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ORIGINAL ARTICLE

Topological Structures of IVF Approximation Spaces

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Abstract An approximation space is a key concept in rough set theory, which plays an important role to approximate reasoning about data. An interval-valued fuzzy (IVF, for short) approximation space is an approximation space in the IVF environment. In this paper, IVF rough approximation operators are investigated with topological structures of IVF approximation spaces given.

Keywords IVF set · IVF relation · IVF approximate space · IVF rough set · IVF topology

1. Introduction

Rough set theory was proposed by Pawlak [10] as a mathematical tool to handle imprecision and uncertainty in data analysis. It has been successfully applied to machine learning, intelligent systems, inductive reasoning, pattern recognition, mereology, image processing, signal analysis, knowledge discovery, decision analysis, expert systems and many other fields [11-14].

The basic structure of rough set theory is an approximation space. Based on it, lower and upper approximations can be induced. Using these approximations, knowledge hidden in information systems may be revealed and expressed in the form of decision rules.

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As a generalization of Zadeh's fuzzy set, IVF sets were introduced by Gorzalczany [3] and Turksen [16], and they were applied to the fields of approximate inference, signal transmission and controller. Mondal et al. [9] defined a topology of IVF sets and studied their properties.

An IVF approximation space is an approximation space under the IVF environment. By replacing crisp relations with IVF relations, Sun et al. [15] introduced the IVF rough sets based on an IVF approximation space, defined IVF information systems and discussed their attribute reduction. Gong et al. [4] presented the IVF rough sets based on approximation spaces and studied the knowledge discovery in IVF information systems. Zhang et al. [21] obtained decomposition theorems on IVF rough approximations.

It is well known that topology is a branch of mathematics, whose concepts exist not only in almost all branches of mathematics but also in many real life applications. Topologies are widely used in the research field of machine learning and cybernetics (see [1, 2, 5-7, 17]). For example, Choudhury and Zaman [2] applied the theory of topology to study the evolutionary impact of learning on social problems; Koretelainen [5] used topologies to detect dependencies of attributes in information systems with respect to gradual rules. The concept of topological structures and their generalizations mean the most powerful notions and give important bases for dealing with data and system analysis.

The purpose of this paper is to investigate topological structures of IVF approximation spaces.

2. Preliminaries

Throughout this paper, "interval-valued fuzzy" will be denoted briefly by "IVF". U denotes a nonempty set called the universe. I denotes $[0,1]$ and $[I]$ denotes $\{[a,b] \mid a,b \in I \text{ and } a \leq b\}$. $F^{(i)}(U)$ denotes the family of all IVF sets in U. $\bar{a}$ denotes $[a,a]$ for each $a \in [0,1]$.

Some relations and operations are defined as follows ([3, 16]): for any $[a_1,b_1]$, $[a_2, b_2] \in [I]$,

$$[a_1,b_1] = [a_2,b_2] \iff a_1 = a_2, b_1 = b_2;$$

$$[a_1,b_1] \leq [a_2,b_2] \iff a_1 \leq a_2, b_1 \leq b_2;$$

$$[a_1,b_1] < [a_2,b_2] \iff [a_1,b_1] \leq [a_2,b_2] \text{ and } [a_1,b_1] \neq [a_2,b_2];$$

$$\bar{1} - [a_1,b_1] \text{ or } [a_1,b_1]^c = [1-b_1, 1-a_1].$$

Obviously, $([a,b]^c)^c = [a,b]$ for each $[a,b] \in [I]$.

Definition 2.1 [3, 16] *For each* $\{[a_j, b_j] \mid j \in J\} \subseteq [I]$, *we define*

$$\bigvee_{j\in J}[a_j, b_j] = [\bigvee_{j\in J} a_j, \bigvee_{j\in J} b_j] \text{ and } \bigwedge_{j\in J}[a_j, b_j] = [\bigwedge_{j\in J} a_j, \bigwedge_{j\in J} b_j],$$

where $\bigvee_{j\in J} a_j = \sup\{a_j \mid j \in J\}$ *and* $\bigwedge_{j\in J} a_j = \inf\{a_j \mid j \in J\}$.

