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Decomposition of Intuitionistic Fuzzy Matrices



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Abstract In this paper, we study some properties of modal operators in intuitionistic fuzzy matrix and we introduce a new composition operator and discuss some of its algebraic properties. Finally, we obtain a decomposition of an intuitionistic fuzzy matrix by using the new composition operator and modal operators.

Keywords Intuitionistic fuzzy matrix · Intuitionistic fuzzy set

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1. Introduction

There have been theories evolved over the years to deal with the various types of uncertainties. These evolved theories are put into practice and when found to be wanting are improved upon, paving the way for new theories to handle the tricky uncertainties. The probability theory is one such important theory concerned with the analysis of random phenomena. In 1965, Zadeh [1] came out with the concept of fuzzy set which is indeed an extension of the classical notion of set. Fuzzy set has been found

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to be an effective tool to deal with fuzziness. However, it often falls short of the expected standard when describing the neutral state. As a result, a new concept namely intuitionistic fuzzy set (IFS) was worked out and the same was introduced in 1983 by Atanassov [2, 3]. Using the concept of IFS, Im et.al [4] studied intuitionistic fuzzy matrix (IFM).

IFM generalizes the fuzzy matrix introduced by Thomson [5] and has been useful in dealing with areas such as decision making, relational equations, clustering analysis etc. A number of authors [6, 7] have effectively presented impressive results using fuzzy matrix. Atanassov [8], using the definition of index matrix, has paved way for intuitionistic fuzzy index matrix and has further extending it to temporal intuitionistic fuzzy index matrix. IFM is also very useful in the discussion of intuitionistic fuzzy relation [9, 10]. Z.S. Xu [11, 12] studied intuitionistic fuzzy value and also IFMs. He defined intuitionistic fuzzy similarity relation and also utilized it in clustering analysis.

A lot of research activities have been carried out over the years on IFMs in [13-17]. The period of powers of square IFMs is discussed at length along with some of the results for the equivalence IFM by Jeong and Park [18] while Pal et al. [19-26] made a comprehensive study and neatly developed IFM in various years. Another researcher namely Mondal [27] attempted a study on the similarity relations, invertibility and eigenvalues of IFM. In [28], a research was carried out on how a transitive IFM decomposed into a sum of nilpotent IFM and symmetric IFM and in [29] how an IFM gets decomposed into a product of idempotent IFM and rectangular IFM.

Atanassov introduced modal operators in [2] which are meaningless in fuzzy set theory and found a promising direction in research. The above operators for IFMs and some results are obtained in [30]. In this paper, some necessary and sufficient conditions are discussed for a transitive and c-transitive closure matrix in terms of modal operators. we explore some more results using modal operators for IFM under max-min composition and discuss similarity relation, idempotents etc. Finally, using modal operators we decompose an IFM by introducing a new composition operator and some properties of that new operator are proved.

2. Preliminaries

We recollect some relevant basic definitions and results will be used later.

Definition 2.1 Let a set $X = \{x_1, x_2, \dots, x_n\}$ be fixed. Then an IFS [2] can be defined as $A = \{\langle x_i, \mu_A(x_i), \nu_A(x_i) \rangle \mid x_i \in X\}$ which assigns to each element x_i a membership degree $\mu_A(x_i)$ and a non membership degree $\nu_A(x_i)$, with the condition $0 \leq \mu_A(x_i) + \nu_A(x_i) \leq 1$ for every $x_i \in X$.

Definition 2.2 Xu and Yager called the 2-tuple $\alpha(x_i) = (\mu_\alpha(x_i), \nu_\alpha(x_i))$ an intuitionistic fuzzy value (IFV) [11, 12] where $\mu_\alpha(x_i) \in [0, 1]$, $\nu_\alpha(x_i) \in [0, 1]$ and $\mu_\alpha(x_i) + \nu_\alpha(x_i) \leq 1$.

Definition 2.3 [2] Let $\langle x, x' \rangle, \langle y, y' \rangle \in IFS$. Then we have

- (i) $\langle x, x' \rangle \vee \langle y, y' \rangle = \langle \max\{x, y\}, \min\{x', y'\} \rangle$.
- (ii) $\langle x, x' \rangle \wedge \langle y, y' \rangle = \langle \min\{x, y\}, \max\{x', y'\} \rangle$.

