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ICT Express 2 (2016) 10–13
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Maximum likelihood localization: When does it fail?☆

Stefania Monica[a,*], Gianluigi Ferrari[b]

[a] *Department of Mathematics and Computer Science, University of Parma, I-43124 Parma, Italy*
[b] *WASNLab, Department of Information Engineering, University of Parma, I-43124 Parma, Italy*

Received 30 October 2015; received in revised form 19 January 2016; accepted 5 February 2016
Available online 15 February 2016

Abstract

Maximum likelihood is a criterion often used to derive localization algorithms. In particular, in this paper we focus on a distance-based algorithm for the localization of nodes in static wireless networks. Assuming that Ultra Wide Band (UWB) signals are used for inter-node communications, we investigate the ill-conditioning of the Two-Stage Maximum-Likelihood (TSML) Time of Arrival (ToA) localization algorithm as the Anchor Nodes (ANs) positions change. We analytically derive novel lower and upper bounds for the localization error and we evaluate them in some localization scenarios as functions of the ANs' positions. We show that particular ANs' configurations intrinsically lead to ill-conditioning of the localization problem, making the TSML-ToA inapplicable. For comparison purposes, we also show, through some examples, that a Particle Swarm Optimization (PSO)-based algorithm guarantees accurate positioning also when the localization problem embedded in the TSML-ToA algorithm is ill-conditioned.

Keywords: Two-Stage Maximum-Likelihood (TSML) algorithms; Time of Arrival (ToA); Localization

1. Introduction

Nowadays, wireless indoor localization is an interesting topic in many applications [1]. Indoor positioning systems aim at providing precise position estimates inside buildings, which is a particularly tricky task, due to phenomena such as non-line-of-sight and multipath, caused by walls and obstacles. In particular, time-based positioning techniques rely on inter-nodes distance estimates evaluated from the times of flight of signals traveling between pairs of nodes. Given the pair-wise distance estimates between a few nodes, denoted as Anchor Nodes (ANs), and a Target Node (TN), the TN's position can be estimated [2]. Among the wide variety of localization algorithms which have been proposed in the literature, in this paper we focus on the Two-Stage Maximum-Likelihood (TSML) Time of Arrival (ToA) algorithm proposed in [3]. This is a well-known algorithm, based on inter-node distance estimates, which yields a closed-form position estimate and can attain the Cramer–Rao Bound [2]. Unfortunately, despite its quasi-optimality, depending on the nodes' relative positions the localization problem "embedded" in the TSML-ToA algorithm can become ill-conditioned, leading to far inaccurate position estimates. This is detrimental in practical applications (e.g., industrial localization), where ANs may not be freely positioned.

In this paper, we investigate the limits of maximum likelihood-based localization techniques. More precisely, we first derive novel lower and upper bounds for the distance between the true TN position and its estimate, i.e., the localization error. For comparison, we investigate the localization accuracy of a Particle Swarm Optimization (PSO)-based localization algorithm. It will be shown that the PSO allows accurate localization even in those scenarios where the TSML-ToA algorithm fails.

This paper is organized as follows. In Section 2, novel lower and upper bounds for the TN localization error are analytically derived. In Section 3, the values of these bounds are evaluated in a few illustrative scenarios. Section 4 concludes the paper.

* Corresponding author.
E-mail addresses: stefania.monica@unipr.it (S. Monica), gianluigi.ferrari@unipr.it (G. Ferrari).

Peer review under responsibility of The Korean Institute of Communications Information Sciences.

