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A new modified Adomian decomposition method and its multistage form for solving nonlinear boundary value problems with Robin boundary conditions

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ABSTRACT

In this paper we propose a new modified recursion scheme for the resolution of boundary value problems (BVPs) for second-order nonlinear ordinary differential equations with Robin boundary conditions by the Adomian decomposition method (ADM). Our modified recursion scheme does not incorporate any undetermined coefficients. We also develop the multistage ADM for BVPs encompassing more general boundary conditions, including Neumann boundary conditions.

1. Introduction

We propose a new resolution method for boundary value problems (BVPs) with Robin boundary conditions, including BVPs with mixed sets of boundary conditions, for nonlinear second-order differential equations by the Adomian decomposition method (ADM) [1–14]. This new approach is based on the Duan–Rach modified recursion scheme for the ADM [15], where we transform the original nonlinear BVP into an equivalent nonlinear Fredholm–Volterra integral equation for the solution before designing the recursion scheme. Our new algorithm for the solution of Robin BVPs subsumes the set of Dirichlet boundary conditions as well as mixed sets of Robin and Dirichlet, Robin and Neumann, Dirichlet and Robin, Dirichlet and Neumann, Neumann and Dirichlet, and Neumann and Robin boundary conditions.

Furthermore we develop a multistage ADM for BVPs through partitioning the domain into two, or more, subdomains, where we compute a separate series in each subdomain using our new modified recursion scheme for nonlinear BVPs. The sub-solutions are combined by applying the conditions of continuity at the interior boundary points in analogy to the multistage ADM for initial value problems (IVPs) [16–23].

We show how our multistage ADM for BVPs can easily treat nonlinear examples when the original series diverges over the specified domain. Another aim of the multistage ADM for BVPs is to solve nonlinear Neumann BVPs relying upon the key concept of converting the original BVP into two sub-BVPs, where each is subject to a mixed set of Neumann and Dirichlet boundary conditions.

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The ADM is a well-known systematic method for practical solution of linear or nonlinear and deterministic or stochastic operator equations, including ordinary differential equations, partial differential equations, integral equations, integro–differential equations, etc. [2–14,24]. Adomian's decomposition method is a powerful technique, which provides efficient algorithms for analytic approximate solutions and numeric simulations for real-world applications in the applied sciences and engineering. The ADM permits us to solve both nonlinear IVPs and BVPs [7,25–41] without unphysical restrictive assumptions such as required by linearization, perturbation, ad hoc assumptions, guessing the initial term or a set of basis functions, and so forth. Furthermore the ADM does not require the use of Green's functions which are not easily determined in most cases. A key notion is the Adomian polynomials, which are tailored to the particular nonlinearity to solve nonlinear differential equations.

Adomian and co-workers have solved nonlinear differential equations with a wide class of nonlinearities, including product [42], polynomials [43], exponential [44], trigonometric [45], hyperbolic [46], composite [47], negative-power [48], radical [49] and even decimal-power nonlinearities [50]. We find that the ADM solves nonlinear operator equations for any analytic nonlinearity, providing us with an easily computable, readily verifiable, rapidly convergent sequence of analytic approximate functions.

Several investigators including Cherruault and co-workers [51–53] among others have previously proved convergence of the Adomian decomposition series and the series of the Adomian polynomials. For example, Abdelrazec and Pelinovsky [54] have published a rigorous proof of convergence for the ADM under the aegis of the Cauchy–Kovalevskaya theorem for IVPs. Furthermore Agarwal [55] has provided the prerequisites for existence and uniqueness of solutions for BVPs, including higher order BVPs. A key concept is that the Adomian decomposition series is a computationally advantageous rearrangement of the Banach-space analog of the Taylor expansion series about the initial solution component function, which permits solution by recursion. A remarkable measure of success of the ADM is demonstrated by its widespread adoption and many adaptations to enhance computability for specific purposes, such as various modified recursion schemes beginning with those by Adomian and Rach [26,27,56], Wazwaz [57], Wazwaz and El-Sayed [58], Duan [59–61], and Duan and Rach [62,15]. The choice of decomposition is nonunique, which provides a valuable advantage to the analyst, permitting the freedom to design modified recursion schemes for ease of computation in realistic systems.

In Section 2, we present a brief review of the ADM for nonlinear IVPs and BVPs to provide a common ground for the sequel. In Section 3, we present a description of our new approach for solving nonlinear Robin BVPs. We develop a systematic algorithm that also encompasses nonlinear BVPs with the set of Dirichlet boundary conditions as well as mixed sets of Robin and Dirichlet, and Robin and Neumann boundary conditions. Furthermore, we propose the multistage modified decomposition solution of nonlinear BVPs. In Section 4, we first consider other inverse linear operators for computational advantage, then apply them to the cases of mixed sets of Dirichlet and Robin, Neumann and Robin, Dirichlet and Neumann, and Neumann and Dirichlet boundary conditions. In Section 5, we next investigate four expository numerical examples including a nonlinear BVP with a set of Neumann boundary conditions. In the sequel, we present our conclusions and summarize our findings.

2. Review of the Adomian decomposition method

We next review the salient features of the Adomian decomposition method in solving IVPs and BVPs for nonlinear deterministic differential equations. Consider the general nonlinear deterministic differential equation in Adomian's operator-theoretic form

$$Lu + Ru + Nu = g, \tag{2.1}$$

where g is the system input and u is the system output, and where L is the linear operator to be inverted, which usually is just the highest order differential operator, R is the linear remainder operator, and N is the nonlinear operator, which is assumed to be analytic. We remark that this choice of the linear operator is designed to yield an easily invertible operator with resulting trivial integrations. Furthermore we emphasize that the choice for L and concomitantly its inverse L^{-1} are determined by the particular equation to be solved, hence the choice is nonunique, e.g. for cases of differential equations with singular coefficients, we choose a different form for the linear operator. Generally we choose $L = \frac{d^p}{dx^p}(\cdot)$ for pth-order differential equations and thus its inverse follows as the p-fold definite integration $L^{-1}(\cdot) = \int_{a_1}^{x} \dots \int_{a_p}^{x} (\cdot) dx \dots dx$. For IVPs, all of the lower limits of integration a_k are equal, i.e. $a_k = a$, which is the initial point, while the values of a_k may differ according to the distinct number of boundary points of the specified BVP. Our overarching aim is to achieve easy-to-integrate series by the ADM for computing the solution approximations.

We first solve Eq. (2.1) for the linear term Lu

$$Lu = g - Ru - Nu. \tag{2.2}$$

Since L has been assumed to be invertible, we apply the inverse linear operator L^{-1} to both sides of Eq. (2.2)

$$L^{-1}Lu = L^{-1}g - L^{-1}Ru - L^{-1}Nu. \tag{2.3}$$

By the definition of integral operators, we also have

$$L^{-1}Lu = u - \Phi, \tag{2.4}$$

where Φ identically satisfies $L\Phi \equiv 0$ and also identically satisfies all of the initial conditions or boundary conditions for either IVPs or BVPs, respectively, i.e.

$$I_j\Phi = c_j \text{ or } B_j\Phi = b_j$$

for $j = 1, 2, \ldots, p$, where the I_j and B_j are the initial value and boundary value operators, respectively. For example, consider the case of second-order differential equations, for IVPs, we have the set of initial conditions

$$I_1 u = u(a), \quad I_2 u = \frac{du}{dx}(a)$$

and for Dirichlet BVPs, we have the set of Dirichlet boundary conditions

$$B_1 u = u(a_1), \quad B_2 u = u(a_2),$$

where the general boundary operators B_j encompass the classic Dirichlet boundary conditions, Neumann boundary conditions, and Robin boundary conditions, as well as mixed sets of boundary condition equations, e.g. a mixed set of Dirichlet and Neumann boundary conditions, and so on.

Hence Φ incorporates the constants of integration to be determined. For the case where L is a simple pth-order differential operator, we say that $[L] = p$, i.e. the order of the linear operator L equals p. If all of the lower limits of integration are chosen to be a, all $a_k = a$, then the inverse linear operator $L^{-1}(\cdot) = \int_a^x \ldots \int_a^x (\cdot) dx \ldots dx$, and thus $\Phi = \sum_{\nu=0}^{p-1} \beta_\nu \frac{(x-a)^\nu}{\nu!}$, and similarly for different choices of the linear differential operator L and for different choices for the values of the lower limits of integration a_k. The β_ν are called the matching coefficients, which are computed from either the initial conditions or boundary conditions, respectively. And therein hangs the tale!

Upon equating the right hand sides of Eqs. (2.3) and (2.4), we have

$$u - \Phi = L^{-1}g - L^{-1}Ru - L^{-1}Nu. \tag{2.5}$$

By adding the homogenous term Φ, with respect to the operator L, to both sides of Eq. (2.5), we have

$$u = \Phi + L^{-1}g - L^{-1}Ru - L^{-1}Nu, \tag{2.6}$$

or

$$u = \gamma - L^{-1}Ru - L^{-1}Nu, \tag{2.7}$$

where we have defined the sum of the a priori known terms as $\gamma = \Phi + L^{-1}g$, which is the equivalent nonlinear Volterra integral equation for the solution for either IVPs or BVPs depending on how we evaluate the constants of integration.

In the ADM, the solution $u(x)$ is represented by the decomposition series and the nonlinearity $Nu(x)$ is represented by the series of the Adomian polynomials that are tailored to the particular nonlinear function

$$u = \sum_{n=0}^{\infty} u_n \text{ and } Nu = \sum_{n=0}^{\infty} A_n, \tag{2.8}$$

respectively, where the Adomian polynomials are dependent upon the solution components from $u_0(x)$ through $u_n(x)$, inclusively, $A_n(x) = A_n(u_0(x), \ldots, u_n(x))$. Adomian and Rach [1] published the definitional formula for the Adomian polynomials in 1983 for the simple nonlinearity $Nu = f(x,u)$, or one-variable nonlinearity with regard to only independent variables, and where f is assumed to be analytic, as

$$A_n(x) = \frac{1}{n!} \frac{\partial^n}{\partial \lambda^n} f(x, u(x;\lambda)) \bigg|_{\lambda=0}, \tag{2.9}$$

where λ is a grouping parameter of convenience, and similarly for more complex nonlinearities. We observe that only the dependent variables, such as the solution $u(x)$ and its derivatives, are parametrized while the independent variables such as x are not parametrized.

