to be an effective tool to deal with fuzziness. However, it often falls short of the expected standard when describing the neutral state. As a result, a new concept namely intuitionistic fuzzy set (IFS) was worked out and the same was introduced in 1983 by Atanassov [2, 3]. Using the concept of IFS, Im et.al [4] studied intuitionistic fuzzy matrix (IFM).

IFM generalizes the fuzzy matrix introduced by Thomson [5] and has been useful in dealing with areas such as decision making, relational equations, clustering analysis etc. A number of authors [6, 7] have effectively presented impressive results using fuzzy matrix. Atanassov [8], using the definition of index matrix, has paved way for intuitionistic fuzzy index matrix and has further extending it to temporal intuitionistic fuzzy index matrix. IFM is also very useful in the discussion of intuitionistic fuzzy relation [9, 10]. Z.S. Xu [11, 12] studied intuitionistic fuzzy value and also IFMs. He defined intuitionistic fuzzy similarity relation and also utilized it in clustering analysis.

A lot of research activities have been carried out over the years on IFMs in [13-17]. The period of powers of square IFMs is discussed at length along with some of the results for the equivalence IFM by Jeong and Park [18] while Pal et al. [19-26] made a comprehensive study and neatly developed IFM in various years. Another researcher namely Mondal [27] attempted a study on the similarity relations, invertibility and eigenvalues of IFM. In [28], a research was carried out on how a transitive IFM decomposed into a sum of nilpotent IFM and symmetric IFM and in [29] how an IFM gets decomposed into a product of idempotent IFM and rectangular IFM.

Atanassov introduced modal operators in [2] which are meaningless in fuzzy set theory and found a promising direction in research. The above operators for IFMs and some results are obtained in [30]. In this paper, some necessary and sufficient conditions are discussed for a transitive and c-transitive closure matrix interms of modal operators. we explore some more results using modal operators for IFM under max-min composition and discuss similarity relation, idempotents etc. Finally, using modal operators we decompose an IFM by introducing a new composition operator and some properties of that new operator are proved.

2. Preliminaries

We recollect some relevant basic definitions and results will be used later.

Definition 2.1 *Let a set* $X = \{x_1, x_2, \cdots x_n\}$ *be fixed. Then an IFS* [2] *can be defined as* $A = \{\langle x_i, \mu_A(x_i), \nu_A(x_i)\rangle \mid x_i \in X\}$ *which assigns to each element* x_i *a membership degree* $\mu_A(x_i)$ *and a non membership degree* $\nu_A(x_i)$*, with the condition* $0 \leq \mu_A(x_i) + \nu_A(x_i) \leq 1$ *for every* $x_i \in X$.

Definition 2.2 *Xu and Yager called the* 2*-tuple* $\alpha(x_i) = (\mu_\alpha(x_i), \nu_\alpha(x_i))$ *an intuitionistic fuzzy value (IFV)* [11, 12] *where* $\mu_\alpha(x_i) \in [0,1]$*,* $\nu_\alpha(x_i)) \in [0,1]$ *and* $\mu_\alpha(x_i) + \nu_\alpha(x_i) \leq 1$.

Definition 2.3 [2] *Let* $\langle x, x'\rangle, \langle y, y'\rangle \in IFS$*. Then we have*

(i) $\langle x, x'\rangle \vee \langle y, y'\rangle = \langle \max\{x, y\}, \min\{x', y'\}\rangle$.

(ii) $\langle x, x'\rangle \wedge \langle y, y'\rangle = \langle \min\{x, y\}, \max\{x', y'\}\rangle$.

(iii) $\langle x, x'\rangle^c = \langle x', x\rangle$.

Definition 2.4 [3] *Let* $\langle x, x'\rangle, \langle y, y'\rangle \in IFS$. *Then*

$$\langle x, x'\rangle \leftarrow \langle y, y'\rangle = \begin{cases} \langle 1, 0\rangle, & \text{if } \langle x, x'\rangle \geq \langle y, y'\rangle, \\ \langle x, x'\rangle, & \text{if } \langle x, x'\rangle \leq \langle y, y'\rangle. \end{cases}$$

Here $\langle x, x'\rangle \geq \langle y, y'\rangle$ means $x \geq y$ and $x' < y'$.

Definition 2.5 [11] *Let* $A = (z_{ij})_{n\times n}$ *be a matrix if all its elements are IFVs. Then A is called an intuitionistic fuzzy matrix.*

Definition 2.6 *An IFM* $J = (\langle 1, 0\rangle)$ *for all entries is known as the universal matrix* [14] *and an IFM* $O = (\langle 0, 1\rangle)$ *for all entries is known as zero matrices. Denote the set of all IFMs of order* $m \times n$ *by* $\mathscr{F}_{mn}$ *and square matrix of order n by* $\mathscr{F}_n$. *The identity IFM* $I = (\langle \delta_{ij}, \delta'_{ij}\rangle)$ *is defined by* $\langle \delta_{ij}, \delta'_{ij}\rangle = \langle 1, 0\rangle$ *if* $i = j$ *and* $\langle \delta_{ij}, \delta'_{ij}\rangle = \langle 0, 1\rangle$ *if* $i \neq j$.