(iii) $\langle x, x' \rangle^c = \langle x', x \rangle$.

Definition 2.4 [3] Let $\langle x, x' \rangle, \langle y, y' \rangle \in IFS$. Then

$$\langle x, x' \rangle \leftarrow \langle y, y' \rangle = \begin{cases} \langle 1, 0 \rangle, & \text{if } \langle x, x' \rangle \geq \langle y, y' \rangle, \\ \langle x, x' \rangle, & \text{if } \langle x, x' \rangle \leq \langle y, y' \rangle. \end{cases}$$

Here $\langle x, x' \rangle \geq \langle y, y' \rangle$ means $x \geq y$ and $x' < y'$.

Definition 2.5 [11] Let $A = (z_{ij})_{n \times n}$ be a matrix if all its elements are IFVs. Then A is called an intuitionistic fuzzy matrix.

Definition 2.6 An IFM $J = (\langle 1, 0 \rangle)$ for all entries is known as the universal matrix [14] and an IFM $O = (\langle 0, 1 \rangle)$ for all entries is known as zero matrices. Denote the set of all IFMs of order $m \times n$ by \mathcal{F}_{mn} and square matrix of order n by \mathcal{F}_n . The identity IFM $I = (\langle \delta_{ij}, \delta'_{ij} \rangle)$ is defined by $\langle \delta_{ij}, \delta'_{ij} \rangle = \langle 1, 0 \rangle$ if $i = j$ and $\langle \delta_{ij}, \delta'_{ij} \rangle = \langle 0, 1 \rangle$ if $i \neq j$.

Definition 2.7 [11, 12] Let $A = (\langle a_{ij}, a'_{ij} \rangle)_{m \times n}$, $B = (\langle b_{ij}, b'_{ij} \rangle)_{m \times n}$ and $C = (\langle c_{ij}, c'_{ij} \rangle)_{n \times p}$ are IFMs. Then

- (i) $A \vee B = (\langle a_{ij}, a'_{ij} \rangle \vee \langle b_{ij}, b'_{ij} \rangle)$.
- (ii) $A \wedge B = (\langle a_{ij}, a'_{ij} \rangle \wedge \langle b_{ij}, b'_{ij} \rangle)$.
- (iii) $AC(\text{max} - \text{min composition}) = (\bigvee_k (\langle a_{ik}, a'_{ik} \rangle \wedge \langle c_{kj}, c'_{kj} \rangle))$.

Definition 2.8 [14] Let $A = (\langle a_{ij}, a'_{ij} \rangle)_{m \times n}$ and $C = (\langle c_{ij}, c'_{ij} \rangle)_{n \times p}$ are IFMs. Then we have

- (i) $A \diamond C(\text{min-max composition}) = (\bigwedge_k (\langle a_{ik}, a'_{ik} \rangle \vee \langle c_{kj}, c'_{kj} \rangle))$.
- (ii) $A^T = (\langle a_{ji}, a'_{ji} \rangle)$ (Transpose of A).
- (iii) $A \leftarrow C = (\bigwedge_k (\langle a_{ik}, a'_{ik} \rangle \leftarrow \langle c_{kj}, c'_{kj} \rangle))$.
- (iv) $A \rightarrow C = (\bigwedge_k (\langle a_{ik}, a'_{ik} \rangle \rightarrow \langle c_{kj}, c'_{kj} \rangle))$.
- (v) $A^c = (\langle a'_{ij}, a_{ij} \rangle)$ (Complement of A).

Also we can use $AC = (\langle \sum_{k=1}^n (a_{ik}c_{kj}), \prod_{k=1}^n (a'_{ik} + c'_{kj}) \rangle)$.

Also $A^2 = AA$, $A^k = A^{k-1}A$ for max-min composition and $A^{[2]} = A \diamond A$, $A^{[k]} = A^{[k-1]} \diamond A$ for min-max composition.