In order to compute the Adomian polynomials in terms of explicit differentiations, we begin with the following definitions

$$u(x;\lambda) = \sum_{n=0}^{\infty} \lambda^n u_n(x), \tag{2.10}$$

$$f(x, u(x;\lambda)) = \sum_{n=0}^{\infty} \lambda^n A_n(x) \tag{2.11}$$

and

$$n! u_n(x) = \frac{\partial^n}{\partial \lambda^n} u(x;\lambda) \bigg|_{\lambda=0}. \tag{2.12}$$

We list the first several Adomian polynomials for the general analytic nonlinearity Nu from A_0 through A_5, inclusively, for convenient reference,

$$
\begin{aligned}
A_0(x) &= f(x,u_0(x)),\\
A_1(x) &= u_1(x)\frac{\partial}{\partial u_0}f(x,u_0(x)),\\
A_2(x) &= u_2(x)\frac{\partial}{\partial u_0}f(x,u_0(x)) + \frac{u_1^2(x)}{2!}\frac{\partial^2}{\partial u_0^2}f(x,u_0(x)),\\
A_3(x) &= u_3(x)\frac{\partial}{\partial u_0}f(x,u_0(x)) + u_1(x)u_2(x)\frac{\partial^2}{\partial u_0^2}f(x,u_0(x)) + \frac{u_1^3(x)}{3!}\frac{\partial^3}{\partial u_0^3}f(x,u_0(x)),\\
A_4(x) &= u_4(x)\frac{\partial}{\partial u_0}f(x,u_0(x)) + \left(\frac{u_2^2(x)}{2!} + u_1(x)u_3(x)\right)\frac{\partial^2}{\partial u_0^2}f(x,u_0(x)) + \frac{u_1^2(x)}{2!}u_2(x)\frac{\partial^3}{\partial u_0^3}f(x,u_0(x)) + \frac{u_1^4}{4!}\frac{\partial^4}{\partial u_0^4}f(x,u_0(x)),\\
A_5(x) &= u_5(x)\frac{\partial}{\partial u_0}f(x,u_0(x)) + (u_2(x)u_3(x) + u_1(x)u_4(x))\frac{\partial^2}{\partial u_0^2}f(x,u_0(x)) + \left(u_1(x)\frac{u_2^2(x)}{2!} + \frac{u_1^2(x)}{2!}u_3(x)\right)\frac{\partial^3}{\partial u_0^3}f(x,u_0(x))\\
&\quad + \frac{u_1^3(x)}{3!}u_2(x)\frac{\partial^4}{\partial u_0^4}f(x,u_0(x)) + \frac{u_1^5(x)}{5!}\frac{\partial^5}{\partial u_0^5}f(x,u_0(x)).
\end{aligned}
$$

We observe that

$$A_0 = f(x,u_0),\ \text{and}\ A_n = \sum_{k=1}^{n} C_n^k f^{(k)}(x,u_0), \quad n \geqslant 1. \tag{2.13}$$

Several convenient algorithms to readily generate the Adomian polynomials have been developed by Adomian and Rach [1,47], Rach [63,64], Wazwaz [11,65], Abdelwahid [66], and several others [67–70]. Recently Duan [59,71,72,61] has developed several new algorithms and subroutines for fast generation of the one-variable and the multi-variable Adomian polynomials.

For fast computer generation, we especially recommend Duan's new Corollary 3 algorithm [72] to generate the coefficients C_n^k and hence the Adomian polynomials quickly and to high orders as

$$
\begin{aligned}
&C_n^1 = u_n,\ \text{for}\ n \geqslant 1,\ \text{and}\\
&C_n^k = \tfrac{1}{n}\sum_{j=0}^{n-k}(j+1)u_{j+1}C_{n-1-j}^{k-1},\ \text{for}\ 2 \leqslant k \leqslant n.
\end{aligned} \tag{2.14}
$$

We favor Duan's Corollary 3 algorithm [72] as it does not involve the differentiation operator, but only requires the analytic operations of addition and multiplication, which is eminently convenient for symbolic implementation by MATHEMATICA, MAPLE or MATLAB as well as for debugging. Furthermore it has been timed to be one of the fastest subroutines on record using a commercially available laptop computer. We list the MATHEMATICA code for this efficient algorithm in Appendix A for convenient reference.

Upon substitution of the Adomian decomposition series for the solution $u(x)$ and the series of Adomian polynomials tailored to the nonlinearity $Nu(x)$ from Eq. (2.8) into Eq. (2.7), we have

$$\sum_{n=0}^{\infty} u_n = \gamma - L^{-1}R\sum_{n=0}^{\infty} u_n - L^{-1}\sum_{n=0}^{\infty} A_n. \tag{2.15}$$

We note that the solution components $u_n(x)$ may be determined by one of several advantageous recursion schemes, which differ from one another by the choice of the initial solution component $u_0(x)$, beginning with the classic Adomian recursion scheme

$$
\begin{aligned}
&u_0(x) = \gamma(x),\\
&u_{n+1}(x) = -L^{-1}Ru_n - L^{-1}A_n, \quad n \geqslant 0,
\end{aligned} \tag{2.16}
$$

where Adomian has chosen the initial solution component as $u_0(x) = \gamma(x)$. By various partitions of the original initial term and then delaying the contribution of its remainder by different algorithms, we can design alternate recursion schemes, such as the Adomian–Rach [26,27], Wazwaz [57], Wazwaz–El-Sayed [58], Duan [59], and Duan–Rach [62,15] modified recursion schemes for different computational advantages.

The new Duan–Rach modified decomposition method (MDM) [15] for solving BVPs relies upon the effective decomposition of the constants of integration, i.e. the matching coefficients β_ν, which features the equivalent nonlinear Fredholm–Volterra integral equation to compute the approximants. The Duan–Rach MDM [15] is almost the same as the ADM except where we replace the classic Adomian recursion scheme with the Duan–Rach modified recursion scheme [15]. In the Duan–Rach MDM [15] we first choose the lower limits of integration for the inverse linear operator L^{-1} so as to incorporate as many of the boundary conditions as possible in order to reduce the number of the unknown constants of integration. For special cases this approach can yield an equivalent nonlinear Volterra integral equation in practice. Otherwise we have at least reduced the number of undetermined coefficients. Then the second feature is that we effectively decompose any remaining constants of integration, i.e. the undetermined coefficients, by appropriate algebraic manipulations and differentiations in order to obtain a system of n-coupled linearly independent equations in n unknowns, where $n < p$ thanks to our first feature. Upon substitution, we have obtained an equivalent nonlinear Fredholm–Volterra integral equation without any undetermined coefficients. Next we substitute the Adomian decomposition series and the series of the Adomian polynomials into the equivalent nonlinear Fredholm–Volterra integral equation for the solution, which, in effect, decomposes the remaining unknown constants of integration.

The Duan–Rach MDM [15] determines the remaining unknown constants of integration before designing an appropriate modified recursion scheme, whereas the Adomian–Rach MDM [26,27] first decomposes the constants of integration and then designs an appropriate modified recursion scheme. In both methods, the components of the constants of integration are computed simultaneously along with the solution components, which is not the case with the method of undetermined coefficients, where the solution approximants are first computed as functions parametrized by the unknown constants of integration, and then the approximate constants of integration are evaluated as the appropriate roots of a system of coupled linearly independent equations. Furthermore we may combine our new modification of the ADM with any one of the Wazwaz modified recursion scheme [57], the Wazwaz–El-Sayed modified recursion scheme [58] or Duan's parametrized recursion scheme [59,62,15].

For a comprehensive bibliography featuring many new engineering applications and a modern review of the ADM see [73,74].

3. Description of our new approach

We consider a second-order nonlinear differential equation of the form

$$\frac{d^2}{dx^2}u(x) - f(u(x)) = 0, \quad a \leqslant x \leqslant b, \tag{3.1}$$

subject to a set of Robin boundary conditions

$$pu(a) + ru'(a) = \alpha, \tag{3.2}$$

$$qu(b) + su'(b) = \beta, \tag{3.3}$$

where $f(u(x))$ is an analytic nonlinearity and p, q, r, s satisfy

$$ps - qr + pq(b-a) \neq 0. \tag{3.4}$$

We note that if $p, q, s \geqslant 0, r \leqslant 0, p, r$ are not all zeroes, q, s are not all zeroes, and p, q are not all zeroes, then we have $ps - qr + pq(b-a) > 0$.

We will treat more general cases for values of p, q, r, s that are not limited by Eq. (3.4), such as when $p = q = 0$, i.e. the Neumann boundary conditions, by the multistage ADM for BVPs, which is introduced later in this section.

We rewrite Eq. (3.1) in Adomian's operator-theoretic form

$$Lu = Nu, \quad a \leqslant x \leqslant b, \tag{3.5}$$

where $L(\cdot) = \frac{d^2}{dx^2}(\cdot)$ is the linear differential operator to be inverted and $Nu = f(u(x))$.

We consider the specific definite integral operators $L_{a,a}^{-1}$, which is defined as

$$L_{a,a}^{-1} = \int_a^x \int_a^x (\cdot) dx dx. \tag{3.6}$$

Applying the operator $L_{a,a}^{-1}$ to both sides of Eq. (3.5) yields

$$u(x) - u(a) - (x-a)u'(a) = L_{a,a}^{-1} Nu. \tag{3.7}$$

Using Eq. (3.7), we evaluate $u(x)$ at $x = b$ to obtain

$$u(b) = u(a) + (b-a)u'(a) + [L_{a,a}^{-1}Nu]_{x=b}, \tag{3.8}$$

where

$$[L_{a,a}^{-1}(\cdot)]_{x=b} = \int_a^b \int_a^x (\cdot) dx dx.$$

Differentiating (3.7) and then evaluating $u'(x)$ at $x = b$ yields

$$u'(b) = u'(a) + \int_a^b Nu dx. \tag{3.9}$$

Substituting Eqs. (3.8) and (3.9) into Eq. (3.3), we obtain

$$qu(a) + (q(b-a)+s)u'(a) = \beta - q[L_{a,a}^{-1}Nu]_{x=b} - s\int_a^b Nudx. \tag{3.10}$$

Eqs. (3.2) and (3.10) constitute a system of two linearly independent equations in the two remaining undetermined coefficients $u(a)$ and $u'(a)$, where the coefficient determinant is

$$\Delta = \begin{vmatrix} p & r \\ q & q(b-a)+s \end{vmatrix} = ps - qr + pq(b-a), \tag{3.11}$$

which is nonzero by our assumption (3.4).