Definition 2.2 [3, 16] *An IVF set A in U is defined by a mapping* $A : U \to [I]$. *Denote*

$$A(x) = [A^-(x), A^+(x)] \quad (x \in U).$$

Then $A^-(x)$ *(resp.* $A^+(x)$ *) is called the lower (resp. upper) degree to which x belongs to A.* A^- *(resp.* A^+ *) is called the lower (resp. upper) IVF set of A.*

The set of all IVF sets in U is denoted by $F^{(i)}(U)$.

Let $a, b \in I$. $\widetilde{[a, b]}$ represent the IVF set which satisfies $\widetilde{[a, b]}(x) = [a, b]$ for each $x \in U$. We denoted $\widetilde{[a, a]}$ by $\tilde{a}$.

We recall some basic operations on $F^{(i)}(U)$ as follows ([3, 16]): For any $A, B \in F^{(i)}(U)$ and $[a, b] \in [I]$,

(1) $A = B \iff A(x) = B(x)$ for each $x \in U$.

(2) $A \subseteq B \iff A(x) \leq B(x)$ for each $x \in U$.

(3) $A = B^c \iff A(x) = B(x)^c$ for each $x \in U$.

(4) $(A \cap B)(x)=A(x) \wedge B(x)$ for each $x \in U$.

(5) $(A \cup B)(x)=A(x) \vee B(x)$ for each $x \in U$.

(6) $([a, b]A)(x) = [a, b] \wedge [A^-(x), A^+(x)]$ for each $x \in U$.

Obviously,

$$A = B \iff A^- = B^- \text{ and } A^+ = B^+ \; ; \; (\widetilde{[a, b]})^c = \widetilde{[a, b]^c} \quad ([a, b] \in [I]).$$

Definition 2.3 [9] $A \in F^{(i)}(U)$ *is called an IVF point in U if there exist* $[a, b] \in [I]-\{\bar{0}\}$ *and* $x \in U$ *such that*

$$A(y) = \begin{cases} [a, b], & y = x, \\ \bar{0}, & y \neq x. \end{cases}$$

We denote this A by $x_{[a,b]}$.

Definition 2.4 [9] $\tau \subseteq F^{(i)}(U)$ *is called an IVF topology on U if*

(i) $\tilde{0}, \tilde{1} \in \tau$,

(ii) $A, B \in \tau \implies A \cap B \in \tau$,

(iii) $\{A_j \mid j \in J\} \subseteq \tau \implies \bigcup_{j\in J} A_j \in \tau$.

The pair (U, τ) is called an IVF topological space. Every member of τ is called an IVF open set in U. Its complement is called an IVF closed set in U.

We denote $\tau^c = \{A \mid A^c \in \tau\}$.

The interior and closure of $A \in F^{(i)}(U)$ denoted respectively by $int(A)$ and $cl(A)$, are defined as follows: $int(A)$ or $int_\tau(A) = \bigcup\{B \in \tau \mid B \subseteq A\}$, $cl(A)$ or $cl_\tau(A) = \bigcap\{B \in \tau^c \mid B \supseteq A\}$.

Proposition 2.1 [9] *Let τ be an IVF topology on U. Then for any $A, B \in F^{(i)}(U)$,*

(1) $int(\tilde{1}) = \tilde{1}$, $cl(\tilde{0}) = \tilde{0}$.

(2) $int(A) \subseteq A \subseteq cl(A)$.

(3) $A \subseteq B \Longrightarrow int(A) \subseteq int(B)$, $cl(A) \subseteq cl(B)$.

(4) $int(A^c) = (cl(A))^c$, $cl(A^c) = (int(A))^c$.

(5) $int(A \cap B) = int(A) \cap int(B)$, $cl(A \cup B) = cl(A) \cup cl(B)$.

