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Buckling and Vibration of Functionally Graded Material Columns Sharing Duncan's Mode Shape, and New Cases

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ABSTRACT

In this study, the closed-form solution for the buckling of an inhomogeneous simply supported column that was uncovered by the noted British engineer Duncan in 1937, is first derived in a straightforward manner. It deals with buckling of a centrally compressed inhomogeneous column. It is also found that there are several other columns with variable axial functionally graded material (FGM) that share the same qualities as Duncan's column. It is then shown that the mode postulated by W.J. Duncan (1894–1970), FRS and the newly found modes, have a greater validity, namely the freely vibrating beam, albeit with different flexural rigidity than the centrally compressed one, may possess the same buckling mode. It is demonstrated also that there exists an inhomogeneous beam under axial compression whose vibration mode coincides with the buckling modes in the previous cases.

1. Introduction

Duncan [3] devoted his study to efficacy of the Bubnov-Galerkin method. *Inter alia*, he communicated, without derivation, closed-form solution for buckling of inhomogeneous columns. As is known, the closed form solutions for inhomogeneous structures are extremely rare. Therefore, it is interesting to know how Duncan obtained his solution. Moreover, a pertinent question arises if there are other columns or beams for which Duncan's mode shape is valid, or if there are other similar examples.

This study addresses above issues. It shows how one can derive Duncan's classic solution, and constructs analogous solutions for the vibration problems. Remarkably, it turns out, that there exists a vibrating column whose vibration mode coincides with Duncan's buckling mode. Note that monograph by Elishakoff [5] contains analysis for other candidate mode shapes of beams in vibratory or buckling conditions. The present study is apparently the first one that addresses Duncan's mode shape directly.

Duncan [3] proposed that the shape of the mode be taken as

$$W(\xi) = 7\xi - 10\xi^3 + 3\xi^5 \quad (1)$$

and this shape satisfies the simple support conditions at the two ends. Later on, Elishakoff [5] suggested another mode

$$W(\xi) = \xi - 2\xi^3 + \xi^4 \quad (2)$$

which has similar properties. These two case can be realized with spatial distribution of material properties that will be given below. The question that one may ask is if there exist other simple shapes that have similar properties. These new cases will yield different buckling loads and spatial distribution of the material properties. In the next section a general derivation is presented for the problem. Other recent studies of inhomogeneous beams and columns include those of Akulenko and Nesterov [1], Caruntu [2], Ece, Ayadoğlu and Taskin [4], Sina and Navazi [10], Gilat, Caliò and Elishakoff [6], Huang and Li [7], Huang and Luo [8], Zarrinzadeh, Attarnejad and Shahba [11], and Maròti [9], among others.

2. Derivation of Duncan's solution and other new solutions

Consider the governing differential equation for the buckling of centrally compressed inhomogeneous column simply supported at its two ends:

$$D(\xi)\frac{d^2W}{d\xi^2} + P_{cr}L^2W = 0. \quad (3)$$

One can show that the function in Eq. (1), postulated by Duncan [3] satisfies the boundary conditions of the simple supports

$$W(0) = D(\xi)W''(0) = W(1) = D(\xi)W''(1) = 0 \quad (4)$$

where the prime denotes the differentiation with respect to ξ. We pose the following question: Is there an inhomogeneous column that has expression in Eq. (1) as its buckling mode? To answer this question, we observe that the second term in Eq. (1), namely, $P_{cr}L^2w$ represents a

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fifth order polynomial. The second derivative of the buckling mode in the first term is a third order polynomial. Therefore, in order for the first term $D(\xi)w''$ to be also a fifth order polynomial, it is sufficient that $D(\xi)$ is a second order polynomial. Hence, we seek $D(\xi)$in the following form

$$D(\xi) = d_0 + d_1\xi + d_2\xi^2. \tag{5}$$

Now we look for other possible fifth order polynomials that can be the mode shape for buckling of a simply supported beam. Then we shall try the following mode

$$W(\xi) = w_0 + w_1\xi + w_2\xi^2 + w_3\xi^3 + w_4\xi^4 + w_5\xi^5. \tag{6}$$

Since we have a simple support at $\xi = 0$ we must have $w_0 = w_2 = 0$. Then we substitute Eqs. (5) and (6) into Eq. (3) and collect terms with the same power of ξ and obtain the following five equations:

$$6w_3 + PL^2 w_1 = 0 \tag{7}$$

$$12w_4 + 6w_3 d_1 = 0 \tag{8}$$

$$6w_3 d_2 + 12w_4 d_1 + 20w_5 + PL^2 w_3 = 0 \tag{9}$$

$$20w_5 d_1 + 12w_4 d_2 + PL^2 w_4 = 0 \tag{10}$$

$$20w_5 d_2 + PL^2 w_5 = 0 \tag{11}$$

and two more equations are obtained from the requirement of zero deflection and moment at $\xi = 1$ as

$$w_1 + w_3 + w_4 + w_5 = 0, \tag{12}$$

$$6w_3 + 12w_4 + 20w_5 = 0. \tag{13}$$

Eqs. ((7)–(13)) represent a set of 7 nonlinear equations with seven unknowns $w_1, w_3, w_4, w_5, d_1, d_2,$ and P. We obtain four solutions (and one trivial solution where all the unknowns are zero).

a. First solution—Duncan's [3] mode shape

$$W(\xi) = 7\xi - 10\xi^3 + 3\xi^5; \quad D(\xi) = d_0\left(1 - \frac{3}{7}\xi^2\right); \quad P_{cr} = \frac{60d_0}{7L^2}. \tag{14}$$

b. Second solution—Elishakoff's [5] mode shape

$$W(\xi) = \xi - 2\xi^3 + \xi^4; \quad D(\xi) = d_0\left(1 + \xi - \xi^2\right); \quad P_{cr} = \frac{12d_0}{L^2}. \tag{15}$$

c. Third solution—First new mode shape

$$W(\xi) = \frac{8}{15}\xi - \frac{4}{3}\xi^3 + \xi^4 - \frac{1}{5}\xi^5; \quad D(\xi) = d_0\left(1 + \frac{3}{2}\xi - \frac{3}{4}\xi^2\right); \quad P_{cr} = \frac{15d_0}{L^2}. \tag{16}$$

d. Fourth solution—Second new mode shape

$$W(\xi) = \frac{1}{15}\xi - \frac{2}{3}\xi^3 + \xi^4 - \frac{2}{5}\xi^5; \quad D(\xi) = d_0\left(1 + 3\xi - 3\xi^2\right); \quad P_{cr} = \frac{60d_0}{L^2}. \tag{17}$$

These four solutions are listed in Table 1 with the modes and the corresponding stiffness distribution along the beam. It is evident that solution (d) above is the second buckling mode as can be also seen from the value of the buckling load that is much higher than the value of the three other solutions. Additionally, the associated mode shape possesses an internal node, serving as an indication that one deals with the second mode-shape.

3. Comparison with uniform column

Let us compare the derived buckling load with that of the associated uniform column. We can introduce the latter columns as that with average flexural rigidity, defined as

$$D_{ave} = \int_0^1 D(\xi)d\xi. \tag{18}$$

In the Duncan's [3] example, the average flexural rigidity, in view of Eq. (14) is

$$D_{ave} = \frac{6}{7}d_0. \tag{19}$$

Thus, $d_0 = 7/6D_{ave}$. The buckling load is from Eq. (14)

$$P_{cr} = \frac{10D_{ave}}{L^2} \tag{20}$$

which is extremely close, from above, to the Euler buckling load of the *uniform* column with flexural rigidity D_{ave}:

$$P_{cr} = \frac{\pi^2 D_{ave}}{L^2}. \tag{21}$$

For the second case (Elishakoff's shape) we have

$$D_{ave} = \frac{7}{6}d_0. \tag{22}$$

The buckling load is from Eq. (15)

$$P_{cr} = \frac{72D_{ave}}{7L^2} \tag{23}$$

which is 4.2% higher than the uniform column.

For the third case (the first new solution) we have

$$D_{ave} = \frac{3}{2}d_0. \tag{24}$$

The buckling load is from Eq. (16)

$$P_{cr} = \frac{10D_{ave}}{L^2} \tag{25}$$

exactly as for the Duncan case.

For the fourth case (second new solution) we have again

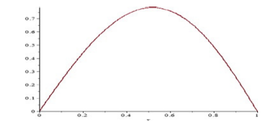
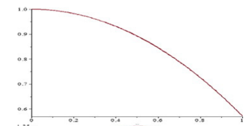
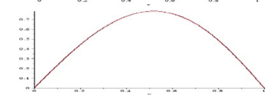
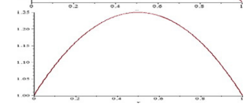
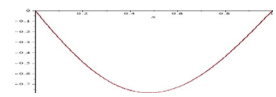
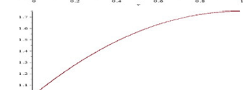
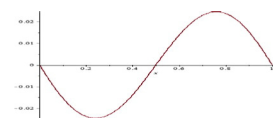
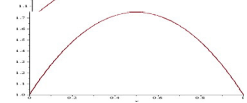
$$D_{ave} = \frac{3}{2}d_0. \tag{26}$$

The buckling load is from Eq. (17)

$$P_{cr} = \frac{40D_{ave}}{L^2} \tag{27}$$

which is the same as the Duncan solution but this time for the second mode. Summarizing these results we see that the Elishakoff mode is the best (by a very slight margin).

