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Interfaces with Other Disciplines

On the strategic behavior of large investors: A mean-variance portfolio approach☆

Marcelo J. Villena*, Lorenzo Reus[1]

Faculty of Engineering and Science, Universidad Adolfo Ibañez, Diagonal Las Torres 2640, Santiago, Chile

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ABSTRACT

One key assumption of Markowitz's model is that all traders act as price takers. In this paper, we extend this mean-variance approach in a setting where large investors can move prices. Instead of having an individual optimization problem, we find the investors' Nash equilibrium and redefine the efficient frontier in this new framework.

We also develop a simplified application of the general model, with two assets and two investors to shed light on the potential strategic behavior of large and atomic investors. Our findings validate the claim that large investors enhance their portfolio performance in relation to perfect market conditions. Besides, we show under which conditions atomic investors can benefit in relation to the standard setting, even if they have not total influence on their eventual performance. The 'two investors-two assets' setting allows us to quantify performance and do sensitivity analysis regarding investors' market power, risk tolerance and price elasticity of demand.

Finally, for a group of well known ETFs, we empirically show how price variations change depending on the volume traded. We also explain how to set up and use our model with real market data.

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1. Introduction

The main economic assumptions in financial markets are perfect competition and symmetric information. Even though, financial markets generally approach perfect competition, in some cases these two assumptions do not hold, especially for powerful investors. Indeed, the investment decisions of institutional investors, who usually run a key part of total assets in the market and cover an even greater portion of the trading volume, can have an important impact on market prices, see Campbell, Grossman, and Wang (1993), Chan and Lakonishok (1995), Llorente, Michaely, Saar, and Wang (2002) and Huang and Heian (2010). Moreover, their private information about the market and their individual trading plans can equally affect the level of competition, see Wang (1994), Foster and Viswanathan (1996), Wang (1998), Dasgupta, Prat, and Verardo (2011).

Clearly, if done via single sell-order a trade of 10,000 shares impacts differently than a 100-share trade. Undoubtedly, the price will be negatively affected and some relevant information could be disclosed. Lo and Wang (2009) point out the theoretical consequences of this important empirical regularity: "That the demand curves of even the most liquid financial securities are downward-sloping for institutional investors, and that the price-discovery process often reveals information, implies that quantities are as fundamental as prices, and equally worthy of investigation". Lo and Wang (2006) built an inter-temporal capital asset pricing model around this empirical fact about investors with some market power.

Thus, the potential existence of market power in financial markets raises important questions about the strategic behavior of big players, and their role in the definition of portfolio allocation.

The literature contains different hypotheses regarding the assumption that prices depend upon trading strategies, giving rise to distinct methodological approaches. For example, in practice, investors may face different trading constraints, such as liquidity, that eventually could explain such deviations from the equilibrium price. Note that transaction costs can influence liquidity and hence market power, since transaction costs influence trading strategies and the bid/ask spread on the asset price, see Davis and Norman (1990) and Jouini and Kallal (1995). Regarding methodologies, Cuoco and Cvitanić (1998) for example considers a price model

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* Corresponding author. Tel.: +56 223311491.
E-mail addresses: marcelo.villena@uai.cl (M.J. Villena), lorenzo.reus@uai.cl (L. Reus).
[1] Tel.: +56 223311349.

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with coefficients depending on large-invest or strategy. In the same line, Ronnie Sircar and Papanicolaou (1998), Bank and Baum (2004) and Cetin, Jarrow, and Protter (2004) develop models where prices depend on strategies using reaction functions.

Nonetheless, examples of strategic models based on game theory in finance are very rare. Kannai and Rosenmüller (2010) developed a financial non-cooperative game in strategic form, where a finite number of players may borrow or deposit money at a central bank and use the cash available to purchase a commodity for immediate consumption. The bank can print money to balance its books and fix interest rates. For this game a pure-strategy Nash equilibrium is found under various assumptions. An extension of this model with multiple periods is presented by Mangoubi (2012).

Regarding portfolio theory, one key assumption of Markowitz's model is that all traders act as price takers, and hence no single one can exercise market power. According to Kolm, Tütüncü, and Fabozzi (2014), the main extensions of the model have been the inclusion of: (i) transaction costs, e.g. Brown and Smith (2011), (ii) different types of specific and institutional constraints, (see Clarke, De Silva, & Thorley, 2002), (iii) modeling and quantification of the impact of estimation errors in risk and return forecasts (via Bayesian techniques, stochastic optimization and robust optimization), (see Ledoit and Wolf, 2004 and Black and Litterman, 1992), and (iv) multi-period modeling, e.g. Merton (1969) and Campbell and Viceira (2002). Thus, despite Markowitz's portfolio selection model for a single period (Markowitz, 1952) having been one of the cornerstones of modern finance – inspiring numerous extensions and applications as those enumerated above – the price taker assumption has not yet been relaxed.

In sum, the financial literature has not directly addressed the issue of strategic behavior of large players in the context of portfolio management. Consequently, possible strategies for atomic players have remained neglected as well.

In this paper, we analyze the strategic behavior of large and atomic investors, using a portfolio optimization model in presence of an oligopolistic financial market. Thus, the ability of large investors to move prices in the traditional single period mean-variance portfolio model is introduced, relaxing one of the key assumptions of Markowitz's model. Under this framework, the Nash equilibrium of both investor types emerges and is compared with standard portfolio results.

This paper is organized as follows. Section 2 describes the general portfolio model considering oligopolistic financial markets. We derived its equilibrium and show how to construct an efficient frontier under this new framework. Section 3 constructs an example of the equilibrium for two risky assets and two types of investors: large and atomic. We analyze and compare performance results between both players and also with respect to results obtained in a perfect market setting. Section 4 shows how the model can be calibrated and applied to real financial data. Finally, some conclusions and potential for further research is presented.

2. The model and its equilibrium

Let us assume a market composed of m investors and n assets. The portfolio return for investor i is defined as:

$$r_p^i := \sum_{j=1}^{n} x_j^i r_j = r' x^i \tag{1}$$

where x_j^i is the fraction allocated in asset j by investor i and r_j is the return of the asset j. From (1), the portfolio mean return and its volatility emerges easily from having each asset's expected return, volatility and correlation between assets:

$$\mu_p^i := E(r_p^i) = \sum_{j=1}^{n} E(r_j x_j^i) = \mu' x^i$$

$$(\sigma_p^i)^2 := Var(r_p^i) := \sum_{j=1}^{n}\sum_{k=1}^{n} x_j^i x_k^i C_{jk} = (x^i)' C x^i$$

with $C_{jk} := \mathrm{cov}(r_j, r_k)$.

In the classical Markowitz problem, each investor determines x_j^i by taking the best compromise between the variance and the expected return of the portfolio, considering the budget constraint $1' x^i = 1$.

Markowitz model assumes a perfect market setting. Investors are price takers, and therefore returns are exogenous to them. In these expressions, returns do not depend on investors' allocations and their wealth is irrelevant when determining optimal allocation.

Now, let us assume participants can individually affect the prevailing market price by modifying the quantity demanded of assets. Following Vath, Mnif, and Pham (2007) and Lo and Wang (2006), a large investor could affect the price of the asset. The stock price rises when a trader buys and falls when s/he sells, and the impact is increasing relative to the size of the order. Specifically, we will assume a positive relationship between the volume of the demand for the asset in the market and its price, i.e., a price mechanism of the form

$$P(Q_j) := P_j^{PM} + \theta_j Q_j \tag{2}$$

where $P(Q_j)$ is the market price of asset j, P_j^{PM} is the price of asset j in a perfect market setting, $\theta_j \geq 0$ is an elasticity measure, or how the price is affected by the volume of assets demanded, and Q_j represents the quantity of asset j demanded in the market. Thus, $\theta_j Q_j$ represents the degree of market power.

If P_j^0 stands for the current price and w^i represents the wealth of investor i, then $Q_j = \frac{\sum_{i=1}^m w^i x_j^i}{P_j^0}$. Hence, the price in (2) becomes

$$P(Q_j) = P_j^{PM} + \theta_j \frac{\sum_{i=1}^m w^i x_j^i}{P_j^0}$$

In this context, the return of asset j is

$$r_j := \frac{P(Q_j)}{P_j^0} - 1 = \frac{P_j^{PM}}{P_j^0} + \frac{\theta_j}{(P_j^0)^2}\sum_{i=1}^{m}\left(w^i x_j^i\right) - 1 = r_j^{PM} + \theta_j' \sum_{i=1}^{m} w^i x_j^i$$

where r_j^{PM} represents the return of the asset in a perfectly competitive market and $\theta_j' = \frac{\theta_j}{(P_j^0)^2}$. Then the expected return of asset j is

$$\mu_j := \bar{r}_j^{PM} + \theta_j' \sum_{i=1}^{m} w^i x_j^i \tag{3}$$

where μ_j^{PM} is the expected return when solving the traditional Markowitz model. Note that r_j can stand above or below $\bar{r}_j^{PM}$ because we allow long and short positions. From now on we denote $\bar{r}_j^{PM}$ as $\bar{r}_j$.

