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Chromatic Number of Resultant of Fuzzy Graphs

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Abstract   Fuzzy graph coloring techniques are used to solve many complex real world problems. The chromatic number of complement of fuzzy graph is obtained and compared with the chromatic number of the corresponding fuzzy graph. The chromatic number of the resultant fuzzy graphs is studied, obtained by various operations on fuzzy graphs like union, join and types of products. A solution to routing problem is suggested by using chromaticity of fuzzy graphs.

1. Introduction

Fuzzy graph theory has numerous applications in modern science and technology especially in the fields of information theory, neural networks, expert systems, cluster analysis, medical diagnosis and control theory. Several papers in related areas of fuzzy graph theory are available in literature [1, 2, 5, 8, 10, 15, 16]. Many practical problems such as scheduling, allocation, network problems etc. can be modeled as coloring problems and hence coloring is one of the most studied areas in the research

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of graph theory. Coloring of fuzzy graphs play a vital role in solving complications in networks. A large number of variations in coloring of fuzzy graphs are available in literature. Coloring of fuzzy graphs were introduced by Monoz et al. [12]. The authors have defined the chromatic number of a fuzzy graph $G = (V, \sigma, \mu)$ as a fuzzy subset of V. Eslahchi and Onagh [6] introduced the concept of fuzzy chromatic number $\chi^f(G)$ as the number of partitions of the color classes. Anjaly and Sunitha introduced chromatic number of fuzzy graphs and developed algorithms to the same [4]. Samanta and Pal introduced fuzzy coloring of fuzzy graphs [14].

Even though many papers are available on coloring of fuzzy graphs and its applications, there are no papers known on the relationship between the chromaticity of fuzzy graphs and resultant fuzzy graphs obtained by various fuzzy graph operations. We provide the relationship between chromaticity of fuzzy graph and that of the resultant fuzzy graphs obtained by performing various operations in fuzzy graphs like union, join and different types of products. We also provide an application to illustrate how chromatic number of fuzzy graphs can be used to solve routing problems.

2. Preliminaries

2.1. Some Definitions

Definition 2.1 [11] *A fuzzy graph is an ordered triple $G : (V, \sigma, \mu)$ where V is a set of vertices $\{u_1, u_2, \cdots, u_n\}$, σ is a fuzzy subset of V, i.e., $\sigma : V \to [0, 1]$ and is denoted by $\sigma = \{(u_1, \sigma(u_1)), (u_2, \sigma(u_2)), \cdots, (u_n, \sigma(u_n))\}$ and μ is a fuzzy relation on σ, i.e., $\mu(u, v) \leq \sigma(u) \wedge \sigma(v) \ \forall u, v \in V$.*

We consider fuzzy graph G with no loops and assume that V is finite and nonempty, μ is reflexive (i.e., $\mu(u, u) = \sigma(u), \forall u$) and symmetric (i.e., $\mu(u, v) = \mu(v, u), \forall(u, v)$). In all the examples σ is chosen suitably. Also, we denote the underlying crisp graph of G by $G^* : (\sigma^*, \mu^*)$, where $\sigma^* = \{u \in V \mid \sigma(u) > 0\}$ and $\mu^* = \{(u, v) \in V \times V \mid \mu(u, v) > 0\}$. Throughout we assume that $\sigma^* = V$.

Definition 2.2 [11] *The level set of fuzzy set σ is defined as $\lambda = \{\alpha \mid \sigma(u) = \alpha$ for some $u \in V\}$. For each $\alpha \in \lambda$, G_α denotes the crisp graph $G_\alpha = (\sigma_\alpha, \mu_\alpha)$, where $\sigma_\alpha = \{u \in V \mid \sigma(u) \geq \alpha\}$, $\mu_\alpha = \{(u, v) \in V \times V \mid \mu(u, v) \geq \alpha\}$.*

Definition 2.3 [9] *An arc (u, v) is called M-strong if $\mu(u, v) = \sigma(u) \wedge \sigma(v)$. A fuzzy graph $G : (V, \sigma, \mu)$ is called an M-strong fuzzy graph if $\mu(u, v) = \sigma(u) \wedge \sigma(v) \forall(u, v) \in \mu^*$.*

Definition 2.4 [13] *The complement of a fuzzy graph $G : (V, \sigma, \mu)$ is the fuzzy graph $\overline{G} : (V, \overline{\sigma}, \overline{\mu})$ with $\overline{\sigma}(u) = \sigma(u)$ and $\overline{\mu}(u, v) = \sigma(u) \wedge \sigma(v) - \mu(u, v)$.*

