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$\mathcal{H}_\infty$ control of 8 degrees of freedom vehicle active suspension system

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Abstract The main objective of this paper is to develop improved robust control techniques for an active suspension system utilizing an improved mathematical model. For that purpose, Euler Lagrange equation is used to obtain a mathematical model for vehicle active suspension system. The dynamics of driver's seat are included to get a more appropriate model. Robust $\mathcal{H}_\infty$ controllers are designed for the system to minimize the effect of road disturbances on vehicle and passengers. The performance of active suspension system is determined by measuring the heave acceleration of driver's seat and rotational acceleration of vehicle around its center of gravity. Effectiveness of the proposed controllers is validated by simulation results.

1. Introduction

According to ISO 2631-1 standard, if human body is continuously exposed to vibrations between 0.5 and 80 Hz, the risk of injury to vertebrae in lumbar region is drastically increased and may cause malfunction of the nerves connected to these segments (Chamseddine et al., 2006). Each one of us daily uses vehicles for traveling and the above fact shows the importance of comfortable ride and the need to minimize the vibrations caused by the irregularities in roads.

The suspension system of vehicle plays a vital role in improving ride quality and ride comfort. It connects vehicle's body to the tires and is a mean to transmit all forces between vehicle's body and road. The desirable characteristics of suspension system are better road handling and ride comfort (Appleyard and Wellstead, 1995). Poor ride quality and ride comfort can harm passengers, vehicle's body and the cargo inside (Granlund, 2008). So the suspension system should be designed to take into account all these constraints.

Suspension systems can be passive, semi-active or active. Passive suspension systems consists of energy storing elements along with dampers with fixed characteristics. Their performance is limited and can only be changed by changing the characteristics of dampers and springs. There is no control over the amount of energy added or dissipated. A heavily damped system will provide good road handling but poor ride

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quality, similarly, a lightly damped system will provide good ride quality but poor road handling. Most of the vehicles in the world are using this suspension system. Compared to that, a semi-active suspension systems allows controlled damping with fixed spring characteristics (Pionke and Bocik, 2011). The idea of semi-active suspension system first came to light in 1970s. These systems are designed to dissipate energy in a controlled manner by changing the damping, however, there is no way to add energy to the system with this suspension technique. Semi-active suspension systems provide better performance compared to passive suspension systems. An active suspension systems (ASS) consist of springs, dampers and force actuators which can dissipate or add energy to the system in a controlled way. Active suspension systems have obvious advantages over the passive and semi-active suspension systems because their actuator can be controlled by controllers to provide ride comfort to the passengers. These systems provide better compromise between road handling and ride quality. Sensors continuously monitor the operating conditions and control units control the active actuators using the information of sensors (Izawa et al., 1997).

Vehicle suspension system models have been proposed regularly with time. But the research work for the analysis and practical implementation of vehicle suspension system started back in 80s. In 1987, it was shown that both ride quality and road handling can be improved by reducing the un-sprung mass (Hrovat, 1988). Design of vehicle suspension systems for ride comfort for frequencies below body structure resonances is discussed by Sharp and Crolla (1987). Due to added advantage of active suspension system, several research articles have also appeared in this domain. The linear quadratic regulator (LQR) control and proportional derivative integral (PID) control techniques are applied to active suspension system in Darus and Enzai (2010). $\mathcal{H}_\infty$ control theory has been utilized to design controller for vehicle active suspension system by Amirifar and Sadati (2006), Yamashita et al. (1994), and Chen and Guo (2005). Adaptive control techniques for active suspension system are discussed by Sun et al. (2013b), Sun et al. (2013b), and Alleyne and Hedrick (1995). Effect of delays in actuator signals are handled by Li et al. (2014) and Du and Zhang (2007). Sampled data control of vehicle ASS is presented by Gao et al. (2010), an $\mathcal{H}_\infty$ approach is adopted therein. In addition to the model based control techniques, artificial intelligence based techniques have also been studied for vehicle ASS (Cao et al., 2010).

The design of vehicle active suspension system has been studied based on three widely used mathematical models, that is, the quarter-car model, half-car model and the full car model. In quarter car model, the suspension of single tire of the car is modeled. Most of the preliminary studies (Alleyne and Hedrick, 1995; Yamashita et al., 1994) and some recent articles (Li et al., 2014; Darus and Enzai, 2010) are based on the quarter car model. The half-car model is an improved mathematical model over the quarter car model. Here, bicycle model is used and vehicle is considered with two tires. Among others, Sun et al. (2013a) and Li et al. (2011) present a design of active suspension system based on half-car model of the system. Full car model of vehicle active suspension system has also been presented in literature and control techniques are presented therein, see for example Darus and Sam (2009), Sun et al. (2013b), and Yagiz and Hacioglu (2008) and references therein. In most of the aforementioned studies on active suspension system, the dynamics of the driver's seat have been ignored. However, these dynamics play an important part regarding ride comfort because passenger is directly affected by the behavior of the seat. Therefore, in some of the recent studies, the dynamics of driver's seat are also incorporated in the mathematical model and some control strategies are proposed (Aly and Salem, 2013; Rahmi, 2003; Guclu, 2004).

The main contribution of this work is that a more detailed derivation of mathematical model of full car active suspension system including dynamics of driver's seat are presented. Furthermore, $\mathcal{H}_\infty$ state feedback control and $\mathcal{H}_\infty$ dynamic output feedback control schemes are devised for the developed model to minimize the effect of terrain irregularities on passenger comfort. The effectiveness is demonstrated by simulation results.