2.1. Optimal allocations in the oligopolistic setting

Following previous definitions, and writing D for the diagonal matrix with $D_{jj} = \theta_j'$, the investor's mean-variance problem becomes

$$\min \left(x^i\right)^T C x^i - \lambda^i \left(\bar{r} + \sum_{k=1}^{m} w^k D x^k\right)^T x^i$$

$$s.t\ 1' x^i = 1$$

with $\lambda > 0$ denoted the risk aversion parameter. Note that Markowitz is recovered when assets are not sensitive to demand $\theta_j = 0\ \forall j$. Denoting market power $mp^i := \frac{w^i}{\sum_k w^k}$, we deem an investor i atomic when it has insignificant market power, that is $mp^i \approx 0$ and large or powerful otherwise. We recover perfect market returns when we assume that all investors are atomic, since $w^i \approx 0$ in such case.

Common wisdom holds that in a perfect market case, allocations with a particular λ can be obtained by setting a portfolio return target R. This target is given by the relationship $\lambda = 2\frac{(R-\frac{g}{h})}{f-\frac{g^2}{h}}$ with $f = \bar{r}'C^{-1}\bar{r}, g = 1'C^{-1}\bar{r}, h = 1'C^{-1}1$. Recall that solution in the perfect market case is given by:

$$\bar{x}^{\lambda} = \frac{\lambda}{2}C^{-1}(\bar{r} - \frac{g}{h}1) + \frac{1}{h}C^{-1}1 \tag{4}$$

We assume each investor knows the wealth of the others and therefore their market power $pw^i := \frac{w^i}{\sum_k w^k}$, just like in a typical Cournot-type game. In this setting, instead of having an individual optimization problem, we have to find the Nash equilibrium of the investors.

Since the problem is convex in x^i, KKT conditions are sufficient to find the optimal strategy. The Lagrangian $L_i(x^i, \rho^i)$ for investor i is

$$L_i\left(x^i, \rho^i\right) = \left(x^i\right)'Cx^i - \lambda^i\left(\bar{r} + \sum_{k\neq i}^{m} w^kDx^k + w^iDx^i\right)'x^i + \rho^i\left(1'x^i - 1\right)$$

The KKT conditions for investor i are

$$2\left(C - \lambda^i w^i D\right)x^i - \lambda^i\left(\bar{r} + \sum_{k\neq i}^{m} w^kDx^k\right) + \rho^i 1 = 0 \tag{5}$$

$$1'x^i = 1 \tag{6}$$

Plugging (5) in (6) and defining $C^i := C - \lambda^i w^i D,\ g^i := 1'C^{i^{-1}}\bar{r}$ and $h^i := 1'C^{i^{-1}}1$, we have

$$\rho^i = \frac{\lambda^i}{h^i}\left(g^i + 1'C^{i^{-1}}\sum_{k\neq i}^{m} w^kDx^k\right) - \frac{2}{h^i} \tag{7}$$

Plugging (4), then (5) becomes

$$\begin{aligned}&2C^ix^i - \lambda^i\sum_{k\neq i}^{m} w^kDx^k + \frac{\lambda^i}{h^i}\sum_{k\neq i}^{m} w^k\left(1'C^{i^{-1}}Dx^k\right)1 \\ &= \lambda^i\bar{r} - \frac{\lambda^i g^i}{h^i}1 + \frac{2}{h^i}1 = 2C^i\bar{x}^{\lambda^i}\end{aligned} \tag{8}$$

Eq. (8) represents the best response equation for each investor, i.e. the best allocation x^i as a function of every other investor's allocations x^k. By solving the mxn system of linear equations defined by (8) the equilibrium of the game arises. Defining $b^i_l := 1'C^{i^{-1}}e_l$, with e_l as the lth canonical vector, $1^TC^{i^{-1}}Dx^k = 1^TC^{i^{-1}}\left(\sum_l e_le_l'\right)Dx^k = \sum_l b^i_l e_l^T Dx^k = \sum_l b^i_l\theta_l x^k_l$. Eq. (8) can be written as:

$$2\sum_{l}^{n} C^i_{jl}x^i_l - \lambda^i\theta_j\sum_{k\neq i}^{m} w^kx^k_j + \frac{\lambda^i}{h^i}\sum_{k\neq i}^{m} w^k\sum_{l}^{n} b^i_l\theta_l x^k_l = 2[C^i\bar{x}^{\lambda^i}]_j \tag{9}$$

Rearranging terms, (9) is equivalent to

$$\begin{aligned}&2\sum_{l}^{n} C^i_{jl}x^i_l + \sum_{k\neq i}^{m}\left[-\lambda^i\theta_j w^k\left(1 - \frac{b^i_j}{h^i}\right)\right]x^k_j + \sum_{k\neq i}^{m}\sum_{l\neq j}^{n}\frac{\lambda^i}{h^i}\theta_l w^k b^i_l x^k_l \\ &= 2[C^i\bar{x}^{\lambda^i}]_j\end{aligned}$$

Hence, we can obtain the values of $x = [x^1_1, x^1_2, \ldots, x^1_n, x^2_1, x^2_2 \ldots x^2_n, \ldots, x^m_1, x^m_2 \ldots, x^m_n]^T$ by solving the linear equation $Ax = c$, with:

$$A_{n(i-1)+j,n(k-1)+l} \leftarrow \begin{cases} C^i_{jl} & k = i \\ -\frac{\lambda^i}{2}\theta_j w^k\left(1 - \frac{b^i_j}{h^i}\right) & k \neq i, l = j \\ \frac{\lambda^i}{2h^i}\theta_l w^k b^i_l & k \neq i, l \neq j \end{cases} \quad c_{n(i-1)+j} \leftarrow [C^i\bar{x}^{\lambda^i}]_j \tag{10}$$

This result represents the Nash equilibrium of the game and the previous equilibrium does not hold for any risk tolerance. The optimization problem a player solves can be unbounded for some λ. Since this situation does not hold in practice, we can add constraints to avoid it. For example, we can add a bound on the total amount of an asset bought in the market or avoid shorting. For either possibility, we can get an equilibrium. However, these equilibriums cannot be obtained in a close form as we did previously. In the two players-two asset example in Section 3, we will revisit and explain how to proceed in such cases.

2.1.1. Special cases

Now we analyze allocation results for some specific cases.

1. Investor have no risk tolerance: That is $\lambda^i = 0$ in which case x^i is the minimum variance portfolio (MVP) allocation of the perfect market setting.

   $$x^i = \bar{x}^0 = \frac{1}{h}C^{-1}1$$

   Note that in this case market power is irrelevant.
2. Investors are identical: That is $w^i = w^k$ and $\lambda^i = \lambda^k$ in which case $x^i = x^k$. To see this, first note that $C^i = C^k$ and hence $g^i = g^k, h^i = h^k$. Thus, Eq. (5) for investor i and k becomes:

   $$i: 2C^ix^i - \lambda^i w^iDx^k + \frac{\lambda^i}{h^i}w^i1'C^{i^{-1}}Dx^k1 = A^d$$

   $$k: 2C^ix^k - \lambda^i w^iDx^i + \frac{\lambda^i}{h^i}w^i1'C^{i^{-1}}Dx^i1 = A^d$$

   With A^d the rest of the terms not depending on x^i and x^k. It emerges that the situation for i equals k and that $x^i = x^k$.
3. All Investors are atomic: In such case we recover Markowitz allocations for each investor, since no investor has the power to move the price of an asset. It has the same effect as the no price elasticity case ($\theta_j = 0\ \forall j$)
4. All Investors are identical: That is, everyone has the same market power and risk tolerance, $mp^i = w\frac{1}{m}, \lambda^i = \lambda\ \forall i$. We know from previous results that allocations are equal. But in this case we have a close solution. From (5) and (6) and defining $C^{eq} := C - \frac{w\lambda}{2}(1 + \frac{1}{m})D, g^{eq} = 1'C^{eq^{-1}}\bar{r}, h^{eq} = 1'C^{eq^{-1}}1$ we get the solution by

   $$x = \frac{\lambda}{2}C^{eq^{-1}}\left(\bar{r} - \frac{g^{eq}}{h^{eq}}1\right) + \frac{1}{h^{eq}}C^{eq^{-1}}1$$