Definition 2.5 *A clique of a simple crisp graph G^* is a subset S of V such that the graph induced by S is complete [7].*

Definition 2.6 [11] *If $G_1 = (V_1, \sigma_1, \mu_1)$ and $G_2 = (V_2, \sigma_2, \mu_2)$ be two fuzzy graphs with $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$, then $G_1 \bigcup G_2$ is the fuzzy graph $(V_1 \bigcup V_2, \sigma, \mu)$, where*

$$\sigma(u) = \begin{cases} \sigma_1(u), & u \in V_1 - V_2, \\ \sigma_2(u), & u \in V_2 - V_1 \end{cases}$$

and

$$\mu(u,v) = \begin{cases} \mu_1(u,v), & (u,v) \in E_1 - E_2, \\ \mu_2(u,v), & (u,v) \in E_2 - E_1. \end{cases}$$

Definition 2.7 [11] *The join of disjoint fuzzy graphs G_1 and G_2 is $G = G_1 + G_2 = (V_1 \bigcup V_2, \sigma, \mu)$, where $\sigma = \sigma_1 + \sigma_2$ and $\mu = \mu_1 + \mu_2$ are fuzzy subsets of $V_1 \bigcup V_2$ and $E_1 \bigcup E_2 \bigcup E'$ respectively, where E' is the set of all edges joining V_1 and V_2,*

$$\sigma(u) = \begin{cases} \sigma_1(u), & u \in V_1, \\ \sigma_2(u), & u \in V_2 \end{cases}$$

and

$$\mu(u,v) = \begin{cases} \mu_1(u,v), & (u,v) \in E_1, \\ \mu_2(u,v), & (u,v) \in E_2, \\ \sigma_1(u) \wedge \sigma_2(v), & (u,v) \in E'. \end{cases}$$

Definition 2.8 [11] *Let $G^* = G_1^* \times G_2^* = (V, E'')$ be the cartesian product of two graphs where $V = V_1 \times V_2$ and $E'' = \{(uu_2, uv_2) \mid u \in V_1, (u_2, v_2) \in E_2\} \bigcup \{(u_1v, v_1v) \mid v \in V_2, (u_1, v_1) \in E_1\}$. Then $G_1 \times G_2$ is the fuzzy graph (V, σ, μ), where*

$$\sigma(u_1 u_2) = \sigma_1(u_1) \wedge \sigma_2(u_2) \,\forall u_1 \in V_1, u_2 \in V_2,$$

$$\mu(uu_2, uv_2) = \sigma_1(u) \wedge \mu_2(u_2, v_2) \,\forall u \in V_1, \forall (u_2, v_2) \in E_2,$$

$$\mu(u_1 v, v_1 v) = \sigma_2(v) \wedge \mu_1(u_1, v_1) \,\forall v \in V_2, \forall (u_1, v_1) \in E_1.$$

3. Coloring of Fuzzy Graphs

The concept of chromatic number of fuzzy graph (Definition 3.1) was introduced by Munoz et al. [12]. The authors considered fuzzy graphs with crisp vertex set, i.e., fuzzy graphs for which $\sigma(x) = 1 \,\forall x \epsilon V$ and edges with membership degree in [0, 1]. The concept of fuzzy chromatic number (Definition 3.2) was defined by Eslahchi and Onagh [6].

Definition 3.1 [12] *If $G : (V, \mu)$ is such a fuzzy graph, where $V = \{1, 2, 3, \cdots, n\}$ and μ is a fuzzy number on the set of all subsets of $V \times V$. Assume $I = A \cup \{0\}$, where $A = \{\alpha_1 < \alpha_2 < \cdots < \alpha_k\}$ is the fundamental set (level set) of G. For each $\alpha \in I, G_\alpha$ denote the crisp graph $G_\alpha = (V, E_\alpha)$, where $E_\alpha = \{ij \mid 1 \le i < j \le n, \mu(i, j) \ge \alpha\}$ and $\chi_\alpha = \chi(G_\alpha)$ denote the chromatic number of crisp graph G_α.*

By this definition the chromatic number of fuzzy graph G is the fuzzy number $\chi(G) = \{(i, v(i)) \mid i \in X\}$, where $v(i) = \max \{\alpha \in I \mid i \in A_\alpha\}$ and $A_\alpha = \{1, 2, 3, \cdots, \chi_\alpha\}$.