The remainder of this paper is organized as follows. In Section 2, the mathematical model of eight degrees of freedom active suspension system of full vehicle model including driver's seat dynamics is developed using Euler Lagrange approach. Section 3 includes the design of robust $\mathcal{H}_\infty$ controllers for suspension system is discussed. These controllers increase passenger comfort by minimizing the effect of road disturbances. The idea of robust controllers is supported by designing state-feedback and dynamic output-feedback controllers. Simulations are carried out to extend the understanding of proposed controllers. Finally, some concluding remarks are presented in Section 4 for further research.

2. Modeling of system

The mathematical model of a full vehicle is developed using the famous Lagrangian equation. The active suspension system considered has eight degrees of freedom. The model of driver's seat is also taken into account because the damping of seats have a vital role in ride quality and comfort provided by the vehicle.

2.1. Euler Lagrange equation

To derive a mathematical model using Euler Lagrange equations, an energy function called *''Lagrangian energy function*'' is defined for the system. The Lagrangian function for a system is the difference of the total kinetic energy and the total potential energy of the system. Based on this function, equations of motion for the system are derived by the following expression;

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} + \frac{\partial D}{\partial \dot{q}_i} = Q_i \quad (1)$$

where L is the Lagrangian energy function, D is the dissipation function of system, q_i is the generalized ith coordinate, $\dot{q}_i$ and Q_i is the force on i_{th} coordinate.

2.2. System description

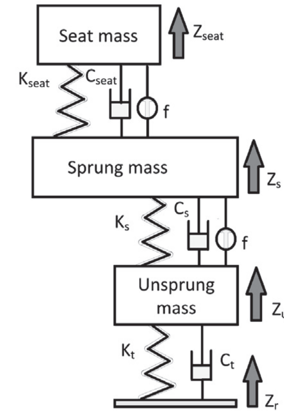
The suspension system considered has eight degrees of freedom, that is, the vehicle's pitch angle, roll angle, displacement of driver's seat, displacement of vehicle's sprung mass and displacement of four unsprung masses. The numerical values for the parameters of the system are presented in Table 1. The derivation of mathematical model is simplified by considering the following assumptions.

Table 1 Numerical values and description of parameters.

Parameter	Description	Value
COG	Center of gravity	
$m_{u(i)}$	ith unsprung mass	40 kg
m_s	Sprung mass	1400 kg
m_{seat}	Mass of seat	90 kg
I_x	Moment of inertia along pitch motion	2100 kg m^2
I_y	Moment of inertia along roll motion	460 kg m^2
$k_{t(i)}$	Spring coefficient of ith tyre	190,000 Nm^{-1}
$k_{s(i)}$	Spring coefficient of sprung mass	25,500 Nm^{-1}
k_{seat}	Spring coefficient of seat	15,000 Nm^{-1}
$c_{t(i)}$	Damping coefficient of ith tyre	0 Nsm^{-1}
$c_{s(i)}$	Damping coefficient of sprung mass	1000 Nsm^{-1}
c_{seat}	Damping coefficient of seat	500 Nsm^{-1}
L_1	Distance of rear tires from COG	1.44 m
L_2	Distance of front tires from COG	0.96 m
w_1	Distance of left tires from COG	1.44 m
w_2	Distance of right tires from COG	0.71 m
s_1	Distance of seat from COG	0.25 m
s_2	Distance of seat from COG	0.3 m

- All components are considered as linear.
- Small displacements are considered.
- Ground contact of vehicle is maintained continuously.
- Vehicle's body is considered rigid.
- The sprung mass can heave, pitch and roll.
- The unsprung masses can only move in vertical direction.

For our case we will consider the driver's seat as well in our model. Fig. 1 shows the simplified model of one corner of the vehicle with seat.



(a) Model of one corner of vehicle with seat

Fig. 1 Model of one corner of vehicle.

2.3. Mathematical model

Including the dynamics of the driver's seat results into eight degrees of freedom. To avoid complex mathematical equations, we define new sate variables as below

$$z_1 \equiv z_s - z_{u(RL)} + w_1\phi - L_1\theta \quad (2)$$

$$z_2 \equiv z_s - z_{u(RR)} - w_2\phi - L_1\theta \quad (3)$$

$$z_3 \equiv z_s - z_{u(FL)} + w_1\phi + L_2\theta \quad (4)$$

$$z_4 \equiv z_s - z_{u(FR)} + L_2\theta - w_2\phi \quad (5)$$

$$z_5 \equiv z_{u(RL)} - z_{r(RL)} \quad (6)$$

$$z_6 \equiv z_{u(RR)} - z_{r(RR)} \quad (7)$$

$$z_7 \equiv z_{u(FL)} - z_{r(FL)} \quad (8)$$

$$z_8 \equiv z_{u(FR)} - z_{r(FR)} \quad (9)$$

$$z_9 \equiv z_{seat} - z_s - s_2\theta + s_1\phi \quad (10)$$

where $z_{u(RR)}$, $z_{u(RL)}$, $z_{u(FR)}$ and $z_{u(FL)}$ are, respectively, the displacements of unsprung mass of rear right, rear left, front right and front left corners of vehicle. z_{seat} represents the displacement of mass of seat, z_s is the displacement of sprung mass of the vehicle. ϕ and θ are, respectively, the rotational displacement of sprung mass along pitch motion and along roll motion.

2.3.1. System energies

Every mechanical system has certain kinds of energies related to it. These energies can be kinetic, potential, translational or rotational. For our case we have the following different energies related to the vehicle suspension system.

