y_{4j_o} is declared as behaving either like an input or output, as its dominant role, we have imposed in (3.11) the constraints $\mu_4^1 \le My$ and $\mu_4^2 \le M(1-y)$, where y is a binary variable and M is a large positive number. Note that if y=0, μ_4^1 will be forced to 0, and the dual role variable will assume an input status. Otherwise, if y=1 then μ_4^2 will be 0, signifying that the dual role variable is behaving like an output.

We point out as well that the constraints (3.6c) have been incorporated in (3.10) and (3.11), representing lower bounds on the weights W_k, W_4^q.

From the optimal solution of (3.11) we can immediately derive the associated splitting variables

$$\hat{\alpha}_{ik} = \hat{\delta}_{ik} / \hat{\upsilon}_i, \hat{\alpha}_{i4}^q = \hat{\delta}_{i4}^q / \hat{\upsilon}_i, \hat{\beta}_4^q = \hat{\xi}_4^{1q} / \hat{\mu}_4^1 \text{ or } \hat{\xi}_4^{2q} / \hat{\mu}_4^2, \hat{\gamma}_4^q = \hat{\rho}_4^q / \hat{\mu}_5 \qquad (3.12)$$

It is observed that the definition of the $\hat{\beta}_4^q$ will depend upon which of $\hat{\mu}_4^1$ or $\hat{\mu}_4^2$ is set to zero via the binary variable y in (3.11).

Assurance Regions Relating γ_4^q ***and*** β_4^q

The parent/offspring relationship between the proportion γ_4^q of by-product arising from a given process q, and the corresponding proportion β_4^q of the parent product created by that same process, would seem to imply the need to impose some form of constraints linking those proportions. A large proportion of the parent arising out of a process q, for example, would generally infer a corresponding large proportion of the total by-product coming from that same process. Assurance Region restrictions (AR) as per Thompson et al. (1990), appear to be a natural choice in this regard, meaning that constraints of the form

$$c_1 \le \gamma_4^q / \beta_4^q \le c_2 \qquad (3.13)$$

would be imposed. Appropriate lower and upper limits can presumably be selected using historical data on observed proportions of parent and offspring products

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generated in each of processes $q = 1,2$. Furthermore, the bounds c_1, c_2 would need to reflect the fact that no by product is created under process 3.

A difficulty with attempting to *directly* impose constraints of the form (3.13) is that γ_4^q and β_4^q do not explicitly appear in model (3.11); in the process of linearizing model (3.10), these proportions have been "covered up" in the sense that they have been rolled into ρ and ξ. One way to *indirectly* impose the restrictions that would be equivalent to (3.13) is by recognizing that since $\rho_4^q = \mu_5 \gamma_4^q$, for example, then knowing the values of any two of these three variables, automatically dictates the value of the third variable. There are only *two degrees of freedom* in this case. Hence, constraints imposed on any two of the variables, has implications about constraints on the *remaining* variable. To formalize this idea, we proceed as follows.

Let us first impose an AR constraint connecting μ_5 and μ_4^1, namely

$$d_1 \le \mu_5 / \mu_4^1 \le d_2 \qquad (3.14)$$

Arguably, since μ_5 and μ_4^1 (assuming μ_4^1 is positive) represent prices on the offspring and parent, respectively, it is reasonable to postulate that appropriate bounds d_1, d_2 can be observed from accounting records. (The case that μ_4^1=0 and μ_4^2 is positive, is discussed below). Now, it is reasonable to ask what AR restrictions on ρ and ξ, namely

$$e_1 \le \rho_4^q / \xi_4^{1q} \le e_2 \qquad (3.15)$$

would need to be imposed, such that when put together with (3.14), would be equivalent to (3.13).

To illustrate this, let us begin with a simple example: Suppose we know that μ_5 / μ_4^1=2, for example, and that ρ_4^q / ξ_4^{1q}=3. Then, observing that $\rho_4^q / \xi_4^{1q} = \mu_5 \gamma_4^q / \mu_4^1 \beta_4^q = 2(\gamma_4^q / \beta_4^q) = 3$, then $\gamma_4^q / \beta_4^q = 1.5$. In summary, if we know any two of the ratios, the third ratio is automatically determined.

