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The critical radius of insulation in thermal radiation environment

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Abstract Critical radius of insulation for a circular tube subjected to radiative and convective heat transfer has been studied analytically. It is assumed that condensation or evaporation takes place inside the circular tube such that the bulk fluid temperature inside the tube remains constant. As the fluid is transported from one end to the other, either an increase or decrease of heat transfer is desired depending on the application. The variation of the rate of heat transfer with respect to the variation of insulation thickness is studied. It is found that an critical insulation thickness may exist such that the heat transfer between the fluid and the radiative environment becomes a maximum. For certain special cases, explicit solutions to the critical insulation thickness are obtained.

Nomenclature

A	dimensionless radiation parameter, $\sigma\varepsilon r T^3_f/k$
b	temperature ratio, T_∞/T_f
h_i	heat transfer coefficient inside the pipe, W/m^{2}K
h_0	heat transfer outside the insulation, W/m^{2}K
k	thermal conductivity, W/mK
k_w	tube wall thermal conductivity, W/mK
q	dimensionless heat transfer rate, $\frac{1}{2\pi k T_f}\frac{\delta \dot{Q}}{dx}$
Q	rate of heat transfer, W
r	radius, m
r_i	inner tube radius, m
r_0	outer radius of insulation, m
R_{tot}	total heat transfer resistance, mK/W
t	thickness, m
t_w	tube wall thickness, m
T_f	fluid bulk temperature, K
T_s	outer insulation surface temperature, K
T_∞	ambient temperature, K
x	axial distance, m
ε	emissivity
λ	dimensionless convection parameter, $h_o r/k$
ρ	dimensionless radius, r_0/L
σ	Stefan-Boltzman constant
τ	dimensionless temperature, T_s/T_f

Subscripts

crit	critical
max	maximum
tot	total

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1 Introduction

Optimum distribution of a finite amount of insulation material over a nonisothermal wall, in order to minimize the total heat loss from the wall to the ambient was studied by Bejan [1]. He showed that, in the case of a single-phase stream suspended in an environment of different temperature, uniform thickness of insulation for circular pipe is the best insulation assuming that the outer radius of insulation is greater than the critical radius. However, it is known that when the insulation thickness is close to the critical thickness value corresponding to the critical radius, heat transfer might be increased as a result of insulation rather than reduction of heat transfer. On the other hand, when the objective is to increase the heat transfer, heat transfer enhancement methods such as extended surfaces are used. A simple alternative to heat transfer enhancement for certain applications could be using critical radius insulation.

Thermal design optimization, configuration, subject to constraints is a very basic problem in research as well as in thermal science education [2–4]. The motivation in pipe insulation is often to minimize total costs. That is, the cost of the insulation, its installation and maintenance as well as the cost of the energy lost. Heat transfer principles and cost information can be used in defining the overall cost function that is to be minimized. All the size (volume, weight) constraints related to manufacturing and installation must also be considered. One may wish to minimize capital and operation costs as well as heat loss. Ito et al. [5] applied multiple objective functions for the design analysis of a piping system to minimize both the heat loss and the amount of insulation used. In these types of problems, a common approach is to sum all objective functions with appropriate weighting factors, and minimize the resulting composite function [6]. However, solving each of the problems separately and using judgment in selecting the solution for the composite problem seems to be a better choice [7].

Industrial plants and chemical processing plants in particular contain intricate and costly piping configurations. Piping systems are also encountered in many other situations including water supply, fire protection, and district cooling/heating applications. Uninsulated steam distribution and condensate return lines are a constant source of wasted energy. Heat loss from an uninsulated pipe of 1 inch diameter and 10 m length through natural convection and 25 °C bulk to ambient temperature difference is estimated to be 50×10^6 kJ/yr. Insulation can typically reduce the energy losses by 90% and help ensure proper steam pressure at plant equipment. Therefore, any surface over 50 °C such as boiler surfaces, steam and condensate return piping, and fittings should be insulated. For proper insulation, however, the critical insulation thickness must be taken into consideration so that no surprises will be faced.

In the present study, the well/known concept of critical radius for convection type of heat transfer analysis is extended to radiation heat transfer applications. In this regard, a circular pipe through which fluid is transported from one end to the other is considered. The outer surface is subjected to both convection and radiation with the surroundings. Possible critical radius for various parametric conditions is studied.