- Translational kinetic energy (KE) due to the heave motion of seat, sprung and unsprung masses.
- Rotational kinetic energy due to pitch and roll motion of sprung mass.
- Potential energy (PE) of tyres due to their stiffness.
- Potential energy of masses due to their height.
- Potential energy of springs.
- Energy dissipation in dampers.

Total kinetic energy of system, T, is the sum of the translational KE of seat, translational KE of unsprung masses, translational KE of sprung mass and the rotational KE of sprung mass, that is,

$$T = \frac{1}{2}m_s\dot{z}_s^2 + \frac{1}{2}I_x\dot{\phi}^2 + \frac{1}{2}I_y\dot{\theta}^2 + \frac{1}{2}m_{u(RR)}\dot{z}_{u(RR)}^2 + \frac{1}{2}m_{seat}\dot{z}_{seat}^2 + \frac{1}{2}m_{u(FR)}\dot{z}_{u(FR)}^2 + \frac{1}{2}m_{u(FL)}\dot{z}_{u(FL)}^2 + \frac{1}{2}m_{u(RL)}\dot{z}_{u(RL)}^2 \quad (11)$$

Total potential energy of the system, V, is the sum of PE of tires due to their stiffness, PE of seat due to its height, PE of sprung mass due to its height, PE of unsprung mass due to its height and PE of suspension springs, that is,

$$V = \frac{1}{2}k_{s(RL)}z_1^2 + \frac{1}{2}k_{t(RL)}z_5^2 + \frac{1}{2}k_{s(RR)}z_2^2 + \frac{1}{2}k_{t(RR)}z_6^2 + \frac{1}{2}k_{s(FL)}z_3^2 + \frac{1}{2}k_{t(FL)}z_7^2 + \frac{1}{2}k_{s(FR)}z_4^2 + \frac{1}{2}k_{t(FR)}z_8^2 + \frac{1}{2}k_{(seat)}z_9^2 - m_{u(RL)}gz_{u(RL)} - m_{u(RR)}gz_{u(RR)} - m_{u(FL)}gz_{u(FL)} - m_{u(FR)}gz_{u(FR)} - m_s z_s g \quad (12)$$

Energy is continuously dissipated in the system due to the dampers associated with suspension system. Total energy dissipation of the system, D, can be represented as,

$$D = \frac{1}{2}c_{s(RL)}\dot{z}_1^2 + \frac{1}{2}c_{t(RL)}\dot{z}_5^2 + \frac{1}{2}c_{s(RR)}\dot{z}_2^2 + \frac{1}{2}c_{t(RR)}\dot{z}_6^2 + \frac{1}{2}c_{s(FL)}\dot{z}_3^2 + \frac{1}{2}c_{t(FL)}\dot{z}_7^2 + \frac{1}{2}c_{s(FR)}\dot{z}_4^2 + \frac{1}{2}c_{t(FR)}\dot{z}_8^2 + \frac{1}{2}c_{(seat)}\dot{z}_9^2 \quad (13)$$

2.3.2. Lagrangian function of system

The Lagrangian function of the system can be obtained by taking the difference of KE and PE of the system. Thus,

$$L = T - V \quad (14)$$

Substituting (11) and (12) in the above equation results in,

$$L = \frac{1}{2}m\dot{z}_s^2 + \frac{1}{2}I_x\dot{\phi}^2 + \frac{1}{2}I_y\dot{\theta}^2 + \frac{1}{2}m_{u(RR)}\dot{z}_{u(RR)}^2 + \frac{1}{2}m_{seat}\dot{z}_{seat}^2 + \frac{1}{2}m_{u(FR)}\dot{z}_{u(FR)}^2 + \frac{1}{2}m_{u(FL)}\dot{z}_{u(FL)}^2 + \frac{1}{2}m_{u(RL)}\dot{z}_{u(RL)}^2 - \frac{1}{2}k_{s(RL)}z_1^2 - \frac{1}{2}k_{t(RL)}z_5^2 - \frac{1}{2}k_{s(RR)}z_2^2 - \frac{1}{2}k_{t(RR)}z_6^2 - \frac{1}{2}k_{s(FL)}z_3^2 - \frac{1}{2}k_{t(FL)}z_7^2 - \frac{1}{2}k_{s(FR)}z_4^2 - \frac{1}{2}k_{t(FR)}z_8^2 - \frac{1}{2}k_{(seat)}z_9^2 + m_s z_s g + m_{u(RL)}g z_{u(RL)} + m_{u(RR)}g z_{u(RR)} + m_{u(FL)}g z_{u(FL)} + m_{u(FR)}g z_{u(FR)} \quad (15)$$

2.3.3. Equations of motion for system

To obtain the equations of motion, we will apply (1) to the Lagrangian function. As we are considering eight degrees of freedom, we get the following eight equations describing the dynamics of vehicle active suspension system.

$$m_{u(RR)}\ddot{z}_{u(RR)} - k_{s(RR)}z_2 + k_{t(RR)}z_6 - m_{u(RR)}g - c_{s(RR)}\dot{z}_2 + c_{t(RR)}\dot{z}_6 = -f_{(RR)} \quad (16)$$

$$m_{u(RL)}\ddot{z}_{u(RL)} - k_{s(RL)}z_1 + k_{t(RL)}z_5 - m_{u(RL)}g - c_{s(RL)}\dot{z}_1 + c_{t(RL)}\dot{z}_5 = -f_{(RL)} \quad (17)$$

$$m_{u(FR)}\ddot{z}_{u(FR)} - k_{s(FR)}z_4 + k_{t(FR)}z_8 - m_{u(FR)}g - c_{s(FR)}\dot{z}_4 + c_{t(FR)}\dot{z}_8 = -f_{(FR)} \quad (18)$$