   Note, we recover the perfect market results when all investors are atomic when ($w = 0$).
5. Monopoly: If one investor has all the market power (*w.l.g.* investor M), then its decisions will not depend on other players, and their allocations are determinable as in the perfect market case. Indeed, Eq. (7) becomes $\rho^M = \frac{\lambda^M g^M}{h^M} - \frac{2}{h^M}$ and plugging this into (8), we have:

   $$x^M := \frac{\lambda^M}{2}C^{M^{-1}}\left(\bar{r} - \frac{g^M}{h^M}1\right) + \frac{1}{h^M}C^{M^{-1}}1$$

For the rest of the players Eq. (7) becomes $\rho^i = \frac{\lambda^i}{h}(g + w^M 1^T C^{-1} D x^M) - \frac{2}{h}$ and plugging this into (8), we get:

$$x^i = \bar{x}^{\lambda^i} + \frac{\lambda^i w^M}{2}\left[C^{-1}Dx^M - \frac{1}{h}(1'C^{-1}Dx^M)C^{-1}1\right]$$

2.2. Oligopolistic efficient frontier

In the perfect market, the efficient frontier appears by solving the Markowitz problem with different values of λ. Its construction does not depend on others investors. Once investors calibrate their risk tolerance, and therefore λ, they can know the portfolio mean return and volatility for their portfolio. The efficient frontier is unique for all investors.

However, in the oligopolistic model described previously, allocations and returns depend on every investor's market power and risk tolerance. Each investor will have their particular efficient frontier, since mean return depends on the particular market power of each investor. This efficient frontier corresponds to the volatility and mean obtained for each different tolerance value λ off all investors.

With this new setting, we can compare the efficient frontier from two perspectives. First, from within the new oligopolistic setting, e.g., between investors with different market power. Second, we can compare the efficient frontier of each investor with respect to the perfect market case. Denoting $\bar{\mu}_p^i$ and $\bar{\sigma}_p^i$ as the portfolio mean return and volatility of player i in a perfect market setting then:

$$\mu_p^i = (\bar{r} + \textstyle\sum_{k=1}^m w^k D x^k)' x^i = \bar{\mu}_p^i + \sum_{k=1}^m w^k (x^k)' D x^i \qquad (11)$$

$$(\sigma_p^i)^2 = (x^i)' C x^i = (\bar{\sigma}_p^i)^2 \qquad (12)$$

For this perspective we compare the mean returns of both type of market under the same volatility, or equivalently under the same allocation. There is no other allocation that can give the same volatility target. To show the latter, suppose that x and y are allocations in perfect and oligopolistic market, respectively. Since C is semi-definite positive, if both have the same portfolio volatility target, that is $x'Cx = y'Cy$, then $x = y$

Hence, we can compare mean returns applying the same allocation x. To compare efficient frontiers between different market structures, we will compare the expected portfolio returns under the same allocation.

Now we show what we know about efficient frontiers in special cases

1. Investor have no risk tolerance: When $\lambda^i = 0$ then investor gets the mean and volatility from the MVP allocation, that is

$$\sigma_p^i = \bar{\sigma}_{MVP} = \frac{1}{\sqrt{h}}$$

$$\mu_p^i = \bar{\mu}_{MVP} + \frac{1}{h}\left(\sum_{k=1}^m w^k x^k\right)' DC^{-1}1$$

Note that μ_p^i can change for other values of λ^k, since x^k changes too. Also note μ_p between two investors (i and k) is equal when both have no risk tolerance, even if both have different market power.

$$\mu_p^i = \bar{\mu}_{MVP} + (w^i + w^k)x'Dx + \sum_{k=1, l\neq i,k}^m w^l x' D x^l = \mu_p^k$$

where $x = \bar{x}^0$.

2. Identical allocations: When two investors hold the same allocation x, then volatility and mean are the same. This is easy to see from (11) and (12). Moreover, when everyone has the same allocation x

$$\mu_p^i = \bar{\mu}_p^i + x'Dx\sum_{k=1}^m w^k \qquad \forall i$$

We have seen above that this situation develops with identical investors, yet it might hold in another situation as well. Hence, investors with less market power could eventually have common points in the efficient frontier with investors with more market power. Further details and examples of this situation appear in the two by two setting.

3. Monopoly: Eq. (11) for the single powerful player M becomes:

$$\mu_p^M = \bar{\mu}_p^M + w^M (x^M)' D x^M > \bar{\mu}_p^M \qquad (13)$$

As expected, a powerful player benefits in this new structure when x^M is the allocation in the perfect market case. The rest will have a return of:

$$\mu_p^i = \bar{\mu}_p^i + w^M (x^i)' D x^M \ \ \forall i \qquad (14)$$

Note $\mu_p^i > \bar{\mu}_p^i$ when atomic investors align with first player allocations, that is when allocations in each asset have the same direction (sign). In short, atomic players should consider to follow the herd.

3. Equilibrium in a market with two assets and two investors

The case of two risky assets and two investors is a handy building block for the general case. It allows us to derive a close formula for optimal allocations and hence to determine under which conditions each type of investor can benefit from oligopolistic market structure.

3.1. Optimal allocations

We assume an atomic and large (powerful) player that is $w^A = 0$ and $w^M = 1$. For simplicity, we assume one asset with and another without elasticity, i.e. $\theta_1 = 0$ and $\theta_2 = \theta > 0$. Denote $\sigma_j := \sqrt{C_{jj}}$ and suppose $\bar{r_1} \le \bar{r_2}$ and $\sigma_1 \le \sigma_2$. Let ρ be the correlation between the two assets. Then $C = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}$. In this case A and c in (10) become

$$A = \begin{bmatrix} C_{11} & C_{12} & 0 & \lambda^A\theta\frac{b_2^A}{2h} \\ C_{12} & C_{22} & 0 & -\lambda^A\theta\frac{b_1^A}{2h} \\ 0 & 0 & C_{11} & C_{12} \\ 0 & 0 & C_{12} & C_{22}^2 \end{bmatrix},$$

$$c = \begin{bmatrix} \frac{\lambda^A}{2}\left(\bar{r_1} - \frac{g}{h}\right) + \frac{1}{h} \\ \frac{\lambda^A}{2}\left(\bar{r_2} - \frac{g}{h}\right) + \frac{1}{h} \\ \frac{\lambda^M}{2}\left(\bar{r_1} - \frac{g^M}{h^M}\right) + \frac{1}{h^M} \\ \frac{\lambda^M}{2}\left(\bar{r_2} - \frac{g^M}{h^M}\right) + \frac{1}{h^M} \end{bmatrix}$$

Since $x_1^i = 1 - x_2^i$ then equations two and four are linearly dependent of one and three. Taking one and three, the system reduces to solve $\widetilde{A}x = \widetilde{c}$ with $x = [x_2^A, x_2^M]'$ and:

$$\widetilde{A} = \begin{bmatrix} C_{12} - C_{11} & \lambda^A\theta \frac{b_2^A}{2h} \\ 0 & C_{12} - C_{11} \end{bmatrix}, \quad \widetilde{c} = \begin{bmatrix} \frac{\lambda^A}{2}(\bar{r_1} - \frac{g}{h}) + \frac{1}{h} - C_{11} \\ \frac{\lambda^M}{2}\left(\bar{r_1} - \frac{g^M}{h^M}\right) + \frac{1}{h^M} - C_{11} \end{bmatrix}$$