Definition 2.7 [11, 12] *Let* $A = (\langle a_{ij}, a'_{ij}\rangle)_{m\times n}$, $B = (\langle b_{ij}, b'_{ij}\rangle)_{m\times n}$ *and* $C = (\langle c_{ij}, c'_{ij}\rangle)_{n\times p}$ *are IFMs. Then*

(i) $A \vee B = (\langle a_{ij}, a'_{ij}\rangle \vee \langle b_{ij}, b'_{ij}\rangle)$.

(ii) $A \wedge B = (\langle a_{ij}, a'_{ij}\rangle \wedge \langle b_{ij}, b'_{ij}\rangle)$.

(iii) $AC(max - mincomposition) = (\bigvee_k (\langle a_{ik}, a'_{ik}\rangle \wedge \langle c_{kj}, c'_{kj}\rangle))$.

Definition 2.8 [14] *Let* $A = (\langle a_{ij}, a'_{ij}\rangle)_{m\times n}$ *and* $C = (\langle c_{ij}, c'_{ij}\rangle)_{n\times p}$ *are IFMs. Then we have*

(i) $A \diamond C$ *(min-max composition)*$= (\bigwedge_k (\langle a_{ik}, a'_{ik}\rangle \vee \langle c_{kj}, c'_{kj}\rangle))$.

(ii) $A^T = (\langle a_{ji}, a'_{ji}\rangle)$ *(Transpose of A).*

(iii) $A \leftarrow C = (\bigwedge_k (\langle a_{ik}, a'_{ik}\rangle \leftarrow \langle c_{kj}, c'_{kj}\rangle))$.

(iv) $A \rightarrow C = (\bigwedge_k (\langle a_{ik}, a'_{ik}\rangle \rightarrow \langle c_{kj}, c'_{kj}\rangle))$.

(v) $A^c = (\langle a'_{ij}, a_{ij}\rangle)$ *(Complement of A).*

Also we can use $AC = (\langle \sum_{k=1}^{n} (a_{ik}c_{kj}), \prod_{k=1}^{n} (a'_{ik} + c'_{kj})\rangle)$.

Also $A^2 = AA$, $A^k = A^{k-1}A$ for max-min composition and $A^{[2]} = A \diamond A, A^{[k]} = A^{[k-1]} \diamond A$ for min-max composition.

Definition 2.9 [14] *For any IFM* $A \in \mathscr{F}_n$,

(i) *A is reflexive if and only if* $A \geqslant I_n$.

(ii) *A is symmetric if and only if* $A = A^T$.

(iii) *A is transitive if and only if* $A \geqslant A^2$.

(iv) *A is idempotent if and only if* $A = A^2$.

(v) *A is irreflexive if* $\langle a_{ii}, a'_{ii}\rangle = \langle 0, 1\rangle$ *for all* $i = j$.

(vi) *A is c-transitive if* $A \le A^{[2]}$.

Definition 2.10 [11] *An IFM A is said to be an intuitionistic fuzzy equivalence matrix if it satisfy reflexivity, symmetry and transitivity.*

Definition 2.11 *For IFS A, Atanassov has defined the modal operators* [2] $\Box(necessity)$ *and* $\Diamond(possibility)$ *in the following way.* $\Box A = \{\langle \mu_A(x), 1 - \mu_A(x)\rangle \mid x \in E$, *the Universal set*$\}$ *and* $\Diamond A = \{\langle 1 - \nu_A(x), \nu_A(x)\rangle \mid x \in E\}$.

Proposition 2.1 [14] $(A \circ B)^c = A^c + B^c$ *for* $A, B \in \mathscr{F}_{mn}$.

Proposition 2.2 [14] $(A + B)^c = A^c \circ B^c$ *for* $A, B \in \mathscr{F}_{mn}$.

In [30], the following are discussed by the authors.

Definition 2.12 *For an IFM A, we have* $\Box A = (\langle a_{ij}, 1 - a_{ij}\rangle)$ *and* $\Diamond A = (\langle 1 - a'_{ij}, a'_{ij}\rangle)$.

Lemma 2.1 $1 - \prod_{k=1}^{n}(a_{ik} + b_{kj}) = \sum_{k=1}^{n}(1 - a_{ik})(1 - b_{kj})$ *for all i, j,* $a_{ij}, b_{ij} \in [0, 1]$.

Lemma 2.2 $1 - \sum_{k=1}^{n} a_{ik}b_{kj} = \prod_{k=1}^{n}((1 - a_{ik}) + (1 - b_{kj}))$ *for all i, j,* $a_{ij}, b_{ij} \in [0, 1]$.