$$m_{u(FL)}\ddot{z}_{u(FL)} - k_{s(FL)}z_3 + k_{t(FL)}z_7 - m_{u(FL)}g - c_{s(FL)}\dot{z}_3 + c_{t(FL)}\dot{z}_7 = -f_{(FL)} \quad (19)$$

$$m_{seat}\ddot{z}_{seat} + m_{seat}g + k_{seat}z_9 + c_{seat}\dot{z}_9 = -f_{seat} \quad (20)$$

$$m_s\ddot{z}_s + k_{s(RL)}z_1 + k_{s(RR)}z_2 + k_{s(FL)}z_3 + k_{s(FR)}z_4 - k_{seat}z_9 + m_s g + c_{s(RL)}\dot{z}_1 + c_{s(RR)}\dot{z}_2 + c_{s(FL)}\dot{z}_3 + c_{s(FR)}\dot{z}_4 - c_{seat}\dot{z}_9 = f_{(FR)} + f_{(RR)} + f_{(FL)} + f_{(RL)} + f_{seat} \quad (21)$$

$$I_x\ddot{\phi} + k_{s(RL)}z_1w_1 + k_{s(FL)}z_3w_1 - k_{s(RR)}z_2w_2 - k_{s(FR)}z_4w_2 + k_{seat}z_9s_1 + c_{s(RL)}\dot{z}_1w_1 + c_{s(FL)}\dot{z}_3w_1 - c_{s(RR)}\dot{z}_2w_2 - c_{s(FR)}\dot{z}_4w_2 + c_{seat}\dot{z}_9s_1 = 0 \quad (22)$$

$$I_y\ddot{\theta} - k_{s(RL)}z_1L_1 + k_{s(FL)}z_3L_2 - k_{s(RR)}z_2L_1 + k_{s(FR)}z_4L_2 - k_{seat}z_9s_2 - c_{s(RL)}\dot{z}_1L_1 + c_{s(FL)}\dot{z}_3L_2 - c_{s(RR)}\dot{z}_2L_1 - c_{s(FR)}\dot{z}_4L_2 - c_{seat}\dot{z}_9s_2 = 0 \quad (23)$$

2.3.4. State-space model for system

After the substitution of numerical values of the system parameters in the differential equations of motion obtained in the previous section, the following state-space model of the system can be obtained.

$$\dot{x} = Ax + Bu + E_d d \quad (24)$$
$$y = Cx + F_d d \quad (25)$$
$$z = C_z x \quad (26)$$

where $x \in \mathbb{R}^{16}$ is the state vector, $u \in \mathbb{R}^5$ is the input vector. For the design of state feedback controller, it is assumed that all states can be measured, that is, the output vector $y \in \mathbb{R}^{16}$. The numerical values of the system matrix A and the input matrix B are not displayed here due to space limitations. It is to be noted that C depends on the number of sensors available. C_z depends on the choice of states to be minimized.

3. $\mathcal{H}_\infty$ control against road disturbances

It is desired to control the active vehicle suspension system in order to improve the ride quality and road handling of the vehicle. It is done by controlling the actuator forces depending on feedback information of the system obtained from sensors. Comfort of passengers is guaranteed by isolating the passengers from undesirable road disturbances. The performance of suspension system is evaluated on the basis of magnitude of acceleration to which passengers are exposed. The key idea in the $\mathcal{H}_\infty$ robust control is to see whether the system performs according to desired criteria even in the worst case. The main focus of this paper will be on the disturbance attenuation problem. Two types of $\mathcal{H}_\infty$ optimized controllers are proposed in this paper which attenuate the effect of road disturbances on heave acceleration of driver's seat and rotational acceleration along pitch and roll motion.

3.1. $\mathcal{H}_\infty$ optimized state feedback control

Using the state feedback control, the closed loop poles of the system can be placed anywhere in the left half s-plane. These closed loop poles determine the eigenvalues of system and indirectly control the stability of system. The control signal u is determined by an instantaneous state. In order to design a state feedback controller, following assumptions are considered.

- (A, B) pair is stabilizable.
- All states are available for feedback.

With the state feedback control law $u = Kx$ in (24), the dynamics of closed loop system are given by

$$\dot{x} = Ax + BKx + E_d d \quad (27)$$
$$z = C_z x \quad (28)$$

where $K \in \mathcal{R}^{q \times n}$ is the state feedback gain matrix. In order to find a suitable controller gain matrix K, so that the effect of disturbances is minimized on selected states, we define the following objective function,

$$\|G_{zd}\|_\infty = sup\left(\frac{\|z\|_2}{\|d\|_2}\right) < \gamma, \quad min(\gamma) \quad (29)$$

This means that the $\mathcal{L}_2$ gain from disturbance to output is less than a positive scalar γ and it is required to minimize γ. The above objective is achieved if the following inequality is satisfied.

$$J_1 = \int (z^T z - \gamma^2 d^T d)dt < 0 \quad (30)$$

Using S-procedure, it can be seen that for, a positive definite $V(t) = x^T(t)Px(t)$, the following inequality guarantees (30).

$$\int (z^T z - \gamma^2 d^T d)dt + V(t) < 0 \quad (31)$$

$$\Leftarrow \int (z^T z - \gamma^2 d^T d + \dot{V}(t))dt < 0 \quad (32)$$

A sufficient condition for (32) is given as,

$$z^T z - \gamma^2 d^T d + \dot{V}(t) < 0$$

$$(Ax + BK + Bd)^T Px + x^T(PAx + BK + PBd) + x^T C_z^T C_z x - \gamma^2 d^T d < 0 \quad (33)$$