Easily observable is that $b_2^M = -\frac{C_{12}-C_{11}}{|C|}$ and $\widetilde{c} = (C_{12} - C_{11})\begin{bmatrix} \frac{\lambda^A}{2}\frac{\bar{r_2}-\bar{r_1}}{|C|h} - \frac{C_{12}-C_{11}}{|C|h} \\ \frac{\lambda^M}{2}\frac{\bar{r_2}-\bar{r_1}}{|C|h-\lambda^M\theta} - \frac{C_{12}-C_{11}}{|C|h-\lambda^M\theta} \end{bmatrix}$. Then the solution is given by:

$$x_2^M = \frac{\lambda^M}{2}\frac{\bar{r_2} - \bar{r_1}}{|C|h - \lambda^M\theta} + \frac{C_{11} - C_{12}}{|C|h - \lambda^M\theta} = \bar{x}_2^{\lambda^M}\left(\frac{|C|h}{|C|h - \lambda^M\theta}\right) \tag{15}$$

$$x_2^A = \frac{\lambda^A}{2}\frac{\bar{r_2} - \bar{r_1}}{|C|h} + \frac{C_{11} - C_{12}}{|C|h} + \frac{\lambda^A\theta}{2|C|h}x_2^M = \bar{x}_2^{\lambda^A} + \frac{\lambda^A\theta}{2|C|h}x_2^M \tag{16}$$

Henceforth, x_2^A and x_2^M are denoted as x_M and x_A, respectively. Likewise, $\bar{x}_2^{\lambda^i}$ is denoted as $\bar{x}_i$

3.2. Powerful player

As laid out in Section 1, this equilibrium does not hold for any λ. To clarify, let us see when the problem is unbounded for the powerful player. The large player solves

$$\min(|C|h - \lambda^M\theta)x_M - (\lambda^M(\bar{r_2} - \bar{r_1}) + 2(C_{11} - C_{12}))x_M - \lambda^M\bar{r_1} + C_{11}$$

When $|C|h - \lambda^M\theta < 0$, the problem is unbounded, i.e. a powerful player can arbitrarily increase the objective function by increasing allocation in asset two. Obviously this does not happen in practice, since even large players have only limited amounts to invest, even if they short in other assets. To simplify analysis, we assume no shorting, that is $0 \le x_M \le 1$. By adding this constraint, we are implicitly bounding the amount bought and asset's two return: $\bar{r_2} + \theta x_M \le \bar{r_2} + \theta$. Now θ represents how much powerful player is able to shift the return.

So when $|C|h - \lambda^M\theta < 0$ then $x_M = 1$. If $|C|h - \lambda^M\theta \ge 0$, then it depends on market conditions and player risk tolerance.

- If $0 \le \frac{\lambda^M}{2} \le \frac{C_{12}-C_{11}}{\bar{r_2}-\bar{r_1}}$ then $x_M = 0$
- If $\frac{C_{12}-C_{11}}{\bar{r_2}-\bar{r_1}} \le \frac{\lambda^M}{2} \le \frac{C_{22}-C_{12}}{\bar{r_2}-\bar{r_1}+2\theta}$ then $x_M = \bar{x}_M(\frac{|C|h}{|C|h-\lambda^M\theta})$
- If $\frac{\lambda^M}{2} > \frac{C_{22}-C_{12}}{\bar{r_2}-\bar{r_1}+2\theta}$ then $x_M = 1$

Allocation of large investor does not depend on atomic decisions. We also observe that $x_M \ge \bar{x}_M$ and hence x_M is increasing in risk tolerance λ^M too. It easily appears that the objective function $z_p^M := (\sigma_p^M)^2 - \lambda^M\mu_p^M$ is better as we increase risk tolerance. Indeed:

$$\frac{\partial z_p^M}{\partial\lambda^M} = -2\theta x_M\frac{\partial x_M}{\partial\lambda^M}\lambda^M - \mu_p^M \le 0$$

The reward for more risk is higher than in the perfect market case, where $\frac{\partial z_p}{\partial\lambda^M} = -\mu_p^M$. The following shows how a powerful player benefits in this new setting.

Proposition 1. *For any risk tolerance λ^M, a powerful player gets a better (at least equal) value of z_p^M in relation to a perfect market setting. If $x_M + \bar{x}_M \le 1$, the benefit increases as we decrease asset's correlation and decreases otherwise.*

Proof. First, let us calculate the difference in volatility and return in both markets:

$$\begin{aligned} &\mu_p^M[x_M] - \bar{\mu}_p^M[\bar{x}_M] \\ &\quad = (1 - x_M)\bar{r_1} + x_M(\bar{r_2} + \theta x_M) - ((1 - \bar{x}_M)\bar{r_1} + \bar{x}_M\bar{r_2}) \\ &\quad = (\bar{r_2} - \bar{r_1})(x_M - \bar{x}_M) + \theta(x_M)^2 \end{aligned}$$

$$\begin{aligned} &(\sigma_p^M[x_M])^2 - (\bar{\sigma}_p^M[\bar{x}_M])^2 \\ &\quad = |C|hx_M^2 - 2(C_{11} - C_{12})x_M + C_{11} \\ &\qquad - (|C|h\bar{x}_M^2 - 2(C_{11} - C_{12})\bar{x}_M + C_{11}) \\ &\quad = (x_M - \bar{x}_M)(|C|h(x_M + \bar{x}_M) - 2(C_{11} - C_{12})) \\ &\quad = (x_M - \bar{x}_M)(|C|h(x_M - \bar{x}_M) + \lambda^M(\bar{r_2} - \bar{r_1})) \end{aligned}$$

Then the difference in z_p^M between both markets is:

$$\begin{aligned} z_p^M[x_M] - \bar{z}_p^M[\bar{x}_M] &= (\sigma_p^M[x_M])^2 - (\bar{\sigma}_p^M[\bar{x}_M])^2 - \lambda(\mu_p^M[x_M] - \bar{\mu}_p^M[\bar{x}_M]) \\ &= |C|h(x_M - \bar{x}_M)^2 - \lambda^M\theta x_M^2 \end{aligned} \tag{17}$$

- If $|C|h - \lambda^M\theta < 0$ then the difference is $|C|h(1 - \bar{x}_M)^2 - \lambda^M\theta \le |C|h - \lambda^M\theta$
- If $|C|h - \lambda^M\theta > 0$, when $x_M = 0$ is easy to see that $\bar{x}_M = 0$ too. Therefore the difference is zero. When $\bar{x}_M = 1$, then $x_M = 1$ and therefore the difference is zero too. Finally when $x_M = \bar{x}_M(\frac{|C|h}{|C|h-\lambda^M\theta})$ then the difference equals $-\frac{\bar{x}_M^2\theta\lambda^M|C|h}{|C|h-\lambda^M\theta} \le 0$

If we differentiate the previous term with respect to ρ we get:

$$\frac{2\bar{x}_M\sigma_1\sigma_2\lambda^M\theta}{(|C|h - \lambda^M\theta)}\{1 - \bar{x}_M - x_M\}$$

Hence, when λ^M is small enough so that $x_M + \bar{x}_M \le 1$, performance difference decreases as we increase correlation, and rises otherwise. □

If we want to compare an efficient frontier of both settings, we already know from (13) that for the same volatility and hence same allocation (x_2), mean portfolio return is higher in the new market. In fact (13) becomes $\mu_p^M = \bar{\mu}_p^M + \theta x_2^2$. Thus, the difference is bigger when we increase the volatility target.

To add some numerical example to the latter results, we construct the following setting: $(\bar{r_1}, \sigma_1) = (5\text{percent}, 15\text{percent})$, $(\bar{r_2}, \sigma_2) = (10\text{percent}, 30\text{percent})$, $\theta = 5\text{percent}$. Fig. 1 shows Proposition 1 results. Note that differences between both markets increases with higher risk tolerance and lower correlation.

Sensitivity analysis in θ is straightforward. As it increases, difference in results accentuates: allocation difference of asset two is larger, as well as the risk-return compromise. Analogously, differences disappear as θ approaches zero.

3.3. Atomic player

For atomic investors, allocation depends on powerful investors. Asset two's allocation still increases in λ and $x_A \ge \bar{x}_A$. To see the difference in performance with respect to the perfect market case, Eq. (17) for an atomic player becomes

$$\begin{aligned} &|C|h(x_A - \bar{x}_A)^2 - \lambda^1\theta x_A x_M \\ &\quad = |C|h\left(\frac{\lambda^A\theta}{2|C|h}x_M\right)^2 - \lambda^1\theta x_A x_M \\ &\quad = \frac{x_M\lambda^A\theta}{(|C|h)}\left\{\frac{\lambda^A\theta x_M}{4} - \frac{\lambda^A}{2}(\bar{r_2} - \bar{r_1} + \theta x_M) + C_{12} - C_{11}\right\} \\ &\quad = \frac{x_M\lambda^A\theta}{(|C|h)}\left\{-\frac{\lambda^A}{2}\left(\bar{r_2} - \bar{r_1} + \frac{\theta}{2}x_M\right) + C_{12} - C_{11}\right\} \end{aligned} \tag{18}$$

So if $x_M = 0$, z_p^A is the same as in perfect market setting. If not, then an improvement exists only when $\frac{\lambda^A}{2} > \frac{C_{12}-C_{11}}{\bar{r_2}-\bar{r_1}+\frac{\theta}{2}x_M}$. The last condition always holds if $C_{12} - C_{11} < 0$, that is when $\rho < \frac{\sigma_1}{\sigma_2}$.