Definition 2.9 [14] For any IFM $A \in \mathcal{F}_n$,

- (i) A is reflexive if and only if $A \geq I_n$.
- (ii) A is symmetric if and only if $A = A^T$.
- (iii) A is transitive if and only if $A \geq A^2$.
- (iv) A is idempotent if and only if $A = A^2$.

(v) A is irreflexive if $\langle a_{ii}, a'_{ii} \rangle = \langle 0, 1 \rangle$ for all $i = j$.

(vi) A is c -transitive if $A \leq A^{[2]}$.

Definition 2.10 [11] An IFM A is said to be an intuitionistic fuzzy equivalence matrix if it satisfy reflexivity, symmetry and transitivity.

Definition 2.11 For IFS A , Atanassov has defined the modal operators [2] \Box (necessity) and \Diamond (possibility) in the following way. $\Box A = \{ \langle \mu_A(x), 1 - \mu_A(x) \rangle \mid x \in E, \text{ the Universal set} \}$ and $\Diamond A = \{ \langle 1 - \nu_A(x), \nu_A(x) \rangle \mid x \in E \}$.

Proposition 2.1 [14] $(A \circ B)^c = A^c + B^c$ for $A, B \in \mathcal{F}_{mn}$.

Proposition 2.2 [14] $(A + B)^c = A^c \circ B^c$ for $A, B \in \mathcal{F}_{mn}$.

In [30], the following are discussed by the authors.

Definition 2.12 For an IFM A , we have $\Box A = (\langle a_{ij}, 1 - a_{ij} \rangle)$ and $\Diamond A = (\langle 1 - a'_{ij}, a'_{ij} \rangle)$.

Lemma 2.1 $1 - \prod_{k=1}^n (a_{ik} + b_{kj}) = \sum_{k=1}^n (1 - a_{ik})(1 - b_{kj})$ for all $i, j, a_{ij}, b_{ij} \in [0, 1]$.

Lemma 2.2 $1 - \sum_{k=1}^n a_{ik}b_{kj} = \prod_{k=1}^n ((1 - a_{ik}) + (1 - b_{kj}))$ for all $i, j, a_{ij}, b_{ij} \in [0, 1]$.

3. More Results of Modal Operators in IFM

Throughout this section, matrices means intuitionistic fuzzy matrices. In this section, some results about modal operators are proved and the definitions of transitive and c -transitive of an IFM A are given.

Lemma 3.1 For any two IFMs A and B ,

$$\Box(\langle a_{ij}, a'_{ij} \rangle \leftarrow \langle b_{ij}, b'_{ij} \rangle) = \Box(\langle a_{ij}, a'_{ij} \rangle) \leftarrow \Box(\langle b_{ij}, b'_{ij} \rangle). \tag{1}$$

Proof (i) If $\langle a_{ij}, a'_{ij} \rangle \geq \langle b_{ij}, b'_{ij} \rangle$, then

$$\Box(\langle a_{ij}, a'_{ij} \rangle \leftarrow \langle b_{ij}, b'_{ij} \rangle) = \Box(\langle 1, 0 \rangle) = \langle 1, 0 \rangle. \tag{2}$$

Since $\langle a_{ij}, a'_{ij} \rangle \geq \langle b_{ij}, b'_{ij} \rangle, a_{ij} \geq b_{ij}$ and $a'_{ij} \leq b'_{ij}$. Therefore, $1 - a_{ij} \leq 1 - b_{ij}$ and $\langle a_{ij}, 1 - a_{ij} \rangle \geq \langle b_{ij}, 1 - b_{ij} \rangle$, so $\Box(\langle a_{ij}, a'_{ij} \rangle) \geq \Box(\langle b_{ij}, b'_{ij} \rangle)$. Thus

$$\Box \langle a_{ij}, a'_{ij} \rangle \leftarrow \Box \langle b_{ij}, b'_{ij} \rangle = \langle 1, 0 \rangle. \tag{3}$$

From (2) and (3), (1) holds.