2 Formulation of the problem

Consider the circular pipe through which a given fluid is transported from one end to the other as shown in Fig. 1. It is assumed that phase change (e.g. condensation) is taking place in the pipe so that the fluid temperature T_f remains constant. The inner and outer radii of the pipe are r_i and r, respectively. The thermal conductivity of the pipe material is k_w. Heat transfer takes place through the inner convective heat transfer coefficient h_i through the inner surface of the pipe and outer combined (convective and radiative) heat transfer coefficient h_o through the outer surface of the insulation. The thickness of insulation t is to be optimized such that the heat transfer from the fluid inside the tube to the environment at temperature T_∞ is made maximum.

The heat loss from the fluid to the environment per unit length of pipe in Fig. 1 is

$$\frac{\delta \dot{Q}}{dx} = \frac{1}{R_{tot}}(T_f - T_\infty) \tag{1}$$

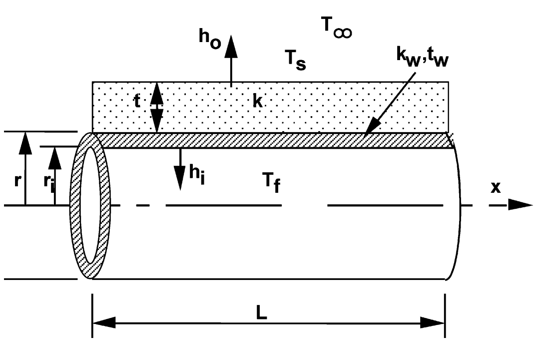

Fig. 1. Sketch of insulated circular duct

where the total thermal resistance is

$$R_{tot} = \frac{1}{2\pi r_i h_i} + \frac{1}{2\pi k_w}\ln\left(\frac{r}{r_i}\right) + \frac{1}{2\pi k}\ln\left(\frac{r_o}{r}\right) + \frac{1}{2\pi r_o(h_o + h_r)} \tag{2}$$

in which $r_i = r - t_w$ is the inner radius of pipe, $r_o = r + t$ is the outer radius of insulation and the radiation heat transfer coefficient is defined as

$$h_r = \frac{\sigma\varepsilon(T_s^4 - T_\infty^4)}{T_s - T_\infty} = \sigma\varepsilon(T_s^2 + T_\infty^2)(T_s + T_\infty). \tag{3}$$

On the other hand, energy balance requires that

$$\frac{T_f - T_\infty}{R_{tot}} = \frac{T_f - T_s}{\frac{1}{2\pi r_i h_i} + \frac{1}{2\pi k_w}\ln\left(\frac{r}{r_i}\right) + \frac{1}{2\pi k}\ln\left(\frac{r_o}{r}\right)} = \frac{T_s - T_\infty}{\frac{1}{2\pi r_o(h_o + h_r)}} \tag{4}$$

The heat transfer per unit length of pipe therefore becomes

$$\frac{1}{2\pi k}\frac{\delta \dot{Q}}{dx} = \frac{T_f - T_s}{R_c + \ln\left(\frac{r_o}{r}\right)} = \frac{T_s - T_\infty}{\frac{k}{r_o(h_o + h_r)}} \tag{5}$$

where the constant terms in the thermal resistance are combined into

$$R_c = \frac{k}{r_i h_i} + \frac{k}{k_w}\ln(r/r_i). \tag{6}$$

The following dimensionless variables can be used

$$\rho = \frac{r_o}{r}, \quad \tau = \frac{T_s}{T_f}, \quad q = \frac{1}{2\pi k T_f}\frac{\delta \dot{Q}}{dx} \tag{7}$$

Thus, using equations (3),(4), and (5) in equation (7), the dimensionless form of the heat transfer becomes

$$q = \frac{1 - \tau}{R_c + \ln\rho} = \lambda\rho(\tau - b) + A\rho(\tau^4 - b^4) \tag{8}$$

where

$$\lambda = \frac{h_o r}{k}, \quad A = \frac{\sigma\varepsilon r T_f^3}{k}, \quad \text{and} \quad b = \frac{T_\infty}{T_f}. \tag{9}$$