This inequality can be conveniently represented in the form of LMI as,

$$\begin{bmatrix} PA + PBK + A^T P + K^T B^T P + C_z^T C_z & PB \\ * & -\gamma^2 I \end{bmatrix} < 0 \quad (34)$$

Applying Schur's compliment to the above inequality (34), it can be seen that

$$\begin{bmatrix} PA + PBK + A^T P + K^T B^T P & PB & C_z^T \\ * & -\gamma^2 I & 0 \\ * & * & -I \end{bmatrix} < 0 \quad (35)$$

Above inequality is nonlinear due to the presence of term PBK. So in order to linearize the inequality, congruence transformation is used. Multiplying with $diag\{P^{-1}, I, I\}$ on left and right sides of inequality and then substituting $P^{-1} = Q$ and $\phi = KQ$, LMI is converted into,

$$\begin{bmatrix} AQ + Q^T A^T + B\phi + \phi^T B^T & E_d & QC_z^T \\ * & -\gamma^2 I & 0 \\ * & * & -I \end{bmatrix} < 0 \quad (36)$$

Theorem 1. *The closed loop system (27) is robust against disturbances with an $\mathcal{H}_\infty$ disturbance attenuation level less than γ ($\gamma > 0$), if the following LMI holds for matrices Q and K of appropriate dimensions.* ($Q > 0$)

$$\begin{bmatrix} AQ + Q^T A^T + BKQ + Q^T K^T B^T & E_d & QC_z^T \\ * & -\gamma^2 I & 0 \\ * & * & -I \end{bmatrix} < 0 \quad (37)$$

3.1.1. Simulation results

The state feedback control matrix can be obtained using the results in Theorem 1 for the vehicle active suspension system. The resulting numerical value of disturbance attenuation constant γ is given by,

$$\gamma = 1.7 \times 10^{-4}$$

Remark 1. From the achieved value of γ, it is obvious that a significant disturbance attenuation can be achieved with the designed controller. On the other hand, the values in controller gain matrix are quite large, which may create difficulties in implementation. A possible solution is to use suboptimal controller gain matrix which will not provide that much disturbance attenuation but will be realizable. This can be done by limiting the number of iterations in solving the LMI in (37).

Fig. 2 shows the disturbance signal used for simulations. First bump is applied to both front tires and second bump is applied to both rear tires. This is the same as the vehicle has passed over some speed breaker in its path. The time delay in the front and rear bumps is created due to the physical distance between front and rear tires. Normal speed breakers are usually 0.1 m high so this magnitude is used in simulations. The heave acceleration of seat, rotational acceleration of vehicle along pitch motion and rotational acceleration of vehicle along roll motion in response to the applied disturbance are shown in Fig. 3. Above results clearly show that the magnitude of acceleration has been drastically decreased in response to road disturbances. So the state feedback controller proposed is valid.

3.2. $\mathcal{H}_\infty$ optimized output feedback control

Sometimes not all of the states are available for feedback. In that case, state feedback controller can't be used and it becomes inevitable to use output feedback controller. In this controller information from system output is fed back to generate the control signal u. Output feedback controller can be of the following two types, that is, the static output feedback controller and the dynamic output feedback controller. Static output feedback controller is generally hard to solve and results in conservative results. So dynamic output feedback controller, which has its own dynamics, will be used in this study. The dynamic output feedback controller is given by,

$$\dot{x}_c = A_c x_c + B_c y \qquad u = C_c x_c \quad (38)$$

Here the A_c, B_c, C_c are the matrices of controller which need to be calculated. With the dynamic output feedback control of (38), the closed loop dynamics of the plant with controller can be written as

$$\dot{x}_{cl} = A_{cl} x_{cl} + B_{cl} d \quad (39)$$

$$z_{cl} = C_{cl} x_{cl} \quad (40)$$

where $x_{cl} = \begin{bmatrix} x \\ x_c \end{bmatrix}$, $A_{cl} = \begin{bmatrix} A & BC_c \\ B_c C & A_c \end{bmatrix}$, $B_{cl} = \begin{bmatrix} E_d \\ 0 \end{bmatrix}$ and $C_{cl} = [C_z \quad 0]$. The main purpose of the controller is to minimize the effect of road disturbances on desired outputs. If (A, B) pair is stabilizable, then the $\mathcal{H}_\infty$ optimized output feedback problem can be stated as,

$$\|G_{z_{cl}d}\|_\infty = sup \frac{\|z_{cl}\|_2}{\|d\|_2} < \gamma, \quad min(\gamma) \quad (41)$$

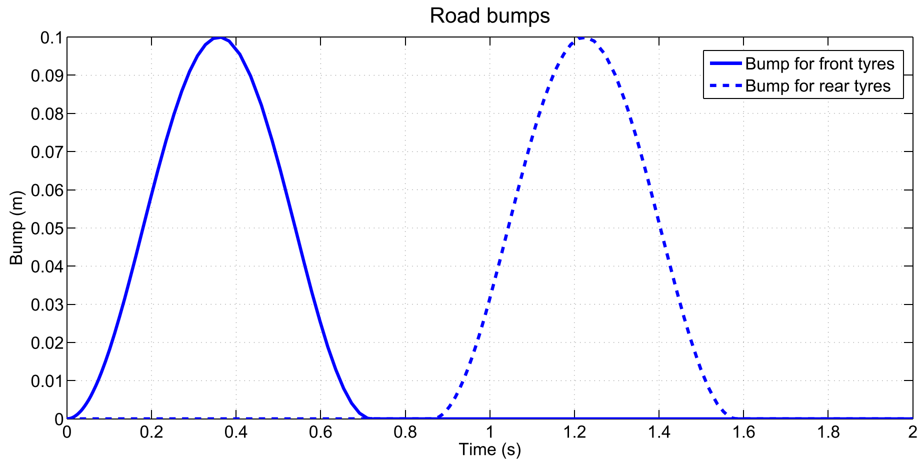


Fig. 2 Road disturbance for vehicle.