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Now assume we can impose lower and upper bounds on μ_5 / μ_4^1 as in (3.14), and let us consider two cases involving (3.13) namely with μ_5 / μ_4^1 set first to its lower bound and second to its upper bound:

Case 1: $\mu_5 = d_1\mu_4^1$ (setting the ratio of the parent/offspring prices to the lower limit): Using this lower bound in (3.14), expression (3.15) becomes

$e_1 \leq \mu_5\gamma_4^q / \mu_4^1\beta_4^q \leq e_2 \Rightarrow e_1 \leq d_1\mu_4^1\gamma_4^q / \mu_4^1\beta_4^q \leq e_2 \Rightarrow e_1 \leq d_1\gamma_4^q / \beta_4^q \leq e_2$ which is equivalent to

$$\frac{e_1}{d_1} \leq \gamma_4^q / \beta_4^q \leq \frac{e_2}{d_1} \tag{3.16}$$

Case 2: $\mu_5 = d_2\mu_4^1$ (setting the ratio of the parent/offspring prices to the upper limit): Here, we arrive at a second set of limits

$$\frac{e_1}{d_2} \leq \gamma_4^q / \beta_4^q \leq \frac{e_2}{d_2} \tag{3.17}$$

We now argue that if both (3.16) and (3.17) must hold, then the AR restrictions connecting γ_4^q and β_4^q must be

$$\frac{e_1}{d_1} \leq \gamma_4^q / \beta_4^q \leq \frac{e_2}{d_2} \tag{3.18}$$

Again, as discussed above, we observe that while we wish to impose three sets of restrictions (3.13), (3.14), (3.15), there are only two *degrees of freedom*, meaning that imposing two of the three sets, determines the third set. Hence, if we regard (3.13) and (3.15) as given requirements, then in restrictions (3.14) we need to define the lower and upper limits as

$$d_1 = \frac{e_1}{c_1} \text{ and } d_2 = \frac{e_2}{c_2} \tag{3.19}$$

respectively. It is important to acknowledge that infeasibility can occur in (3.19) in the sense that because $d_1 \leq d_2$ must hold, then the bounds in (3.13) and (3.15) must be selected such that $\frac{e_1}{e_2} \leq \frac{c_1}{c_2}$ holds.

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To summarize, if we wish to implement constraints (3.13), we do so by imposing the two sets of constraints (3.14) and (3.15), where we define the lower and upper limits in (3.14) by (3.19).

In case $\mu_4^1 = 0$ and μ_4^2 is positive then we would replace (3.14) and (3.15), respectively by

$$d_1 \le \mu_5 / \mu_4^2 \le d_2 \qquad (3.14a)$$

$$e_1 \le \rho_4^q / \xi_4^{2q} \le e_2 \qquad (3.15a)$$

It is important to emphasize that while we refer to the Step 1 problem as having a two-stage like profile (which we refer to as conditional two-stage), we do not actually utilize any of the conventional two-stage solution methods as described in Cook et al. (2010). Mathematically, rather than treating the parent product as having an output role in the first stage (where we create the parent product) and a *discretionary* input role in the second stage (where we create the by-product), we instead treat the parent as a *nondiscretionary* input to that second stage. This means that in that latter role it is placed, with a negative sign, on the output side of the efficiency ratio, hence reducing the two stage problem to a single stage problem.

We now move to the problem of deriving the efficiencies of the subunits and sub bundles.

Step 2: Deriving Subunit and Sub-bundle Efficiency Scores

The outcomes from solving the aggregate model (3.11) (Step 1), are the optimal values for the splitting variables $\hat{\alpha}, \hat{\beta}, \hat{\gamma}$ (see (3.12)). Recall that such a set is derived for each DMU j_o. These can now be used to derive the inputs and outputs for the subunits and sub-bundles, specifically:

$$x_{ikj_o} = \hat{\alpha}_{ik} x_{ij_o}, k = 1,2,3; \quad x_{i4j_o}^q = \hat{\alpha}_{i4}^q x_{ij_o}; \quad y_{4j_o}^q = \hat{\beta}_4^q y_{4j_o}; \quad z_{4j_o}^q = \hat{\gamma}_4^q z_{4j_o} \qquad (3.20)$$