☆ This paper is part of a special issue entitled "Positioning Techniques and Applications" guest edited by Prof. Sunwoo Kim, Prof. Dong-Soo Han, Prof. Chansu Yu, Dr. Francesco Potorti, Prof. Seung-Hyun Kong and Prof. Shiho Kim.

http://dx.doi.org/10.1016/j.icte.2016.02.004

2. Problem localization conditioning

We assume to know the positions of three ANs, denoted as $\{\underline{s}_i = [x_i, y_i]^T\}_{i=1}^3$ and we aim at localizing a TN with coordinates $\underline{u} = [x, y]^T$. In the following, without leading the generality of the derivation, we assume that $x \neq 0$ and $y \neq 0$. Given the three exact distances $\{r_i\}_{i=1}^3 = \{\|\underline{s}_i - \underline{u}\|\}_{i=1}^3$, the TN position could be determined by simply intersecting the three circumferences centered in $\{\underline{s}_i\}_{i=1}^3$ with radii $\{r_i\}_{i=1}^3$, respectively. In the following, the TSML-ToA algorithm is first briefly outlined and, then, lower and upper bounds for the positioning error are derived.

2.1. The TSML-ToA algorithm

The *first phase* of the TSML-ToA algorithm involves the solution of the system of equations:

$$\underline{\underline{G}}\,\underline{\omega} = \underline{h} \tag{1}$$

where: $\underline{\omega} = [x, y, n]$, $n \triangleq \|\underline{u}\|^2$,

$$\underline{\underline{G}} \triangleq \begin{pmatrix} x_1 & y_1 & -0.5 \\ x_2 & y_2 & -0.5 \\ x_3 & y_3 & -0.5 \end{pmatrix} \qquad \underline{h} \triangleq \frac{1}{2}\begin{pmatrix} K_1 - r_1^2 \\ K_2 - r_2^2 \\ K_3 - r_3^2 \end{pmatrix} \tag{2}$$

and $\{K_i\}_{i=1}^3 \triangleq \{\|s_i\|^2\}_{i=1}^3$. Observe that $\underline{\underline{G}}$ is ill-conditioned when: (i) the three ANs lie nearly on the same line (corresponding to two columns nearly linearly dependent); (ii) at least two ANs are very close (namely, two rows are similar).

Since the true distance measurements $\{r_i\}_{i=1}^3$ are not available, one can only rely on their noisy estimates, which will be denoted as

$$\hat{r}_i \triangleq r_i + \delta r_i \quad i \in \{1, 2, 3\} \tag{3}$$

where $\{\delta r_i\}_{i=1}^3$ are the estimation errors. Hence, instead of (1), one is left with

$$\underline{\underline{G}}\,\hat{\underline{\omega}} = \hat{\underline{h}} \tag{4}$$

where $\hat{\underline{\omega}} \triangleq \underline{\omega} + \underline{\delta\omega}$ and $\hat{\underline{h}} \triangleq \underline{h} + \underline{\delta h}$. Observe that, from (3), it follows that

$$\underline{\delta h} = -\begin{pmatrix} r_1\delta r_1 + 0.5(\delta r_1)^2 \\ r_2\delta r_2 + 0.5(\delta r_2)^2 \\ r_3\delta r_3 + 0.5(\delta r_3)^2 \end{pmatrix} \simeq -\begin{pmatrix} r_1\delta r_1 \\ r_2\delta r_2 \\ r_3\delta r_3 \end{pmatrix} \tag{5}$$

where the last approximation has been obtained neglecting quadratic perturbations—the approximation holds if the perturbations are sufficiently small.

The *second phase* of the TSML-ToA algorithm is meant to take into account the dependence of n on the other two unknowns of (1) and involves solving:

$$\underline{\underline{G}}'\,\underline{\phi} = \underline{h}' \tag{6}$$

where:

$$\underline{\underline{G}}' \triangleq \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \qquad \underline{\phi} \triangleq \begin{pmatrix} x^2 \\ y^2 \end{pmatrix} \qquad \underline{h}' \triangleq \begin{pmatrix} \omega_1^2 \\ \omega_2^2 \\ \omega_3 \end{pmatrix}$$

and ω_j, $j \in \{1, 2, 3\}$, denotes the jth component of $\underline{\omega}$. Assuming that only $\hat{\underline{\omega}}$ is known, one is left with