This means that the $\mathcal{L}_2$ gain from disturbance to output is less than a positive scalar γ and it is required to minimize γ. For that purpose, an objective function J_2 is defined as follows,

$$J_2 = \int (z_{cl}^T z_{cl} - \gamma^2 d^T d)dt < 0 \tag{42}$$

It is obvious that $J_2 < 0$ guarantees (41). Using S-procedure, it can be seen that with a positive definite $V(t)$, (42) is equivalent to,

$$\int (z_{cl}^T z_{cl} - \gamma^2 d^T d)dt + V(t) < 0 \tag{43}$$

$$\int (z_{cl}^T z_{cl} - \gamma^2 d^T d + \dot{V}(t))dt < 0 \tag{44}$$

A sufficient condition for (43) is given by,

$$z_{cl}^T z_{cl} - \gamma^2 d^T d + \dot{V}(t) < 0 \tag{45}$$

with $V(t) = x_{cl}^T P x_{cl}$, (45) is transformed into,

$$(A_{cl}x_{cl} + B_{cl}d)^T P x_{cl} + x_{cl}^T P(A_{cl}x_{cl} + B_{cl}d) + x_{cl}^T C_{cl}^T C_{cl} x_{cl} - \gamma^2 d^T d < 0 \tag{46}$$

This inequality can be conveniently represented in the form of LMI as,

$$\begin{bmatrix} PA_{cl} + A_{cl}^T P + C_{cl}^T C_{cl} & PB_{cl} \\ * & -\gamma^2 I \end{bmatrix} < 0 \tag{47}$$

Applying Schur's compliment to inequality (47), LMI becomes

$$\begin{bmatrix} PA_{cl} + A_{cl}^T P & PB_{cl} & C_{cl}^T \\ * & -\gamma^2 I & 0 \\ * & * & -I \end{bmatrix} < 0 \tag{48}$$

This LMI is quite complex, so in order to simplify it, some substitutions and transformations are used. Let $P = \begin{bmatrix} P_1 & P_2 \\ * & P_3 \end{bmatrix}$, $P^{-1} = \begin{bmatrix} P_{-1} & P_{-2} \\ * & P_{-3} \end{bmatrix}$ and $\pi = \begin{bmatrix} P_{-1} & I \\ P_{-2}^T & 0 \end{bmatrix}$. As $P > 0$ so $\pi^T P \pi > 0$. Now multiplying (48) with $diag\{\pi^T, I, I\}$ on the left and right, we get

$$\begin{bmatrix} \phi_{11} & \phi_{12} & E_d & \phi_{14} \\ * & \phi_{22} & P_1 E_d & C_z^T \\ * & * & -\gamma^2 I & 0 \\ * & * & * & -I \end{bmatrix} < 0 \tag{49}$$

where

$$\phi_{11} = AP_{-1} + P_{-1}A^T + B\phi + \phi^T B^T$$
$$\phi_{12} = A + X^T$$
$$\phi_{14} = P_{-1}C_z^T$$
$$\phi_{22} = AP_1 + P_1 A^T + \lambda C + C^T \lambda^T$$
$$X = P_1 A P_{-1} + P_1 B\phi + \lambda C P_{-1} + P_2 A_c P_{-2}^T$$

The controller matrices can be obtained as

$$A_c = P_2^{-1}(X - P_1 A P_{-1} - P_1 B\phi - \lambda C P_{-1})P_{-2}^{-T} \tag{50}$$

$$B_c = P_2^{-1}\lambda \tag{51}$$

$$C_c = \phi P_{-2}^{-T} \tag{52}$$

Remark 2. As $PP^{-1} = I$, it gives $P_1 P_{-1} + P_2 P_{-2}^T = I$. After solving inequality (49), P_2 and P_{-2} can be obtained by singular value decomposition of $I - P_1 P_{-1}$.

The derivations above lead to the following theorem.

Theorem 2. *The closed loop system (39) is robust against disturbances with an $\mathcal{H}_\infty$ disturbance attenuation level less than γ ($\gamma > 0$), if the following LMI's hold for matrices P_1, P_{-1}, P_{-2}, λ and ϕ of appropriate dimensions.*

$$\begin{bmatrix} P_{-1} & I \\ I & P_1 \end{bmatrix} > 0 \tag{53}$$

$$\begin{bmatrix} \phi_{11} & \phi_{12} & E_d & \phi_{13} \\ * & \phi_{21} & P_1 E_d & C_z^T \\ * & * & -\gamma^2 I & 0 \\ * & * & * & -I \end{bmatrix} < 0 \tag{54}$$