2.1 Optimization problem

The heat transfer in equation (8) is a function of ρ and τ. On the other hand ρ and τ are inter-related as shown also in equation (8). For possible extremum in heat transfer q with respect to the insulation thickness ρ, the first derivative of q with respect to ρ is set equal to zero. Since from equation (8)

$$q = \lambda\rho(\tau - b) + A\rho(\tau^4 - b^4) \tag{10}$$

the first derivative is

$$\frac{dq}{d\rho} = \lambda(\tau - b) + A(\tau^4 - b^4) + (\lambda\rho + 4A\rho\tau^3)\frac{d\tau}{d\rho} \tag{11}$$

On the other hand, the inter-relationship between ρ and τ according to equation (8) is

$$\frac{1-\tau}{R_c+\ln\rho}=\lambda\rho(\tau-b)+A\rho(\tau^4-b^4) \tag{12}$$

Therefore,

$$\frac{d\tau}{d\rho}=-\frac{\lambda(\tau-b)+A(\tau^4-b^4)}{1+(4A\tau^3+\lambda)(R_c+\ln\rho)}(1+R_c+\ln\rho) \tag{13}$$

Substituting equation (13) into equation (11) it can be shown that

$$\frac{dq}{d\rho}=\frac{[\lambda(\tau-b)+A(\tau^4-b^4)]\times[1-\rho(4A\tau^3+\lambda)]}{1+\rho(4A\tau^3+\lambda)(R_c+\ln\rho)}. \tag{14}$$

Setting equation (14) equal to zero the critical insulation thickness for the extremum heat transfer can be found

$$\left.\frac{dq}{d\rho}\right|_{\rho=\rho_{crit}}=0 \tag{15}$$

since $\tau = b$ is a trivial solution, therefore

$$\rho_{crit}=\frac{1}{4A\tau^3+\lambda} \tag{16}$$

where τ is obtained by substituting equation (16) into equation (12). That is, τ is the root of

$$\frac{1-\tau}{R_c-\ln(4A\tau^3+\lambda)}=\frac{\lambda(\tau-b)+A(\tau^4-b^4)}{4A\tau^3+\lambda} \tag{17}$$

The heat transfer at the critical insulation thickness given by equation (16) can be shown to be a maximum. For this reason, equation (14) can be re-written more conveniently as

$$\frac{dq}{d\rho}=\frac{E\times F}{G} \tag{18}$$

where

$$E=\lambda(\tau-b)+A(\tau^4-b^4) \tag{19}$$

$$F=1-\rho(4A\tau^3+\lambda) \tag{20}$$

and

$$G=1+(4A\tau^3+\lambda)(R_c+\ln\rho) \tag{21}$$

Now, the second derivative of heat transfer q with respect to ρ is

$$\left.\frac{d^2q}{d\rho^2}\right|_{\rho=\rho_{crit}}=\left.\frac{F}{G}\frac{\partial E}{\partial\rho}\right|_{\rho=\rho_{crit}}+\left.\frac{E}{G}\frac{\partial F}{\partial\rho}\right|_{\rho=\rho_{crit}}-\left.\frac{EF}{G^2}\frac{\partial G}{\partial\rho}\right|_{\rho=\rho_{crit}} \tag{22}$$

noting from equation (16) and (20) that $F|_{\rho=\rho_{crit}}=0$. Thus,

$$\begin{aligned}\left.\frac{d^2q}{d\rho^2}\right|_{\rho=\rho_{crit}}&=\left.\frac{E}{G}\frac{\partial F}{\partial\rho}\right|_{\rho=\rho_{crit}}\\&=\frac{E}{G}\left.\left[-(4A\tau^3+\lambda)-12A\tau^2\rho\frac{\partial\tau}{\partial\rho}\right]\right|_{\rho=\rho_{crit}}\end{aligned} \tag{23}$$