(ii) If $\langle a_{ij}, a'_{ij} \rangle \leq \langle b_{ij}, b'_{ij} \rangle$, then

$$\Box(\langle a_{ij}, a'_{ij} \rangle \leftarrow \langle b_{ij}, b'_{ij} \rangle) = \Box \langle a_{ij}, a'_{ij} \rangle = \langle a_{ij}, 1 - a_{ij} \rangle \tag{4}$$

$$\Box \langle a_{ij}, a'_{ij} \rangle \leftarrow \Box \langle b_{ij}, b'_{ij} \rangle = \langle a_{ij}, 1 - a_{ij} \rangle \leftarrow \langle b_{ij}, 1 - b_{ij} \rangle = \langle a_{ij}, 1 - a_{ij} \rangle \tag{5}$$

Clearly, from (4) and (5), (1) holds.

Lemma 3.2 For any two IFMs A and B ,

$$\Diamond(\langle a_{ij}, a'_{ij} \rangle \leftarrow \langle b_{ij}, b'_{ij} \rangle) = \Diamond(\langle a_{ij}, a'_{ij} \rangle) \leftarrow \Diamond(\langle b_{ij}, b'_{ij} \rangle). \tag{6}$$

Proof (i) If $\langle a_{ij}, a'_{ij} \rangle \geq \langle b_{ij}, b'_{ij} \rangle$, then

$$\Diamond(\langle a_{ij}, a'_{ij} \rangle \leftarrow \langle b_{ij}, b'_{ij} \rangle) = \Diamond(\langle 1, 0 \rangle) = \langle 1, 0 \rangle. \tag{7}$$

Since $\langle a_{ij}, a'_{ij} \rangle \geq \langle b_{ij}, b'_{ij} \rangle, a_{ij} \geq b_{ij}$ and $a'_{ij} \leq b'_{ij}$. Therefore $1 - a_{ij} \leq 1 - b_{ij}$ and $\langle a_{ij}, 1 - a_{ij} \rangle \geq \langle b_{ij}, 1 - b_{ij} \rangle$, so $\Diamond(\langle a_{ij}, a'_{ij} \rangle) \geq \Diamond(\langle b_{ij}, b'_{ij} \rangle)$. Thus

$$\diamond \langle a_{ij}, a'_{ij} \rangle \leftarrow \diamond \langle b_{ij}, b'_{ij} \rangle = \langle 1, 0 \rangle. \tag{8}$$

From (7) and (8), (6) holds.

(ii) If $\langle a_{ij}, a'_{ij} \rangle < \langle b_{ij}, b'_{ij} \rangle$, then

$$\diamond \langle \langle a_{ij}, a'_{ij} \rangle \leftarrow \langle b_{ij}, b'_{ij} \rangle \rangle = \diamond \langle a_{ij}, a'_{ij} \rangle = \langle 1 - a'_{ij}, a'_{ij} \rangle, \tag{9}$$

$$\diamond \langle a_{ij}, a'_{ij} \rangle \leftarrow \diamond \langle b_{ij}, b'_{ij} \rangle = \langle 1 - a'_{ij}, a'_{ij} \rangle \leftarrow \langle 1 - b'_{ij}, b'_{ij} \rangle = \langle 1 - a'_{ij}, a'_{ij} \rangle. \tag{10}$$

Clearly from (9) and (10), (6) holds.

Lemma 3.3 *A is reflexive matrix if and only if $\square A$ is reflexive matrix.*

Proof A is reflexive $\Leftrightarrow A \geq I \Leftrightarrow \langle a_{ij}, a'_{ij} \rangle \geq \langle \delta_{ij}, \delta'_{ij} \rangle$ for all i, j .

$$\Leftrightarrow \langle a_{ij}, 1 - a'_{ij} \rangle \geq \langle \delta_{ij}, 1 - \delta_{ij} \rangle \text{ for all } i, j$$

$$\Leftrightarrow \square A \geq \square I \Leftrightarrow \square A \text{ is reflexive.}$$

In dual way we can prove the following lemma.

Lemma 3.4 *A is reflexive matrix if and only if $\diamond A$ is reflexive matrix.*

Lemma 3.5 *A is reflexive if and only if $\square A^c$ is irreflexive.*

Proof It is evident that if A is reflexive if and only if A^c is irreflexive and so $\square A^c$.

Similarly, $\diamond A^c$ is irreflexive if and only if A is reflexive.