$$\underline{\underline{G}}'\,\hat{\underline{\phi}} = \hat{\underline{h}}' \tag{7}$$

where $\hat{\underline{\phi}} \triangleq \underline{\phi} + \underline{\delta\phi}$, $\hat{\underline{h}}' \triangleq \underline{h}' + \underline{\delta h}'$, and

$$\underline{\delta h}' = \begin{pmatrix} 2\omega_1\delta\omega_1 + (\delta\omega_1)^2 \\ 2\omega_2\delta\omega_2 + (\delta\omega_2)^2 \\ \delta\omega_3 \end{pmatrix} \simeq \begin{pmatrix} 2\omega_1\delta\omega_1 \\ 2\omega_2\delta\omega_2 \\ \delta\omega_3 \end{pmatrix} \tag{8}$$

where $\{\delta\omega_i\}_{i=1}^3$ denote the ith component of $\underline{\delta\omega}$. The last approximation in (8) has been obtained, as in (5), neglecting quadratic perturbations.

The final position estimate $\hat{\underline{u}}$ is $\hat{\underline{u}} = \underline{\underline{U}}\sqrt{\hat{\underline{\phi}}}$ where $\underline{\underline{U}} = \mathrm{diag}(\mathrm{sign}(\hat{\underline{\omega}}_1))$ [3]. Denoting as $\underline{\delta u} = \hat{\underline{u}} - \underline{u}$ the error on the position estimate, we derive lower and upper bounds for the localization error $\|\underline{\delta u}\|$.

2.2. Bounds for position estimation error

First, lower and upper bounds for the norms $\|\underline{\delta\omega}\|$ and $\|\underline{\delta\phi}\|$ of the errors on the solution of (4) and (7), respectively, are derived. The results are then combined together to finally obtain bounds on the norm of the localization error $\|\underline{\delta u}\|$.

From (4) and (1), one obtains $\underline{\underline{G}}\,\underline{\delta\omega} = \underline{\delta h}$ and taking the norm of both sides, it follows:

$$\|\underline{\underline{G}}\,\underline{\delta\omega}\| = \|\underline{\delta h}\| \leq \|\underline{\underline{G}}\|\,\|\underline{\delta\omega}\|. \tag{9}$$

Assuming that $\underline{\underline{G}}$ is not singular, one can conclude that:

$$\|\underline{\delta\omega}\| = \|\underline{\underline{G}}^{-1}\,\underline{\delta h}\| \leq \|\underline{\underline{G}}^{-1}\|\,\|\underline{\delta h}\|. \tag{10}$$

Finally, from (9) and (10), the following bounds for $\|\underline{\delta\omega}\|$ can be derived:

$$\|\underline{\underline{G}}\|^{-1}\|\underline{\delta h}\| \leq \|\underline{\delta\omega}\| \leq \|\underline{\underline{G}}^{-1}\|\,\|\underline{\delta h}\|. \tag{11}$$

From (7) and (6), one obtains $\underline{\underline{G}}'\,\underline{\delta\phi} = \underline{\delta h}'$ and, taking the norm of both sides, one derives:

$$\|\underline{\underline{G}}'\,\underline{\delta\phi}\| = \|\underline{\delta h}'\| \leq \|\underline{\underline{G}}'\|\,\|\underline{\delta\phi}\|. \tag{12}$$

Moreover, defining

$$\underline{\underline{H}} \triangleq \underline{\underline{G}}'^T\,\underline{\underline{G}}' \qquad \underline{\delta\ell} \triangleq \underline{\underline{G}}'^T\,\underline{\delta h}' \tag{13}$$

one obtains $\underline{\delta\phi} = \underline{\underline{H}}^{-1}\,\underline{\delta\ell}$ and, hence,

$$\|\underline{\delta\phi}\| \leq \|\underline{\underline{H}}^{-1}\|\,\|\underline{\delta\ell}\|. \tag{14}$$