Thus upon appropriate algebraic manipulation, we have derived $u(a)$ and $u'(a)$ in terms of the specified values of the system parameters $\alpha, \beta, a, b, p, q, r$ and s as

$$u(a) = \frac{1}{\Delta}\left[q\alpha(b-a) + s\alpha - r\beta + qr[L_{a,a}^{-1}Nu]_{x=b} + rs\int_a^b Nudx\right], \tag{3.12}$$

$$u'(a) = \frac{1}{\Delta}\left[p\beta - q\alpha - pq[L_{a,a}^{-1}Nu]_{x=b} - ps\int_a^b Nudx\right]. \tag{3.13}$$

Substituting Eqs. (3.12) and (3.13) into Eq. (3.7) yields

$$u(x) = \frac{1}{\Delta}[s\alpha - r\beta + p\beta(x-a) + q\alpha(b-x)] + L_{a,a}^{-1}Nu - \frac{p(x-a)-r}{\Delta}\left[q[L_{a,a}^{-1}Nu]_{x=b} + s\int_a^b Nudx\right], \tag{3.14}$$

which now is free of any undetermined coefficients. We emphasize that Eq. (3.14) is an equivalent nonlinear Fredholm–Volterra integral equation for the solution of the second-order nonlinear differential Eq. (3.1) subject to the Robin boundary conditions (3.2) and (3.3).

We next apply the decomposition of the solution $u(x)$ and the nonlinearity $Nu(x)$,

$$u(x) = \sum_{n=0}^{\infty} u_n(x), \quad Nu(x) = \sum_{n=0}^{\infty} A_n(x), \tag{3.15}$$

respectively, where the $A_n(x)$ are the Adomian polynomials.

Inserting Eqs. in (3.15) into Eq. (3.14), we obtain the solution components as determined by the modified recursion scheme,

$$u_0(x) = \frac{1}{\Delta}[s\alpha - r\beta + p\beta(x-a) + q\alpha(b-x)], \tag{3.16}$$

$$u_n(x) = L_{a,a}^{-1}A_{n-1} - \frac{p(x-a)-r}{\Delta}\left[q[L_{a,a}^{-1}A_{n-1}]_{x=b} + s\int_a^b A_{n-1}dx\right], \quad n \geqslant 1, \tag{3.17}$$

where, of course, we suppose the resulting integrals exist.

The mth-stage approximation obtained by the ADM is the truncated decomposition series

$$\varphi_m(x) = \sum_{n=0}^{m-1} u_n(x). \tag{3.18}$$

We have checked that each approximation $\varphi_m(x), m \geqslant 1$, exactly satisfies the boundary conditions (3.2) and (3.3) by our design.

We note that Adomian and Rach [7,26,27,75] proposed a modified recursion scheme of the ADM, alias the double decomposition method, for nonlinear second-order BVPs for the purpose of determining the constants of integration. Other techniques for solving BVPs by using the ADM, such as the method of undetermined coefficients, were previously considered by Wazwaz and others in [11,28,29,76–80,33,34]. Duan and Rach [15] presented a new modified recursion scheme of the ADM for nonlinear higher order BVPs, where a convergence parameter [59,62,15] can be embedded in the solution components, thus developing a parametrized recursion scheme to extend the effective interval of convergence.

Furthermore we consider the multistage form of our modified ADM for BVPs. Consider the nonlinear BVP (3.1)–(3.3), but where the condition (3.4) does not hold, instead we suppose that p, r are not all zeroes and q, s are not all zeroes. In particular, we consider the case of the Neumann boundary conditions, i.e. $p = q = 0, r \neq 0, s \neq 0$.

We insert $N-1$ points in the specified interval $[a, b]$,

$$a = x_0 < x_1 < \ldots < x_i < \ldots < x_{N-1} < x_N = b.$$

Suppose the values of the solution at the interior points are $u(x_i) = \eta_i$, for $i = 1, 2, \ldots, N-1$, which represent $N-1$ undetermined coefficients.

On the left hand subinterval $[a, x_1]$, we solve the nonlinear BVP with a mixed set of Robin and Dirichlet boundary conditions

$$Lu = Nu, \quad a \leqslant x \leqslant x_1, \tag{3.19}$$

$$pu(a) + ru'(a) = \alpha, \quad u(x_1) = \eta_1 \tag{3.20}$$

and denote the mth-stage approximation as

$$\varphi_m^{(1)}(x) = \varphi_m^{(1)}(x; \eta_1) = \sum_{k=0}^{m-1} u_k^{(1)}(x).$$

On the interior subintervals $[x_{i-1}, x_i], i = 2, 3, \ldots, N-1$, we solve the nonlinear BVP with a set of Dirichlet boundary conditions

$$Lu = Nu, \quad x_{i-1} \leqslant x \leqslant x_i, \tag{3.21}$$

$$u(x_{i-1}) = \eta_{i-1}, \quad u(x_i) = \eta_i \tag{3.22}$$

and denote the mth-stage approximation as

$$\varphi_m^{(i)}(x) = \varphi_m^{(i)}(x; \eta_{i-1}, \eta_i) = \sum_{k=0}^{m-1} u_k^{(i)}(x), \quad i = 2, 3, \ldots, N-1.$$

On the right hand subinterval $[x_{N-1}, b]$, we solve the nonlinear BVP with a mixed set of Dirichlet and Robin boundary conditions

$$Lu = Nu, \quad x_{N-1} \leqslant x \leqslant b, \tag{3.23}$$

$$u(x_{N-1}) = \eta_{N-1}, \quad qu(b) + su'(b) = \beta \tag{3.24}$$

and denote the mth-stage approximation as

$$\varphi_m^{(N)}(x) = \varphi_m^{(N)}(x; \eta_{N-1}) = \sum_{k=0}^{m-1} u_k^{(N)}(x).$$

Matching the N approximations $\varphi_m^{(i)}(x),\ i = 1, 2, \ldots, N$ at the interior points by the continuity condition for the flux,

$$\frac{d\varphi_m^{(i)}}{dx}(x_i) = \frac{d\varphi_m^{(i+1)}}{dx}(x_i), \quad i = 1, 2, \ldots, N-1, \tag{3.25}$$

determines the $N-1$ coefficients $\eta_1, \eta_2, \ldots, \eta_{N-1}$. If, as m increases, there exist solutions for the coefficients $\eta_1, \eta_2, \ldots, \eta_{N-1}$ approaching fixed values, we take them as the approximants for the η_k.

We denote by $\eta_k[\varphi_m]$ the approximate value of η_k obtained by matching $\varphi_m^{(i)}(x),\ i = 1, 2, \ldots, N$, according to Eqs. (3.25). Combining $\varphi_m^{(1)}(x; \eta_1[\varphi_m]), \varphi_m^{(2)}(x; \eta_1[\varphi_m], \eta_2[\varphi_m]), \ldots, \varphi_m^{(N)}(x; \eta_{N-1}[\varphi_m])$ yields the matched mth-stage approximation of the solution on the entire interval $[a, b]$. For our examples in Section 5, we find that two subintervals, i.e. $N = 2$, are sufficient.

We can use the boxcar function to express the mth-stage solution approximant as

$$\varphi_m(x) = \sum_{i=1}^{N} \varphi_m^{(i)}(x) \Pi(x; x_{i-1}, x_i), \tag{3.26}$$

where the boxcar function may be defined in terms of the Heaviside unit step function [81]

$$\Pi(x; c, d) = H(x; c) - H(x; d) \tag{3.27}$$

and where

$$H(x; h) = \begin{cases} 0 & x < h, \\ 1 & x \geqslant h. \end{cases} \tag{3.28}$$

For convenient reference in solving various sub-BVPs in our examples using the multistage ADM for BVPs, we list the following special cases of the boundary conditions (3.2) and (3.3) and their corresponding equivalent nonlinear Fredholm–Volterra integral equations.

Case 1. The Dirichlet boundary conditions

$$u(a) = \alpha, \quad u(b) = \beta, \tag{3.29}$$

correspond to the case of $p = q = 1, r = s = 0$ in Eqs. (3.2) and (3.3). Hence we have $\Delta = b - a$, and Eq. (3.14) becomes

$$u(x) = \frac{\beta(x-a) + \alpha(b-x)}{b-a} + L_{a,a}^{-1}Nu(x) - \frac{x-a}{b-a}[L_{a,a}^{-1}Nu(x)]_{x=b}. \tag{3.30}$$

Case 2. The mixed set of Robin and Dirichlet boundary conditions

$$pu(a) + ru'(a) = \alpha, \quad u(b) = \beta, \tag{3.31}$$

corresponds to the case of $q = 1, s = 0$ in Eqs. (3.2) and (3.3). Hence we have $\Delta = p(b-a) - r$, and Eq. (3.14) becomes

$$u(x) = \frac{\alpha(b-x) + p\beta(x-a) - r\beta}{p(b-a) - r} + L_{a,a}^{-1}Nu(x) - \frac{p(x-a) - r}{p(b-a) - r}[L_{a,a}^{-1}Nu(x)]_{x=b}. \tag{3.32}$$

Case 3. The mixed set of Robin and Neumann boundary conditions

$$pu(a) + ru'(a) = \alpha, \quad u'(b) = \beta, \tag{3.33}$$

corresponds to the case of $q = 0, s = 1$ in Eqs. (3.2) and (3.3). Hence we have $\Delta = p$, and Eq. (3.14) becomes

$$u(x) = \beta(x-a) + \frac{\alpha - r\beta}{p} + L_{a,a}^{-1}Nu - \frac{p(x-a) - r}{p}\int_a^b Nudx. \tag{3.34}$$

We shall present several other special cases in the next section.

4. Other inverse linear operators

In this section we also consider the inverse linear operators

$$L_{b,b}^{-1}(\cdot) = \int_b^x \int_b^x (\cdot)dxdx, \tag{4.1}$$

$$L_{a,b}^{-1}(\cdot) = \int_a^x \int_b^x (\cdot)dxdx \tag{4.2}$$

and

$$L_{b,a}^{-1}(\cdot) = \int_b^x \int_a^x (\cdot)dxdx. \tag{4.3}$$

We suppose that the following resulting integrals exist.