If we want to compare an efficient frontier of both settings, we already know from (14) that for the same volatility, mean portfolio return is higher in the new market, as long as allocation of the

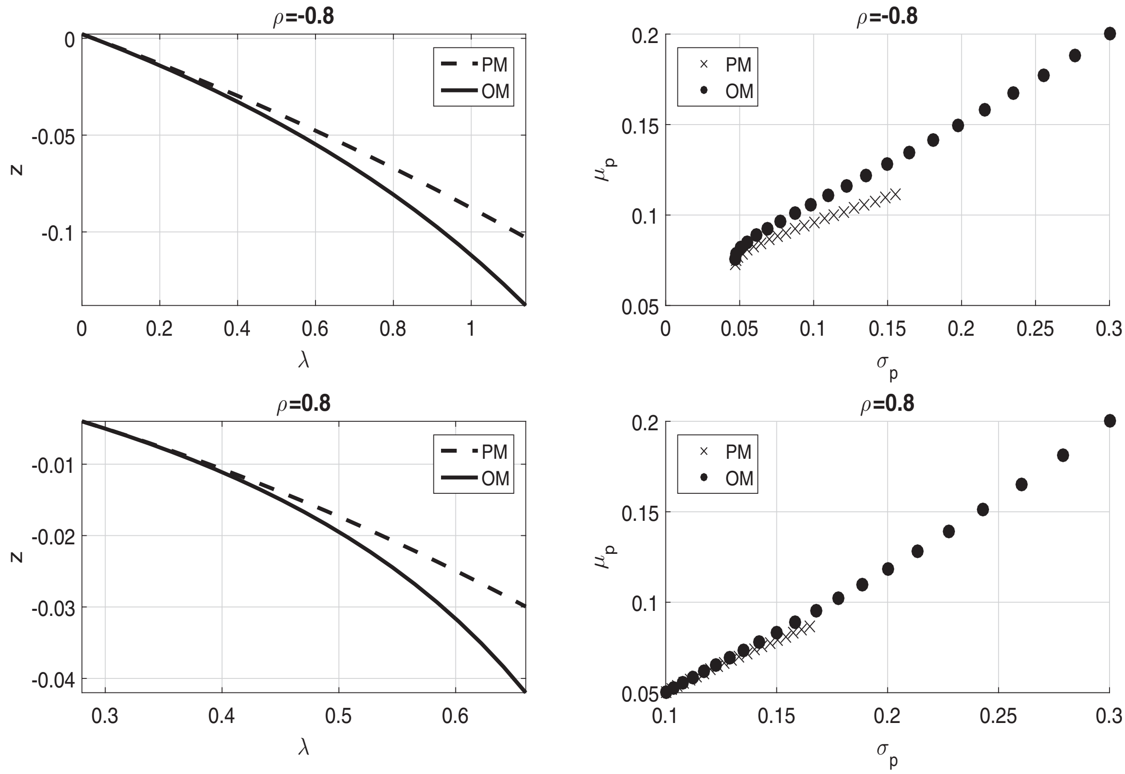

Fig. 1. Risk-return compromise of powerful player in perfect market (PM) and oligopolistic market (OM). Figure at the top (bottom) is set with $\rho = -.8$ ($\rho = .8$). For both situations, $\frac{\lambda^M}{2} \in [\max\{0, \frac{C_{12}-C_{11}}{\bar{r_2}-\bar{r_1}}\}, \frac{C_{22}-C_{12}}{\bar{r_2}-\bar{r_1}+2\theta}]$.

atomic player is positive. In fact, for atomic allocation x_2 in asset two, (14) turns into $\mu_p^A = \bar{\mu}_p^A + \theta x_2 x_M$. Its difference is bigger when we increase each player's volatility target. Note that when $\rho < \frac{\sigma_1}{\sigma_2}$, allocation in assets two is always positive, so in this case the difference is positive. When not, this happens if $\frac{\lambda^A}{2} > \frac{C_{12}-C_{11}}{\bar{r_2}-\bar{r_1}}$.

Fig. 2 shows the ultimate results with the same numerical example as in the powerful player example. For $\rho = -.8$ we are certain the atomic player will improve z_p^A and the volatility–mean relationship. With $\rho = .8$, we have better z_p^A when $\frac{\lambda^A}{2} > \frac{C_{12}-C_{11}}{\bar{r_2}-\bar{r_1}+\frac{\theta}{2}x_M} = .24$ and better volatility-mean relationship when $\frac{\lambda^A}{2} > \frac{C_{12}-C_{11}}{\bar{r_2}-\bar{r_1}} = .14$. If we increase x_M this threshold also increases. That is, when the powerful player tolerates more risk, the atomic player does likewise to finish better off than in a perfect market case.

3.4. Powerful player vs. atomic player

Previously, we compared each player's performance with respect to the perfect market case. Now we compare the performance of both players within the new market setting. To begin, we work out the difference between z_p^M and z_p^A:

$$\begin{aligned} z_p^M[x_M] - z_p^A[x_A] &= (\sigma_p^M[x_M])^2 - (\sigma_p^A[x_A])^2 \\ &\quad - (\lambda^M \mu_p^M[x_M] - \lambda^A \mu_p^A[x_A]) \\ &= (x_M - x_A)(|C|h(x_M + x_A) - 2(C_{11} - C_{12})) \\ &\quad - ((\lambda^M - \lambda^M)\bar{r_1} + (\bar{r_2} - \bar{r_1})(\lambda^M x_M - \lambda^A x_A) \\ &\quad + \theta(\lambda^M x_M^2 - \lambda^A x_A^2)) \end{aligned} \tag{19}$$

Fig. 3 shows the numerical example results for Eq. (19) for different equilibriums. As expected, the atomic player can only get better z_p than the powerful player when the latter is more risk averse and the former increases risk tolerance. In that case the atomic player can get a greater benefit with the increase in asset two's return

Apparently, no close exists for the relationship between λ^M and λ^A that could determine the sign of Eq. (19). However, in some cases we can determine when the atomic player may eventually perform better than the powerful player. Note, when $x_M = x_A$, (19) equals $-(\lambda^2 - \lambda^1)\mu_p[x_A]$. So in these cases it suffices to have $\lambda^1 > \lambda^2$. Recalling from previous results:

- In $R_1 := \{\frac{\lambda^M}{2} > \min\{\frac{|C|h}{2\theta}, \frac{C_{22}-C_{12}}{\bar{r_2}-\bar{r_1}+2\theta}\}, \frac{\lambda^A}{2} > \frac{C_{22}-C_{12}}{\bar{r_2}-\bar{r_1}+\theta}\}$ then $x_M = x_A = 1$
- If $\rho > \frac{\sigma_1}{\sigma_2}$, $R_2 := \{\frac{\lambda^M}{2} < \frac{C_{12}-C_{11}}{\bar{r_2}-\bar{r_1}}, \frac{\lambda^A}{2} < \frac{C_{12}-C_{11}}{\bar{r_2}-\bar{r_1}}\}$ then $x_M = x_A = 0$.

In other cases, we have $x_M = x_A$ when $\bar{x}_M(\frac{|C|h}{|C|h-\lambda^M\theta}) = \bar{x}_A + \frac{\lambda^A\theta}{2|C|h}x_M$. It is not hard to see that this equation turns into:

$$\begin{aligned} R_3 : \theta(\bar{r_2} - \bar{r_1})\frac{\lambda^A}{2}\frac{\lambda^M}{2} &- [|C|h(\bar{r_2} - \bar{r_1}) + \theta(C_{11} - C_{12})]\frac{\lambda^A}{2} \\ &+ [|C|h(\bar{r_2} - \bar{r_1}) + 2\theta(C_{11} - C_{12})]\frac{\lambda^M}{2} = 0 \end{aligned}$$

So in region $R_A = R_1 \cup R_2 \cup R_3 \cup \{\lambda^A > \lambda^M\}$ the atomic player can certainly achieve a better risk-return compromise than the powerful player. Analogously $R_M = R_1 \cup R_2 \cup R_3 \cup \{\lambda^A < \lambda^M\}$ is where the large investor receives a higher benefit.