From equations (13),(19), and (21)

$$\frac{\partial\tau}{\partial\rho}=\frac{-E(\ln\rho+R_c+1)}{G} \tag{24}$$

Substituting equation (24) into equation (23) and organizing the resulting equation, we, after some lengthy algebra, obtain

$$\left.\frac{d^2q}{d\rho^2}\right|_{\rho=\rho_{crit}}=-\left.\frac{E}{G^2}\right|_{\rho=\rho_{crit}}\left[\frac{1+R_c-\ln(4A\tau^3+\lambda)}{4A\tau^3+\lambda}\right]\left[\lambda^2(1-2A\tau^2)^2+12Ab\tau^2(Ab^3+\lambda)\right] \tag{25}$$

Some remarks on equation (25):

- Since $\lambda > 0$, $A > 0$, and $\tau > b$, from equation (19), $E > 0$. Thus, the first term in equation (25) is positive. That is,

  $(E/G^2)|_{\rho=\rho_{crit}} > 0.$

- For physically meaningful solution, $\rho_{crit} > 1 \rightarrow 0 < (4A\tau^3 + \lambda) < 1 \rightarrow \ln(4A\tau^3 + \lambda) < 0$. Noting that $R_c > 0$, then, the second term in equation (25) is also positive. That is,

  $\frac{1+R_c-\ln(4A\tau^3+\lambda)}{4A\tau^3+\lambda} > 0.$

- Since $\lambda > 0$, $A > 0$, and $b > 0$, clearly, the third term in equation (25) is positive. That is,

  $\lambda^2(1-2A\tau^2)^2+12Ab\tau^2(Ab^3+\lambda) > 0.$

  Therefore, equation (25) indicates that

$$\left.\frac{d^2q}{d\rho^2}\right|_{\rho=\rho_{crit}}<0 \tag{26}$$

This means that the heat transfer for $\rho = \rho_{crit}$ is a maximum.

3
Results and discussion

Critical insulation thickness that will result in the maximum heat transfer is shown to exist in the previous section. This insulation thickness is found to be a function of A (representing radiation), λ (convection), b (surrounding to bulk fluid temperature difference) and R_c as can be seen from equations (16) and (17). Some special cases are discussed in the following.

Case 1: Convection only:

$A = 0$ When the radiation is negligible, the critical insulation thickness as given in equation (16) becomes

$$\rho_{crit}=\rho_1=\frac{1}{\lambda}=\frac{k}{h_0r} \tag{27}$$

This result is well known from the literature [8, 9].The dimensionless outer surface temperature of the insulation in this case is obtained explicitly using equation (17) as

$$\tau = \frac{1 - b(\ln\lambda - R_c)}{1 - (\ln\lambda - R_c)} \tag{28}$$

Using equation (10) the maximum heat transfer is evaluated to be

$$q_{\max} = q|_{\rho=\rho_1} = \frac{1 - b}{1 - \ln\lambda + R_c} \tag{29}$$

The locus of maximum heat transfer is obtained by eliminating from both equations (27) and (29)

$$q_{\max} = \frac{1 - b}{1 + \ln\rho_1 + R_c} \tag{30}$$

For an optimum to exist, $\rho_1 > 1$ i.e. $\lambda < 1$. Therefore, for the limiting case where $\rho_1 = 1$ (i.e., zero critical insulation thickness), the surface temperature becomes

$$\tau|_{\rho_1=1} = \frac{1 + bR_c}{1 + R_c} \tag{31}$$

The corresponding heat transfer in this case is obtained to be

$$q|_{\rho_1=1} = \frac{1 - b}{1 + R_c} \tag{32}$$

The constant term R_c does not affect the qualitative assessment of the optimization problem; therefore, the value of R_c is set to zero in the following discussion for convenience. Figure 2 shows the variation of heat transfer as function of insulation thickness for several cases of convection parameter λ and $b = 0.3$. The cases where $\lambda < 1$ are considered for which optimum solutions exist. The locus of the maximum heat transfer is also added to the figure in a dashed line.