(6) $int(int(A)) = int(A)$, $cl(cl(A)) = cl(B)$.

Definition 2.5 [19] *Let R be a crisp relation on U. For each $x \in U$, denote*

$$R_p(x) = \{y \in U \mid (y, x) \in R\} \text{ and } R_s(x) = \{y \in U \mid (x, y) \in R\}.$$

$R_p(x)$ and $R_s(x)$ are called the predecessor and successor neighborhood of x, respectively.

3. IVF Approximation Spaces and the IVF Rough Sets

Recall that R is called an IVF relation on U if $R \in F^{(i)}(U \times U)$.

Definition 3.1 [15] *Suppose that R is an IVF relation on U. Then R is called*

(1) *reflexive if $R(x, x) = \bar{1}$ for each $x \in U$.*

(2) *symmetric if $R(x, y) = R(y, x)$ for any $x, y \in U$.*

(3) *transitive if $R(x, z) \geq R(x, y) \wedge R(y, z)$ for any $x, y, z \in U$.*

Given R is an IVF relation on U. R is called a preorder (resp. equivalence) if R is reflexive and transitive (resp. reflexive, symmetric and transitive).

Definition 3.2 [15] *Let R be an IVF relation on U. The pair (U, R) is called an IVF approximation space. For each $A \in F^{(i)}(U)$, the IVF lower and IVF upper approximation of A with respect to (U, R), denoted by $\underline{R}(A)$ and $\overline{R}(A)$, are two IVF sets and are respectively defined as follows:*

$$\underline{R}(A)(x) = \bigwedge_{y\in U} (A(y) \vee (\bar{1} - R(x, y))) \quad (x \in U)$$

and

$$\overline{R}(A)(x) = \bigvee_{y\in U}(A(y) \wedge R(x,y)) \qquad (x \in U).$$

The pair $(\underline{R}(A), \overline{R}(A))$ *is called the IVF rough set of A with respect to* (U, R)*.*

Remark 3.1 Let (U, R) be an IVF approximation space. Then

(1) for each $x, y \in U$,

$$\overline{R}(x_{\bar{1}})(y) = R(y,x) \ \ \textit{and} \ \ \underline{R}((x_{\bar{1}})^c)(y) = \bar{1} - R(y,x);$$

(2) for each $[a,b] \in [I]$, $\underline{R}(\widetilde{[a,b]}) \supseteq \widetilde{[a,b]} \supseteq \overline{R}(\widetilde{[a,b]})$.

Proposition 3.1 [15] *Let* (U, R) *be an IVF approximation space. Then for each* $A \in F^{(i)}(U)$*,*

$$(\underline{R}(A))^- = \underline{R^+}(A^-),\ (\underline{R}(A))^+ = \underline{R^-}(A^+),$$
$$(\overline{R}(A))^- = \overline{R^-}(A^-)\ \ \textit{and}\ \ (\overline{R}(A))^+ = \overline{R^+}(A^+).$$

Proposition 3.2 [20] *Let* (U, R) *be an IVF approximation space. Then for any* $A, B \in F^{(i)}(U)$*,* $\{A_j \mid j \in J\} \subseteq F^{(i)}(U)$ *and* $[a,b] \in [I]$*,*

(1) $\underline{R}(\tilde{1}) = \tilde{1}, \overline{R}(\tilde{0}) = \tilde{0}$.

(2) $A \subseteq B \Longrightarrow \underline{R}(A) \subseteq \underline{R}(B), \overline{R}(A) \subseteq \overline{R}(B)$.

(3) $\underline{R}(A^c) = (\overline{R}(A))^c, \overline{R}(A^c) = (\underline{R}(A))^c$.

(4) $\underline{R}(\bigcap_{j\in J} A_j) = \bigcap_{j\in J} \underline{R}(A_j), \overline{R}(\bigcup_{j\in J} A_j) = \bigcup_{j\in J} \overline{R}(A_j)$.