3.5. No adaptation cost

We would like to quantify the cost for an atomic player when it is assumed that all players are atomic but in fact are not. Specifically, we want to compare z_p^A when allocating as in the perfect market case, instead of allocating as Eq. (16). The cost is the following:

$$\begin{aligned} c^A &:= z_p^A[\bar{x}_A] - z_p^A[x_A] \\ &= (\sigma_p^A[\bar{x}_A])^2 - (\sigma_p^A[x_A])^2 - \lambda^A(\mu_p^A([\bar{x}_A]) - \mu_p^A[x_A]) \end{aligned}$$

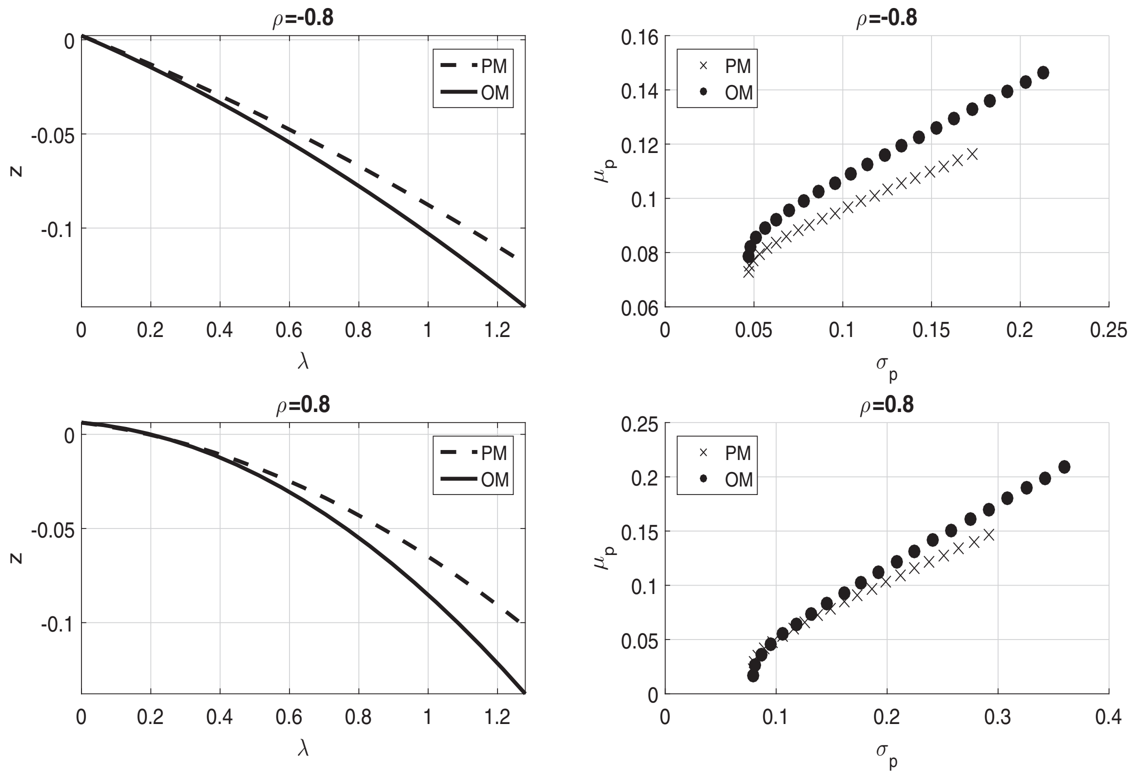

Fig. 2. Risk-return compromise of atomic player in a perfect market (PM) and an oligopolistic market (OM). Figure at the top (bottom) is set with $\rho = -.8$ and $\rho = .8$. For both situations, $x_M = 0.5$ and $\frac{\lambda^A}{2} \in [0, \frac{C_{22}-C_{12}}{\bar{r}_2-\bar{r}_1+\theta x_M}]$.

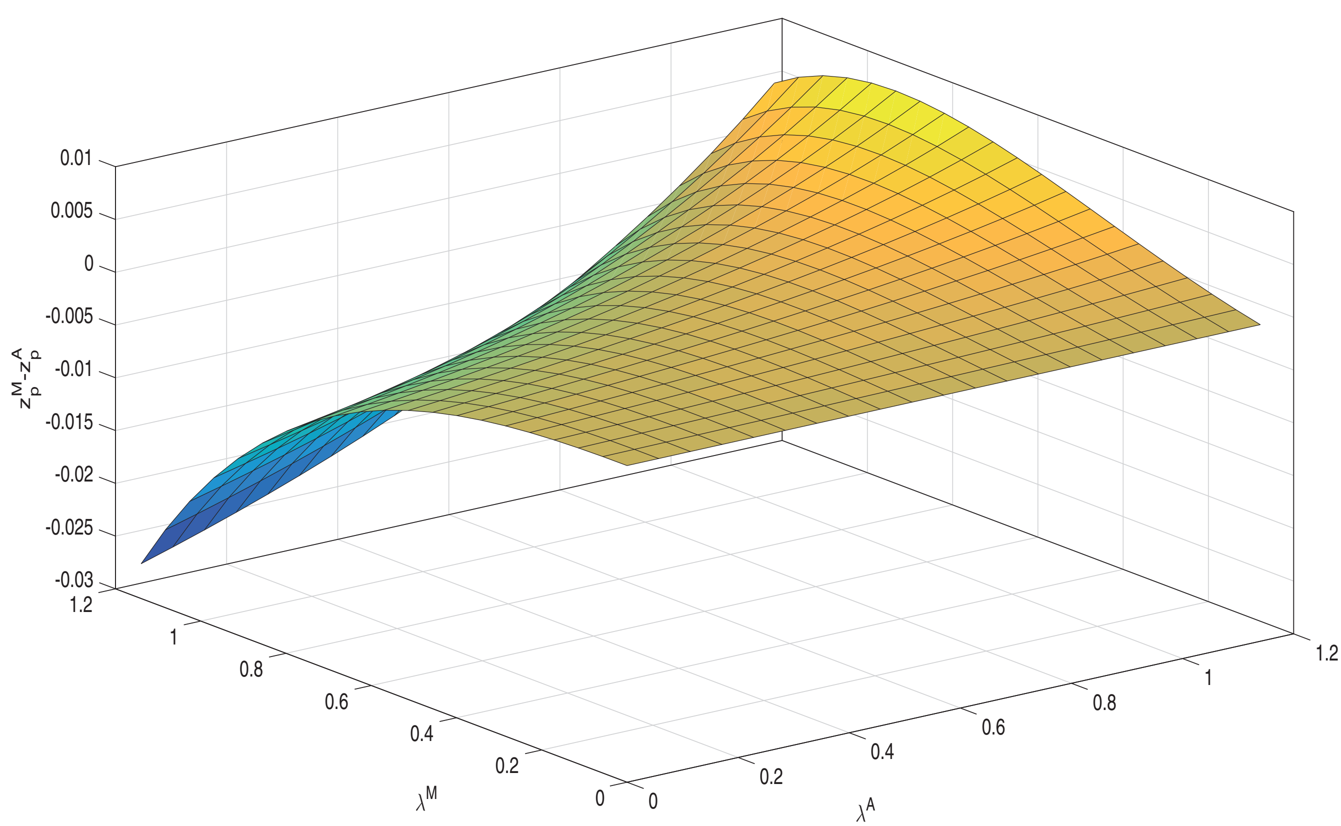

Fig. 3. z_p difference between powerful and atomic player for different risk tolerance levels. $\rho = -.8$.