Case 2: Radiation only: $\lambda = 0$

In this case, radiation is considered to be the dominant mode of heat transfer and therefore the convection mode is neglected ($\lambda = 0$). The critical insulation thickness in this case, according to equation (16) is

$$\rho_{crit} = \rho_2 = \frac{1}{4A\tau^3} \tag{33}$$

where the corresponding outer surface temperature of the insulation τ is the root of

$$\frac{1 - \tau}{R_c + \ln(\rho_2)} = \frac{1 - \tau}{R_c - \ln(4A\tau^3)} = \frac{\tau^4 - b^4}{4\tau^3} \tag{34}$$

Using equation (8), the maximum heat transfer in this case becomes

$$q_{\max} = q|_{\rho=\rho_2} = \frac{1 - \tau}{R_c - \ln(4A\tau^3)} = \frac{\tau^4 - b^4}{4\tau^3} \tag{35}$$

The locus of maximum heat transfer in this case is very difficult to express explicitly. However, one of the roots of the following fourth order polynomial is the maximum heat transfer for any critical insulation thickness.

$$[1 - q_{\max}(R_c + \ln\rho_2)]^4 - 4q_{\max}[1 - q_{\max}(R_c + \ln\rho_2)]^3 - b^4 = 0 \tag{36}$$

For an optimum to exist, it is required that

$$\rho_2 > 1, \quad \text{i.e.,}\ A < \frac{1}{4\tau^3} \tag{37}$$

Therefore, in the limit where $\rho_2 = 1$, the surface temperature of the pipe with zero insulation thickness, denoted as τ_2, is found to be the root of

$$\frac{1 - \tau_2}{R_c} = \frac{\tau_2^4 - b^4}{4\tau_2^3} \tag{38}$$

where equation (38) is obtained by setting $\rho_2 = 1$ in equation (34). The heat transfer in this case becomes

$$q|_{\rho_2=1} = \frac{1 - \tau_2}{R_c} = \frac{\tau_2^4 - b^4}{4\tau_2^3} \tag{39}$$

where τ_2 is the root of equation (38).

Figure 3 shows the variation of heat transfer as function of insulation thickness for several cases of radiation parameter A and $b = 0.3$. The cases where $A < \frac{1}{4\tau^3}$]] >are considered for which optimum solutions exist. The locus of the maximum heat transfer is also added to the figure in a dashed line.

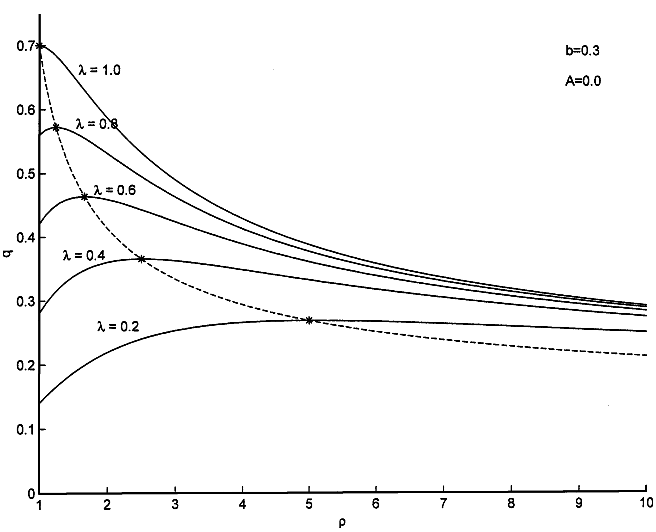

Fig. 2. The variation of heat transfer as function of insulation thickness for several cases of convection parameter λ with no radiation and $b = 0.3$

Case 3: Space Applications: $\lambda = 0$ and $b = 0$

In an environment with zero surrounding temperature and no convection, the critical insulation thickness, according to equation (16), is

$$\rho_{crit} = \rho_3 = \frac{1}{4A\tau^3} \tag{40}$$

where the corresponding outer surface temperature of the insulation τ is the root of

$$\frac{1 - \tau}{R_c + \ln(\rho_3)} = \frac{1 - \tau}{R_c - \ln(4A\tau^3)} = \frac{\tau}{4} \tag{41}$$