(5) $\underline{R}(\widetilde{[a,b]} \cup A) = \widetilde{[a,b]} \cup \underline{R}(A), \overline{R}([a,b]A) = [a,b]\overline{R}(A)$.

Proposition 3.3 [20] *Let* (U, R) *be an IVF approximation space. Then*

$$R \textit{ is reflexive} \iff (ILR)\ \forall A \in F^{(i)}(U), \underline{R}(A) \subseteq A.$$
$$\iff (IUR)\ \forall A \in F^{(i)}(U), A \subseteq \overline{R}(A).$$

4. Topological Structures of IVF Approximation Spaces

Let (U, R) be an IVF approximation space. We denote

$$\tau_R = \{A \in F^{(i)}(U) \mid \underline{R}(A) = A\},$$
$$s_R = \bigwedge_{x,y\in U} R(x,y), \quad t_R = \bigvee_{x,y\in U, x\neq y} R(x,y).$$

Theorem 4.1 *Let* (U, R) *be an IVF approximation space. If R is reflexive, then*

(1) τ_R *is an IVF topology on U;*

(2) *for each* $A \in F^{(i)}(U)$,

$$int_{\tau_R}(A) \subseteq \underline{R}(A) \subseteq A \subseteq \overline{R}(A) \subseteq cl_{\tau_R}(A);$$

(3) *for each* $[a,b] \in [I]$, $\widetilde{[a,b]} \in \tau_R \cap \tau_R^c$.

Proof (1) (i) By Proposition 3.2(1), $\underline{R}(\tilde{1}) = \tilde{1}$. Then $\tilde{1} \in \tau_R$.
By Proposition 3.3, $\underline{R}(\widetilde{0}) \subseteq \widetilde{0}$. Then $\underline{R}(\widetilde{0}) = \widetilde{0}$. So $\tilde{0} \in \tau_R$.

(ii) Let $A, B \in \tau_R$. By Proposition 3.2(4),

$$\underline{R}(A \cap B) = \underline{R}(A) \cap \underline{R}(B).$$

Then $\underline{R}(A \cap B) = A \cap B$. Thus $A \cap B \in \tau_R$.

(iii) Let $\{A_j \mid j \in J\} \subseteq \tau_R$. Then $\underline{R}(A_j) = A_j$ for each $j \in J$. By Proposition 3.2(2),

$$\underline{R}(\bigcup_{j\in J} A_j) \supseteq \bigcup_{j\in J} \underline{R}(A_j) = \bigcup_{j\in J} A_j.$$

By Proposition 3.3, $\underline{R}(\bigcup_{j\in J} A_j) \subseteq \bigcup_{j\in J} A_j$.
Then $\underline{R}(\bigcup_{j\in J} A_j) = \bigcup_{j\in J} A_j$ and so $\bigcup_{j\in J} A_j \in \tau_R$.
Thus, τ_R is an IVF topology on U.

(2) For each $A \in F^{(i)}(U)$, we have

$$int_{\tau_R}(A) = \bigcup\{B \mid B \in \tau_R \text{ and } B \subseteq A\} = \bigcup\{\underline{R}(B) \mid B \in \tau_R \text{ and } B \subseteq A\} \subseteq \underline{R}(A).$$

By Propositions 2.1(4) and 3.2(3),

$$cl_{\tau_R}(A) = (int_{\tau_R}(A^c))^c \supseteq (\underline{R}(A^c))^c = \overline{R}(A).$$

By Proposition 3.3,

$$int_{\tau_R}(A) \subseteq \underline{R}(A) \subseteq A \subseteq \overline{R}(A) \subseteq cl_{\tau_R}(A).$$

(3) For each $[a,b] \in [I]$, by Proposition 3.3, $\underline{R}(\widetilde{[a,b]}) = \widetilde{[a,b]}$. Then $\widetilde{[a,b]} \in \tau_R$. By Proposition 2.1(4),

$$cl_{\tau_R}(\widetilde{[a,b]}) = (int_{\tau_R}((\widetilde{[a,b]})^c))^c = (int_{\tau_R}(\widetilde{[a,b]}^c))^c = (\widetilde{[a,b]}^c)^c = \widetilde{[a,b]}.$$

So $\widetilde{[a,b]} \in \tau_R^c$.