Finally, from (12) and (14), the following bounds can be derived:

$$\|\underline{\underline{G}}'\|^{-1}\|\underline{\delta h}'\| \leq \|\underline{\delta\phi}\| \leq \|\underline{\underline{H}}^{-1}\|\,\|\underline{\delta\ell}\|. \tag{15}$$

From (13):

$$\underline{\delta\ell} = \underline{\underline{G}}'^T\,\underline{\delta h}' = \begin{pmatrix} 2\omega_1\delta\omega_1 + \delta\omega_3 \\ 2\omega_2\delta\omega_2 + \delta\omega_3 \end{pmatrix} = \underline{\underline{C}}\,\underline{\delta\omega} \tag{16}$$

where

$$\underline{\underline{C}} \triangleq \begin{pmatrix} 2x & 0 & 1 \\ 0 & 2y & 1 \end{pmatrix}.$$

From (16) it holds that

$$\|\delta\underline{\ell}\| \leq \|\underline{\underline{C}}\|\|\delta\underline{\omega}\|. \tag{17}$$

From (8),

$$\delta\underline{\omega} = \underline{\underline{B}}\,\delta\underline{h}' \tag{18}$$

where (recall that $x \neq 0$ and $y \neq 0$ in our assumptions) matrix $\underline{\underline{B}}$ is defined as

$$\underline{\underline{B}} \triangleq \begin{pmatrix} \frac{1}{2x} & 0 & 0 \\ 0 & \frac{1}{2y} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Hence, from (18):

$$\|\delta\underline{h}'\| \geq \|\underline{\underline{B}}\|^{-1}\|\delta\underline{\omega}\|. \tag{19}$$

From (15), using (17), (19) and (11), one obtains

$$\begin{aligned} \|\delta\underline{\phi}\| &\geq \|\underline{\underline{G}}'\|^{-1}\|\underline{\underline{B}}\|^{-1}\|\underline{\underline{G}}\|^{-1}\|\delta\underline{h}\| \\ \|\delta\underline{\phi}\| &\leq \|\underline{\underline{H}}^{-1}\|\|\underline{\underline{C}}\|\|\underline{\underline{G}}^{-1}\|\,\|\delta\underline{h}\|. \end{aligned} \tag{20}$$

Denoting $\hat{\underline{u}} \triangleq [\hat{x}, \hat{y}]^T = [x + \delta x, y + \delta y]^T$, one can write:

$$\delta\underline{\phi} = \begin{pmatrix} \hat{x}^2 - x^2 \\ \hat{y}^2 - y^2 \end{pmatrix} = \begin{pmatrix} 2x\delta x + (\delta x)^2 \\ 2y\delta y + (\delta y)^2 \end{pmatrix}. \tag{21}$$

Neglecting non-linear perturbations in (21), one obtains:

$$\delta\underline{\phi} \simeq \begin{pmatrix} 2x\delta x \\ 2y\delta y \end{pmatrix} = \begin{pmatrix} 2x & 0 \\ 0 & 2y \end{pmatrix} \delta\underline{u} = \underline{\underline{A}}\,\delta\underline{u}. \tag{22}$$

From (22), it can be concluded that

$$\|\delta\underline{\phi}\| \leq \|\underline{\underline{A}}\|\|\delta\underline{u}\|. \tag{23}$$

Moreover, since $\underline{\underline{A}}$ is not singular,[1] from (22) it can be derived that $\delta\underline{u} = \underline{\underline{A}}^{-1}\delta\underline{\phi}$ so that

$$\|\delta\underline{u}\| \leq \|\underline{\underline{A}}^{-1}\|\|\delta\underline{\phi}\|. \tag{24}$$