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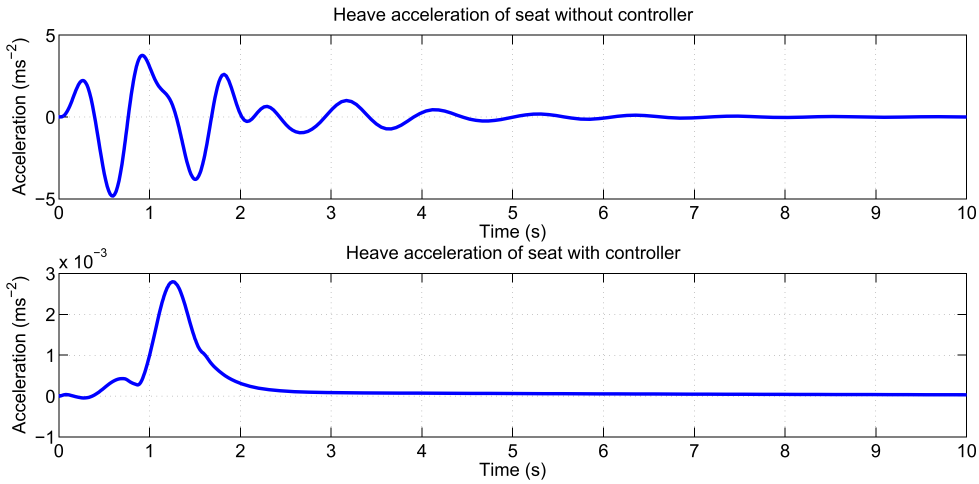
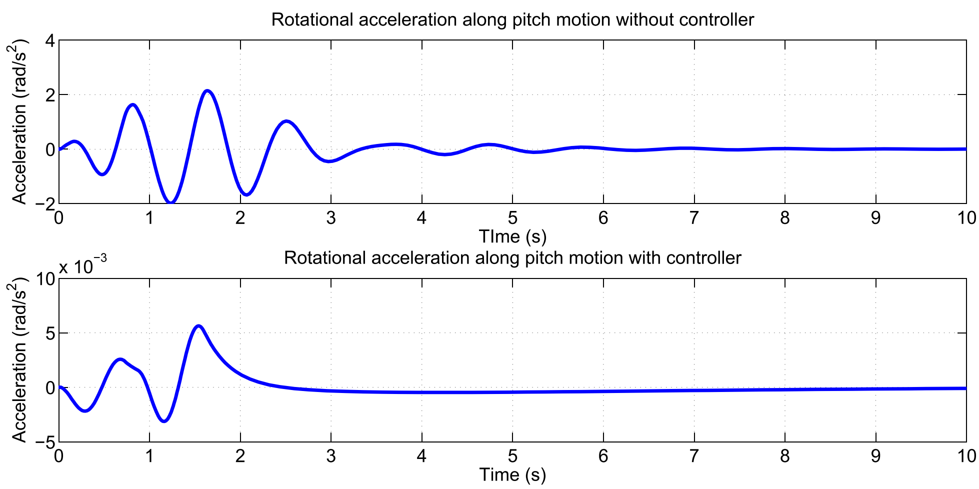
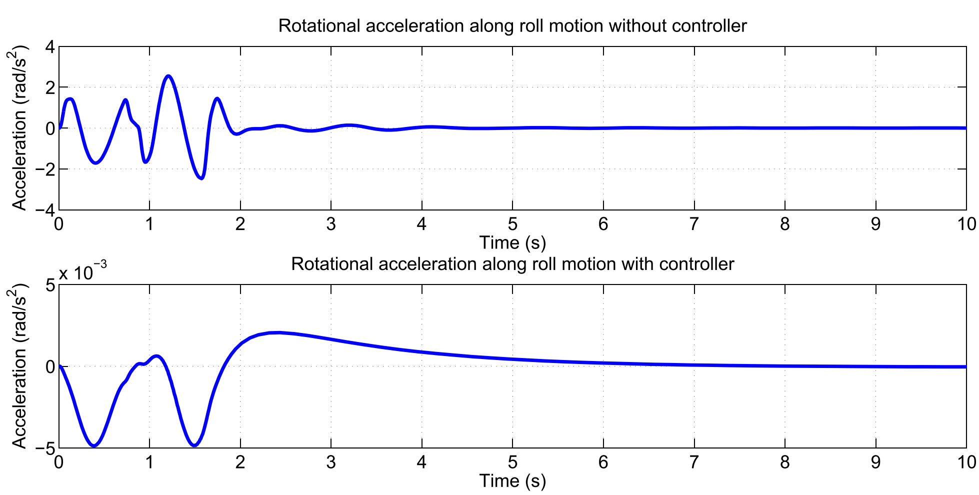


(a) Heave acceleration of seat



(b) Rotational acceleration of vehicle along pitch motion



(c) Rotational acceleration of vehicle along roll motion

Fig. 3 Performance of state feedback controller.