Theorem 4.2 *Let* (U, R_1) *and* (U, R_2) *be two IVF approximation spaces. If* R_1 *and* R_2 *are preorders, then*

(1) $R_1 \subseteq R_2 \Longrightarrow \tau_{R_2} \subseteq \tau_{R_1}$.

(2) $\tau_{R_1} = \tau_{R_2} \Longleftrightarrow R_1 = R_2$.

Proof (1) Let $R_1 \subseteq R_2$. For each $A \in \tau_{R_2}$, $\underline{R_2}(A) = A$. For each $x \in U$, by the transitivity of R_2,

$$\begin{aligned}
\underline{R_1}(A)(x) &= \underline{R_1}(\underline{R_2}(A))(x) \\
&= \bigwedge_{y\in U}(\underline{R_2}(A)(y) \vee (\bar{1} - R_1(x,y))) \\
&= \bigwedge_{y\in U}((\bigwedge_{z\in U}(A(z) \vee (\bar{1} - R_2(y,z)))) \vee (\bar{1} - R_1(x,y))) \\
&= \bigwedge_{y\in U}(\bigwedge_{z\in U}((A(z) \vee (\bar{1} - R_2(y,z))) \vee (\bar{1} - R_1(x,y)))) \\
&= \bigwedge_{y\in U}(\bigwedge_{z\in U}(A(z) \vee ((\bar{1} - R_2(y,z)) \vee (\bar{1} - R_1(x,y))))) \\
&\geq \bigwedge_{y\in U}(\bigwedge_{z\in U}(A(z) \vee ((\bar{1} - R_2(y,z)) \vee (\bar{1} - R_2(x,y))))) \\
&= \bigwedge_{y\in U}(\bigwedge_{z\in U}(A(z) \vee (\bar{1} - R_2(x,y) \wedge R_2(y,z)))) \\
&\geq \bigwedge_{y\in U}(\bigwedge_{z\in U}(A(z) \vee (\bar{1} - R_2(x,z)))) \\
&= \bigwedge_{z\in U}(A(z) \vee (\bar{1} - R_2(x,z))) \\
&= \underline{R_2}(A)(x) = A(x).
\end{aligned}$$

Then $\underline{R_1}(A) \supseteq A$.

By Theorem 3.7(2), $\underline{R_1}(A) \subseteq A$.

Then $\underline{R_1}(A) = A$ and so $A \in \tau_{R_1}$. Thus $\tau_{R_2} \subseteq \tau_{R_1}$.

(2) Let $\tau_{R_1} = \tau_{R_2}$. By Remark 3.1(1) and Theorem 4.3(2),

$$R_1(x,y) = \overline{R_1}(y_{\bar{1}})(x) = cl_{R_{\tau_1}}(y_{\bar{1}})(x) = cl_{R_{\tau_2}}(y_{\bar{1}})(x) = R_2(x,y)$$

for any $x, y \in U$. Then $R_1 = R_2$.

The converse implication is trivial.

Definition 4.1 *Let R be an IVF relation on U. R is called pseudo-constant if there exists* $[a,b] \in [I]$ *such that for any* $x, y \in U$,

$$R(x,y) = \begin{cases} \bar{1}, & \text{if } x = y, \\ [a,b], & \text{if } x \neq y. \end{cases}$$

We write this R by $[a,b]^*$.

Obviously, every IVF pseudo-constant relation is an IVF equivalence relation.

Remark 4.1 (1) For any $[a_1,b_1], [a_2,b_2] \in [I]$,

$$[a_1,b_1] \leq [a_2,b_2] \text{ implies } [a_1,b_1]^* \subseteq [a_2,b_2]^*.$$

(2) If R is a reflexive IVF relation on U, then $s_R^* \subseteq R \subseteq t_R^*$.