These are now used to derive the standard CRS efficiency scores for each subunit and sub-bundle q. In the case of subunit k one solves the problem:

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$$
\begin{aligned}
&\max\ \mu_k y_{kj_0} \\
&\textit{subject to} \\
&\sum_{i\in I_k} \upsilon_i x_{ikj_0} = 1 \\
&\mu_k y_{kj} - \sum_{i\in I_k} \upsilon_i x_{ikj} \le 0, \quad \forall j \\
&\mu_k, \upsilon_i \ge 0
\end{aligned}
\tag{3.21}
$$

In the case of the three sub-bundles, it is necessary to view these as a group. Furthermore, we must be careful to properly represent for each DMU j, which of the two roles (output or input) that DMU received in model (3.11). Let J_1, J_2 denote the sets of DMUs designated in the solution of model (3.11) as outputs and inputs, respectively. Thus, the overall sub-bundle efficiency can be computed from problem (3.22).

While we do not display herein the individual measures of efficiency for each of the three processes, we note that from the aggregate efficiency, the sub bundles can be split apart to get an approximation of an optimal score for each of the processes by using the left side of the constraints in (3.22).

$$
\begin{aligned}
&\max\ [\mu_4^1 y_{4j_o} + \mu_5 z_{4j_o}\ \textit{for}\ \ j_o \in J_1]\ \textit{or}\ [-\mu_4^2(1-2\beta_4^3) y_{4j_o} + \mu_5 z_{4j_o}\ \textit{for}\ \ j_o \in J_2] \\
&\textit{subject to} \\
&\sum_{1\in I_4} \upsilon_i \Big(\sum_{q=1}^{3} x_{14j_o}^q\Big) = 1 \\
&\mu_4^1 y_{4j}^q + \mu_5 z_{4j}^q - \sum_{i\in I_4} \upsilon_i x_{i4j}^q \le 0, \quad q = 1,2,\ j \in J_1 \\
&-\mu_4^2 y_{4j}^q + \mu_5 z_{4j}^q - \sum_{i\in I_4} \upsilon_i x_{i4j}^q \le 0, \quad q = 1,2,\ j \in J_2 \\
&\mu_4^1 y_{4j}^3 - \sum_{i\in I_4} \upsilon_i x_{i4j}^3 \le 0, \quad j \in J_1 \\
&\mu_4^2 y_{4j}^3 - \sum_{i\in I_4} \upsilon_i x_{i4j}^3 \le 0, \quad j \in J_2 \\
&\mu_4^1, \mu_4^2, \mu_5, \upsilon_i \ge 0
\end{aligned}
\tag{3.22}
$$

***Step 3**: The Overall Efficiency Scores*

In this stage the overall score is computed as the weighted average of the subunit and aggregate sub-bundle scores, using the weights shown in (3.6a) and (3.6b).

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4. Application

Table 1 displays the input and output data for the 20 steel fabrication plants, including by-product quantities. While the aggregate model (3.11) requires that it be viewed as a mixed integer programming problem, we take a slightly different approach, namely solving the model twice for each DMU; once for each value of the binary variable *y*=1 and *y*=0. Whichever value of *y* yields the largest efficiency score, determines whether that DMU will be designated as having the bearings behave like an output or input, hence determining whether the DMU belongs to J_1 or J_2. Regardless of the designation, the splitting variables are derived, and are shown (for the output case) in Table 2. Note that we have imposed a lower limit of 0.1 on all splitting variables. Specifically, in the case of the input variables α_{ik}, for example, the share of input *i* assigned to a subunit *k* or sub bundle *q* is required to be at least 0.1.

As well we point out that given the nonlinear nature of problem (3.11), it has been solved by treating β_4^3 as a parameter and searching over the range $b_4^3 \le \beta_4^3 \le \overline{b}_4^3$ as per (3.10).

It happens to turn out in the case of the 20 plants that all but three of the plants have the bearings y_4 designated as an outputs; the remaining 3 plants had identical efficiency scores for each of the input and output designations, hence we were able to treat all DMUs as belonging to J_1.