3. More Results of Modal Operators in IFM

Throughout this section, matrices means intuitionistic fuzzy matrices. In this section, some results about modal operators are proved and the definitions of transitive and c-transitive of an IFM A are given.

Lemma 3.1 *For any two IFMs A and B,*

$$\Box(\langle a_{ij}, a'_{ij}\rangle \leftarrow \langle b_{ij}, b'_{ij}\rangle) = \Box(\langle a_{ij}, a'_{ij}\rangle) \leftarrow \Box(\langle b_{ij}, b'_{ij}\rangle). \tag{1}$$

Proof (i) If $\langle a_{ij}, a'_{ij}\rangle \geqslant \langle b_{ij}, b'_{ij}\rangle$, then

$$\Box(\langle a_{ij}, a'_{ij}\rangle \leftarrow \langle b_{ij}, b'_{ij}\rangle) = \Box(\langle 1, 0\rangle) = \langle 1, 0\rangle. \tag{2}$$

Since $\langle a_{ij}, a'_{ij}\rangle \geqslant \langle b_{ij}, b'_{ij}\rangle, a_{ij} \geqslant b_{ij}$ and $a'_{ij} \leqslant b'_{ij}$. Therefore, $1 - a_{ij} \leqslant 1 - b_{ij}$ and $\langle a_{ij}, 1 - a_{ij}\rangle \geqslant \langle b_{ij}, 1 - b_{ij}\rangle$, so $\Box(\langle a_{ij}, a'_{ij}\rangle) \geqslant \Box(\langle b_{ij}, b'_{ij}\rangle)$. Thus

$$\Box\langle a_{ij}, a'_{ij}\rangle \leftarrow \Box\langle b_{ij}, b'_{ij}\rangle = \langle 1, 0\rangle. \tag{3}$$

From (2) and (3), (1) holds.

(ii) If $\langle a_{ij}, a'_{ij}\rangle \le \langle b_{ij}, b'_{ij}\rangle$, then

$$\Box(\langle a_{ij}, a'_{ij}\rangle \leftarrow \langle b_{ij}, b'_{ij}\rangle) = \Box\langle a_{ij}, a'_{ij}\rangle = \langle a_{ij}, 1 - a_{ij}\rangle \tag{4}$$

$$\Box\langle a_{ij}, a'_{ij}\rangle \leftarrow \Box\langle b_{ij}, b'_{ij}\rangle = \langle a_{ij}, 1 - a_{ij}\rangle \leftarrow \langle b_{ij}, 1 - b_{ij}\rangle = \langle a_{ij}, 1 - a_{ij}\rangle \tag{5}$$

Clearly, from (4) and (5), (1) holds.

Lemma 3.2 *For any two IFMs A and B,*

$$\Diamond(\langle a_{ij}, a'_{ij}\rangle \leftarrow \langle b_{ij}, b'_{ij}\rangle) = \Diamond(\langle a_{ij}, a'_{ij}\rangle) \leftarrow \Diamond(\langle b_{ij}, b'_{ij}\rangle). \tag{6}$$

Proof (i) If $\langle a_{ij}, a'_{ij}\rangle \geqslant \langle b_{ij}, b'_{ij}\rangle$, then

$$\Diamond(\langle a_{ij}, a'_{ij}\rangle \leftarrow \langle b_{ij}, b'_{ij}\rangle) = \Diamond(\langle 1, 0\rangle) = \langle 1, 0\rangle. \tag{7}$$

Since $\langle a_{ij}, a'_{ij}\rangle \geqslant \langle b_{ij}, b'_{ij}\rangle, a_{ij} \geqslant b_{ij}$ and $a'_{ij} \leqslant b'_{ij}$. Therefore $1 - a_{ij} \leqslant 1 - b_{ij}$ and $\langle a_{ij}, 1 - a_{ij}\rangle \geqslant \langle b_{ij}, 1 - b_{ij}\rangle$, so $\Diamond(\langle a_{ij}, a'_{ij}\rangle) \geqslant \Diamond(\langle b_{ij}, b'_{ij}\rangle)$. Thus

$$\diamond\langle a_{ij}, a'_{ij}\rangle \leftarrow \diamond\langle b_{ij}, b'_{ij}\rangle = \langle 1, 0\rangle. \tag{8}$$

From (7) and (8), (6) holds.