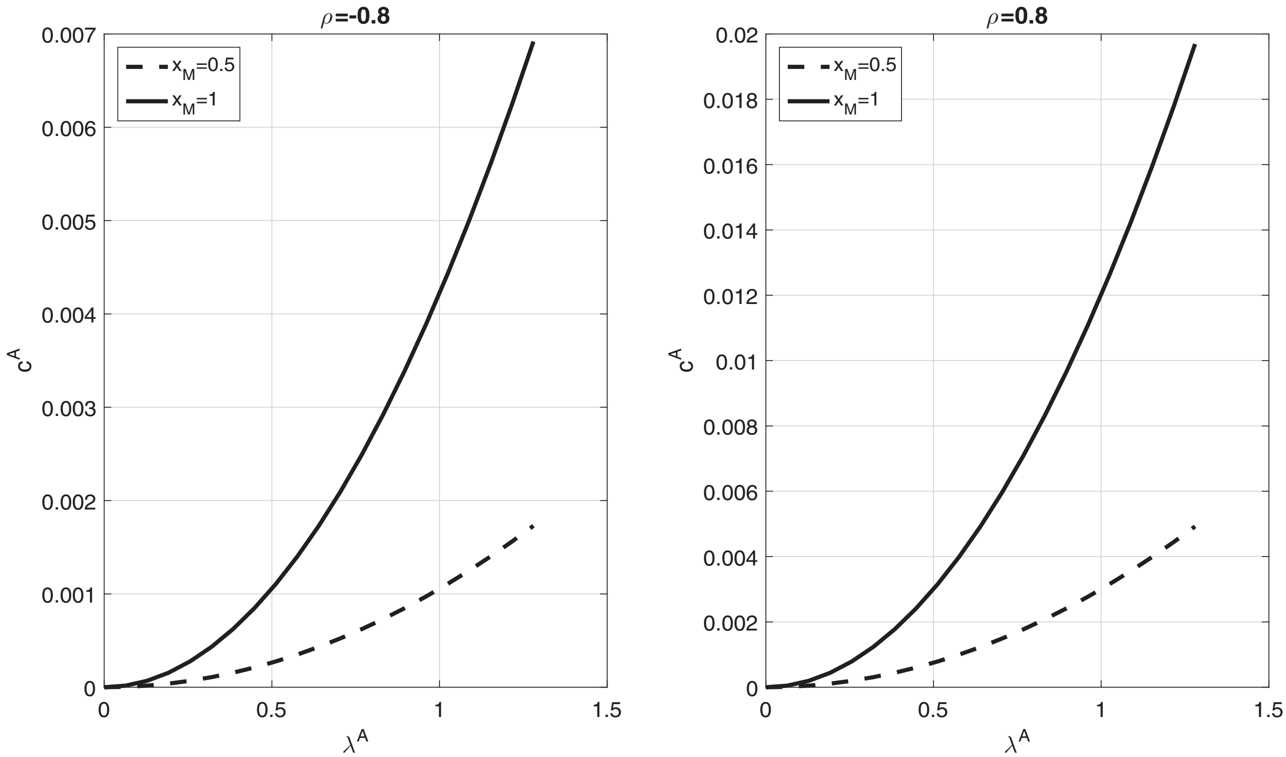

Fig. 4. Atomic player cost when not considering the effect of a powerful investor on assets two's return.

$$= (\bar{x}_A - x_A)(|C|h(x_A - \bar{x}_A) - \lambda^A\theta x_M)$$
$$= \frac{\lambda^A\theta x_M}{2|C|h}\left(\frac{\lambda^A\theta x_M}{2} - \lambda^A\theta x_M\right) = \frac{(\lambda^A\theta x_M)^2}{4|C|h}$$

This represents the cost for an atomic player to allocate with perfect market information each time a powerful investor decides to invest and affect asset two's return. Note, the cost is an even function in terms of x_M. This means that the cost remains the same no matter if powerful investment is buying or selling asset two.

As expected, the cost increases when more risk is taken by either player and also when elasticity is higher. To know whether less correlation decreases the cost, it is easy to see that

$$\frac{\partial c^A}{\partial \rho} = \frac{(\lambda^A\theta)^2 x_M\sigma_1\sigma_2}{2|C|h(Ch - \lambda^M\theta)^2}[-1 + \bar{x}_A + 2x_A]$$

The previous term is certainly positive with $\bar{x}_A \geq \frac{1}{3}$, that is less correlation decreases cost. Fig. 4 shows this cost for the numeric example and also shows how the cost increases with higher correlation and allocation of the powerful investor in asset two.

4. Application with OTC data

In this section, we first present empirical evidence of how the amount traded is correlated with price variations (returns). Then we show a way to estimate the effect a powerful player has on assets.

We select historical data from six known ETF, each from a different asset class. For each asset, we classify data in two groups, according to the volume traded V on that day. If V is below the first quartile, data goes into the first group ($G1$) and if it is above the third quartile into the second ($G2$). Table 1 and Fig. 5 shows the clear difference between the return's distribution of the two groups. Except for money market ETF (BIL), its clear that returns from group two have thicker tails and higher volatility. BIL is also affected, but in the other direction. Therefore, variations in days with more transactions are different (higher except for BIL) than days with less transactions.

Table 1
Mean and test results of daily returns, classified according to volume traded. Data daily taken from Google finance database from July 2006 until Oct 2015. V_i stands for the ith percentile of shares traded in millions. σ_{Gi} is the standard deviation of returns in group i (basis points). Pk_{Gi} is the kth percentile of returns in group i (basis points).

	SPY	EFA	EMM	TLT	GSC	BIL
V_{25}	47	8	30	2	0.1	0.08
V_{75}	161	20	69	7	0.3	0.5
σ_{G1}	136	127	165	70	110	5
σ_{G2}	183	262	309	118	199	4
$P5_{G1}$	−200	−209	−231	−123	−164	−6.5
$P5_{G2}$	−300	−417	−451	−197	−341	−4
$P95_{G1}$	181	216	241	116	191	6.5
$P95_{G2}$	288	385	523	168	303	2

With this data, we can estimate the mean return from Eq. (3). We can argue that on days belonging to group two, allocations of players are moving prices beyond the prices without their participation. For the two players case, we can aggregate these allocations and think it as an allocation of a single powerful player.

If we assume that investors are price takers, it is common to estimate the mean return of an asset with the sample mean $\bar{r}$. In our model, we have to classify returns on whether powerful players allocate or not. If we assume that returns belonging to $G2$ are the returns when this happens, then we can estimate the effect of these investors on asset's returns.

First we classify returns of $G2$ in two groups: $G2^+$ is the group for positive and $G2^-$ with negative price variations. We also denote $G2^C$ as the returns not belonging to $G2$. Then, the term $\theta_j' \sum_{i=1}^m w^i x_j^i$ in Eq. (3) can be estimated with $\bar{r}^+_{G2} - \bar{r}^C_{G2}$ when powerful investors buy and with $\bar{r}^-_{G2} - \bar{r}^C_{G2}$ when they sell. Table 2 shows the estimation for all ETF.

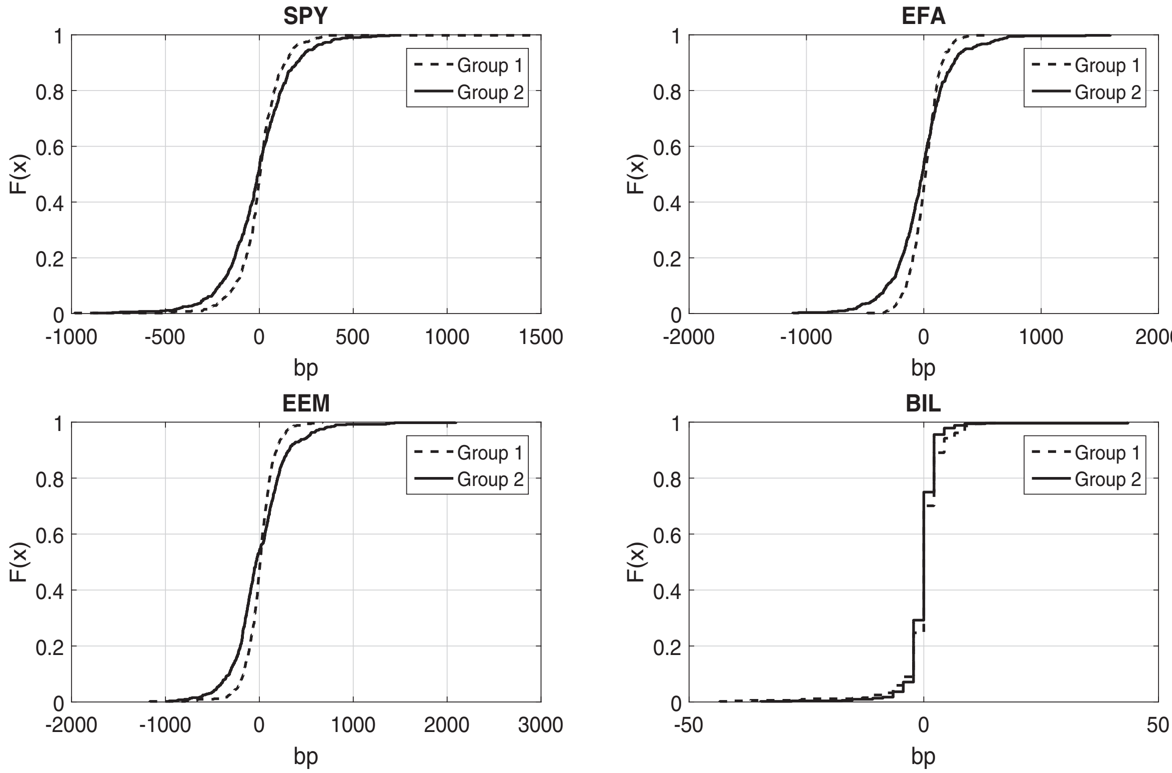

Fig. 5. Cumulative distribution function for daily returns classified in group 1 (less volume traded) and group 2. The ETFs not shown in the figure have the same patterns as SPY, EEM and EFA.