Applying the operator $L_{b,b}^{-1}(\cdot)$ to Eq. (3.5) we have

$$u - u(b) - (x-b)u'(b) = L_{b,b}^{-1}Nu. \tag{4.4}$$

Using a similar method as in Section 3, we obtain

$$u(x) = \frac{1}{\Delta}[s\alpha - r\beta + p\beta(x-a) + q\alpha(b-x)] + L_{b,b}^{-1}Nu + \frac{q(x-b) - s}{\Delta}\left[p[L_{b,b}^{-1}Nu]_{x=a} + r\int_b^a Nudx\right], \tag{4.5}$$

from which we obtain the modified recursion scheme corresponding to (3.16) and (3.17)

$$u_0(x) = \frac{1}{\Delta}[s\alpha - r\beta + p\beta(x-a) + q\alpha(b-x)], \tag{4.6}$$

$$u_n(x) = L_{b,b}^{-1}A_{n-1} + \frac{q(x-b) - s}{\Delta}\left[p[L_{b,b}^{-1}A_{n-1}]_{x=a} + r\int_b^a A_{n-1}dx\right], \quad n \geqslant 1. \tag{4.7}$$

Applying the operator $L_{a,b}^{-1}(\cdot)$ to Eq. (3.5) we have

$$u - u(a) - (x-a)u'(b) = L_{a,b}^{-1}Nu. \tag{4.8}$$

Letting $x = b$, we have

$$u(b) = u(a) + (b-a)u'(b) + [L_{a,b}^{-1}Nu]_{x=b}. \tag{4.9}$$

Differentiating Eq. (4.8) and then letting $x = a$, we obtain

$$u'(a) = u'(b) + \int_b^a Nudx. \tag{4.10}$$

Substituting Eqs. (4.9) and (4.10) into the boundary conditions (3.2),(3.3) we obtain

$$pu(a) + ru'(b) = \alpha - r\int_b^a Nudx, \tag{4.11}$$

$$qu(a) + (s + q(b-a))u'(b) = \beta - qL_{a,b}^{-1}Nu|_{x=b}. \tag{4.12}$$

Solving for $u(a)$ and $u'(b)$, and then inserting their results into Eq. (4.8), we obtain the equivalent nonlinear Fredholm–Volterra integral equation as

$$u(x) = \frac{1}{\Delta}[s\alpha - r\beta + p\beta(x-a) + q\alpha(b-x)] + L_{a,b}^{-1}Nu - \frac{pq(x-a)-qr}{\Delta}[L_{a,b}^{-1}Nu]_{x=b} - \frac{rs+qr(b-x)}{\Delta}\int_b^a Nudx. \tag{4.13}$$

From Eq. (4.13), we obtain the modified recursion scheme corresponding to (3.16) and (3.17) as

$$u_0(x) = \frac{1}{\Delta}[s\alpha - r\beta + p\beta(x-a) + q\alpha(b-x)], \tag{4.14}$$

$$u_n(x) = L_{a,b}^{-1}A_{n-1} - \frac{1}{\Delta}[q(p(x-a)-r)[L_{a,b}^{-1}A_{n-1}]_{x=b} + r(q(b-x)+s)\int_b^a A_{n-1}dx], \quad n \geqslant 1. \tag{4.15}$$

Applying the operator $L_{b,a}^{-1}(\cdot)$ to Eq. (3.5), we have

$$u - u(b) - (x-b)u'(a) = L_{b,a}^{-1}Nu. \tag{4.16}$$

Using a similar method as for operator $L_{a,b}^{-1}(\cdot)$, we obtain the equivalent nonlinear Fredholm–Volterra integral equation

$$u(x) = \frac{1}{\Delta}[s\alpha - r\beta + p\beta(x-a) + q\alpha(b-x)] + L_{b,a}^{-1}Nu + \frac{pq(x-b)-ps}{\Delta}[L_{b,a}^{-1}Nu]_{x=a} + \frac{ps(a-x)+rs}{\Delta}\int_a^b Nudx. \tag{4.17}$$

From Eq. (4.17) we obtain the modified recursion scheme corresponding to (3.16) and (3.17) as

$$u_0(x) = \frac{1}{\Delta}[s\alpha - r\beta + p\beta(x-a) + q\alpha(b-x)], \tag{4.18}$$

$$u_n(x) = L_{b,a}^{-1}A_{n-1} + \frac{1}{\Delta}\left[p(q(x-b)-s)[L_{b,a}^{-1}A_{n-1}]_{x=a} + s(p(a-x)+r)\int_a^b A_{n-1}dx\right], \quad n \geqslant 1. \tag{4.19}$$

We find that the initial components in the above four recursion schemes are the same, and by the exchanging $a \leftrightarrow b, p \leftrightarrow q$, $r \leftrightarrow s$ and $\Delta \leftrightarrow -\Delta$, we find that the Eqs. (3.17) and (4.7) and Eqs. (4.15) and (4.19) are symmetric.

Furthermore, we have checked that the solution components obtained by the four recursion schemes are identical. Here we just outline this derivation for the operators $L_{a,b}^{-1}$ and $L_{b,a}^{-1}$.

We denote

$$h^{[1]}(x) = \int A_{n-1}dx, \tag{4.20}$$

$$h^{[2]}(x) = \int h^{[1]}(x)dx, \tag{4.21}$$

where the right hand sides denote pure integration, i.e. without constants of integration. From Eq. (4.15) we have

$$\begin{aligned} u_n(x) = {} & h^{[2]}(x) - h^{[2]}(a) - h^{[1]}(b)(x-a) - \frac{1}{\Delta}[(pq(x-a)-qr)(h^{[2]}(b) - h^{[2]}(a) - h^{[1]}(b)(b-a)) + (qr(b-x)+rs) \\ & \times (h^{[1]}(a) - h^{[1]}(b))]. \end{aligned} \tag{4.22}$$

From Eq. (4.19) we have

$$\begin{aligned} \bar{u}_n(x) = {} & h^{[2]}(x) - h^{[2]}(b) - h^{[1]}(a)(x-b) + \frac{1}{\Delta}[(pq(x-b)-ps)(h^{[2]}(a) - h^{[2]}(b) - h^{[1]}(a)(a-b)) + (ps(a-x)+rs) \\ & \times (h^{[1]}(b) - h^{[1]}(a))], \end{aligned} \tag{4.23}$$

where we use a bar to distinguish between the two solutions. By direct verification of Eqs. (4.22) and (4.23), we have the identity $u_n(x) - \bar{u}_n(x) = 0$.

We list the other special cases of these boundary conditions using the inverse linear operators in this section.

Case 4. The mixed set of Dirichlet and Robin boundary conditions

$$u(a) = \alpha, \quad qu(b) + su'(b) = \beta, \tag{4.24}$$

corresponds to the case of $p = 1, r = 0$ in Eqs. (3.2) and (3.3). Hence we have $\Delta = q(b - a) + s$, and from Eq. (4.5) we derive the equivalent nonlinear Fredholm–Volterra integral equation

$$u(x) = \frac{\beta(a - x) + q\alpha(x - b) - s\alpha}{q(a - b) - s} + L_{b,b}^{-1}Nu(x) - \frac{q(x - b) - s}{q(a - b) - s}[L_{b,b}^{-1}Nu(x)]_{x=a}. \tag{4.25}$$

Case 5. The mixed set of Neumann and Robin boundary conditions

$$u'(a) = \alpha, \quad qu(b) + su'(b) = \beta, \tag{4.26}$$

corresponds to the case of $p = 0, r = 1$ in Eqs. (3.2) and (3.3). Hence we have $\Delta = -q$, and Eq. (4.5) becomes the equivalent nonlinear Fredholm–Volterra integral equation

$$u(x) = \alpha(x - b) + \frac{\beta - s\alpha}{q} + L_{b,b}^{-1}Nu - \frac{q(x - b) - s}{q}\int_b^a Nudx. \tag{4.27}$$

Case 6. The mixed set of Dirichlet and Neumann boundary conditions

$$u(a) = \alpha, \quad u'(b) = \beta, \tag{4.28}$$

corresponds to the case of $p = s = 1, q = r = 0$ in Eqs. (3.2) and (3.3). Hence we have the equivalent nonlinear Volterra integral equation as

$$u(x) = \alpha + \beta(x - a) + L_{a,b}^{-1}Nu(x), \tag{4.29}$$

from Eq. (4.8).

Case 7. The mixed set of Neumann and Dirichlet boundary conditions

$$u'(a) = \alpha, \quad u(b) = \beta, \tag{4.30}$$

corresponds to the case of $p = s = 0, q = r = 1$ in Eqs. (3.2) and (3.3). Hence we have the equivalent nonlinear Volterra integral equation as

$$u(x) = \beta + \alpha(x - b) + L_{b,a}^{-1}Nu(x), \tag{4.31}$$

from Eq. (4.16).

5. Numerical examples

In the computation of the Adomian polynomials we will use the following properties.

$$A_n[h(x)Nu] = h(x)A_n[Nu], \tag{5.1}$$

$$A_n[N_1u + N_2u] = A_n[N_1u] + A_n[N_2u]. \tag{5.2}$$

Example 1. Consider the linear BVP with variable coefficients and subject to a set of Robin boundary conditions

$$u'' = \frac{1}{1 + x}u + \frac{x}{1 + x}u', \quad 0 \leqslant x \leqslant 1, \tag{5.3}$$

$$u(0) - 2u'(0) = -1, \quad u(1) + 2u'(1) = 3e. \tag{5.4}$$

The exact solution of the BVP is $u^*(x) = e^x$. For this BVP, we have $a = 0, b = 1, \alpha = -1, \beta = 3e, p = 1, r = -2, q = 1, s = 2$, and $\Delta = 5$. Thus Eq. (3.14) becomes

$$u(x) = \frac{1}{5}\left[6e - 3 + (1 + 3e)x - (2 + x)[L_{0,0}^{-1}Nu]_{x=1} - (4 + 2x)\int_0^1 Nudx\right] + L_{0,0}^{-1}Nu.$$

For this example Nu degenerates to the sum of linear terms $Nu = \frac{1}{1+x}u + \frac{x}{1+x}u'$. Thus in the linear case, the Adomian polynomials are

$$A_n = \frac{1}{1+x}u_n + \frac{x}{1+x}u_n', \quad n = 0,1,2,\ldots.$$

From the recursion scheme (3.16), (3.17) we have

$$u_0(x) = \frac{1}{5}[6e - 3 + (1+3e)x], \tag{5.5}$$

$$u_n(x) = \frac{1}{5}\left[-(2+x)[L_{0,0}^{-1}A_{n-1}]_{x=1} - (4+2x)\int_0^1 A_{n-1}dx\right] + L_{0,0}^{-1}A_{n-1}, \quad n \geqslant 1. \tag{5.6}$$

The solution components are computed as

$$u_1 = \left(\frac{1}{5} + \frac{3e}{5}\right)x^2 + x\left(-\log(x+1) - \frac{3e}{5} + \frac{3}{5} - \frac{\log(4)}{5} + \frac{\log(64)}{5}\right) - \log(x+1) - \frac{6e}{5} - \frac{4}{5} + \frac{\log(256)}{5},$$

$$\begin{aligned} u_2 = {} & \left(\frac{1}{10} + \frac{3e}{10}\right)x^3 + x^2\left(-\log(x+1) - \frac{3e}{2} + \frac{13}{10} + \frac{\log(256)}{10}\right) \\ & + x\left(\frac{1}{2}\log^2(x+1) - \frac{17}{5}\log(x+1) + \frac{9}{5}e\log(x+1) - \frac{9e}{25} + \frac{22}{25} - \frac{1}{5}3\log^2(2) \right. \\ & \left. + \frac{1}{25}\log(32)\log(2) - \frac{32\log(2)}{25} + \frac{3\log(64)}{5} - \frac{3}{25}e\log(4096)\right) \\ & + \frac{1}{2}\log^2(x+1) - \frac{12}{5}\log(x+1) + \frac{9}{5}e\log(x+1) - \frac{76}{25} + \frac{72e}{25} - \frac{6\log^2(2)}{5} \\ & + \frac{2}{25}\log(2)\log(32) + \frac{6\log(64)}{5} - \frac{6}{25}e\log(4096) - \frac{64\log(2)}{25}, \end{aligned}$$

$\cdots$

We consider the maximal error parameters

$$ME_n = \max_{0\leqslant x\leqslant 1}|\varphi_n(x) - u^*(x)| \tag{5.7}$$

for the nth-stage solution approximation $\varphi_n(x) = \sum_{k=0}^{n-1}u_k$ by using the MATHEMATICA command 'NMaximize'.

