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Beauty contest, bounded rationality, and sentiment pricing dynamics

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ABSTRACT

We present a dynamic asset pricing model that incorporates investor sentiment, bounded rationality and higher-order expectations to study how these factors affect asset pricing equilibrium. In the model, we utilize a two-period trading market and investors make decisions based on the heterogeneous expectations principle and the "sparsity-based bounded rational" sentiment. We find that bounded rationality results in mispricing and reduces it in next period. Investor sentiment produces more significant effects than private signals, optimistic investor sentiment increases hedging demand, thus causing prices to soar. Higher-order investors are more rational and attentive to the strategies of other participants rather than private signals. This model also derives the dampening effect of higher-order expectations to price volatility and the heterogeneity expectation depicts inconsistent investor behavior in financial markets. In the model, investors' expectations about future price is distorted by their sentiment and bounded rationality, so they obtain a biased mean from the signal extraction.

1. Introduction

According to Keynes' "Beauty Contest" view of financial markets (see Keynes (2006)), investment decisions are driven by the investors' anticipation of their peers' changing whims rather than actual knowledge and expectations of the investments they trade. This type of behavior introduces a particular form of informational inefficiency whereby investors tend to place a disproportionate weight on public signals for their forecast of asset prices (see Allen et al. (2006)). Furthermore, we show that the beauty contest analogy for financial markets explains only a portion of the issue because when "sparsity-based bounded rational" sentiment investors' demand shocks are persistent, prices reflect average expectations of not only the fundamental value but also the market sentiment and the bounded capacity of cognition.

Traditional asset pricing theory implies that changes in asset prices are dependent on fundamental changes. However, according to the beauty contest theory, Allen et al. (2006) study the role of higher-order expectations (HOEs) in asset pricing and demonstrate the failure of the law of iterated expectations; they find that prices (i) are driven by higher-order expectations about fundamentals, (ii) underweight private information (with respect to the optimal statistical weight), and (iii) are further from fundamentals than investors' consensus. In addition, higher-order expectations differ from first-order average expectations of the asset's payoff. Our paper confirms other findings in the current literature regarding Keynes' beauty contest theory. Bacchetta and van Wincoop (2006, 2008) illustrate the impact of higher-order expectations in the foreign exchange market and in asset pricing. Banerjee et al. (2009) state that higher-order expectations may explain price drift in the stock market and indicate that it is necessary to generate price drift for heterogeneous beliefs. Kondor (2012) proposes that the reason for forming higher-order expectations is that early investors are forced to make guesses regarding information that later investors have obtained. Kondor further proposed that public information polarized higher-order expectations without polarizing first-order expectations. Yang and Cai (2014) study the effect of higher-order expectations on a static sentiment asset pricing model. Cespa and Vives (2015) suggest that short-termism does not necessarily breed informational price inefficiency even when generating beauty contests. Specifically, the primary conclusion of the literature discussed above is that when compared to first order expectations, higher-order expectations will result in significant advantages.

As an alternative explanation to the argument that financial markets are not always informationally efficient and rational arbitrage cannot completely eliminate irrational effects on asset prices, much of the current literature relies on behavioral finance theories, which often differ from traditional assumptions of strict rationality or unlimited computational capacity on investors. Behavioral finance offers two new theories to explain this deviation: investor sentiment and bounded rationality.

One possible explanation for the deviation is that investor sentiment impacts asset price, hence sentimental investors produce a biased

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valuation of the asset (Baker et al., 2012; Kumar and Lee, 2006; Lee et al., 2002; Brown and Cliff, 2004, 2005; Yu and Yuan, 2011; Seybert and Yang, 2012; Cen et al., 2013; Yang and Zhang, 2013; Zhu, 2013; Kim et al., 2014; Ni and Wang et al., 2015). Shleifer and Vishny (1997) and Hong et al. (2012) demonstrate that greater pessimistic or optimistic shocks result in the failure of arbitrageurs to price at the fundamental value. Barber et al. (2009) argue that sentimental investors produce mispricing, such that imbalances in buyer- and seller-initiated small trades alter prices and result in deviations. Empirically, stocks are difficult to estimate and arbitrageurs earn high subsequent returns when sentiment proxies are low (Baker and Wurgler, 2006, 2007); high sentiment increases the profitability of the short cross-sectional return anomalies (Stambaugh et al., 2012).