Table 1
Buckling solutions.

#	Mode name	Mode shape function	Stiffness distribution	Normalized buckling load	Mode shape	Stiffness variation
1	Duncan	$W(\xi)=7\xi-10\xi^3+3\xi^5$	$D(\xi)=d_0(1-\frac{3}{7}\xi^2)$	$P_{cr}=\frac{60d_0}{7L^2}$		
2	Elishakoff	$W(\xi)=\xi-2\xi^3+\xi^4$	$D(\xi)=d_0(1+\xi-\xi^2)$	$P_{cr}=\frac{12d_0}{L^2}$		
3	New (I)	$W(\xi)=\frac{8}{15}\xi-\frac{4}{3}\xi^3+\xi^4-\frac{1}{5}\xi^5$	$D(\xi)=d_0(1+\frac{3}{2}\xi-\frac{3}{4}\xi^2)$	$P_{cr}=\frac{15d_0}{L^2}$		
4	New (II)	$W(\xi)=\frac{1}{15}\xi-\frac{2}{3}\xi^3+\xi^4-\frac{2}{5}\xi^5$	$D(\xi)=d_0(1+3\xi-3\xi^2)$	$P_{cr}=\frac{60d_0}{L^2}$		

The natural question arises if the centrally compressed column is the only problem which possesses, as its mode shape, an expression given in Eq. (2). In the next section we consider the vibrations of a simply-supported beam.

4. Vibration of inhomogeneous beams

The vibration of an inhomogeneous beam is governed by the following differential equation

$$\frac{d^2}{d\xi^2}\left[D(\xi)\frac{d^2W}{d\xi^2}\right]-VW=0 \tag{28}$$

$$V=\rho A\omega^2L^4. \tag{29}$$

Here we pose the following question: Is there a simply-supported inhomogeneous beam that possesses the vibration mode as given in Eqs. ((14)–(17)), i.e. which coincides with the buckling mode of the previously considered cases?

To explore this possibility, we observe that in Eq. (28) the second term is the fifth order polynomial (at the most) for ρA = const. Hence, in order for the first term to also constitute the same order polynomial, it is sufficient that $D(\xi)$ is the fourth order polynomial

$$D(\xi)=d_0+d_1\xi+d_2\xi^2+d_3\xi^3+d_4\xi^4. \tag{30}$$

(a) Duncan's mode

In this case we obtain the set of five equations as follows:

$$360d_0-360d_2-7V=0 \tag{31}$$

$$d_1-d_3=0 \tag{32}$$

$$120d_2-120d_4+V=0 \tag{33}$$

$$d_3=0 \tag{34}$$

$$840d_4-V=0 \tag{35}$$

and the solution is

$$D(\xi)=d_0\left(1-\frac{18}{31}\xi^2+\frac{3}{31}\xi^4\right) \tag{36}$$

$$\omega^2=\frac{2520}{31}\frac{d_0}{\rho AL^4}. \tag{37}$$

(b) Elishakoff's mode

In this case we obtain the set of four equations as follows:

$$72d_1-72d_2-V=0 \tag{38}$$

$$d_2-d_3=0 \tag{39}$$

$$120d_3-120d_4+V=0 \tag{40}$$

$$360d_4-V=0 \tag{41}$$

and the solution is

$$D(\xi)=d_0\left(1+\xi-\frac{2}{3}\xi^2-\frac{2}{3}\xi^3+\frac{1}{3}\xi^4\right) \tag{42}$$

$$\omega^2=120\frac{d_4}{\rho AL^4}. \tag{43}$$

(c) First new mode

In this case we obtain the set of five equations as follows:

$$-24d_0+72d_1-48d_2-\frac{8}{15}V=0 \tag{44}$$

$$d_1-3d_2+2d_3=0 \tag{45}$$

$$60d_2-180d_3+120d_4-V=0 \tag{46}$$

$$120d_3 - 360d_4 + V = 0 \tag{47}$$

$$840d_4 - V = 0 \tag{48}$$

and the solution is

$$D(\xi) = d_0\left(1 + \frac{3}{2}\xi - \frac{3}{4}\xi^3 + \frac{3}{16}\xi^4\right) \tag{49}$$