In Table 1 we list the maximal error parameters ME_n for $n = 1, 2, \ldots, 12$. The logarithmic plots for these ME_n are shown in Fig. 1, which demonstrates an exponential rate of convergence.

We plot the curves of the exact solution $u^*(x) = e^x$ and the nth-stage approximations $\varphi_n(x)$ for $n = 1$ through $n = 6$ in Fig. 2, and the curves of the exact solution $u^*(x) = e^x$ and the nth-stage approximations $\varphi_n(x)$ for $n = 7$ through $n = 12$ in Fig. 3 in different magnifications of the vertical scale. We observe in Figs. 2 and 3 that the sequences of the even and odd numbered approximations bound the exact solution and thus their averages can yield a better approximation.

Table 1
The maximal error parameters ME_n for $n = 1, 2, \ldots, 12$.

n	1	2	3	4	5	6	7	8	9	10	11	12
ME_n	1.93842	1.41912	0.985009	0.677592	0.465683	0.320021	0.21992	0.15113	0.103857	0.0713709	0.0490464	0.0337049

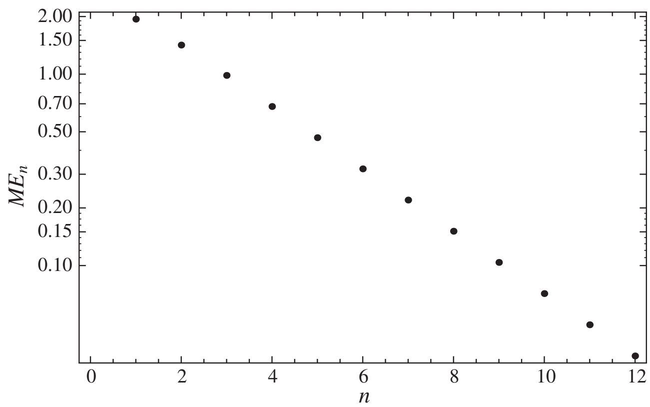

Fig. 1. Logarithmic plots of ME_n for $n = 1$ through 12.

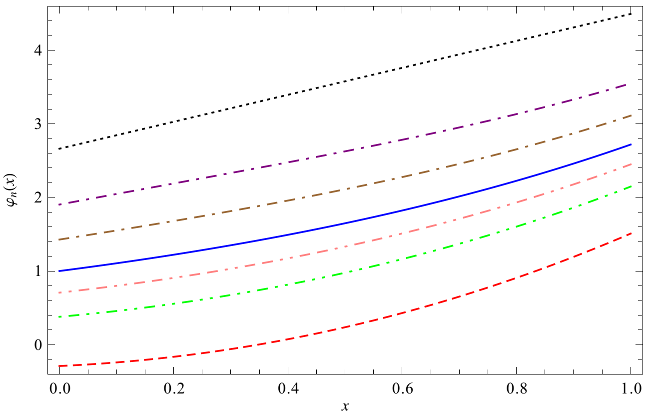

Fig. 2. Exact solution $u^*(x)$ (solid line) and approximations $\varphi_n(x)$ for $n = 1$ (dot line), $n = 2$ (dashed line), $n = 3$ (dot-side line), $n = 4$ (dot-dot-side line), $n = 5$ (dot-side-side line), and $n = 6$ (dot-dot-side-side line).

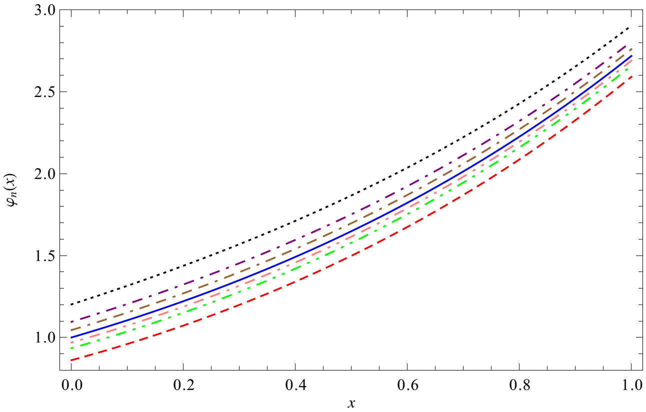

Fig. 3. Exact solution $u^*(x)$ (solid line) and approximations $\varphi_n(x)$ for $n = 7$ (dot line), $n = 8$ (dashed line), $n = 9$ (dot-side line), $n = 10$ (dot-dot-side line), $n = 11$ (dot-side-side line), and $n = 12$ (dot-dot-side-side line).

Example 2. Consider the nonlinear BVP with an exponential nonlinearity in the solution and a quadratic nonlinearity in the derivative and subject to a set of Robin boundary conditions

$$u'' = -\frac{1}{8}(e^{-2u} + 4(u')^2), \quad 0 \leqslant x \leqslant 1, \tag{5.8}$$

$$u(0) - 2u'(0) = -1, \quad u(1) + 2u'(1) = \frac{2}{3} + \log\frac{3}{2}, \tag{5.9}$$

The exact solution of the BVP is $u^*(x) = \log\frac{2+x}{2}$. For this BVP we have $a = 0, b = 1, \alpha = -1, \beta = \frac{2}{3} + \log\frac{3}{2}, p = 1, r = -2, q = 1, s = 1$, and $\Delta = 5$. Eq. (3.14) becomes

$$u(x) = -\frac{1}{3} + \frac{2}{5}\log\frac{3}{2} + x\left(\frac{1}{3} + \frac{1}{5}\log\frac{3}{2}\right) - \frac{1}{8}L_{0,0}^{-1}Nu + \frac{2+x}{40}[L_{0,0}^{-1}Nu]_{x=1} + \frac{2+x}{20}\int_0^1 Nudx,$$

where $Nu = e^{-2u} + 4(u')^2$.

Decompose the solution and the nonlinearity Nu as $u(x) = \sum_{n=0}^{\infty} u_n(x)$ and $Nu(x) = \sum_{n=0}^{\infty} A_n(x)$, where the Adomian polynomials are

$$\begin{aligned}
A_0 &= e^{-2u_0} + 4(u_0')^2, \\
A_1 &= -2e^{-2u_0}u_1 + 8u_0'u_1', \\
A_2 &= 2e^{-2u_0}u_1^2 - 2e^{-2u_0}u_2 + 4(u_1')^2 + 8u_0'u_2', \\
A_3 &= -\frac{4}{3}e^{-2u_0}u_1^3 + 4e^{-2u_0}u_1u_2 - 2e^{-2u_0}u_3 + 8u_1'u_2' + 8u_0'u_3', \\
&\ldots
\end{aligned}$$

Table 2
The maximal error parameter ME_n for $n = 2$ through 19.

n	2	3	4	5	6	7	8	9	10
ME_n	0.0640962	0.086546	0.0222792	0.00933165	0.00405274	0.00342178	0.00277756	0.00190359	0.00142094
n	11	12	13	14	15	16	17	18	19
ME_n	0.00108823	0.000835303	0.000643241	0.000502041	0.000397089	0.000315939	0.000252689	0.000203294	0.000164427

By the parametrized recursion scheme

$$\begin{aligned}
u_0 &= c,\\
u_1 &= -c - \frac{1}{3} + \frac{2}{5}\log\frac{3}{2} + x\left(\frac{1}{3} + \frac{1}{5}\log\frac{3}{2}\right) - \frac{1}{8}L_{0,0}^{-1}A_0 + \frac{2+x}{40}[L_{0,0}^{-1}A_0]_{x=1} + \frac{2+x}{20}\int_0^1 A_0\,dx,\\
u_n &= -\frac{1}{8}L_{0,0}^{-1}A_{n-1} + \frac{2+x}{40}[L_{0,0}^{-1}A_{n-1}]_{x=1} + \frac{2+x}{20}\int_0^1 A_{n-1}\,dx, \quad n \geqslant 2,
\end{aligned}$$

where u_0 is taken as a constant in order to lead to easy integrations and c is a predetermined parameter, we obtain the solution components. We take $c = 0.06$ and check the approximate solutions $\varphi_n(x) = \sum_{k=0}^{n-1} u_k(x)$ with the help of MATHEMATICA. In Table 2 we list the maximal error parameters $ME_n = \max_{0\leqslant x\leqslant 1}|\varphi_n(x) - u^*(x)|$, for $n = 2$ through 19, which is calculated by using the MATHEMATICA command 'NMaximize'. The logarithmic plots of these values of ME_n are displayed in Fig. 4, which demonstrates, after $n = 5$, an approximate exponential rate of convergence.

In order to observe intuitively the behavior of the approximate solutions we plot the approximations $\varphi_n(x)$ for $n = 2$ through 6 in Fig. 5.

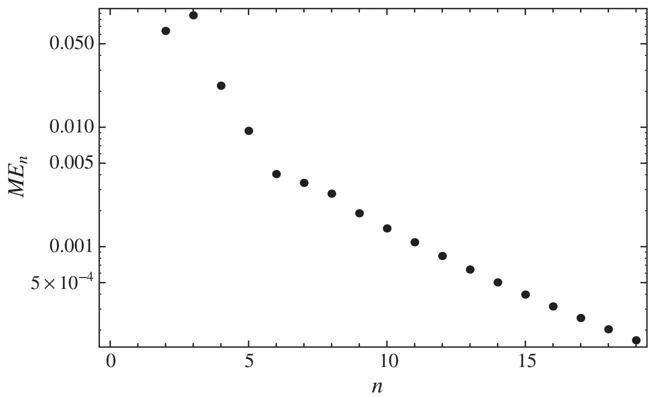

Fig. 4. Logarithmic plots of ME_n for $n = 2$ through 19.