(ii) If $\langle a_{ij}, a'_{ij}\rangle < \langle b_{ij}, b'_{ij}\rangle$, then

$$\diamond(\langle a_{ij}, a'_{ij}\rangle \leftarrow \langle b_{ij}, b'_{ij}\rangle) = \diamond\langle a_{ij}, a'_{ij}\rangle = \langle 1 - a'_{ij}, a'_{ij}\rangle, \tag{9}$$

$$\diamond\langle a_{ij}, a'_{ij}\rangle \leftarrow \diamond\langle b_{ij}, b'_{ij}\rangle = \langle 1 - a'_{ij}, a'_{ij}\rangle \leftarrow \langle 1 - b'_{ij}, b'_{ij}\rangle = \langle 1 - a'_{ij}, a'_{ij}\rangle. \tag{10}$$

Clearly from (9) and (10), (6) holds.

Lemma 3.3 *A is reflexive matrix if and only if $\Box A$ is reflexive matrix.*

Proof A is reflexive $\Leftrightarrow A \geqslant I \Leftrightarrow \langle a_{ij}, a'_{ij}\rangle \geqslant \langle \delta_{ij}, \delta'_{ij}\rangle$ for all i, j.
$\Leftrightarrow \langle a_{ij}, 1 - a'_{ij}\rangle \geqslant \langle \delta_{ij}, 1 - \delta_{ij}\rangle$ for all i, j
$\Leftrightarrow \Box A \geqslant \Box I \Leftrightarrow \Box A$ is reflexive.

In dual way we can prove the following lemma.

Lemma 3.4 *A is reflexive matrix if and only if $\diamond A$ is reflexive matrix.*

Lemma 3.5 *A is reflexive if and only if $\Box A^c$ is irreflexive.*

Proof It is evident that if A is reflexive if and only if A^c is irreflexive and so $\Box A^c$. Similarly, $\diamond A^c$ is irreflexive if and only if A is reflexive.

Lemma 3.6 *A is symmetric matrix if and only if $\Box A$ is symmetric matrix and so $\Box A^c$.*

Proof A is symmetric$\Leftrightarrow \langle a_{ij}, a'_{ij}\rangle = \langle a_{ji}, a'_{ji}\rangle$ for all $i, j \Leftrightarrow \langle a_{ij}, 1 - a_{ij}\rangle = \langle a_{ji}, 1 - a_{ji}\rangle \Leftrightarrow \Box A = (\Box A)^T$. Thus A is symmetric if and only if $\Box A$ is symmetric.

Similarly, we can prove the following lemma.

Lemma 3.7 *A is symmetric matrix if and only if $\diamond A$ is reflexive matrix.*

Lemma 3.8 *A is transitive matrix if and only if $\Box A$ is transitive matrix.*

Proof A is transitive $\Leftrightarrow A \geqslant A^2 \Leftrightarrow \langle a_{ij}, a'_{ij}\rangle \geqslant \langle \sum_{k=1}^{n}(a_{ik}a_{kj}), \prod_{k=1}^{n}(a'_{ik}+a'_{kj})\rangle$ for all$i, j \Leftrightarrow$ $a_{ij} \geqslant \sum_{k=1}^{n}(a_{ik}a_{kj}, a'_{ij}) \leqslant \prod_{k=1}^{n}(a'_{ik} + b'_{kj}) \Leftrightarrow a_{ij} \geqslant \sum_{k=1}^{n}(a_{ik}a_{kj}), 1 - a_{ij} \leqslant 1 - \sum_{k=1}^{n}(a_{ik}a_{kj}) \Leftrightarrow$ $\langle a_{ij}, 1 - a_{ij}\rangle \geqslant \langle \sum_{k=1}^{n} a_{ik}a_{kj}, 1 - \sum_{k=1}^{n} a_{ik}a_{kj}\rangle = \langle \sum_{k=1}^{n} a_{ik}a_{kj}, \prod_{k=1}^{n}((1 - a_{ik}) + (1 - a_{kj}))\rangle$ by Lemma 2.2.

Similarly, we can prove the following lemma.

Lemma 3.9 *A is transitive matrix if and only if $\diamond A$ is transitive matrix.*

Lemma 3.10 *A is idempotent matrix if and only if $\Box A$ is idempotent matrix.*

Proof A idempotent $\Leftrightarrow A = A^2 \Leftrightarrow \langle a_{ij}, a'_{ij}\rangle = \langle \sum_{k=1}^{n}(a_{ik}a_{kj}), \prod_{k=1}^{n}(a_{ik} + a_{kj})\rangle$ for all i, j. $\Leftrightarrow \langle a_{ij}, 1-a_{ij}\rangle = \langle \sum_{k=1}^{n}(a_{ik}a_{kj}), 1-\sum_{k=1}^{n}(a_{ik}a_{kj})\rangle \Leftrightarrow \langle \sum_{k=1}^{n}(a_{ik}a_{kj}), \prod_{k=1}^{n}((1-a_{ik})+1-a_{kj})\rangle$ by Lemma 2.2 $\Leftrightarrow \Box A = (\Box A)^2$. Thus A is idempotent $\Leftrightarrow \Box A$ is idempotent.

The following lemma is trivial from the above.