For each $A \in F^{(i)}(U)$, we denote

$$R_A = \{(x,y) \in U \times U \mid A(x) \neq A(y)\},$$

$$R_{A^-} = \{(x,y) \in U \times U \mid A^-(x) > A^-(y)\},$$

$$R_{A^+} = \{(x,y) \in U \times U \mid A^+(x) > A^+(y)\}.$$

Obviously,

$$R_{A^-} = \emptyset \iff A^- = \bar{a} \text{ for some } a \in I.$$

$$R_{A^+} = \emptyset \iff A^+ = \bar{a} \text{ for some } a \in I.$$

$$R_A = \emptyset \iff R_{A^-} = R_{A^+} = \emptyset \iff A = \widetilde{[a,b]} \text{ for some } [a,b] \in [I].$$

Lemma 4.1 *Let R be a reflexive IVF relation on U. Then for each $A \in F^{(i)}(U)$,*

(1) *if $R_{A^-} \neq \emptyset$, then*

a) $(\underline{R}(A))^- = A^- \iff (ILL)\forall (x,y) \in R_{A^-}, 1 - R^+(x,y) \geq A^-(x) \vee A^-(y)$,

b) $(\overline{R}(A))^- = A^- \iff (IUL)\forall (x,y) \in R_{A^-}, R^-(y,x) \leq A^-(x) \wedge A^-(y)$;

(2) *if $R_{A^+} \neq \emptyset$, then*

a) $(\underline{R}(A))^+ = A^+ \iff (ILU)\forall (x,y) \in R_{A^+}, 1 - R^-(x,y) \geq A^+(x) \vee A^+(y)$,

b) $(\overline{R}(A))^+ = A^+ \iff (IUU)\forall (x,y) \in R_{A^+}, R^+(y,x) \leq A^+(x) \wedge A^+(y)$.

Proof (1) a) Necessity. Suppose that $(\underline{R}(A))^- = A^-$. For each $x \in U$, by Proposition 3.2,

$$\bigwedge_{y \in U} (A^-(y) \vee (\bar{1} - R^+(x,y))) = (\underline{R}(A))^-(y) = A^-(x).$$

Then

$$A^-(y) \vee (\bar{1} - R^+(x,y)) \geq A^-(x) \ \ (x,y \in U). \tag{$*$}$$

For each $(x,y) \in R_{A^-}$, $A^-(x) > A^-(y)$. By $(*)$,

$$1 - R^+(x,y) \geq A^-(x) = A^-(x) \vee A^-(y).$$

Sufficiency. Suppose that (ILL) holds. If $x \in U$ and $y \in (R_{A^-})_s(x)$, then

$$A^-(y) \vee (1 - R^+(x, y)) \geq A^-(y) \vee (A^-(x) \vee A^-(y)) \geq A^-(x).$$

If $x \in U$ and $y \notin (R_{A^-})_s(x)$, then $A^-(y) \geq A^-(x)$ and so

$$A^-(y) \vee (1 - R^+(x, y)) \geq A^-(y) \geq A^-(x).$$

Thus for each $x \in U$,

$$(\underline{R}(A))^-(x) = \bigwedge_{y \in U}(A^-(y) \vee (1 - R^+(x, y))) \geq A^-(x).$$

This implies that $(\underline{R}(A))^- \supseteq A^-$.

By Proposition 3.3, $\underline{R}(A) \subseteq A$. Then $(\underline{R}(A))^- \subseteq A^-$.