Inserting (20) into (23) and (24), the following lower bound (LB) and upper bound (UB) for the norm of the positioning error $\delta\underline{u}$ are found

$$\begin{aligned} \text{LB} &\triangleq \|\underline{\underline{A}}\|^{-1}\|\|\underline{\underline{G}}'\|^{-1}\|\underline{\underline{B}}\|^{-1}\|\underline{\underline{G}}\|^{-1}\|\delta\underline{h}\| \\ \text{UB} &\triangleq \|\underline{\underline{A}}^{-1}\|\|\underline{\underline{H}}^{-1}\|\|\underline{\underline{C}}\|\|\underline{\underline{G}}^{-1}\|\,\|\delta\underline{h}\|. \end{aligned} \tag{25}$$

As expected, the novel bounds in (25) depend on: the distance errors $\{\delta r_i\}_{i=1}^3$ (through $\|\delta\underline{h}\|$); the ANs' coordinates (through $\underline{\underline{G}}$); the TN's coordinates (through $\underline{\underline{A}}$, $\underline{\underline{B}}$, and $\underline{\underline{C}}$). **Such bounds are not necessary to perform localization, but they are useful to investigate the ill-conditioning of the considered ML approach.**

[1] This is true if $x \neq 0$ and $y \neq 0$, as assumed before.

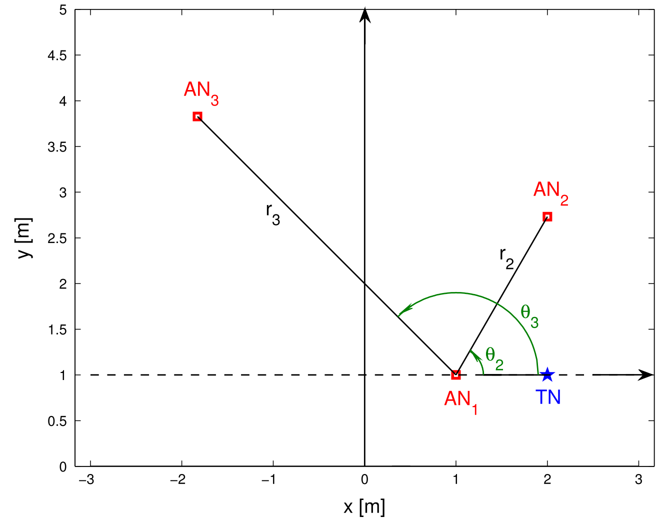

Fig. 1. System configuration.

3. Numerical results

The considered (very general) system configuration is shown in Fig. 1. The TN's coordinates (dimension: [m]) are $\underline{u} = [1, 1]^T$ and the coordinates of the first AN are $\underline{s}_1 = [2, 1]^T$. The positions of the remaining ANs have the following expressions

$$\underline{s}_i = \underline{u} + [r_i \cos\theta_i^{(k)}, r_i \sin\theta_i^{(k)}]^T \quad i \in \{2, 3\} \tag{26}$$

where: $r_2 = 2$ m; $r_3 = 3$ m; the values of $\{\theta_i^{(k)}\}_{i=2}^3$ vary in $[0, 2\pi)$ according to $\{\theta_i^{(k)}\}_{i=2}^3 = 2k\pi/180$, $k \in \{0, \ldots, 179\}$. In other words, $\{\theta_i^{(k)}\}_{i=2}^3$ vary, at steps multiple of $\pi/90$, between 0 and $2\pi - \pi/90$. We assume that the inter-node distance estimates are based on UWB signaling and the distance estimation errors $\{\delta r_i\}_{i=1}^3$ can be modeled as $\delta r_i = 0.016 r_i - 0.15$ [m], as shown in [4].

The values of LB and UB as functions of $\theta_2^{(k)}$ and $\theta_3^{(h)}$ are shown in Fig. 2(a) and Fig. 2(b), respectively. From Fig. 2(a), it can be observed that the values of LB are always smaller than 5 cm and are approximately constant. At the opposite, the values of UB strongly depend on the ANs' positions and reach values up to 10 m, as shown in Fig. 2(b). For each pair $(\theta_2^{(k)}, \theta_3^{(h)})$, we also perform localization using the TSML-ToA algorithm: in Fig. 2(c) the pairs $(\theta_2^{(k)}, \theta_3^{(h)})$ corresponding to $\|\delta\underline{u}\| > 2$ m are shown: the agreement with the derived UB in Fig. 2(b) is evident.