$$\omega^2 = \frac{315}{2}\frac{d_0}{\rho A L^4}. \tag{50}$$

(d) Second new mode

In this case we obtain the set of five equations as follows:

$$48d_0 - 72d_1 + 24d_2 + \frac{1}{15}V = 0 \tag{51}$$

$$2d_1 - 3d_2 + d_3 = 0 \tag{52}$$

$$240d_2 - 360d_3 + 120d_4 - V = 0 \tag{53}$$

$$240d_3 - 360d_4 + V = 0 \tag{54}$$

$$840d_4 - V = 0 \tag{55}$$

and the solution is

$$D(\xi) = d_0\left(1 + 3\xi - 6\xi^3 + 3\xi^4\right) \tag{56}$$

$$\omega^2 = 2520\frac{d_0}{\rho A L^4}. \tag{57}$$

These four solutions are listed in Table 2 with the modes and the stiffness distribution along the beam.

5. Vibration in the presence of axial force

Consider now the vibration of the beam in the presence of the axial force P. The governing differential equation reads:

$$\frac{d^2}{d\xi^2}\left[D(\xi)\frac{d^2W}{d\xi^2}\right] + PL^2\frac{d^2W}{d\xi^2} - VW = 0. \tag{58}$$

We again look for the possibility that the four buckling mode, as derived earlier, will serve as the vibration mode of the beam. The flexural rigidity will be taken in the form as given in Eqs. ((14)–(17)).

(a) —Duncan's mode

In this case we obtain the set of five equations as follows:

$$360d_0 - 360d_2 - 7V - 60P = 0 \tag{59}$$

$$d_1 - d_3 = 0 \tag{60}$$

$$120d_2 - 120d_4 + V + 6P = 0 \tag{61}$$

$$d_3 = 0 \tag{62}$$

$$840d_4 - V = 0 \tag{63}$$

and the solution is

$$D(\xi) = \frac{7}{60}P + \frac{31}{3}d_4 - \left(6d_4 + \frac{P}{20}\right)\xi^2 + d_4\xi^4 \tag{64}$$

$$\omega^2 = 840\frac{d_4}{\rho A L^4}. \tag{65}$$

(b) —Elishakoff's mode

In this case we obtain the set of four equations as follows:

$$72d_1 - 72d_2 - V - 12P = 0 \tag{66}$$

$$12d_2 - 12d_3 + P = 0 \tag{67}$$

$$120d_3 - 120d_4 + V = 0 \tag{68}$$

Table 2
Vibration solutions.

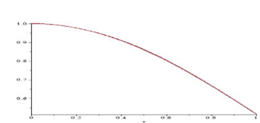
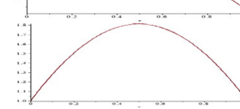
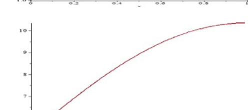
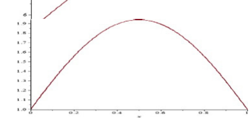
	Mode name	Mode shape function	Stiffness distribution	Normalized frequency	Stiffness variation
1	Duncan	$W(\xi) = 7\xi - 10\xi^3 + 3\xi^5$	$D(\xi) = d_0(1 - \frac{18}{31}\xi^2 + \frac{3}{31}\xi^4)$	$V = \frac{2520}{31}d_0$	
2	Elishakoff	$W(\xi) = \xi - 2\xi^3 + \xi^4$	$D(\xi) = d_0(1 + \xi - \frac{2}{3}\xi^2 - \frac{2}{3}\xi^3 + \frac{1}{3}\xi^4)$	$V = 120d_4$	
3	New (I)	$W(\xi) = \frac{8}{15}\xi - \frac{4}{3}\xi^3 + \xi^4 - \frac{1}{5}\xi^5$	$D(\xi) = d_0(1 + \frac{3}{2}\xi - \frac{3}{4}\xi^3 + \frac{3}{16}\xi^4)$	$V = \frac{315}{2}d_0$	
4	New (II)	$W(\xi) = \frac{1}{15}\xi - \frac{2}{3}\xi^3 + \xi^4 - \frac{2}{5}\xi^5$	$D(\xi) = d_0(1 + 3\xi - 6\xi^3 + 3\xi^4)$	$V = 2520d_0$	

Table 3
Vibration solutions in the presence of axial loading.