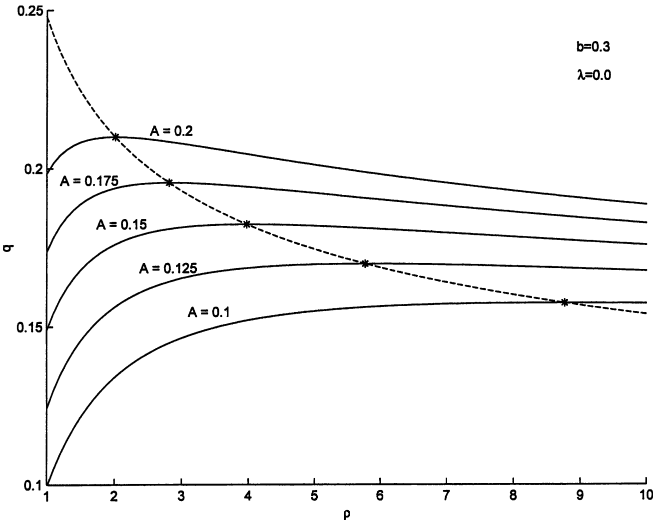

Fig. 3. Variation of the heat transfer as function of insulation thickness for several cases of radiation parameter A with no convection and $b = 0.3$

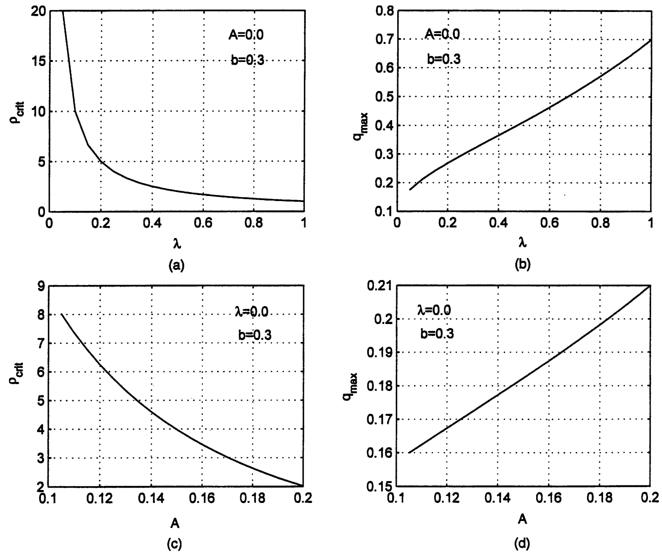

Fig. 5. Variations of the critical insulation thickness and the maximum heat transfer as function of convection λ and the radiation A parameters

Maximum heat transfer in this case becomes

$$q_{\max} = q|_{\rho=\rho_3} = \frac{1-\tau}{R_c - \ln(4A\tau^3)} = \frac{\tau}{4} \tag{42}$$

and the locus of maximum heat transfer is obtained to be

$$q_{\max} = \frac{1}{4 + R_c + \ln \rho_3} \tag{43}$$

For an optimum to exist in this case $\rho_3 > 1$, i.e., $A < \frac{1}{4\tau^3}$. Therefore, in the limit where $\rho_3 = 1$ (zero insulation thickness), the corresponding outer surface temperature, denoted as τ_3, of the pipe is

$$\tau_3 = \frac{4}{4 + R_c} \tag{44}$$

where equation (44) is obtained by setting $\rho_3 = 1$ in equation (41). The heat transfer for this particular case becomes

$$q_{\max} = \frac{1}{4 + R_c}. \tag{45}$$

Figure 4 shows the variation of heat transfer as function of insulation thickness for several cases of radiation parameter A. As for the previous case, values of $A < \frac{1}{4\tau^3}$ are considered for which optimum solutions exist. The locus of the maximum heat transfer is also added to the figure in a dashed line.

In all the three cases discussed above, the critical insulation thickness decreases as the parameter of convection λ or that of the radiation A increase as shown in Fig. 5. On the other hand, it is clear also from Fig. 5 that

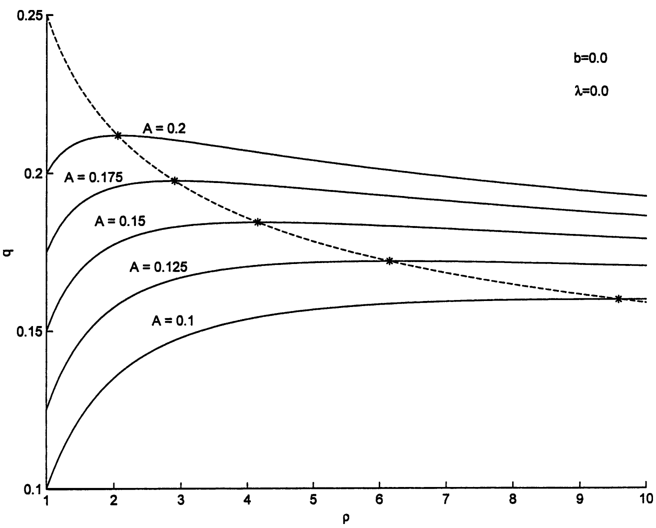

Fig. 4. Variation of the heat transfer as function of insulation thickness for several cases of radiation parameter A with no convection and $b = 0$