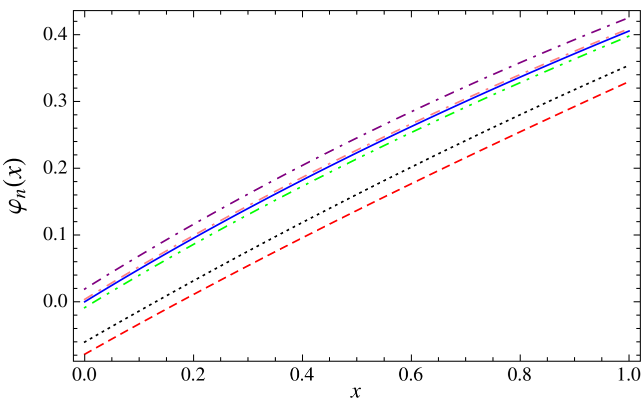

Fig. 5. Exact solution $u^*(x)$ (solid line) and approximations $\varphi_n(x)$ for $n = 2$ (dot line), $n = 3$ (dashed line), $n = 4$ (dot-side line), $n = 5$ (dot-dot-side line), and $n = 6$ (dot-side-side line).

The MATHEMATICA code used in this example is provided in Appendix B. Furthermore we have checked that if we take $c = -0.03, 0, 0.1, 0.2, 0.3$, similar approximate solutions are obtained.

Example 3. Consider the nonlinear BVP with a variable coefficient and subject to a set of Robin boundary conditions

$$u'' = \frac{1}{2}e^{-x}Nu, \quad 0 \leqslant x \leqslant 1, \tag{5.10}$$

$$u(0) - u'(0) = 0, \quad u(1) + u'(1) = 2e, \tag{5.11}$$

where $Nu = u^2 + (u')^2$, i.e. the nonlinear differential equation is quadratic in the solution and quadratic in its derivative.

The exact solution of this nonlinear BVP is $u^*(x) = e^x$. For this BVP we have $a = 0, b = 1, \alpha = 0, \beta = 2e, p = 1, r = -1, q = 1, s = 1$, and $\Delta = 3$. Thus Eq. (3.14) becomes

$$u(x) = \frac{2e + 2ex}{3} - \frac{1+x}{6}[L_{0,0}^{-1}e^{-x}Nu]_{x=1} - \frac{1+x}{6}\int_0^1 e^{-x}Nudx + \frac{1}{2}L_{0,0}^{-1}e^{-x}Nu.$$

We use the following parametrized recursion scheme

$$\begin{aligned}
u_0 &= \frac{2e+2ex}{3} - c_0 - c_1 x, \\
u_1 &= c_0 + c_1 x - \frac{1+x}{6}[L_{0,0}^{-1}e^{-x}A_0]_{x=1} - \frac{1+x}{6}\int_0^1 e^{-x}A_0 dx + \frac{1}{2}L_{0,0}^{-1}e^{-x}A_0, \\
u_n &= -\frac{1+x}{6}[L_{0,0}^{-1}e^{-x}A_{n-1}]_{x=1} - \frac{1+x}{6}\int_0^1 e^{-x}A_{n-1}dx + \frac{1}{2}L_{0,0}^{-1}e^{-x}A_{n-1}, \quad n \geqslant 2,
\end{aligned}$$

where c_0 and c_1 are two predetermined parameters, and the Adomian polynomials for the nonlinearity $Nu = u^2 + (u')^2$ are

$$\begin{aligned}
A_0 &= u_0^2 + (u_0')^2, \\
A_1 &= 2u_0u_1 + 2u_0'u_1', \\
A_2 &= 2u_0u_2 + u_1^2 + 2u_0'u_2' + (u_1')^2, \\
A_3 &= 2u_0u_3 + 2u_1u_2 + 2u_0'u_3' + 2u_1'u_2', \\
&\ldots
\end{aligned}$$

If we take $c_0 = 0.5, c_1 = 1$, we compute the nth-stage approximation $\varphi_n(x) = \sum_{k=0}^{n-1}u_k(x)$ with the help of the MATHEMATICA. In Fig. 6 we plot the exact solution and the approximations $\varphi_n(x)$ for $n = 1$ through 5, where the last two approximations are very closed to the exact solution. In Fig. 7 we plot the exact solution and the approximations $\varphi_n(x)$ for $n = 5$ through 9 in different magnifications of the vertical scale, where the last two approximations and the exact solution overlap.

We have checked that other values of the parameters c_0 and c_1, such as $c_0 = 0.5, c_1 = 0.5$, can also obtain a similar sequence of approximate solutions.

We note that if we take $c_0 = 0, c_1 = 0$, which correspond to the recursion scheme without parametrization, the values of $M_n = \max_{0 \leqslant x \leqslant 1}|u_n(x)|, n = 1, 2, \ldots, 8$, as computed by using the MATHEMATICA command 'NMaximize', increase monotonically. Next we treat this example by the multistage ADM for BVPs by partitioning the interval $[0, 1]$ into two subintervals $[0, 0.5]$ and $[0.5, 1]$.

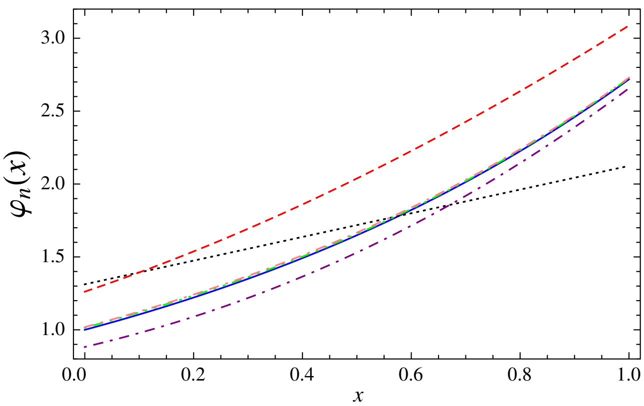

Fig. 6. Exact solution $u^*(x)$ (solid line) and the nth-stage approximations $\varphi_n(x)$ for $n = 1$ (dot line), $n = 2$ (dashed line), $n = 3$ (dot-side line), $n = 4$ (dot-dot-side line) and $n = 5$ (dot-side-side line).

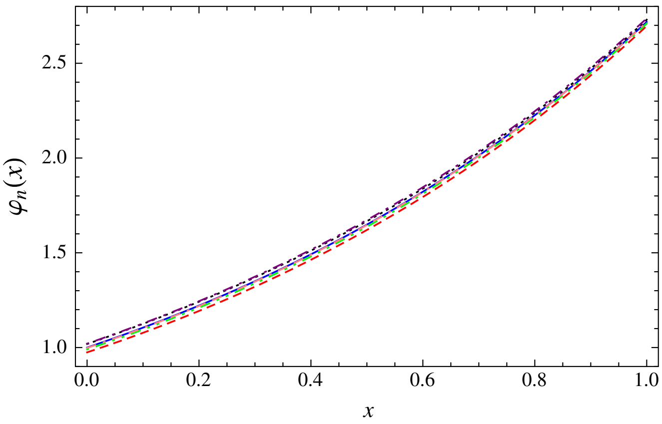

Fig. 7. Exact solution $u^*(x)$ (solid line) and the nth-stage approximations $\varphi_n(x)$ for $n=5$ (dot line), $n=6$ (dashed line), $n=7$ (dot-side line), $n=8$ (dot-dot-side line) and $n=9$ (dot-side-side line).

Let $u(0.5)=\eta$. We then solve the two nonlinear sub-BVPs

$$u''=\frac{1}{2}e^{-x}Nu,\quad 0\leqslant x\leqslant 0.5, \tag{5.12}$$

$$u(0)-u'(0)=0,\quad u(0.5)=\eta, \tag{5.13}$$

which is subject to a mixed set of Robin and Dirichlet boundary conditions, and

$$u''=\frac{1}{2}e^{-x}Nu,\quad 0.5\leqslant x\leqslant 1, \tag{5.14}$$

$$u(0.5)=\eta,\quad u(1)+u'(1)=2e, \tag{5.15}$$

which is subject to a mixed set of Dirichlet and Robin boundary conditions. For the nonlinear BVP (5.12) and (5.13) on the subinterval $[0,0.5]$, we have the equivalent nonlinear Fredholm–Volterra integral equation from (3.32) in Case 2,

$$u=\frac{\eta(1+x)}{1.5}+\frac{1}{2}L_{0,0}^{-1}e^{-x}Nu-\frac{1+x}{3}[L_{0,0}^{-1}e^{-x}Nu]_{x=0.5}. \tag{5.16}$$

The solution components on the subinterval $[0,0.5]$ are given by the modified recursion scheme

$$u_0=\frac{\eta(1+x)}{1.5}, \tag{5.17}$$

$$u_n=\frac{1}{2}L_{0,0}^{-1}e^{-x}A_{n-1}-\frac{1+x}{3}[L_{0,0}^{-1}e^{-x}A_{n-1}]_{x=0.5},\quad n\geqslant 1. \tag{5.18}$$

We list the first two calculated solution components

$$u_1=\left(-\frac{4}{3}-\frac{61}{27\sqrt{e}}+\frac{8e^{-x}}{3}+\frac{8x}{3}-\frac{61x}{27\sqrt{e}}+\frac{4e^{-x}x}{3}+\frac{2}{9}e^{-x}x^2\right)\eta^2,$$

$$u_2=\left(-\frac{1139}{162}+\frac{13693}{972e}-\frac{76}{27\sqrt{e}}-\frac{488}{27}e^{-\frac{1}{2}-x}+\frac{32e^{-2x}}{27}+\frac{40e^{-x}}{3}+\frac{1139x}{81}+\frac{13693x}{972e}-\frac{808x}{27\sqrt{e}}-\frac{244}{27}e^{-\frac{1}{2}-x}x\right.$$
$$\left.+\frac{61}{54}e^{-2x}x+8e^{-x}x-\frac{122}{81}e^{-\frac{1}{2}-x}x^2+\frac{1}{3}e^{-2x}x^2+\frac{16}{9}e^{-x}x^2+\frac{1}{27}e^{-2x}x^3\right)\eta^3,$$

where the nth-stage approximation is $\varphi_n^{(1)}(x;\eta)=\sum_{k=0}^{n-1}u_k(x)$.