Definition 3.2 [6] *A family $\Gamma = \{\gamma_1, \gamma_2, \cdots, \gamma_k\}$ of fuzzy sets on a set V is called a k-fuzzy coloring of $G = (V, \sigma, \mu)$ if*

  (i) $\vee \Gamma = \sigma$,

  (ii) $\gamma_i \wedge \gamma_j = 0$,

(iii) *for every strong edge* (x, y) *(i.e.,* $\mu(x, y) > 0$ *) of* G, $\min \{\gamma_i(x), \gamma_i(y)\} = 0$, $(1 \le i \le k)$.

The minimum number k for which there exists a k-fuzzy coloring is called the fuzzy chromatic number of G, denoted as $\chi^f(G)$.

Incorporating the features of the above Definitions 3.1 and 3.2, the chromatic number $\chi(G)$ of fuzzy graph is modified by Anjaly and Sunitha [4] as follows.

Definition 3.3 [4] *For each* $\alpha \in I$, G_α *denote the crisp graph* $G_\alpha = (\sigma_\alpha, \mu_\alpha)$ *and* $\chi_\alpha = \chi(G_\alpha)$ *denote the chromatic number of crisp graph* G_α. *The chromatic number of fuzzy graph* G *is the number* $\chi(G) = \max \{(\chi(G_\alpha) \mid \alpha \in \lambda\}$.

Theorem 3.1 [4] *For a fuzzy graph* $G : (V, \sigma, \mu)$, *the chromatic number* $\chi(G) = \chi^f(G)$.

Remark 3.1 [4] The fundamental set I of G is the set $I = \{\alpha/$ either $\sigma(u) = \alpha$, for some $u \in V$; or $\mu(u, v) = \alpha$, for some $(u, v) \in V \times V\}$. We choose $I = \{\alpha_1 < \alpha_2 < \cdots < \alpha_k\}$.

Remark 3.2 [4] If $\alpha_i \le \alpha_j$, then $\chi(G(\alpha_i)) \ge \chi(G(\alpha_j))$.

Theorem 3.2 *For a fuzzy graph* $G : (V, \sigma, \mu)$ *except M-strong fuzzy graph,* $\chi(\overline{G}) > \chi(G)$ *and* $\chi(\overline{G}) = \chi(G)$ *if G is self complementary.*

Proof   For all graphs except M-strong fuzzy graphs $\overline{\mu}(u, v) = \sigma(u) \wedge \sigma(v) - \mu(u, v) > 0 \; \forall(u, v)$. We know $\mu(u, v) \le \sigma(u) \wedge \sigma(v) \; \forall u, v \in V$ and equality holds if the fuzzy graph is M-strong. Hence the corresponding crisp graph $\overline{G}_\alpha$ of $\overline{G}$ will have more number of adjacencies than G_α of G. The equality is obvious.

Theorem 3.3 *For a fuzzy graph* $G : (V, \sigma, \mu)$, $\chi(\overline{G}) + \chi(G) \ge n$ *where n is the number of nodes. Equality holds for fuzzy graphs G, such that* $G^* = C_4$.

Proof   For all fuzzy graphs, $\overline{G}_\alpha$ and G_α for the corresponding minimum values of α satisfies the relation $\chi(\overline{G}_\alpha) + \chi(G_\alpha) > n$. This can be proved by considering the following cases.

Case I   If G is a complete fuzzy graph on n nodes, then $\overline{G}$ is a disconnected graph with n isolated nodes. Hence $\chi(K_n) = n$ and $\chi(\overline{K_n}) = 1$.

Case II   If for all arcs (u, v), $\mu(u, v) \le \sigma(u) \wedge \sigma(v)$, then $\overline{G}_\alpha$ is a graph with the same or more number of adjacencies than G_α. If $\mu(u, v) < \sigma(u) \wedge \sigma(v)$, then u and v are assigned different colors in $\overline{G}_\alpha$ and in G_α. If $\mu(u, v) = \sigma(u) \wedge \sigma(v)$, then u and v are assigned the same color in $\overline{G}_\alpha$ and different in G_α. Hence in either case, either in $\chi(\overline{G}_\alpha)$ or in $\chi(G_\alpha)$ two colors are required to color u and v. This is repeated for all adjacent nodes.

Theorem 3.4 *For fuzzy graphs* $G_1 = (V_1, \sigma_1, \mu_1)$ *and* $G_2 = (V_2, \sigma_2, \mu_2)$, $\chi(G_1 \bigcup G_2) = \max \{\chi(G_1), \chi(G_2)\}$.