Lemma 3.11 *A is idempotent matrix if and only if $\diamond A$ is idempotent matrix.*

Remark 3.1 If A is an intuitionistic fuzzy equivalence matrix, then $\Box A$ and $\Diamond A$ are also intuitionistic fuzzy equivalence matrices.

Definition 3.1 *Let $A \in \mathscr{F}_n$, the transitive closure and c-transitive closure of A is defined by $A^\infty = A \vee A^2 \vee A^3 \vee \cdots \vee A^n$ and $A_\infty = A^c \wedge (A^c)^{[2]} \wedge \cdots \wedge (A^c)^{[n]}$ respectively.*

Theorem 3.1 *For $A \in \mathscr{F}_n, A_\infty = (A^\infty)^c$.*

Proof By Definition 3.1, $(A^\infty)^c = (A \vee A^2 \vee A^3 \vee \cdots \vee A^n)^c = (A^c \wedge (A^2)^c \wedge \cdots \wedge (A^n)^c)$. First let us prove $(A^2)^c = (A^c)^{[2]}$.

We know that $A^2 = \langle \sum_{k=1}^{n} (a_{ik}a_{kj}), \prod_{k=1}^{n} (a'_{ik} + a'_{kj}) \rangle$ and so

$$(A^2)^c = \langle \prod_{k=1}^{n} (a'_{ik} + a'_{kj}), \sum_{k=1}^{n} (a_{ik}a_{kj}) \rangle. \tag{11}$$

Also $A^c = \langle a'_{ij}, a_{ij} \rangle$ gives by the definition of $A^{[2]}$,

$$(A^c)^{[2]} = \langle \prod_{k=1}^{n} (a'_{ik} + a'_{kj}), \sum_{k=1}^{n} (a_{ik}a_{kj}) \rangle. \tag{12}$$

Thus by (11) and (12) $(A^2)^c = (A^c)^{[2]}$, so in general $(A^n)^c = (A^c)^{[n]}$.
By Definition 3.1,

$$\begin{aligned}(A^\infty)^c &= (A \vee A^2 \vee A^3 \vee \cdots \vee A^n)^c \\ &= (A^c \wedge (A^2)^c \wedge \cdots \wedge (A^n)^c \\ &= (A^c) \wedge (A^c)^{[2]} \wedge \cdots \wedge (A^c)^{[n]}) = A_\infty.\end{aligned}$$

Lemma 3.12 *A is transitive if and only if A^c is c-transitive and so $\Box A^c$ is.*

Proof It is evident from the definition of transitive and c-transitive.

Lemma 3.13 *If A is reflexive IFM, then*

(i) *A^T is reflexive.*

(ii) *$A \vee B$ is reflexive.*

(iii) *$A \wedge B$ is reflexive if and only if B is reflexive.*

Proof (i) and (ii) are obvious from the definition of reflexive.

(iii) If B is not reflexive, then $\langle b_{ii}, b'_{ii} \rangle \neq \langle 1, 0 \rangle$ for at least one i, that is $\langle b_{ii}, b'_{ii} \rangle < \langle 1, 0 \rangle$. Thus $\langle a_{ii}, a'_{ii} \rangle \wedge \langle b_{ii}, b'_{ii} \rangle < \langle 1, 0 \rangle$. Therefore the condition B is reflexive is necessary, the sufficient part is trivial.

Theorem 3.2 *If $A, B \in \mathscr{F}_n$ where A is reflexive and symmetric, B is reflexive, symmetric and transitive and $A \leq B$, then $A^\infty \leq B$.*

Proof For $A = (\langle a_{ij}, a'_{ij} \rangle), B = (\langle b_{ij}, b'_{ij} \rangle), AB = (\langle \sum_{k=1}^{n} (a_{ik}b_{kj}), \prod_{k=1}^{n} (a'_{ik} + b'_{kj}) \rangle)$ and each

$$\langle \sum_{k=1}^{n} (a_{ik}b_{kj}), \prod_{k=1}^{n} (a'_{ik} + b'_{kj}) \rangle = \begin{cases} \langle 1, 0 \rangle, & \text{if } i = j, \\ \langle b_{ij}, b'_{ij} \rangle, & \text{if } i \neq j. \end{cases}$$

Thus $AB = B \Rightarrow AA \leq AB = B$. That is $A^2 \leq B$. Continuing in this way, we have $A^3 \leq B, A^4 \leq B \cdots$ and also $A \vee A^2 \vee A^3 \cdots \vee A^n \leq B$ and hence $A^\infty \leq B$.

Lemma 3.14 *If A^∞ is the transitive closure of A, then the transitive closure of $\Box A$ is $\Box A^\infty$.*

Proof Now $\Box A^\infty = \Box[A \vee A^2 \vee \cdots \vee A^n] = \Box A \vee \Box A^2 \vee \cdots \Box A^n = \Box A \vee (\Box A)^2 \vee \cdots (\Box A)^n = (\Box A)^\infty$. Similarly, the following results are also true.