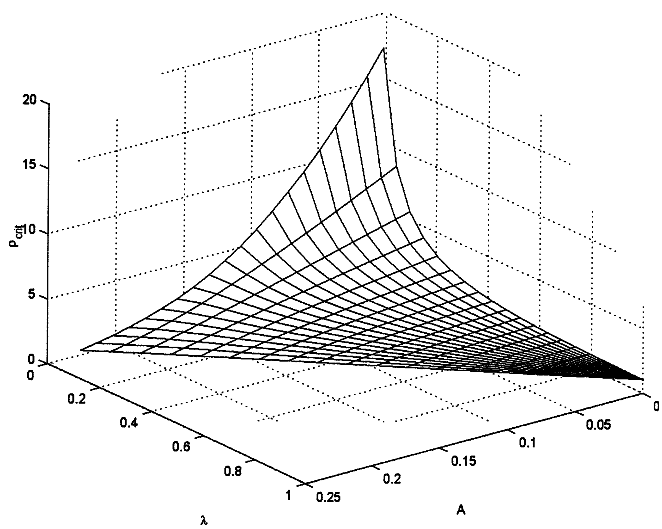

Fig. 6. The variation of the critical insulation thickness as function of λ and A for $b = 0.3$

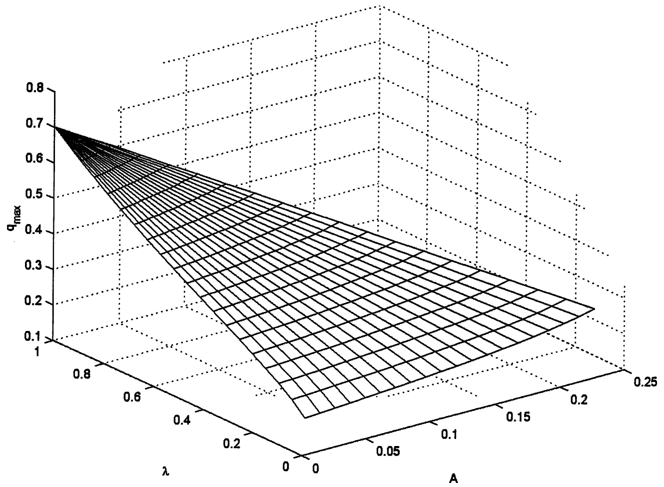

Fig. 7. The variation of the maximum heat transfer as function of λ and A for $b = 0.3$

the maximum heat transfer increases as both the convection and radiation parameter increases.

Figures 6 and 7 show the variation of the critical insulation thickness and maximum heat transfer, respectively, as function of λ and A. The critical insulation thickness is found to increase as both convection and radiation decreases. On the other hand, the maximum heat transfer decreases when convection or radiation decreases, as expected. This is due to the increase of the critical insulation thickness.

4
Conclusions

The following conclusions can be derived from the present study.

1. A critical radius of insulation can be found for circular tube subjected to radiative and convective heat transfer environment provided that certain constraints are satisfied.
2. The critical radius of insulation increases when either the convection or radiation decreases. Therefore, the existence of the critical radius depends on both the convection and radiation parameters. As the critical insulation radius increases for low convection and radiation heat transfers, it becomes feasible to use critical radius of insulation for heat transfer enhancement only for high convection or radiation heat transfer environment.
3. The maximum heat transfer is a function of radiation and convection parameters. For the increase of both the radiative and convective parameters, the maximum heat transfer also increases.
4. Due to the nonlinear nature of the problem, the solution for the critical radiation is obtained by numerical means, however, explicit analytical solutions are obtained for certain special cases.

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