For the nonlinear BVP (5.14) and (5.15) on the subinterval $[0.5,1]$, we have the equivalent nonlinear Fredholm–Volterra integral equation from (4.25) in Case 4,

$$u=\frac{\eta(2-x)+e(2x-1)}{1.5}+\frac{1}{2}L_{1,1}^{-1}e^{-x}Nu+\frac{x-2}{3}[L_{1,1}^{-1}e^{-x}Nu]_{x=0.5}. \tag{5.19}$$

The solution components on the subinterval $[0.5,1]$ are given by the modified recursion scheme

$$u_0=\frac{\eta(2-x)+e(2x-1)}{1.5}, \tag{5.20}$$

$$u_n=\frac{1}{2}L_{1,1}^{-1}e^{-x}A_{n-1}+\frac{x-2}{3}[L_{1,1}^{-1}e^{-x}A_{n-1}]_{x=0.5},\ n\geqslant 1. \tag{5.21}$$

We list the first two computed solution components

$$\begin{aligned}u_1=&\frac{40e}{27}-\frac{224e^{3/2}}{27}+\frac{14e^{2-x}}{3}-\frac{80ex}{27}+\frac{112}{27}e^{3/2}x+\frac{8}{3}e^{2-x}x+\frac{8}{9}e^{2-x}x^2\\&+\left(-\frac{28}{27}+\frac{128\sqrt{e}}{27}-\frac{8e^{1-x}}{3}+\frac{56x}{27}-\frac{64\sqrt{e}x}{27}-\frac{4}{3}e^{1-x}x-\frac{8}{9}e^{1-x}x^2\right)\eta\\&+\left(\frac{4}{27e}-\frac{26}{27\sqrt{e}}+\frac{2e^{-x}}{3}-\frac{8x}{27e}+\frac{13x}{27\sqrt{e}}+\frac{2}{9}e^{-x}x^2\right)\eta^2,\end{aligned}$$

$$\begin{aligned}u_2=&-\frac{1519e}{243}+\frac{10528e^{3/2}}{243}-\frac{8080e^2}{243}+\frac{25}{27}e^{3-2x}-\frac{560e^{2-x}}{27}+\frac{448}{27}e^{\frac{5}{2}-x}\\&+\frac{3038ex}{243}-\frac{7616}{243}e^{3/2}x+\frac{4040e^2x}{243}+\frac{82}{27}e^{3-2x}x-\frac{320}{27}e^{2-x}x+\frac{224}{27}e^{\frac{5}{2}-x}x\\&+\frac{4}{3}e^{3-2x}x^2-\frac{320}{81}e^{2-x}x^2+\frac{448}{81}e^{\frac{5}{2}-x}x^2+\frac{8}{27}e^{3-2x}x^3+\left(\frac{476}{81}-\frac{3392\sqrt{e}}{81}\right.\\&+\frac{2420e}{81}+\frac{8}{9}e^{2-2x}+\frac{184e^{1-x}}{9}-\frac{160}{9}e^{\frac{3}{2}-x}-\frac{952x}{81}+\frac{2368\sqrt{e}x}{81}-\frac{1210ex}{81}\\&-\frac{23}{9}e^{2-2x}x+\frac{304}{27}e^{1-x}x-\frac{128}{27}e^{\frac{3}{2}-x}x-\frac{4}{3}e^{2-2x}x^2+\frac{128}{27}e^{1-x}x^2\\&\left.-\frac{160}{27}e^{\frac{3}{2}-x}x^2-\frac{4}{9}e^{2-2x}x^3\right)\eta+\left(-\frac{796}{81}-\frac{281}{162e}+\frac{1042}{81\sqrt{e}}-\frac{5}{6}e^{1-2x}\right.\\&+\frac{20}{3}e^{\frac{1}{2}-x}-\frac{56e^{-x}}{9}+\frac{398x}{81}+\frac{281x}{81e}-\frac{740x}{81\sqrt{e}}+\frac{7}{9}e^{1-2x}x+\frac{26}{27}e^{\frac{1}{2}-x}x\\&\left.-\frac{88e^{-x}x}{27}+\frac{1}{3}e^{1-2x}x^2+\frac{20}{9}e^{\frac{1}{2}-x}x^2-\frac{16}{9}e^{-x}x^2+\frac{2}{9}e^{1-2x}x^3\right)\eta^2\\&+\left(\frac{67}{486e^2}-\frac{308}{243e^{3/2}}+\frac{589}{486e}+\frac{16e^{-1-x}}{27}-\frac{26}{27}e^{-\frac{1}{2}-x}+\frac{11e^{-2x}}{54}-\frac{67x}{243e^2}\right.\\&+\frac{232x}{243e^{3/2}}-\frac{589x}{972e}+\frac{8}{27}e^{-1-x}x-\frac{7}{54}e^{-2x}x+\frac{16}{81}e^{-1-x}x^2-\frac{26}{81}e^{-\frac{1}{2}-x}x^2\\&\left.-\frac{1}{27}e^{-2x}x^3\right)\eta^3,\\&\ldots\end{aligned}$$

where we denote $\varphi_n^{(2)}(x;\eta)=\sum_{k=0}^{n-1}u_k(x)$.

By solving the matching equation for the flux at the interior point

$$\left.\frac{d\varphi_n^{(1)}(x;\eta)}{dx}\right|_{x=0.5}=\left.\frac{d\varphi_n^{(2)}(x;\eta)}{dx}\right|_{x=0.5}, \tag{5.22}$$

we can compute the sequence of approximate values for the undetermined coefficient η.

Using the MATHEMATICA command 'NSolve' for $n=1$, we obtain the root $\eta[\varphi_1]=2.71828$. For $n=2$, we solve the matching equations by the command 'FindRoot' with the initial value $\eta[\varphi_1]$ and obtain $\eta[\varphi_2]=1.53737$. Then taking the value $\eta[\varphi_2]$ as the initial value in the command 'FindRoot' we obtain $\eta[\varphi_3]$ from the matching equations in $n=3$. In a similar manner, we obtain the values of $\eta[\varphi_n]$ for $n=4,5,\ldots$.

In Table 3, we list these values of $\eta[\varphi_n]$ for $n=1,2,\ldots,8$. We note that the true value of η is $e^{1/2}=1.64872\ldots$.

Thus we express the matched nth-stage approximant as

$$\varphi_n(x)=\varphi_n^{(1)}(x;\eta[\varphi_n])\Pi(x;0,0.5)+\varphi_n^{(2)}(x;\eta[\varphi_n])\Pi(x;0.5,1).$$

Table 3
The values of $\eta[\varphi_n]$ for $n=1,2,\ldots,8$.

n	1	2	3	4	5	6	7	8
$\eta[\varphi_n]$	2.71828	1.53737	1.69831	1.63356	1.6542	1.6468	1.64941	1.64848

The maximal error parameters $ME_n = \max_{0\leqslant x\leqslant 1}|\varphi_n(x) - u^*(x)|, n = 1, 2, \ldots, 8$, are computed and listed in Table 4. The logarithmic plots of these values of ME_n are displayed in Fig. 8, which demonstrates, after $n = 1$, an approximate exponential rate of convergence.

In the sequel we treat an example subject to a set of Neumann boundary conditions by our multistage ADM for BVPs.

Example 4. Consider the nonlinear BVP with an exponential nonlinearity in the solution and subject to a set of Neumann boundary conditions

$$u'' = -e^{-2u}, \quad 0 \leqslant x \leqslant 1, \tag{5.23}$$

$$u'(0) = 1, \quad u'(1) = 1/2. \tag{5.24}$$

The exact solution of this nonlinear BVP is $u^*(x) = \log(1 + x)$.

We rewrite Eq. (5.23) as $Lu = Nu$, and partition the interval $[0, 1]$ into two subintervals $[0, 0.5]$ and $[0.5, 1]$. We suppose that $u(0.5) = \eta$.

On the subdomain $[0, 0.5]$, we solve the nonlinear sub-BVP with a mixed set of Neumann and Dirichlet boundary conditions

$$Lu = Nu, \quad 0 \leqslant x \leqslant 0.5, \tag{5.25}$$

$$u'(0) = 1, \quad u(0.5) = \eta. \tag{5.26}$$

From Eq. (4.31) we have an equivalent nonlinear Volterra integral equation

$$u = \eta + x - 0.5 + L_{0.5,0}^{-1} Nu,$$

from which we derive the modified recursion scheme

$$\begin{aligned} &u_0 = \eta - 0.5 + x, \\ &u_n = L_{0.5,0}^{-1} A_{n-1}, \quad n \geqslant 1, \end{aligned}$$

where the Adomian polynomials for the nonlinearity $Nu = -e^{-2u}$ are

$$\begin{aligned} &A_0 = -e^{-2u_0}, \\ &A_1 = 2e^{-2u_0} u_1, \\ &A_2 = -2e^{-2u_0} u_1^2 + 2e^{-2u_0} u_2, \\ &\quad \cdots \end{aligned}$$

We list the first two calculated solution components

Table 4
The maximal error parameter ME_n.

n	1	2	3	4	5	6	7	8
ME_n	1.07741	0.129275	0.0526345	0.0161081	0.00573991	0.00199712	0.000709397	0.000250784

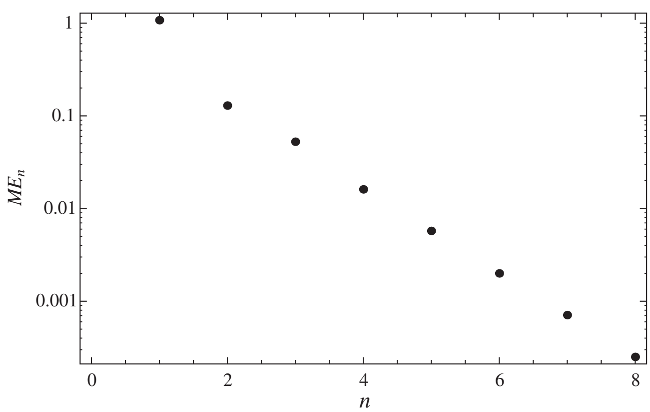

Fig. 8. Logarithmic plots of ME_n for $n = 1$ through 8.

$$u_1 = \frac{1}{4}e^{1-2\eta} - \frac{1}{4}e^{1-2x-2\eta} + \frac{e^{-2\eta}}{4} - \frac{1}{2}e^{1-2\eta}x,$$
$$u_2 = \frac{1}{8}e^{1-4\eta} + \frac{1}{16}e^{2-4\eta} - \frac{1}{32}e^{2-4x-4\eta} + \frac{1}{8}e^{1-2x-4\eta} - \frac{1}{8}e^{2-2x-4\eta} - \frac{3e^{-4\eta}}{32} + \frac{1}{4}e^{1-4\eta}x - \frac{1}{8}e^{2-4\eta}x - \frac{1}{4}e^{2-2x-4\eta}x,$$
$$\ldots$$

where we denote the nth-stage solution approximation as $\varphi_n^{(1)}(x;\eta) = \sum_{k=0}^{n-1} u_k$.