Second, Sargent (1993) and Kahneman (2003) provide an introduction to bounded rationality. Researchers utilize a limited number of variables when analyzing a specific problem (Miller 1956). Kahneman further utilize two systems 1 and 2, where system 1 is the intuitive, largely unconscious system, and system 2 is the analytical, conscious system, that makes use of “mental operations”. This decision-making system (mixed systems 1 and 2) is not taken into account when an investor has no time to think and will thus rely on defaults (Gennaioli and Shleifer, 2010). Then, the investor may anchor on a default value and make a partial adjustment toward it (Tversky and Kahneman, 1974). Ding et al. (2014) proposed a dynamic system of investment game which played by two firms with bounded rationality, they found that time-delayed feedback control can be used to control system chaos. Gabaix (2014) and other relevant literature (Gabaix, 2016a, 2016b) propose tractable models of bounded rationality that include all of the above characteristics. Empirical findings demonstrate that a bounded rationality model with cognitive limitation provides a reasonable fit to auto- and cross-covariances of the data, mainly driven by a high degree of intrinsic persistence in output and inflation gap on economic dynamics (Jang and Sacht, 2016).

However, few of the sentiment pricing models consider the higher order expectations or bounded rationality to study how they make effects on sentiment asset pricing, and none of them attempt to combine the two ideas jointly. While this work makes some theoretical contributions to previous static models such as Yang and Cai (2014) by generalizing multi-trading, time-varying sentiment and bounded information between different periods.

This study provides three primary contributions to current literature: **First,** in contrast to previous literature on the sentiment asset pricing model, we present an innovative sentiment asset pricing dynamic model with higher-order expectations to analyze how higher-order expectations impact sentiment asset pricing. Nevertheless, when considering the impact of time-varying sentiment effects on the equilibrium prices, the model with dynamic setting has a better capacity to capture changes in the market than static models. **Second,** our model demonstrates the importance of incorporating the expectation heterogeneity of two types of investors with different orders into the asset pricing model, which focuses on inconsistent investor behavior in financial markets. This contribution distinguishes our model from many other models that contain only single first-order or higher-order expectations investors; we detail interactive trading behavior between heterogeneous expectation order investors and determine who gains or loses from trading on a different expectations order. This is a vital issue that may not be explained by ordinary single expectation order investor models. **Third,** we employ “sparsity-based bounded rationality” to further characterize the irrationality of investors and are able to identify which investors have bounded information, a sparse view of the world, and bounded computational capacity. In our model, the investor weighs the cost of having an imperfect decision against the benefits of saving on “thinking costs” (see Yang and Liang (2016)).

The remainder of this paper is organized as follows: We set up a benchmark model utilizing a first-order expectations investor. Section 2 introduces the “sparsity-based” bounded rationality operator. Section 3 builds a second-order expectations dynamic model to illustrate the role of investor sentiment and bounded rationality on second-order expectations equilibrium, and describes the equilibrium characterizations. Section 4 demonstrates how investor expectations heterogeneity of investor sentiment and bounded rationality impacts asset prices. Section 5 presents comparative statics, while Section 6 provides concluding remarks.

2. Benchmark case: A first-order expectations investor model

2.1. Economy

Our goal is to tackles at an essentially difficult problem of asset pricing with higher-order expectations to formalize the discussions outlined in the introduction. One advantage of this model is that we try to incorporate higher-order expectations and bounded rationality in a dynamic setting. The starting point here is Allen, Morris and Shin (AMS) model (see Allen et al. (2006)), where it is shown that when investors form expectations about expectations of others, the law of iterative expectations will fail and consequently price will deviate from the fundamental value in the way that it reacts to the changes slower than under rational (Bayesian) reasoning.