In order to overcome the errors due to ill-conditioned topologies, a PSO-based approach to localization can be considered. **The PSO algorithm is iterative and does not suffer from ill-conditioning as it does not involve matrix calculations.** The reader is referred to [5] for a detailed description of the use of the PSO algorithm for localization purposes. We now consider a few illustrative ANs' configurations which lead to position estimates with $\|\delta\underline{u}\| > 2$ m. With these ANs configurations, we estimate the TN positions using the PSO algorithm. The obtained values of $\|\delta\underline{u}\|$ are shown in Table 1. Observe that, in all cases, the values of $\|\delta\underline{u}\|$ obtained with the PSO algorithm

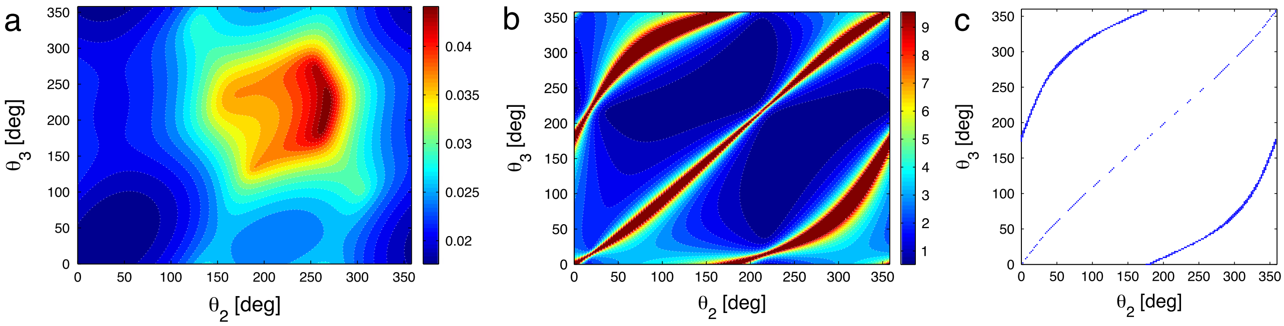

Fig. 2. (a) LB and (b) UB as functions of $\theta_2^{(k)}$ and $\theta_3^{(k)}$ for $r_2 = 2$ m and $r_3 = 3$ m; (c) values of $\theta_2^{(k)}$ and $\theta_3^{(k)}$ which correspond to $\|\underline{\delta u}\| > 2$ m (blue dots) in (b). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

Table 1
The values of $\underline{\delta u}$ are shown when using the TSML and the PSO algorithms (third and fourth columns) for different values of $\theta_2^{(k)}$ and $\theta_3^{(h)}$ (first and second columns).

$\theta_2^{(k)}$	$\theta_3^{(h)}$	$\|\underline{\delta u}\|$ — TSML [m]	$\|\underline{\delta u}\|$— PSO [m]
π	0	$5.99 \cdot 10^{14}$	0.19
$3\pi/2$	$\pi/4$	49	0.13
0	π	$8.40 \cdot 10^{14}$	0.25

are sufficiently accurate for many applications, whereas TSML-ToA algorithm fails.

4. Conclusion

In this paper, we have studied the conditioning of the TSML-ToA localization algorithm. Using norm inequalities, we have derived novel lower and upper bounds for the positioning error. Then, we have considered scenarios with different ANs' positions and we have shown how the lower and upper bounds behave as functions of the ANs' positions. Moreover, for each ANs' configuration, we have solved the localization problem by means of the TSML-ToA algorithm, showing that the obtained position estimates can be far inaccurate. Finally, we have shown that the localization errors obtained with the TSML-ToA algorithm can be avoided using a localization approach based on the PSO algorithm.

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