Hence $(\underline{R}(A))^- = A^-$.

b) Necessity. Suppose that $(\overline{R}(A))^- = A^-$. For each $y \in U$, by Proposition 3.2,

$$\bigvee_{x \in U}(A^-(x) \wedge R^-(y, x)) = (\overline{R}(A))^-(y) = A^-(y).$$

Then

$$A^-(x) \wedge R^-(y, x) \leq A^-(y) \ \ (x, y \in U). \qquad (**)$$

For each $(x, y) \in R_{A^-}$, $A^-(x) > A^-(y)$. By $(**)$,

$$R^-(y, x) \leq A^-(y) = A^-(x) \wedge A^-(y).$$

Sufficiency. Suppose that (ILL) holds. If $y \in U$ and $x \in (R_A)_p(y)$, then

$$A^-(x) \vee R^-(y, x) \leq A^-(x) \wedge (A^-(x) \wedge A^-(y)) \leq A^-(y).$$

If $y \in U$ and $x \notin (R_A)_p(y)$, then $A^-(x) \leq A^-(y)$ and so

$$A^-(x) \wedge R^-(y, x) \leq A^-(x) \leq A^-(y).$$

Thus for each $y \in U$,

$$(\overline{R}(A))^-(y) = \bigvee_{x \in U}(A^-(x) \wedge R^-(y, x)) \leq A^-(y).$$

This implies that $(\overline{R}(A))^- \subseteq A^-$.

By Proposition 3.3, $\overline{R}(A) \supseteq A$. Then $(\overline{R}(A))^- \supseteq A^-$.

Hence $(\overline{R}(A))^- = A^-$.

(2) The proof is similar to (1).

Theorems 4.3 and 4.4 below give topological structures of IVF approximation spaces.

Theorem 4.3 *Let* (U, R) *be an IVF approximation space. If* R *is reflexive, then* $\tau_R = \{A \in F^{(i)}(U) \mid R_{A^-} = \emptyset \ or\ A\ satisfies\ (ILL),\ R_{A^+} = \emptyset \ or\ A\ satisfies\ (ILU)\}$.

Proof This holds by Lemma 4.1 and Theorem 4.1.

Theorem 4.4 *Let* (U, R) *be an IVF approximation space. If* R *is reflexive, then*

(1) $\tau_{t_R^*} \subseteq \tau_R \subseteq \tau_{s_R^*}$.

(2) *For each* $[a, b] \in [I]$ *with* $a \leq b$, $\tau_{[a,b]^*} = \{A \in F^{(i)}(U) \mid R_{A^-} = \emptyset \ or \ \bigvee_{x \in U} A^-(x) \leq 1 - b,\ R_{A^+} = \emptyset \ or \ \bigvee_{x \in U} A^+(x) \leq 1 - a\}$.

(3) $\tau_{[0,0]^*} = F^{(i)}(U)$, $\tau_{[1,1]^*} = \{\widetilde{[a, b]} \mid [a, b] \in [I]\}$.

(4) *If* $\{x_{\bar{1}} \mid x \in U\} \subseteq {\tau_R}^c$, *then* $\tau_R = F^{(i)}(U)$.

Proof (1) This holds by Theorem 4.2(1) and Remark 4.1(2).

(2) This holds by Theorem 4.3.

(3) This holds by (2).

(4) Let $\{x_{\bar{1}} \mid x \in U\} \subseteq {\tau_R}^c$. For any $x, y \in U$, by Theorem 4.1, $\overline{R}(x_{\bar{1}})(y) \subseteq cl_{\tau_R}(x_{\bar{1}})(y) = x_{\bar{1}}(y)$. By the reflexivity of R and Proposition 3.3,

$$\overline{R}(x_{\bar{1}})(y) \supseteq x_{\bar{1}}(y).$$

Then $\overline{R}(x_{\bar{1}})(y) = x_{\bar{1}}(y)$. By Remark 3.1(1),

$$R(y, x) = \overline{R}(x_{\bar{1}})(y) = x_{\bar{1}}(y) = \begin{cases} \bar{1}, & y = x, \\ \bar{0}, & y \neq x. \end{cases}$$

Thus $R = [0, 0]^*$. By (3), $\tau_R = F^{(i)}(U)$.

5. Conclusion

Topology and rough set theory are widely used in the research for computer science. In this paper, we have studied topological structures of IVF approximation spaces and gave connections between topology and rough set theory, which may improve our understandings of the two theories. In the future, we will consider some concrete applications of our results.

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