(i) $\Box A_\infty = (\Box A)_\infty$.

(ii) $\Diamond A^\infty = (\Diamond A)^\infty$.

(iii) $\Diamond A_\infty = (\Diamond A)_\infty$.

Lemma 3.15 *For an IFM $A \in \mathscr{F}_n$, $[(\Box A)^c]^\infty = [(\Box A)_\infty]^c$.*

Proof As we know $(\Box A)^c = \Diamond A^c$, $[(\Box A)^c]^\infty = [\Diamond A^c]^\infty = \Diamond A^c \vee (\Diamond A^c)^2 \cdots \vee (\Diamond A^c)^n$.

$$\begin{aligned}(\Diamond A^c)^2 &= (\langle \sum_{k=1}^{n}(1-a_{ik})(1-a_{kj}), \prod_{k=1}^{n}(a_{ik}+a_{kj})\rangle) \\ &= (\langle 1-\prod_{k=1}^{n}(a_{ik}+a_{kj}), \prod_{k=1}^{n}(a_{ik}+a_{kj})\rangle).\end{aligned} \tag{13}$$

By definition $A^{[2]} = (\langle \prod_{k=1}^{n}(a_{ik}+a_{kj}), \sum_{k=1}^{n}(a'_{ik}a_{kj})'\rangle)$ and so

$$\Box A^{[2]} = (\langle \prod_{k=1}^{n}(a_{ik}+a_{kj}), 1-\prod_{k=1}^{n}(a_{ik}+a_{kj})\rangle).$$

Which yields

$$(\Box A^{[2]})^c = (\langle 1-\prod_{k=1}^{n}(a_{ik}+a_{kj}), \prod_{k=1}^{n}(a_{ik}+a_{kj})\rangle).$$

Therefore, $(\Diamond A^c)^2 = (\Box A^{[2]})^c$, so in general $(\Diamond A^c)^n = (\Box A^{[n]})^c$

$$\begin{aligned}[(\Box A)^c]^\infty = [\Diamond A^c]^\infty &= \Diamond A^c \vee (\Diamond A^c)^2 \cdots \vee (\Diamond A^c)^n \\ &= (\Box A)^c \vee (\Box A^{[2]})^c \vee \cdots (\Box A^{[n]})^c \\ &= (\Box A \wedge \Box A^{[2]} \wedge \cdots \wedge \Box A^{[n]})^c = (\Box A_\infty)^c.\end{aligned}$$

In dual fashion, one can prove the following lemma.

Lemma 3.16 *For an IFM $A \in \mathscr{F}_n$, $((\Diamond A)_\infty)^c = ((\Diamond A)^c)^\infty$.*

Definition 3.2 *For any two elements $\langle x, x'\rangle, \langle y, y'\rangle \in IFS$, we introduce the operation $'\wedge'_m$ as $\langle x, x'\rangle \wedge_m \langle y, y'\rangle = \langle \min\{x, y\}, \min\{x', y'\}\rangle$.*

Using this definition the following lemmas are trivial.

Lemma 3.17 *The operation $\wedge_m$ is commutative on IFS.*

Lemma 3.18 *The operation $\wedge_m$ is associative on IFS.*

Lemma 3.19 *The operation $\wedge_m$ is distributive over addition in IFSs.*

Proof For any $\langle x, x'\rangle, \langle y, y'\rangle, \langle z, z'\rangle \in IFS$

$$\begin{aligned}(\langle x, x'\rangle + \langle y, y'\rangle) \wedge_m \langle z, z'\rangle &= \langle \max\{x, y\}, \min\{x', y'\}\rangle \wedge_m \langle z, z'\rangle \\ &= \langle \min\{\max\{x, y\}, z\}, \min\{\min\{x', y'\}, z'\}\rangle.\end{aligned} \tag{14}$$

Case (1) If $\langle x, x'\rangle \geq \langle y, y'\rangle$ and $\langle x, x'\rangle \geq \langle z, z'\rangle$, then right hand side of (14) is $\langle z, x'\rangle$. Now consider

$$(\langle x,x^{'}\rangle \wedge_m \langle z,z^{'}\rangle) + (\langle y,y^{'}\rangle \wedge_m \langle z,z^{'}\rangle)$$
$$= \langle z,x^{'}\rangle + \begin{cases}\langle z,y^{'}\rangle \text{ if } \langle z,z^{'}\rangle \le \langle y,y^{'}\rangle \\ \langle y,z^{'}\rangle \text{ if } \langle y,y^{'}\rangle \le \langle z,z^{'}\rangle\end{cases} = \langle z,x^{'}\rangle. \tag{15}$$

In this case, it is distributive.