Table 3 displays the final efficiency scores for all 20 plants. Columns 2, 3 and 4 labelled as e1, e2, e3 are the efficiency scores for the three subunits *k*=1, 2 and 3. These are derived from the solution of model (3.21). The 5th column, labelled e4 provides an efficiency score for the combined 3 processes or sub-bundles. Note again that because it is necessary to view the three sub-bundles as a group, individual *optimal* scores for each of those three sub-bundles are not immediately available, although approximations of these scores can be derived from the left sides of the constraints in (3.22).

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As indicated earlier, the aggregate efficiency score e^0 for each DMU is taken to be a convex combination (weighted average) of the scores e^1, e^2, e^3, e^4.

5. Conclusions and Further Research

In the original Data Envelopment Analysis (DEA) model that is used to measure the relative efficiencies of peer decision-making units (DMUs), it is assumed that in a multiple input, multiple output setting, all members of the input bundle affect the entire output bundle. There are many situations in real world, however, where this assumption does not hold, and where partial input to output interactions occur. Earlier work by Beasley (1995), Molinero (1996), Zu et al. (2013), Cook et al. (2013) and Imanirad et al. (2013) examined various aspects of the partial input/output problem, which was later extended in Li et al. (2015) to include multiple processes. The application used to develop the ideas in Cook et al. (2013) and Imanirad et al. (2013) involved measuring efficiencies of a set of steel fabrication plants.

An important issue arising in many settings, particularly in manufacturing, is the need to deal with by-products. The main problem surrounding the presence of byproducts is that of having to address the accompanying "parent-offspring" phenomenon. The principle complication relating to the by-product phenomenon is the dual role played by the parent products. Specifically, it is necessary to allow for the fact that the parent or principle product is an output from the manufacturing process, along with its offspring, but at the same time is a type of input that influences the amount of by-product being generated. In the paper we develop what is referred to as a *conditional* two-stage DEA model to allow for the evaluation when byproducts (and multiple processes) are present. We point out, however, that our approach to solving this model differs from the conventional approaches in two-stage DEA, which involves the development of efficiency scores for each stage, and then combining those scores to arrive at an aggregate score for the combined stages.

The methodology developed herein has the potential for a number of important applications. The methodology may have important implications regarding multistage processes in general, but specifically including supply chains. The argument is as follows: One of the shortcomings of the current approaches used to

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solve, for example, two-stage processes (assume an input-oriented technology), is that outputs from stage 1 are treated in two different ways. Specifically, to arrive at a stage 1 score, outputs from that stage are treated as being non-discretionary (one keeps them fixed and reduces the inputs in order to project to the frontier). However, as inputs to stage 2, those same variables are viewed as being discretionary, meaning that in deriving a score for stage 2, they are decreased while projecting to the frontier. Therefore, one might argue that there is an inconsistency in the current approaches to two-stage efficiency measurement; allowing certain variables to be treated in two different manners (discretionary versus nondiscretionary). An alternative approach, along the lines of the previous section, would be to deem the outputs from the first stage as *nondiscretionary* inputs to the second stage. This provides for the necessary consistency that is lost using current methodologies.

We must point out, however, that the suggestion above comes with its own set of challenges. Research is currently proceeding on this approach.

Another important topic for further research is that where the nature of by-products and their parents is context-dependent. For example, certain parent-offspring relations may apply only to a subset of the DMUs, with completely different by-product situations (including the case where no by-products are present) in the case of other DMUs. Further, while some manufacturing facilities may have the practice of utilizing subcontractors, other facilities may not.

Another direction still is that where by-products can have positive value up to a certain point, but beyond this point there are disposal costs that must be considered.

These and other similar directions are the subject of future research.