	Mode name	Mode shape function	Stiffness distribution	Normalized frequency
1	Duncan	$W(\xi)=7\xi-10\xi^3+3\xi^5$	$D(\xi)=\frac{7}{60}P+\frac{31}{3}d_4-(6d_4+\frac{P}{20})\xi^2+d_4\xi^4$	$V=840d_4$
2	Elishakoff	$W(\xi)=\xi-2\xi^3+\xi^4$	$D(\xi)=\frac{P}{12}+3d_4+(\frac{P}{12}+3d_4)\xi-(\frac{P}{12}+2d_4)\xi^2-2d_4\xi^3+d_4\xi^4)$	$V=360d_4$
3	New (I)	$W(\xi)=\frac{8}{15}\xi-\frac{4}{3}\xi^3+\xi^4-\frac{1}{5}\xi^5$	$D(\xi)=\frac{P}{15}+\frac{16}{3}d_4+(\frac{P}{10}+8d_4)^4\xi-\frac{P}{20}\xi^2-4d_4\xi^3+d_4\xi^4)$	$V=840d_4$
4	New (II)	$W(\xi)=\frac{1}{15}\xi-\frac{2}{3}\xi^3+\xi^4-\frac{2}{5}\xi^5$	$D(\xi)=d_0+3d_0\xi-\frac{P}{20}\xi^2-(6d_0-\frac{P}{10})\xi^3+(3d_0-\frac{P}{20})\xi^4)$	$V=2520d_0-42P$

$$360d_4-V=0 \quad (69)$$

and the solution is

$$D(\xi)=\frac{P}{12}+3d_4+\left(\frac{P}{12}+3d_4\right)\xi-\left(\frac{P}{12}+2d_4\right)\xi^2-2d_4\xi^3+d_4\xi^4) \quad (70)$$

$$\omega^2=360\frac{d_4}{\rho AL^4}. \quad (71)$$

(c) —First new mode

In this case we obtain the set of five equations as follows:

$$-24d_0+72d_1-48d_2-\frac{8}{15}V-8P=0 \quad (72)$$

$$4d_1-12d_2+8d_3-P=0 \quad (73)$$

$$60d_2-180d_3+120d_4-V+3P=0 \quad (74)$$

$$120d_3-360d_4+V=0 \quad (75)$$

$$840d_4-V=0 \quad (76)$$

and the solution is

$$D(\xi)=\frac{P}{15}+\frac{16}{3}d_4+\left(\frac{P}{10}+8d_4\right)^4\xi-\frac{P}{20}\xi^2-4d_4\xi^3+d_4\xi^4) \quad (77)$$

$$\omega^2=840\frac{d_4}{\rho AL^4}. \quad (78)$$

(d) —Second new mode

In this case we obtain the set of five equations as follows:

$$48d_0-72d_1+24d_2+\frac{1}{15}V+4P=0 \quad (79)$$

$$8d_1-12d_2+4d_3-P=0 \quad (80)$$

$$240d_2-360d_3+120d_4-V+12P=0 \quad (81)$$

$$240d_3-360d_4+V=0 \quad (82)$$

$$840d_4-V=0 \quad (83)$$

and the solution is

$$D(\xi)=d_0+3d_0\xi-\frac{P}{20}\xi^2-\left(6d_0-\frac{P}{10}\right)\xi^3+\left(3d_0-\frac{P}{20}\right)\xi^4 \quad (84)$$

$$\omega^2=\frac{2520d_0-42P}{\rho AL^4}. \quad (85)$$

These four solutions are listed in Table 3.

6. Summary

In this paper Duncan's [3] classic solution was derived systematically from the basic buckling equation of a member with axial FGM properties. Then, three other cases are derived using the same methodology: the first in known as Elishakoff's [5] shape another new first buckling mode, and a fourth one which is a second mode buckling case. Then the same method was extended to two other similar problems: vibrations and vibrations in the presence of axial loading, for the same four mode shapes. All three problems, although associated with different flexural rigidities of the corresponding beams share the same mode shapes. Explicit variation of the FGM properties for all the above cases was found analytically. The improvement in the magnitude of the buckling load can be utilized in all cases as compared to simply supported columns with constant properties. The improvement is small (less than 5% increase) and will be justified only in special cases where performance is the major consideration.

It is remarkable that in all three problems, although associated with different flexural rigidities of the corresponding beams, they all share the same eigenfunction.

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