Proof   We know, $G_1 \bigcup G_2 = (V_1 \bigcup V_2, \sigma, \mu)$ with $\mu(x, y) > 0$ if and only if $\mu_1(x, y) > 0$ or $\mu_2(x, y) > 0$. For each α in the fundamental set of $G_1 \bigcup G_2$, either α belongs to the fundamental set of G_1 or to the fundamental set of G_2. Hence the minimum value of

α in the fundamental set of $G_1 \bigcup G_2$ is the minimum among that in the fundamental sets of G_1 and G_2. Let it be α_1, the minimum value of α in the fundamental set of G_1. Then $(G_1 \bigcup G_2)_{\alpha_1}$ has all the adjacencies of G_1 and G_2. By Definition 3.3, $\chi(G_1 \bigcup G_2) = \chi(G_1 \bigcup G_2)_{\alpha_1} = \max \{\chi(G_1), \chi(G_2)\}$.

Theorem 3.5 *For fuzzy graphs $G_1 = (V_1, \sigma_1, \mu_1)$ and $G_2 = (V_2, \sigma_2, \mu_2)$, $\chi(G_1 + G_2) \leq n$ where $n = |V_1| + |V_2|$. Equality holds if G_1^* and G_2^* are complete graphs.*

Proof   By the definition of join, the support of join becomes a complete graph if and only if both G_1^* and G_2^* are complete graphs.

Now we find the chromatic number of products of fuzzy graphs.

Definition 3.4 (Direct Product[3]) *If $G_1 = (V_1, \sigma_1, \mu_1)$ and $G_2 = (V_2, \sigma_2, \mu_2)$, then the direct product $G_1 \sqcap G_2$ is the fuzzy graph $G = (V_1 \times V_2, \sigma, \mu)$, where $E = \{(u_1v_1, u_2v_2) \mid (u_1, u_2) \in E_1, (v_1, v_2) \in E_2\}$,*

$$\sigma(uv) = \min \{\sigma_1(u), \sigma_2(v)\} \ \forall uv \in V_1 \times V_2,$$

$$\mu(u_1v_1, u_2v_2) = \mu_1(u_1, u_2) \wedge \mu_2(v_1, v_2).$$

Theorem 3.6 *For fuzzy graphs $G_1 = (V_1, \sigma_1, \mu_1)$ and $G_2 = (V_2, \sigma_2, \mu_2)$, the chromatic number of the direct product, $\chi(G_1 \sqcap G_2) = \min \{\chi(G_1), \chi(G_2)\}$.*

Proof   By the definition of direct product, any two nodes are adjacent if and only if the corresponding nodes in G_1 and G_2 are adjacent. Since chromaticity depends on adjacency, the number of colors required to color the nodes of $G_1 \sqcap G_2$ is minimum of $\chi(G_1)$ and $\chi(G_2)$.

Definition 3.5 (Semi Product [3]) *If $G_1 = (V_1, \sigma_1, \mu_1)$ and $G_2 = (V_2, \sigma_2, \mu_2)$, then the semi product $G_1 \diamond G_2$ is the fuzzy graph $G = (V_1 \times V_2, \sigma, \mu)$, where*
$E = \{(uv_1, uv_2) \mid u \in V_1, (v_1, v_2) \in E_2\} \bigcup \{(u_1v_1, u_2v_2) \mid (u_1, u_2) \in E_1, (v_1, v_2) \in E_2\}$,

$$\sigma(uv) = \min \{\sigma_1(u), \sigma_2(v)\} \ \forall uv \in V_1 \times V_2,$$

$$\mu(uv_1, uv_2) = \sigma_1(u) \wedge \mu_2(v_1, v_2),$$

$$\mu(u_1v_1, u_2v_2) = \mu_1(u_1, u_2) \wedge \mu_2(v_1, v_2).$$

Remark 3.3   A clique with maximum number of vertices is a maximal clique of the graph G^*.

Theorem 3.7 *For fuzzy graphs $G_1 = (V_1, \sigma_1, \mu_1)$ and $G_2 = (V_2, \sigma_2, \mu_2)$, the chromatic number of the semi product, $\chi(G_1 \diamond G_2) = |V_C|$, where C is a maximal clique of $(G_1 \diamond G_2)^*$.*

Proof   By the definition of semi product, the largest complete subgraph of G is the graph with corresponding adjacent nodes u_iv_j and u_iv_k, where $u_i \in V_1$ and $(v_j, v_k) \in E_2$ and (u_iv_j, u_mv_k), where $(u_i, u_m) \in E_1$ and $(v_j, v_k) \in E_2$, i.e., the largest complete subgraph is the graph with arcs of the form (u_1v_j, u_1v_k) with u_1 as a node of maximum degree in V_1 and arcs of the form (u_1v_j, u_mv_k), where $(u_1, u_m) \in E_1$ and $(v_j, v_k) \in E_2$.