Case (2) If $\langle x,x^{'}\rangle \le \langle y,y^{'}\rangle$ and $\langle x,x^{'}\rangle \le \langle z,z^{'}\rangle$, then the left hand side of (15) reduces to $\langle y,y^{'}\rangle \wedge_m \langle z,z^{'}\rangle$,

Subcase (2.1) If $\langle z,z^{'}\rangle \le \langle y,y^{'}\rangle$, then $\langle z,z^{'}\rangle \wedge_m \langle y,y^{'}\rangle = \langle z,y^{'}\rangle$. Now

$$(\langle x,x^{'}\rangle \wedge_m \langle z,z^{'}\rangle) + (\langle x,x^{'}\rangle \wedge_m \langle z,z^{'}\rangle) = \langle x,z^{'}\rangle + \langle z,y^{'}\rangle = \langle z,y^{'}\rangle.$$

Thus distributivity holds.

Subcase (2.2) If $\langle z,z^{'}\rangle \ge \langle y,y^{'}\rangle$, then left hand side of (15) becomes $\langle y,z^{'}\rangle$ and right hand side of (15) becomes

$$(\langle x,x^{'}\rangle \wedge_m \langle z,z^{'}\rangle) + (\langle x,x^{'}\rangle \wedge_m \langle z,z^{'}\rangle) = \langle x,z^{'}\rangle + \langle y,z^{'}\rangle = \langle y,z^{'}\rangle.$$

Thus it is distributive in this case also.

Case (3) If $\langle y,y^{'}\rangle \le \langle x,x^{'}\rangle \le \langle z,z^{'}\rangle$, then the left hand side becomes

$$(\langle x,x^{'}\rangle + \langle y,y^{'}\rangle) \wedge_m \langle z,z^{'}\rangle = \langle x,x^{'}\rangle \wedge_m \langle z,z^{'}\rangle = \langle x,z^{'}\rangle.$$

Also $(\langle x,x^{'}\rangle \wedge_m \langle z,z^{'}\rangle) + (\langle x,x^{'}\rangle \wedge_m \langle z,z^{'}\rangle) = \langle x,z^{'}\rangle + \langle y,z^{'}\rangle = \langle x,z^{'}\rangle$.

So it is distributive in this case too.

Case (4) If $\langle z,z^{'}\rangle \ge \langle x,x^{'}\rangle \ge \langle y,y^{'}\rangle$, then the left hand side reduces to

$$\langle y,y^{'}\rangle \wedge_m \langle z,z^{'}\rangle = \langle z,y^{'}\rangle.$$

And

$$(\langle x,x^{'}\rangle \wedge_m \langle z,z^{'}\rangle) + (\langle x,x^{'}\rangle \wedge_m \langle z,z^{'}\rangle) = \langle z,x^{'}\rangle + \langle z,y^{'}\rangle = \langle z,y^{'}\rangle.$$

Thus distributivity holds for all cases.

Definition 3.3 *For any two elements $\langle x,x^{'}\rangle, \langle y,y^{'}\rangle \in IFS$, we define the inequality $' \le '$ as $\langle x,x^{'}\rangle \le \langle y,y^{'}\rangle$ means $x \le y$ and $x^{'} \le y^{'}$.*

Remark 3.2 The elements in the set $\{\langle y,y^{'}\rangle \in IFS \mid \langle x,x^{'}\rangle \le \langle y,y^{'}\rangle\}$ are identity element of $\langle x,x^{'}\rangle$ with respect to $\wedge_m$. That is we have multiple identity element.

Remark 3.3 Any IFM A can be decomposed into two intutionistic fuzzy matrices $\Box A$ and $\Diamond A$ by means of $\wedge_m$. That is

$$A = (\Box A) \wedge_m (\Diamond A).$$

Remark 3.4 For any two IFMs A and B, $(A \vee B) \wedge_m (A \wedge B) = (A \wedge_m B)$.

4. Conclusion

Any fuzzy matrix $A = (a_{ij})$ is an IFM in the form of $A = (a_{ij}, 1 - a_{ij})$. The matrices $\Box A$ and$\Diamond A$ using necessity and possibility operators denoted as $\Box$ and $\Diamond$ are formed, in which we consider only either membership or nonmembership of any IFM A gives a fuzzy matrix. Here we present some results of the above said operators with other operators using illustration. Transitive and c-transitive closures are defined and some results are proved on IFM. Finally, using a new operator we express an IFM in terms of fuzzy matrix.