And then in this section, we study a first-order expectations investor (henceforth first-order investor) dynamic to develop our intuition and key insights. The model here involves three periods and the focus is on the price in the period 2, the last period before the fundamental value is revealed. The first idea that we explore is the impact of the higher order expectation on price.

Consider an economy with a continuum of investors of unit measures indexed by i. Time is discrete and there are three periods t=0, 1, 2 and 3. There is one single risky asset that trade over time and will be liquidated on date 3.

The liquidation value of the asset θ is determined prior to trading on date 0 and is a normally distributed random variable with mean P_0 and variance$1/\alpha$, where the initial P_0 is also the price of the risky asset, and α describes the precision of the public signal. In period 0, all investors share the same initial public signal. In period 1, investor i may observe a private signal about θ that satisfies: $v_{1i}^{S_1} = \theta + IS_1 + \varepsilon_{1i}^{S}$, the precision of the private signal on date 1 is $\tau_v^{S_1}$, P_1 is the price of the asset on date 1. This investor may also observe a private signal that satisfies $v_{2i}^{S_2} = \theta + IS_2 + \varepsilon_{2i}^{S}$ in period 2, where the precision is $\tau_v^{S_2}$. Clearly, in the model, the precision of the private signal is the reciprocal value of the variance.

As a convention, we hold that$P_3 = \theta$, where the sentiment terms IS_1, $IS_2 \sim N(0, 1/\sigma)$ are i.i.d. across investors i. As in our previous static model (see Yang and Cai (2014)), IS represents the measure of average errors in forming a private signal, which is normally distributed with mean 0 and variance $1/\sigma$ and monotonically increases with increases in the market sentiment S as follows[2]:

(1) If $S > 0$,$IS > 0$, investor$i$is optimistic and forms an overconfident private signal, $v_i^S > v_i^R$.
(2) If $S < 0$,$IS < 0$, investor i is pessimistic and forms a low mean private signal, $v_i^S < v_i^R$.
(3) If $S = 0$,$IS = 0$, investor i is rational and $v_i^S = v_i^R$.

Similarly, we also set the cognitive fundamental of investors are changing when investors are impacted by different market sentiment, thus the aggregation of private signals deviates in the influence of market sentiment.

[2] Notice that S here is the market sentiment, while investor sentiment SI is an independent realization from S.

Fig. 1. Timeline of events.

In this case, ε_i^S represents a normally distributed noise term of real private signal with mean 0 and variance $1/\beta$.

No other signal source may be acquired by the investors, and the private signal of each investor cannot be observed by others. Based on the private signal and the public signal, investor i forms an expectation of θ. Hence, as shown in Fig. 1, the information set of investor i on date t is $\{P_0, v_{1i}^S, v_{2i}^S, P_1\}$. To forecast the liquidation value based on investors' expectations, we set P_1 as a parameter that uses the information set for a signal.

The current literature, including Grossman (1976), Allen et al. (2006) and Kondor (2012), demonstrates that equilibrium may be introduced when prices play an informational role in competitive markets with a heterogeneous signal. Past researchers often assume that prices satisfied a linear equation that utilizes some variables and accurately describes the relationship between prices and expectations, as well as providing variance. Similar to above, we assume that P_1 satisfies

$$P_1 = a(\theta + IS_1) \tag{1}$$

where a is a constant and the above equation may also be written as

$$\frac{P_1}{a} = \theta + IS_1 \tag{2}$$

For $IS_1 \sim \mathrm{N}(0, 1/\sigma)$, we use $\frac{P_1}{a}$ to signal the liquidation value, which has a variance of $1/\sigma$.