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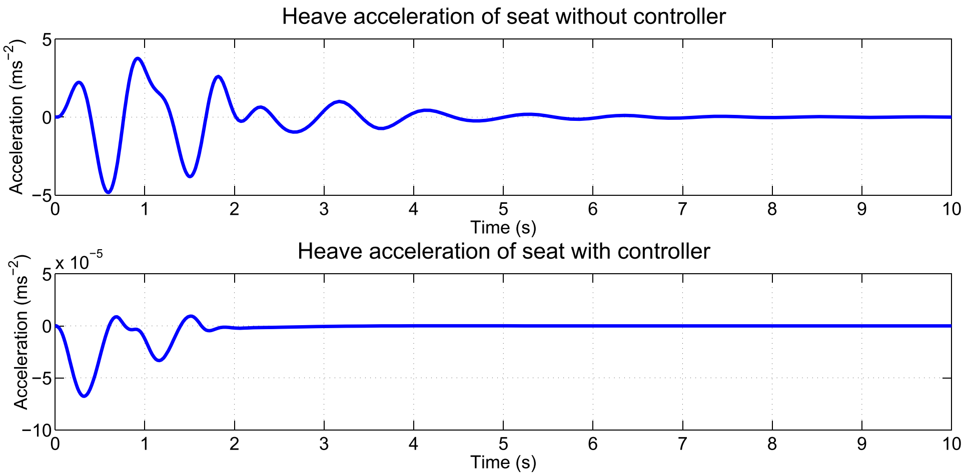
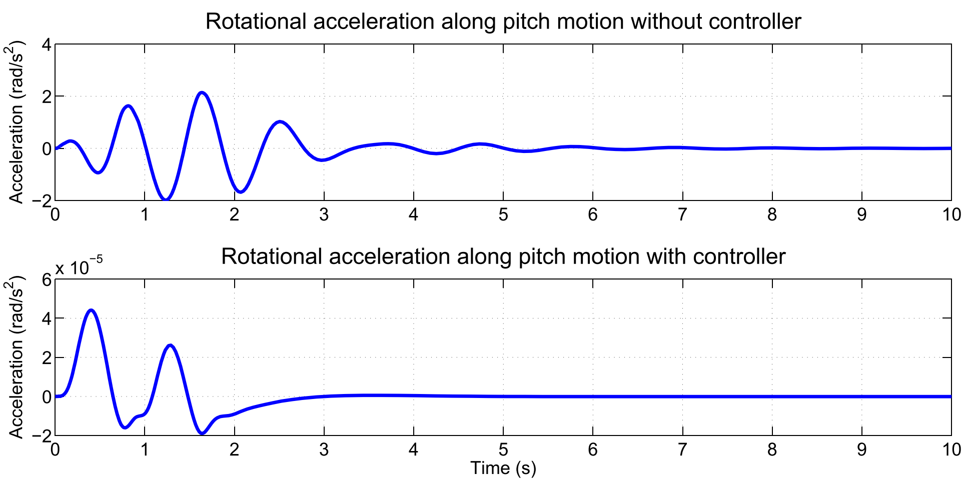
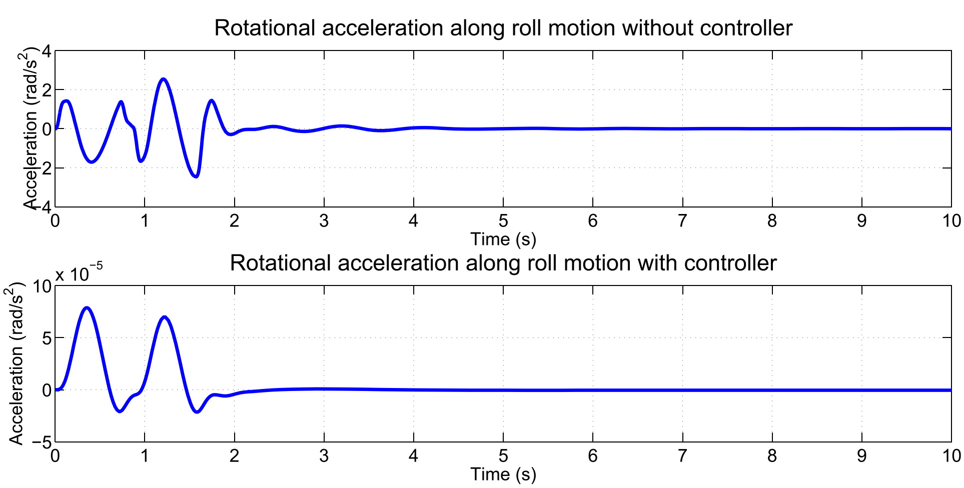


(a) Heave acceleration of seat



(b) Rotational acceleration of vehicle along pitch motion



(c) Rotational acceleration of vehicle along roll motion

Fig. 4 Performance of dynamic output feedback controller.

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3.2.1. Simulation results

Using the results of Theorem 2, the LMIs (53) and (54) are solved for the vehicle active suspension system. The achieved value of disturbance attenuation level $\gamma = 1.81 \times 10^{-4}$, which is quite small. This means that the designed controller will significantly attenuate the effect of road disturbances on the driver's seat. The simulation results for vehicle active suspension system with the dynamic output feedback controller are shown in Fig. 4. The system was subjected to the disturbance of Fig. 2. Above results clearly show that the magnitude of acceleration has been drastically decreased in response to road disturbances. So the output feedback controller proposed is valid.

Remark 3. It can be seen from the simulation results that the output feedback control law is, at least, as good as the state feedback control presented in Section 3.1. Thus, in case that not all states are being measured, the output feedback can be utilized instead of the state feedback control.

It is important to note that vehicle active suspension system has multiple actuators, the redundancy in the actuators can be effectively utilized to add fault tolerant control in the system. Design of fault tolerant control for the system is a future work. For that purpose, reference model approach proposed by Nazir et al. (2015) will be utilized for fault detection and robust control theory will be studied to re-configure controllers in the presence of faults. In addition, application of artificial intelligence techniques, including fuzzy techniques proposed in Abadi and Khooban (2015), and study of the effect of time delays in actuators by following the approach of Elnaggar and Khalil (2016) are our future study.

4. Conclusion

With the continuously increasing technology, the complexity of every system is also increasing at the same rate. Vehicles are an integral part of everybody's daily routine. To get better ride quality and ride comfort research has been going on for decades. Robust control of suspension system of vehicle is important to provide good ride comfort to passengers. So the unwanted accelerations need to be minimized. In this paper, a more detailed mathematical model for vehicle active suspension system, including the dynamics of driver, is derived. State feedback and output feedback controllers are proposed and simulations are carried out to demonstrate that the designed controllers provide required performance.

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