Hence the chromatic number of the semi product is the chromatic number corresponding to this largest complete subgraph.

Definition 3.6 (Strong Product [3]) *If $G_1 = (V_1, \sigma_1, \mu_1)$ and $G_2 = (V_2, \sigma_2, \mu_2)$, then the strong product $G_1 \otimes G_2$ is the fuzzy graph $G = (V_1 \times V_2, \sigma, \mu)$, where*

$$E = \{(uv_1, uv_2) \mid u \in V_1, (v_1, v_2) \in E_2\} \bigcup \{(u_1w, u_2w) \mid (u_1, u_2) \in E_1, w \in V_2\}$$
$$\bigcup \{(u_1v_1, u_2v_2) \mid (u_1, u_2) \in E_1, (v_1, v_2) \in E_2\},$$

$$\sigma(uv) = \min\{\sigma_1(u), \sigma_2(v)\} \ \forall uv \in V_1 \times V_2,$$

$$\mu(uv_1, uv_2) = \sigma_1(u) \wedge \mu_2(v_1, v_2),$$

$$\mu(u_1w, u_2w) = \mu_1(u_1, u_2) \wedge \sigma_2(w),$$

$$\mu(u_1v_1, u_2v_2) = \mu_1(u_1, u_2) \wedge \mu_2(v_1, v_2).$$

Theorem 3.8 *If $\mu_1(u_1, u_2) > 0$ and $\mu_2(v_1, v_2) > 0 \ \forall u_1, u_2 \in V_1$ and $v_1, v_2 \in V_2$, the chromatic number of the strong product, $\chi(G_1 \otimes G_2) = n_1n_2$, where $n_1 = |V_1|$ and $n_2 = |V_2|$.*

Proof   By the definition of strong product, if $\mu_1(u_1, u_2) > 0$ and $\mu_2(v_1, v_2) > 0$ $\forall u_1, u_2 \in V_1$ and $v_1, v_2 \in V_2$, then $(G_1 \otimes G_2)^*$ is a complete graph with n_1n_2 nodes. Hence $\chi(G_1 \otimes G_2) = n_1n_2$.

5. Application of Chromatic Number of Fuzzy Graphs

Consider a routing problem with places represented as nodes u_1, u_2, u_3, u_4 of a fuzzy graph. Let arcs represent approachability between pairs of nodes with strengths denoting difficulty in traveling modes. Depending on the tourist demand, the nodes are assigned strengths as High (0.9), Low(0.1) and Medium(0.5). Let the following fuzzy graph G in (a) which represents the situation.
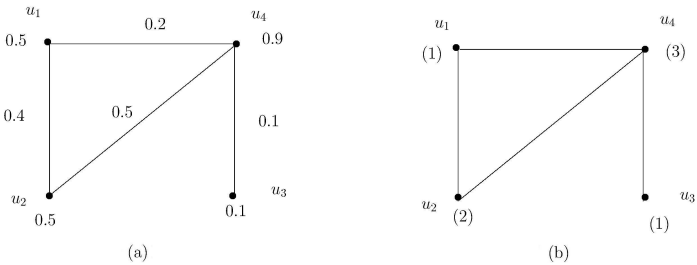

Fig. 1 Fuzzy graph G

Here the level set is $\{0.1, 0.2, 0.4, 0.5, 0.9\}$. Now consider G_α for $\alpha = 0.1, 0.2, 0.4$ and 0.5, the arc strengths of G. The graph in (b) represents $G_{0.1}$. Then by Definition 3.3, $\chi(G) = \chi(G_{0.1})$. The numbers inside brackets shows the corresponding colors of nodes in $G_{0.1}$. Hence $\chi(G) = \chi(G_{0.1}) = 3$. The chromatic number 3 indicates that

maximum 3 places can be reached from one another. That is, maximum number of places, which are directly linked by road is 3, represented by nodes u_1, u_2 and u_4 which are adjacent to each other.

6. Conclusion

The concept of chromatic number of fuzzy graph is studied in the resultant fuzzy graph of different operations of fuzzy graphs. The relation between chromatic number of a fuzzy graph and its complement is studied. The chromaticity of the resultant of the union of two fuzzy graphs, join and different types of products are also obtained. Solution of routing problem using chromaticity of fuzzy graphs is proposed. The solution suggested in the present approach gives the number of places which are linked by road directly. One can apply different coloring approaches by considering arc strengths to find the most effective route between different places.

Acknowledgements

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