In our model, we define first- (second-, or higher-) order expectations in a manner that is similar to Allen et al. (2006). We defined the first-order expectations $\mathrm{E}_{ti}(\theta)$, which stand for investor $i's$ estimates about θ and are the only conditions on the investor information set on date t. Here the investor takes the return of assets into account. In addition, $\overline{\mathrm{E}}_t(\theta)$ denotes the average of first-order expectations, $\mathrm{E}_{ti}^k(\theta)$ is k-order expectations of investor i, and $\overline{\mathrm{E}}_t^k(\theta)$ is the average k-order expectations. All calculations adopt the Bayesian information updating rule, where $\mathrm{var}_{ti}^k(\theta)$ is the variance of $\overline{\mathrm{E}}_{ti}^{k-1}(\theta)$ based on the same information set as above, which may be expressed as follows:

$$\mathrm{E}_{2i}^1(\theta) = \frac{\tau_v^{S_1}}{\tau'} v_{1i}^{S_1} + \frac{\tau_v^{S_2}}{\tau'} v_{2i}^{S_2} + \frac{\sigma}{\tau'}\frac{P_1}{a} \tag{3}$$

$$\mathrm{var}_{2i}^1(\theta) = \frac{1}{\tau'} = \frac{1}{\tau_0 + \tau_v^{S_1} + \tau_v^{S_2} + \sigma} \tag{4}$$

where $\tau^{(\bullet)}$ denotes the precisions of private signal, then investor $i's$ demand on date 2 is denoted by

$$\begin{aligned} x_{2i}^1 &= \frac{[\mathrm{E}_{2i}^1(\theta) - P_2]}{\gamma \mathrm{var}_{2i}^1(\theta)} = \frac{\tau'}{\gamma}\left(\frac{\tau_v^{S_1}}{\tau'} v_{1i}^{S_1} + \frac{\tau_v^{S_2}}{\tau'} v_{2i}^{S_2} + \frac{\sigma P_1}{\tau' a} - P_2\right) \\ &= \frac{1}{\gamma}\left(\tau_v^{S_1} v_{1i}^{S_1} + \tau_v^{S_2} v_{2i}^{S_2} + \frac{\sigma P_1}{a} - \tau' P_2\right) \\ &= \frac{1}{\gamma}\left[\tau_0(P_0 - P_2) + \tau_v^{S_1}(v_{1i}^{S_1} - P_2) + \tau_v^{S_2}(v_{2i}^{S_2} - P_2) + \sigma\left(\frac{P_1}{a} - P_2\right)\right]. \end{aligned} \tag{5}$$

Here, we use $P_0 = 0$ as in the static model by Yang and Cai (2014) to illustrate the relationship between the final price and the original price.

As indicated by Eq. (5), the demand of investor i on date 2 can be divided into two parts: arbitrage demand between the original price on date 0 and the prices on date 1 and 2; and the speculative demand between the private signal and the price on date 2.

Summing Eq. (5) across investors, the aggregated demands on date 2 is explained by

$$\begin{aligned} \sum x_{2i}^1 &= \sum \frac{\tau'}{\gamma}\left(\frac{\tau_v^{S_1}}{\tau'} v_{1i}^{S_1} + \frac{\tau_v^{S_2}}{\tau'} v_{2i}^{S_2} + \frac{\sigma P_1}{\tau' a} - P_2\right) \\ &= \frac{\tau'}{\gamma}\left[\frac{\tau_v^{S_1}}{\tau'}(\theta + IS_1) + \frac{\tau_v^{S_2}}{\tau'}(\theta + IS_2) + \frac{\sigma P_1}{\tau' a} - P_2\right] \qquad = 0. \end{aligned} \tag{6}$$

Market clearing then implies that

$$P_2{}^* = \frac{\tau_v^{S_1} + \sigma}{a\tau'} P_1 + \frac{\tau_v^{S_2}}{\tau'}(\theta + IS_2). \tag{7}$$

The first term of the above equation is the effect of the date 1 price, and the second term stands for the aggregation of the private signal. Clearly, optimistic investor sentiment will result in an increase in equilibrium prices.