Lemma 3.6 *A is symmetric matrix if and only if $\square A$ is symmetric matrix and so $\square A^c$.*

Proof A is symmetric $\Leftrightarrow \langle a_{ij}, a'_{ij} \rangle = \langle a_{ji}, a'_{ji} \rangle$ for all $i, j \Leftrightarrow \langle a_{ij}, 1 - a_{ij} \rangle = \langle a_{ji}, 1 - a_{ji} \rangle \Leftrightarrow \square A = (\square A)^T$. Thus A is symmetric if and only if $\square A$ is symmetric.

Similarly, we can prove the following lemma.

Lemma 3.7 *A is symmetric matrix if and only if $\diamond A$ is reflexive matrix.*

Lemma 3.8 *A is transitive matrix if and only if $\square A$ is transitive matrix.*

Proof A is transitive $\Leftrightarrow A \geq A^2 \Leftrightarrow \langle a_{ij}, a'_{ij} \rangle \geq \langle \sum_{k=1}^n (a_{ik}a_{kj}), \prod_{k=1}^n (a'_{ik} + a'_{kj}) \rangle$ for all $i, j \Leftrightarrow$

$$a_{ij} \geq \sum_{k=1}^n (a_{ik}a_{kj}, a'_{ij}) \leq \prod_{k=1}^n (a'_{ik} + b'_{kj}) \Leftrightarrow a_{ij} \geq \sum_{k=1}^n (a_{ik}a_{kj}), 1 - a_{ij} \leq 1 - \sum_{k=1}^n (a_{ik}a_{kj}) \Leftrightarrow$$

$$\langle a_{ij}, 1 - a_{ij} \rangle \geq \langle \sum_{k=1}^n a_{ik}a_{kj}, 1 - \sum_{k=1}^n a_{ik}a_{kj} \rangle = \langle \sum_{k=1}^n a_{ik}a_{kj}, \prod_{k=1}^n ((1 - a_{ik}) + (1 - a_{kj})) \rangle \text{ by Lemma 2.2.}$$

Similarly, we can prove the following lemma.

Lemma 3.9 *A is transitive matrix if and only if $\diamond A$ is transitive matrix.*

Lemma 3.10 *A is idempotent matrix if and only if $\square A$ is idempotent matrix.*

Proof A idempotent $\Leftrightarrow A = A^2 \Leftrightarrow \langle a_{ij}, a'_{ij} \rangle = \langle \sum_{k=1}^n (a_{ik}a_{kj}), \prod_{k=1}^n (a_{ik} + a_{kj}) \rangle$ for all

$$i, j. \Leftrightarrow \langle a_{ij}, 1 - a_{ij} \rangle = \langle \sum_{k=1}^n (a_{ik}a_{kj}), 1 - \sum_{k=1}^n (a_{ik}a_{kj}) \rangle \Leftrightarrow \langle \sum_{k=1}^n (a_{ik}a_{kj}), \prod_{k=1}^n ((1 - a_{ik}) + 1 - a_{kj}) \rangle$$

by Lemma 2.2 $\Leftrightarrow \square A = (\square A)^2$. Thus A is idempotent $\Leftrightarrow \square A$ is idempotent.

The following lemma is trivial from the above.

Lemma 3.11 *A is idempotent matrix if and only if $\diamond A$ is idempotent matrix.*

Remark 3.1 If A is an intuitionistic fuzzy equivalence matrix, then $\Box A$ and $\Diamond A$ are also intuitionistic fuzzy equivalence matrices.

Definition 3.1 Let $A \in \mathcal{F}_n$, the transitive closure and c -transitive closure of A is defined by $A^\infty = A \vee A^2 \vee A^3 \vee \dots \vee A^n$ and $A_\infty = A^c \wedge (A^c)^{[2]} \wedge \dots \wedge (A^c)^{[n]}$ respectively.

Theorem 3.1 For $A \in \mathcal{F}_n, A_\infty = (A^\infty)^c$.

Proof By Definition 3.1, $(A^\infty)^c = (A \vee A^2 \vee A^3 \vee \dots \vee A^n)^c = (A^c \wedge (A^2)^c \wedge \dots \wedge (A^n)^c)$. First let us prove $(A^2)^c = (A^c)^{[2]}$.