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Table 1: Input and Output Data for the Steel Fabrication Plants

	OUTPUTS					INPUTS				
	Sheet Steel	Flat Bar	Pipes/ Cylinders	Bearings	By-Product	Labor	Shears	Presses	**Lathes**	**Subcontract**
DMU	Y1	Y2	Y3	**Y4**	**Z4**	**X1**	**X2**	**X3**	**X4**	**X5**
1	70	103	100	80	40	30	5	5	12	15
2	60	125	90	90	65	40	4	4	14	20
3	50	110	105	85	45	35	5.2	4.2	8	10
4	80	80	110	90	30	38	7	4.6	6.5	10
5	56	40	60	55	38	28	9	5.5	10	12.5
6	40	95	120	110	72	37	4.2	3.8	11	15
7	100	180	200	210	55	31	6	4.1	8	15
8	25	55	180	160	65	35	5	5	12	15
9	65	150	125	145	46	25	6.2	4.8	15	20
10	40	110	70	115	49	30	3	3.2	17	20
11	70	117	122	115	38	25	4	4	9.5	12.5
12	92	135	89	64	**66**	45	5	3.3	18	20
13	88	47	57	109	61	35	4.1	6	16.5	20
14	48	68	146	99	23	32	5.3	3.4	9	10
15	79	123	220	122	46	26	7.7	4.3	12.3	15
16	99	114	89	49	52	19	5.3	4.2	10	10
17	97	101	88	55	66	25	8	3	7	12
18	55	55	132	116	44	32	6	2.8	5.3	7.5
19	80	97	142	168	34	33	2.8	3.9	11	12
20	97	68	209	122	37	27	3.3	4.3	17	20

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Table2: Splitting Variables When Parent y_4 is Treated as an Output

DMU	β_4^{11}	β_4^{12}	β_4^{13}	γ_4^1	γ_4^2	α_{11}	α_{12}	α_{13}	α_{14}^1	α_{14}^2
1	0.45	0.45	0.10	0.50	0.50	0.10	0.60	0.10	0.10	0.10
2	0.80	0.10	0.10	0.90	0.10	0.10	0.10	0.10	0.10	0.60
3	0.26	0.10	0.64	0.10	0.90	0.10	0.10	0.10	0.10	0.60
4	0.80	0.10	0.10	0.90	0.10	0.17	0.11	0.11	0.11	0.50
5	0.10	0.78	0.12	0.12	0.88	0.10	0.10	0.10	0.18	0.52
6	0.20	0.37	0.43	0.10	0.90	0.10	0.10	0.10	0.10	0.60
7	0.80	0.10	0.10	0.90	0.10	0.10	0.10	0.10	0.10	0.60
8	0.10	0.78	0.12	0.10	0.90	0.10	0.10	0.10	0.17	0.53
9	0.45	0.45	0.10	0.50	0.50	0.10	0.60	0.10	0.10	0.10
10	0.39	0.43	0.18	0.50	0.50	0.11	0.24	0.11	0.27	0.27
11	0.60	0.30	0.10	0.74	0.26	0.10	0.60	0.10	0.10	0.10
12	0.30	0.36	0.33	0.40	0.60	0.33	0.11	0.13	0.25	0.18
13	0.42	0.47	0.11	0.48	0.52	0.11	0.10	0.10	0.39	0.29
14	0.10	0.80	0.10	0.40	0.60	0.10	0.10	0.59	0.10	0.11
15	0.43	0.47	0.10	0.53	0.47	0.10	0.10	0.60	0.10	0.10
16	0.29	0.59	0.12	0.50	0.50	0.11	0.59	0.10	0.10	0.10
17	0.29	0.25	0.46	0.22	0.78	0.13	0.10	0.11	0.31	0.36
18	0.10	0.17	0.73	0.10	0.90	0.14	0.12	0.11	0.33	0.30
19	0.80	0.10	0.10	0.76	0.24	0.10	0.10	0.10	0.10	0.60
20	0.43	0.45	0.12	0.51	0.49	0.45	0.10	0.22	0.11	0.11

Table2 Continued:

DMU	α_{21}	α_{22}	α_{31}	α_{32}	α_{42}	α_{43}	α_{44}^1	α_{54}^2	α_{54}^3
1	0.10	0.90	0.81	0.19	0.79	0.10	0.11	0.10	0.90
2	0.10	0.90	0.10	0.90	0.80	0.10	0.10	0.76	0.24
3	0.18	0.82	0.34	0.66	0.10	0.10	0.80	0.90	0.10
4	0.88	0.12	0.89	0.11	0.10	0.80	0.10	0.67	0.33
5	0.59	0.41	0.62	0.38	0.10	0.10	0.80	0.90	0.10
6	0.22	0.78	0.41	0.59	0.10	0.10	0.80	0.90	0.10
7	0.15	0.85	0.10	0.90	0.46	0.44	0.10	0.53	0.47
8	0.17	0.83	0.27	0.73	0.10	0.10	0.80	0.90	0.10
9	0.10	0.90	0.15	0.85	0.54	0.22	0.24	0.10	0.90
10	0.10	0.90	0.10	0.90	0.75	0.12	0.13	0.31	0.69
11	0.10	0.90	0.56	0.44	0.80	0.10	0.10	0.10	0.90
12	0.90	0.10	0.90	0.10	0.16	0.30	0.54	0.58	0.42
13	0.90	0.10	0.82	0.18	0.19	0.14	0.67	0.74	0.26
14	0.57	0.43	0.79	0.21	0.10	0.80	0.10	0.87	0.13
15	0.51	0.49	0.64	0.36	0.10	0.80	0.10	0.10	0.90
16	0.90	0.10	0.90	0.10	0.15	0.12	0.73	0.88	0.12
17	0.88	0.12	0.90	0.10	0.10	0.13	0.77	0.89	0.11
18	0.64	0.36	0.82	0.18	0.10	0.10	0.80	0.63	0.37
19	0.10	0.90	0.42	0.58	0.80	0.10	0.10	0.45	0.55
20	0.90	0.10	0.90	0.10	0.25	0.50	0.25	0.22	0.78

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Table 3: Efficiency Scores

DMU	e^1	e^2	e^3	e^4	e^o	w1	w2	w3	w4
1	0.8269	0.2289	0.6364	0.4273	0.3194	0.0978	0.7768	0.0344	0.0910
2	1	0.6643	0.4365	0.1643	0.6529	0.0807	0.8311	0.0206	0.0676
3	0.4449	0.953	0.6951	0.115	0.1600	0.0166	0.0280	0.0278	0.9277
4	0.3394	0.7921	0.4159	0.0919	0.4117	0.0159	0.0974	0.7644	0.1224
5	0.4889	0.3344	0.4147	0.1214	0.1667	0.0513	0.0523	0.0523	0.8441
6	0.3519	0.6406	0.6738	0.1741	0.2071	0.0195	0.0306	0.0306	0.9193
7	1	1	1	0.1809	0.8682	0.0140	0.4354	0.3897	0.1609
8	0.2291	0.3734	1	0.1662	0.2243	0.0525	0.0526	0.0526	0.8423
9	0.8566	0.2548	0.775	0.5762	0.4240	0.0970	0.6122	0.0870	0.2037
10	0.8889	0.389	0.3801	0.202	0.4254	0.0951	0.8299	0.0159	0.0590
11	1	0.2325	0.9317	0.4918	0.3269	0.0891	0.8429	0.0191	0.0488
12	0.1922	1	0.2432	0.2164	0.2724	0.8127	0.0959	0.0208	0.0707
13	0.6282	0.4814	0.2809	0.1585	0.5331	0.7260	0.0970	0.0184	0.1586
14	0.4096	0.5397	0.1491	0.2277	0.2224	0.0444	0.0657	0.4306	0.4593
15	0.7691	1	0.251	0.5625	0.4381	0.0922	0.0978	0.5979	0.2122
16	1	1	0.833	0.857	0.9542	0.3903	0.2975	0.0489	0.2633
17	0.8197	1	0.6424	0.2475	0.7250	0.7171	0.0840	0.0100	0.1889
18	0.3462	0.7624	1	0.1415	0.1621	0.0100	0.0132	0.0121	0.9648
19	1	0.6542	0.8481	0.1043	0.6597	0.0830	0.8512	0.0174	0.0484
20	0.26	0.9	0.54	0.3799	0.3328	0.8328	0.0970	0.0142	0.0559

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Highlights

.Conventional applications of DEA assume all inputs affect all outputs

. This paper considers settings with multiple processes and partial input/output impacts

. As well, we consider situations involving by-products and parent-offspring relations

. This gives rise to parent products acting in dual roles

. We develop a DEA-based model for handling such complexities

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