On the subdomain $[0.5, 1]$, we solve the nonlinear sub-BVP with a mixed set of Dirichlet and Neumann boundary conditions

$$Lu = Nu, \quad 0.5 \leqslant x \leqslant 1, \tag{5.27}$$

$$u(0.5) = \eta, \quad u'(1) = 1/2. \tag{5.28}$$

We have from Eq. (4.29)

$$u = \eta + \frac{1}{2}(x - 0.5) + L_{0.5,1}^{-1} Nu$$

and the modified recursion scheme

$$u_0 = \eta + \frac{1}{2}(x - 0.5),$$

$$u_n = L_{0.5,1}^{-1} A_{n-1}, \quad n \geqslant 1.$$

Furthermore we obtain the first two calculated solution components

$$u_1 = \frac{1}{2}e^{-\frac{1}{2}-2\eta} - e^{\frac{1}{2}-x-2\eta} + e^{-2\eta} - e^{-\frac{1}{2}-2\eta}x,$$
$$u_2 = 2e^{-1-4\eta} + 3e^{-\frac{1}{2}-4\eta} - \frac{1}{2}e^{1-2x-4\eta} - 3e^{-x-4\eta} + 2e^{\frac{1}{2}-x-4\eta} - \frac{3e^{-4\eta}}{2} - 4e^{-1-4\eta}x + 2e^{-\frac{1}{2}-4\eta}x - 2e^{-x-4\eta}x,$$
$$\ldots,$$

where we denote the nth-stage solution approximation $\varphi_n^{(2)}(x;\eta) = \sum_{k=0}^{n-1} u_k$. Solving the matching equation for the flux

$$\left.\frac{d\varphi_n^{(1)}(x;\eta)}{dx}\right|_{x=0.5} = \left.\frac{d\varphi_n^{(2)}(x;\eta)}{dx}\right|_{x=0.5}, \tag{5.29}$$

we compute the sequence of approximate values for the undetermined coefficient η. Starting with $n = 2$ and using a similar method as in the last example, we obtain the values of $\eta[\varphi_n]$ for $n = 2, 3, \ldots, 10$ and list them in Table 5. We observe that the sequence of $\eta[\varphi_n]$ approaches the value $0.4054\ldots$. We note that the true value of η is $\log(1.5) = 0.405465\ldots$.

Thus we express the matched nth-stage approximation as

$$\varphi_n(x) = \varphi_n^{(1)}(x;\eta[\varphi_n])\Pi(x;0,0.5) + \varphi_n^{(2)}(x;\eta[\varphi_n])\Pi(x;0.5,1).$$

The maximal error parameters $ME_n = \max_{0\leqslant x\leqslant 1}|\varphi_n(x) - u^*(x)|$ are computed and listed in Table 6. The logarithmic plots of these values of ME_n are displayed in Fig. 9, which demonstrates, after $n = 3$, an approximate exponential rate of convergence. In Fig. 10 we show the curves of the exact solution and the matched nth-stage approximations $\varphi_n(x)$ for $n = 2, 3, 4, 5$, where $\varphi_5(x)$ and the exact solution overlap.

Table 5
The values of $\eta[\varphi_n]$ for $n = 2, 3, \ldots, 10$.

n	2	3	4	5	6	7	8	9	10
$\eta[\varphi_n]$	0.459188	0.389312	0.410025	0.403754	0.406114	0.405198	0.405577	0.405417	0.405486

Table 6
The maximal error parameter ME_n.

n	2	3	4	5	6	7	8	9	10
ME_n	0.05898	0.0184545	0.00527023	0.00199138	0.00075682	0.000311709	0.000130807	0.0000567361	0.0000250119

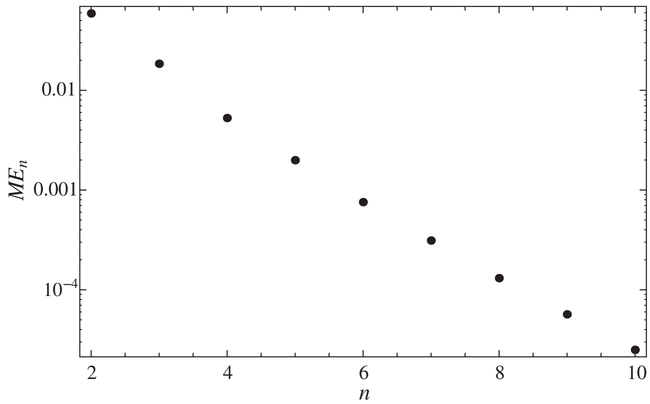

Fig. 9. Logarithmic plots of ME_n for $n = 2$ through 10.

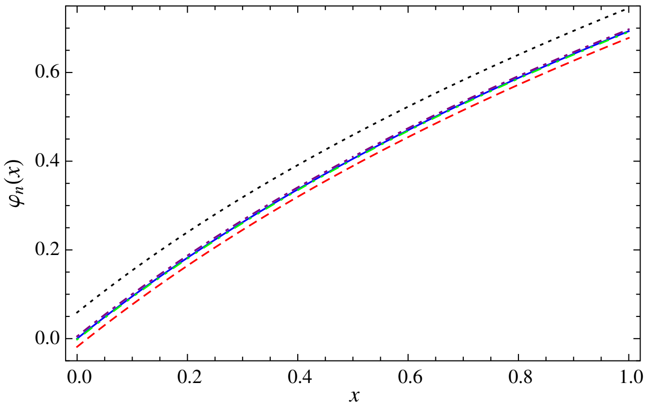

Fig. 10. Exact solution $u^*(x)$ (solid line) and the matched nth-stage approximations $\varphi_n(x)$ for $n = 2$ (dot line), $n = 3$ (dashed line), $n = 4$ (dot-side line), and $n = 5$ (dot-dot-side line).

6. Conclusion

We have presented a new approach using the ADM to systematically solve linear and nonlinear BVPs with Robin boundary conditions, and also developed the multistage ADM for BVPs. We have derived general formulas for the solution's equivalent nonlinear Fredholm–Volterra integral equation, and its corresponding modified recursion scheme, which subsumes various special cases. Furthermore we apply the multistage ADM for BVPs to treat two important cases, when the original computed series does not converge over the specified domain, or when the nonlinear differential equation is subject to a set of Neumann boundary conditions. Our new approach yields an analytic, readily verifiable, rapidly convergent approximation. We have demonstrated the practicality and efficiency of our new modification of the ADM by four numerical expository examples for a variety of nonlinear BVPs. Example 1 is a linear Robin BVP with variable coefficients, which we readily solved by our new algorithm. Example 2 is a nonlinear Robin BVP with an exponential nonlinearity of the solution and a quadratic nonlinearity of the derivative, which we solved with our new algorithm combined with a parametrized recursion scheme. Example 3 is a nonlinear Robin BVP with a variable coefficient and a quadratic nonlinearity of the solution and a quadratic nonlinearity of the derivative, which demonstrates the advantages of the multistage ADM for BVPs in treating cases where the original computed series exhibits divergence. Example 4 is a nonlinear Neumann BVP with an exponential nonlinearity, which demonstrates how to solve nonlinear Neumann BVPs by the multistage ADM for BVPs. We also emphasized the logarithmic plots of the maximal error parameters, which demonstrates an exponential rate of convergence for our computed solutions of nonlinear BVPs.

Acknowledgements

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Appendix A. MATHEMATICA code generating the first $M+1$ Adomian polynomials based on Duan's Corollary 3 algorithm [72]

```
Ado[M_]:=Module[{c,n,k,j,der},Table[c[n,k],{n,1,M},{k,1,n}];
A[0]=f[u[0]];
For[n = 1,n<=M,n++,c[n,1]=u[n];
For[k = 2,k<=n,k++,
c[n,k]=Expand[1/n*Sum[(j + 1)*u[j + 1]* c[n-1-j,k-1],{j,0,n-k}]]]];
der = Table[D[f[u[0]],{u[0],k}],{k,1,M}];
A[n]=Take[der,n].Table[c[n,k],{k,1,n}]];
Table[A[n],{n,0,M}]]
```

Appendix B. Appendix B. MATHEMATICA code used in Example 2

```
AP[m_]:=Module[{F},
f[u_]:=E^(-2 u) + 4 * D[u,x]^2;
Subscript[F, 0]=f[Sum[Subscript[u, k][x]*s^k,{k,0,m}]];
For [i = 0,i<=m,i++,
A[i]= Expand[Subscript[F, i]/.s->0];
Subscript[F, i + 1]=1/(i + 1)*D[Subscript[F, i],s]]];
AP[17]; c = 0.06; Subscript[u, 0][x_]=c;
Subscript[u, 1][x_]=N[-c-1/3 + 2/5 Log[3/2]+x (1/3 + 1/5 Log[3/2])+
(2 + x)/40* Integrate[Integrate[A[0],{x,0,x}],{x,0,1}]+
(2 + x)/20* Integrate[A[0],{x,0,1}]-1/8*
Integrate[Integrate[A[0],{x,0,x}],{x,0,x}]];
For [k = 2,k<=18,k++,
Subscript[u, k][x_]=
(2 + x)/40* Integrate[Integrate[A[k-1],{x,0,x}],{x,0,1}]
+(2 + x)/20* Integrate[A[k-1],{x,0,1}]-1/8*
Integrate[Integrate[A[k-1],{x,0,x}],{x,0,x}]];
ph[n_]:=Sum[Subscript[u, k][x],k,0,n-1]; ue = Log[(2 + x)/2];
ME = Table[NMaximize[{Abs[ph[n]-ue],0<=x<=1},x][[1]],{n,2,19}];
n = Table[k,{k,2,19}];
Fig4=ListLogPlot[Table[{n[[k]],ME[[k]]},{k,1,18}],
PlotRange->All,Frame->True];
Fig5=Plot[{ue,ph[2],ph[3],ph[4],ph[5],ph[6]},{x,0,1},
Frame->True, Axes->None,
PlotStyle->{{Blue},{Black},{Red},{Purple},{Green},{Pink}}];
ME
Fig4
Fig5
```

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