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References

[1] L.A. Zadeh, Fuzzy sets, Journal of Information and Control 8 (1965) 338-353.
[2] K. Atanassov, Intuitionistic Fuzzy Sets, VII ITKR's Section, Sofia (Deposed in Central Sci.-Techn. Library of Bulg. Acad. of Sci.,1697/84) June 1983.
[3] K. Atanassov, Intuitionistic fuzzy sets, Fuzzy Sets and Systems 20(1) (1986) 87-96.
[4] Y.B. Im, E.P. Lee, S.W. Park, The determinant of square intuitionistic fuzzy matrices, Far East Journal of Mathematical Sciences 3(5) (2001) 789-796.
[5] M.G. Thomson, Convergence of powers of a fuzzy matrix, Journal of Mathematical Analysis and Applications 57 (1977) 476-480.
[6] H. Hashimoto, Decomposition of fuzzy matrices, Siam Journal on Algebraic & Discrete Methods 6(1) (1985) 32-38.
[7] K.H. Kim, R.W. Roush, Generalized fuzzy matrices, Fuzzy Sets and Systems 4 (1980) 293-315.
[8] K. Atanassov, Index matrices: towards an augmented matrix calculus, Studies in Computational Intelligence 573 (2014) doi: 10.1007/978-3-319-10945-9.
[9] H. Bustince, P.Burillo, Structures on intuitionistic fuzzy relations, Fuzzy Sets and Systems 78(3) (1996) 293-303.
[10] A.R. Meenakshi, T.Gandhimathi, Intuitionistic fuzzy relational equations, Advances in Fuzzy Mathematics 5(2) (2010) 239-244.
[11] Z.S. Xu, Intuitionistic fuzzy aggregation and clustering, Studies in Fuzziness and soft computing 275 (2012) 159-190.
[12] X. Zhang, A new method for ranking intuitionistic fuzzy values and its applications in multi attribute decision making, Fuzzy Optimization Decision Making 11(2) (2012) 135-146.
[13] Y.B. IM, E.P. Lee, S.W. Park, The adjoint of square intuitionistic fuzzy matrices, Journal of Applied Math and Computing (series A), 11(2003),No 1-2, 401-412.
[14] P. Murugadas, Contribution to a study on generalized fuzzy matrices, Ph.D Thesis, Department of Mathematics, Annamalai University, July-2011.
[15] M. Pal, S.K. Khan, Amiya K Shyamal, Intuitionistic fuzzy matrices, Notes on Intuitionistic Fuzzy Sets 8(2) (2002) 51-62.
[16] R. Pradhan, M. Pal, Some results on generalized inverse of intuitionistic fuzzy matrices, Fuzzy Information and Engineering, 6(2) (2014) 133-145.
[17] Z. Wang, Z.S. Xu, S.S. Liu, J. Tang, A netting clustering analysis method under intuitionistic environment, Applied Soft Computing 11(8) (2011) 5558-5564.
[18] N.G. Jeong, S.W. Park, The equivalence intuitionistic fuzzy matrix and idempotent, Far East Journal of Mathematical Sciences 11(3) (2003) 355-365.
[19] A.K. Adak, M. Pal, M. Bhowmik, Distributive lattice over intuitionistic fuzzy matrices, The Journal of Fuzzy Mathematics 21(2) (2013) 401-416.
[20] M. Bhowmik, M. Pal, Some results on intuitionistic fuzzy matrices and circulant intuitionistic fuzzy matrices, International Journal of Mathematical Sciences 7(1-2) (2008) 81-96.
[21] M. Bhowmik, M. Pal, Generalized intuitionistic fuzzy matrices, Far East Journal of Mathematical Sciences 29(3) (2008) 533-554.
[22] M. Bhowmik, M. Pal, Intuitionistic neutrosophic set, Journal of Computing & Information Science in Engineering 4(2) (2009) 142-152.
[23] S. Mondal, M. Pal, Intuitionistic fuzzy incline matrix and determinant, Annals of Fuzzy Mathematics and Informatics 8(1) (2014) 19-32.
[24] M. Pal, Intuitionistic fuzzy determinant, V.U.J. Physical Sciences 7 (2001) 65-73.
[25] M. Pal, Interval-valued fuzzy matrices with interval-valued fuzzy rows and columns, Fuzzy Information and Engineering 7(3) (2015) 335-368.
[26] R. Pradhan, M. Pal, The generalized inverse of Atanassov's intuitionistic fuzzy matrices, International Journal of Computational Intelligence Systems 7(6) (2014) 1083-1095.

[27] S. Mondal, M. Pal, Similarity relations, invertibility and eigenvalues of intuitoinistic fuzzy matrix, Fuzzy Information and Engineering 5(4) (2013) 431-443.

[28] N.G. Jeong, H.Y. Lee, Canonical form of transitive intuitionistic fuzzy matrices, Honam Mathematical Journal 27(4) (2005) 543-550.

[29] P. Murugadas, K.Lalitha, Decomposition of an intuitionistic fuzzy matrices using implication operators, Annals of Fuzzy Mathematics and Informatics 11(1) (2015) 11-18.

[30] P. Murugadas, S. Sriram, T. Muthuraji, Modal operators in intuitionistic fuzzy matrices, International Journal of Computer Application 90(17) (2014) 1-4.