We know that $A^2 = \langle \sum_{k=1}^n (a_{ik}a_{kj}), \prod_{k=1}^n (a'_{ik} + a'_{kj}) \rangle$ and so

$$(A^2)^c = \langle \prod_{k=1}^n (a'_{ik} + a'_{kj}), \sum_{k=1}^n (a_{ik}a_{kj}) \rangle. \tag{11}$$

Also $A^c = \langle a'_{ij}, a_{ij} \rangle$ gives by the definition of $A^{[2]}$,

$$(A^c)^{[2]} = \langle \prod_{k=1}^n (a'_{ik} + a'_{kj}), \sum_{k=1}^n (a_{ik}a_{kj}) \rangle. \tag{12}$$

Thus by (11) and (12) $(A^2)^c = (A^c)^{[2]}$, so in general $(A^n)^c = (A^c)^{[n]}$.

By Definition 3.1,

$$\begin{aligned} (A^\infty)^c &= (A \vee A^2 \vee A^3 \vee \dots \vee A^n)^c \\ &= (A^c \wedge (A^2)^c \wedge \dots \wedge (A^n)^c)^c \\ &= (A^c) \wedge (A^c)^{[2]} \wedge \dots \wedge (A^c)^{[n]} = A_\infty. \end{aligned}$$

Lemma 3.12 A is transitive if and only if A^c is c -transitive and so $\Box A^c$ is.

Proof It is evident from the definition of transitive and c -transitive.

Lemma 3.13 If A is reflexive IFM, then

- (i) A^T is reflexive.
- (ii) $A \vee B$ is reflexive.
- (iii) $A \wedge B$ is reflexive if and only if B is reflexive.

Proof (i) and (ii) are obvious from the definition of reflexive.

(iii) If B is not reflexive, then $\langle b_{ii}, b'_{ii} \rangle \neq \langle 1, 0 \rangle$ for at least one i , that is $\langle b_{ii}, b'_{ii} \rangle < \langle 1, 0 \rangle$. Thus $\langle a_{ii}, a'_{ii} \rangle \wedge \langle b_{ii}, b'_{ii} \rangle < \langle 1, 0 \rangle$. Therefore the condition B is reflexive is necessary, the sufficient part is trivial.

Theorem 3.2 If $A, B \in \mathcal{F}_n$ where A is reflexive and symmetric, B is reflexive, symmetric and transitive and $A \leq B$, then $A^\infty \leq B$.

Proof For $A = \langle (a_{ij}, a'_{ij}) \rangle, B = \langle (b_{ij}, b'_{ij}) \rangle, AB = \langle (\sum_{k=1}^n (a_{ik}b_{kj}), \prod_{k=1}^n (a'_{ik} + b'_{kj})) \rangle$ and each

$$\langle \sum_{k=1}^n (a_{ik}b_{kj}), \prod_{k=1}^n (a'_{ik} + b'_{kj}) \rangle = \begin{cases} \langle 1, 0 \rangle, & \text{if } i = j, \\ \langle b_{ij}, b'_{ij} \rangle, & \text{if } i \neq j. \end{cases}$$

Thus $AB = B \Rightarrow AA \leq AB = B$. That is $A^2 \leq B$. Continuing in this way, we have $A^3 \leq B, A^4 \leq B \dots$ and also $A \vee A^2 \vee A^3 \dots \vee A^n \leq B$ and hence $A^\infty \leq B$.

Lemma 3.14 *If A^∞ is the transitive closure of A , then the transitive closure of $\Box A$ is $\Box A^\infty$.*

Proof Now $\Box A^\infty = \Box[A \vee A^2 \vee \dots \vee A^n] = \Box A \vee \Box A^2 \vee \dots \vee \Box A^n = \Box A \vee (\Box A)^2 \vee \dots \vee (\Box A)^n = (\Box A)^\infty$. Similarly, the following results are also true.

- (i) $\Box A_\infty = (\Box A)_\infty$.
- (ii) $\Diamond A^\infty = (\Diamond A)^\infty$.
- (iii) $\Diamond A_\infty = (\Diamond A)_\infty$.