Table 2
Powerful players' influence in ETF daily returns (basis points). For BIL, the difference sign is changed, since price variation is inversely related with traded volume.

	SPY	EFA	EMM	TLT	GSC	BIL
$\bar{r}^C_{G2}$	5.4	5.1	1.8	4.5	0.9	0.1
$\bar{r}^+_{G2}$	130	179.3	242.1	91.9	142	3.4
$\bar{r}^-_{G2}$	−133.2	188	−210.7	−102.1	−155.1	−3.6
$\bar{r}^+_{G2}-\bar{r}^C_{G2}$	124.6	174.2	240.3	87.4	141.1	−3.4
$\bar{r}^-_{G2}-\bar{r}^C_{G2}$	−138.6	−193	−212.6	−106.6	−156	3.6

Table 3
Annualized mean returns, volatility and correlation with BIL. When large investors buy, $\mu = \bar{r}^C_{G2} + p(G2^+)(\bar{r}^+_{G2} - \bar{r}^C_{G2})$. Analogously when large investors sell $\mu = \bar{r}^C_{G2} + p(G2^-)(\bar{r}^-_{G2} - \bar{r}^C_{G2})$. With $x_M = 1$, θ is estimated by $\mu - \bar{r}^C_{G2}$. Returns are not in basis points and the model is not applied to BIL.

	SPY	EFA	EEM	TLT	GSC	BIL
ρ	−.16	−.16	−.16	.07	−.08	1
σ (percent)	22	27	35	16	25	0.8
$\bar{r}(=\bar{r}^C_{G2})$ (percent)	13	13	5	11	2	.1
$p(G2^+)$ (percent)	11.9	11.6	11.5	12.6	11.7	–
μ (percent)	51	63	74	39	43	.1
$\theta(x_M = 1)$ (percent)	38	50	69	28	41	0
$p(G2^-)$ (percent)	13	13.2	13.3	12.3	13.1	–
μ (percent)	−31	−51	−66	−21	−49	.1
$\theta(x_M = 1)$ (percent)	−44	−64	−71	−33	−51	0

For the two player-two asset example, the difference is equal to θx_M. To illustrate how this affects atomic player decisions, Fig. 6 shows the analogous result shown in Fig. 2 for an atomic player with respect to the perfect market case. But now we use the BIL-SPY and BIL-TLT pairs. To compute the annualized mean return μ of Eq. (3) from daily estimations, we simply annualized the value of $\bar{r}^C_{G2}$ + $\bar{r}^+_{G2} - \bar{r}^C_{G2}$ (multiply by 250). However, the last term has to be multiply by the probability that powerful investors decide to buy. That last probability can be estimated by counting the amount of data belonging to $\bar{r}^+_{G2}$ from all the data. Analogously, we can estimate μ when large investors decide to sell. Table 3 shows the latter estimations and the rest of input needed to construct the ETF pairs example.

5. Conclusions and further work

We have successfully addressed strategic behavior of large and atomic players in the context of portfolio management. Our model permits to find the optimal portfolio for each investor, some of them capable of moving asset prices when trading. In this framework, we also explained that the efficient frontier is different for each player and depends on size and degree of risk tolerance of the remaining investors. We show how to compare this efficient frontier with the perfect market frontier.

The two investor-two asset example allows to quantify and analyze how both investors are affected in this new market setting. We compare results with respect to standard settings and also between both players. As expected, the large investor always benefits (at least achieves equal results) in terms of risk-return performance. Atomic player can also benefit if it emulates the strategic behavior of the large investor. This expected pattern of results obtained in risk-return performance helps to validate the model constructed. The example also allows to do sensitivity analysis with respect to risk tolerance and price elasticity of volume traded.

We have empirically shown how price variations changed depending on volume traded, which also validates the claim that large investors can eventually affect the price of an asset. Finally, we have implemented the model with real market data, by estimating the influence of large investors on prices.

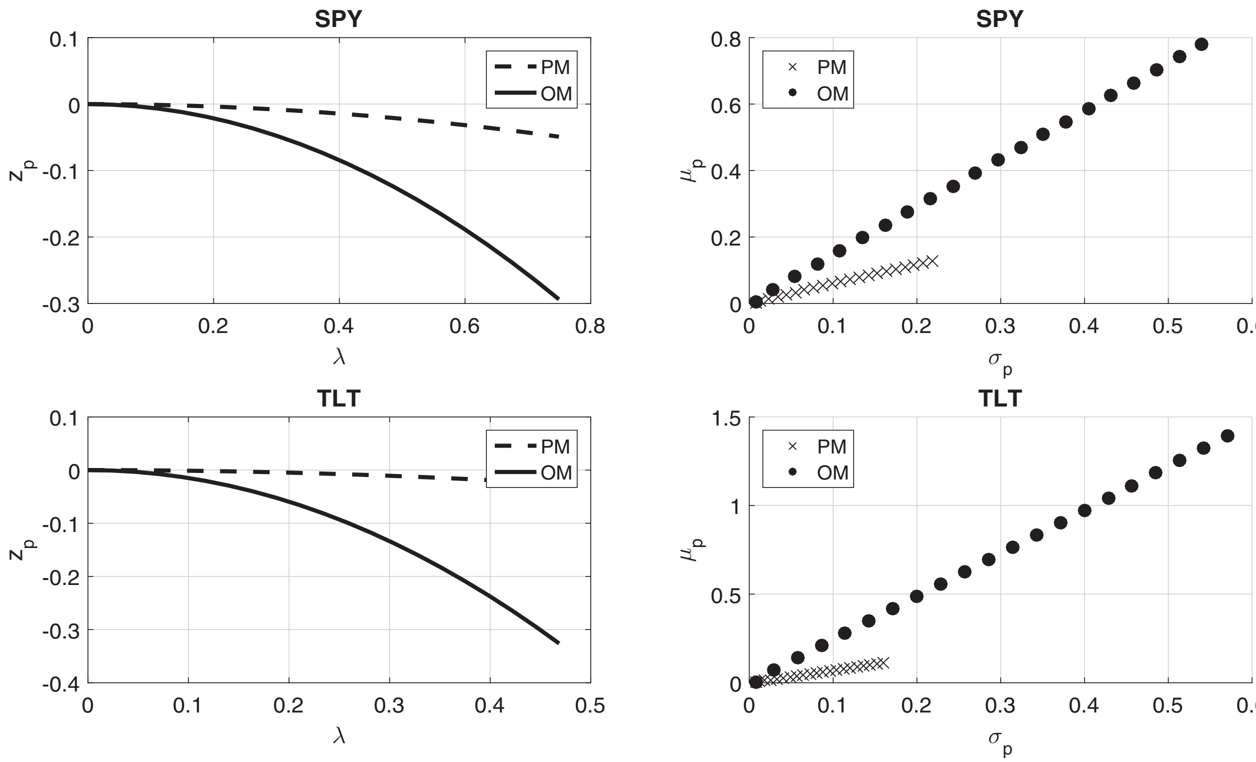

Fig. 6. Risk-return compromise of an atomic player in perfect market (PM) and oligopolistic market (OM) for two pairs: BIL-SPY and BIL-TLT. θ used is the one shown in Table 3 when large investors go long in asset two. Other pairs, such as BIL-EFA, BIL-EMM and BIL-GSC are omitted because they follow the same pattern as shown pairs.

In this paper we have combined mean-variance portfolio optimization and game theory to determine investors' allocation in oligopolistic markets, opening an interesting new line of research. One natural extension of this work is to add studied changes made to Markowitz model into our model. Another interesting extension is to analyze investors' behavior when the game is played in time. The model can be also used to measure the impact of collusion between large investors or to understand herd behavior of atomic players.

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