Lemma 3.15 *For an IFM $A \in \mathcal{F}_n$, $[(\Box A)^c]^\infty = [(\Box A)_\infty]^c$.*

Proof As we know $(\Box A)^c = \Diamond A^c$, $[(\Box A)^c]^\infty = [\Diamond A^c]^\infty = \Diamond A^c \vee (\Diamond A^c)^2 \dots \vee (\Diamond A^c)^n$.

$$\begin{aligned}
 (\Diamond A^c)^2 &= (\langle \sum_{k=1}^n (1 - a_{ik})(1 - a_{kj}), \prod_{k=1}^n (a_{ik} + a_{kj}) \rangle) \\
 &= (\langle 1 - \prod_{k=1}^n (a_{ik} + a_{kj}), \prod_{k=1}^n (a_{ik} + a_{kj}) \rangle). \tag{13}
 \end{aligned}$$

By definition $A^{[2]} = (\langle \prod_{k=1}^n (a_{ik} + a_{kj}), \sum_{k=1}^n (a'_{ik} a_{kj}) \rangle)$ and so

$$\Box A^{[2]} = (\langle \prod_{k=1}^n (a_{ik} + a_{kj}), 1 - \prod_{k=1}^n (a_{ik} + a_{kj}) \rangle).$$

Which yields

$$(\Box A^{[2]})^c = (\langle 1 - \prod_{k=1}^n (a_{ik} + a_{kj}), \prod_{k=1}^n (a_{ik} + a_{kj}) \rangle).$$

Therefore, $(\Diamond A^c)^2 = (\Box A^{[2]})^c$, so in general $(\Diamond A^c)^n = (\Box A^{[n]})^c$

$$\begin{aligned}
 [(\Box A)^c]^\infty &= [\Diamond A^c]^\infty = \Diamond A^c \vee (\Diamond A^c)^2 \dots \vee (\Diamond A^c)^n \\
 &= (\Box A)^c \vee (\Box A^{[2]})^c \vee \dots \vee (\Box A^{[n]})^c \\
 &= (\Box A \wedge \Box A^{[2]} \wedge \dots \wedge \Box A^{[n]})^c = (\Box A_\infty)^c.
 \end{aligned}$$

In dual fashion, one can prove the following lemma.

Lemma 3.16 *For an IFM $A \in \mathcal{F}_n$, $((\Diamond A)_\infty)^c = ((\Diamond A)^c)^\infty$.*

Definition 3.2 *For any two elements $\langle x, x' \rangle, \langle y, y' \rangle \in IFS$, we introduce the operation \wedge'_m as $\langle x, x' \rangle \wedge'_m \langle y, y' \rangle = \langle \min\{x, y\}, \min\{x', y'\} \rangle$.*

Using this definition the following lemmas are trivial.

Lemma 3.17 *The operation \wedge_m is commutative on IFS.*

Lemma 3.18 *The operation \wedge_m is associative on IFS.*

Lemma 3.19 *The operation \wedge_m is distributive over addition in IFS s.*

Proof For any $\langle x, x' \rangle, \langle y, y' \rangle, \langle z, z' \rangle \in IFS$

$$\begin{aligned}
 (\langle x, x' \rangle + \langle y, y' \rangle) \wedge_m \langle z, z' \rangle &= \langle \max\{x, y\}, \min\{x', y'\} \rangle \wedge_m \langle z, z' \rangle \\
 &= \langle \min\{\max\{x, y\}, z\}, \min\{\min\{x', y'\}, z'\} \rangle. \tag{14}
 \end{aligned}$$

Case (1) If $\langle x, x' \rangle \geq \langle y, y' \rangle$ and $\langle x, x' \rangle \geq \langle z, z' \rangle$, then right hand side of (14) is $\langle z, z' \rangle$. Now consider

$$\begin{aligned}
 & (\langle x, x' \rangle \wedge_m \langle z, z' \rangle) + (\langle y, y' \rangle \wedge_m \langle z, z' \rangle) \\
 & = \langle z, x' \rangle + \begin{cases} \langle z, y' \rangle & \text{if } \langle z, z' \rangle \leq \langle y, y' \rangle \\ \langle y, z' \rangle & \text{if } \langle y, y' \rangle \leq \langle z, z' \rangle \end{cases} = \langle z, x' \rangle. \tag{15}
 \end{aligned}$$

In this case, it is distributive.

Case (2) If $\langle x, x' \rangle \leq \langle y, y' \rangle$ and $\langle x, x' \rangle \leq \langle z, z' \rangle$, then the left hand side of (15) reduces to $\langle y, y' \rangle \wedge_m \langle z, z' \rangle$,

Subcase (2.1) If $\langle z, z' \rangle \leq \langle y, y' \rangle$, then $\langle z, z' \rangle \wedge_m \langle y, y' \rangle = \langle z, y' \rangle$. Now $(\langle x, x' \rangle \wedge_m \langle z, z' \rangle) + (\langle x, x' \rangle \wedge_m \langle z, z' \rangle) = \langle x, z' \rangle + \langle z, y' \rangle = \langle z, y' \rangle$.

Thus distributivity holds.

Subcase (2.2) If $\langle z, z' \rangle \geq \langle y, y' \rangle$, then left hand side of (15) becomes $\langle y, z' \rangle$ and right hand side of (15) becomes

$$(\langle x, x' \rangle \wedge_m \langle z, z' \rangle) + (\langle x, x' \rangle \wedge_m \langle z, z' \rangle) = \langle x, z' \rangle + \langle y, z' \rangle = \langle y, z' \rangle.$$

Thus it is distributive in this case also.

Case (3) If $\langle y, y' \rangle \leq \langle x, x' \rangle \leq \langle z, z' \rangle$, then the left hand side becomes

$$(\langle x, x' \rangle + \langle y, y' \rangle) \wedge_m \langle z, z' \rangle = \langle x, x' \rangle \wedge_m \langle z, z' \rangle = \langle x, z' \rangle.$$

Also $(\langle x, x' \rangle \wedge_m \langle z, z' \rangle) + (\langle x, x' \rangle \wedge_m \langle z, z' \rangle) = \langle x, z' \rangle + \langle y, z' \rangle = \langle x, z' \rangle$.

So it is distributive in this case too.

Case (4) If $\langle z, z' \rangle \geq \langle x, x' \rangle \geq \langle y, y' \rangle$, then the left hand side reduces to

$$\langle y, y' \rangle \wedge_m \langle z, z' \rangle = \langle z, y' \rangle.$$

And

$$(\langle x, x' \rangle \wedge_m \langle z, z' \rangle) + (\langle x, x' \rangle \wedge_m \langle z, z' \rangle) = \langle z, x' \rangle + \langle z, y' \rangle = \langle z, y' \rangle.$$

Thus distributivity holds for all cases.

Definition 3.3 For any two elements $\langle x, x' \rangle, \langle y, y' \rangle \in IFS$, we define the inequality ' \leq' ' as $\langle x, x' \rangle \leq \langle y, y' \rangle$ means $x \leq y$ and $x' \leq y'$.

Remark 3.2 The elements in the set $\{\langle y, y' \rangle \in IFS \mid \langle x, x' \rangle \leq \langle y, y' \rangle\}$ are identity element of $\langle x, x' \rangle$ with respect to \wedge_m . That is we have multiple identity element.

Remark 3.3 Any IFM A can be decomposed into two intuitionistic fuzzy matrices $\square A$ and $\diamond A$ by means of \wedge_m . That is

$$A = (\square A) \wedge_m (\diamond A).$$

Remark 3.4 For any two IFMs A and B , $(A \vee B) \wedge_m (A \wedge B) = (A \wedge_m B)$.

4. Conclusion

Any fuzzy matrix $A = (a_{ij})$ is an IFM in the form of $A = (a_{ij}, 1 - a_{ij})$. The matrices $\square A$ and $\diamond A$ using necessity and possibility operators denoted as \square and \diamond are formed, in which we consider only either membership or nonmembership of any IFM A gives a fuzzy matrix. Here we present some results of the above said operators with other operators using illustration. Transitive and c-transitive closures are defined and some results are proved on IFM. Finally, using a new operator we express an IFM in terms of fuzzy matrix.

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