2.2. Bounded rationality operator

Gabaix (2014) and the relevant literature propose a tractable model of bounded rationality. This model demonstrates the "sparsity-based" bounded rationality operator, which is a behavioral generalization of the traditional optimization method to capture some type of "sparsity-based" bounded rationality. Gabaix assumes that agents do not pay attention to all the aspects of the problem at hand, but only to some of them, those that matter for their decision most. In solving any problem they proceed in two steps. In the first step, they choose which aspects of the problem will be worthwhile to consider by minimizing the losses from possible non-optimal decision but also taking the costs of thinking into account. This benefit-cost analysis allows them to choose an optimal level of attention for the problem that they may face in the future, treating it at this stage as the problem whose parameters are random variables. At the second stage, the optimal level of attention is chosen, the values of the parameters are realized and so the problem at hand can be solved. Since the level of attention is never full (due to cost), the solution is not optimal.

In our model, we assume that investors are boundedly rational as in Gabaix's paper at the first period, because this agent knows that the average expectations affect the prices. Furthermore, we include bounded information, sparse view of the world, and the bounded ability of calculation to some extent and refer to it as "sparsity-based bounded rationality" according to convention. Tversky and Kahneman (1974) first described two aspects of investor psychology as "anchoring and adjustment". Their study indicates anchoring on a default value (assumes equal to 1 in our model) and partial adjustment towards the truth. The "sparsity-based bounded rational" investor weighs the cost of having an imperfect decision against the benefits of saving on "thinking costs". Gabaix (2014) states that choosing "what comes to mind" is a decision-making style that operates in the unconscious background which selects what is brought to the conscious mind.

Daniel et al. (1998) find that investors may underestimate the variance of private signals. Similarly, we deduce that investors subjectively assess the signal precision and follow a "sparsity-based bounded rational" method (for details, see Yang and Liang (2016)), knows that the average expectations affect the prices. In this model we defined the operator ϕ^{BR} as follows, which reflects the (bounded) rationality of the investor. A higher ϕ^{BR} indicates that the investor has acquired more resources for higher rationality, such as paying more attention and improving the calculation. However, the investor does so based on an exogenous level of bounded rationality and the object (variable) itself, not without restrictions.

Definition 1. *The "sparsity-based" bounded rationality operator* ϕ^{BR} *is.*

$$\phi^{BR} = \arg\min_{\phi}\left[\underbrace{\frac{A}{2}\left(\frac{\phi-\mu}{\sigma}\right)^2}_{\text{loss of imperfect decision}} + \underbrace{\xi\frac{|\phi|}{\sigma}}_{\text{cost of rationality}}\right]\Bigg|_{\phi\in[0,\mu]} \tag{8}$$

while ξ in this function is a constant which is exogenous and reflects the level of bounded rationality of this investor, we call it "coefficient of irrationality". σ is the standard deviation of variable which in view of sparsity.

Here, we discuss the interpretation of the "sparsity-based bounded rational" principle.

The first term in the above function represents a loss between biased cognition and 'default value' (entirely rational cognition value μ), so we define it as 'loss of imperfect decision' as does Gabaix (2014). This expression is somewhat standardized by units and scaling, as is the invariant by reparametrization (divided by σ). These calculations add to the robustness of the model.

The second term in the above function estimates the costs of entirely rational access in the default value by investor and is defined as 'cost of rationality'. This expression is also independent of the unit of the sparse variable (in our model, private signal precisions $\tau_v^{S_2}$).

Apparently, when the investors' bounded rationality level increases while the loss of imperfect decision-making decreases, the cost of rationality simultaneously increases. To clarify, when the first term decreases, the second term increases by the same levels, and then, the scales are tipped.

As mentioned in our earlier studies (see Yang and Liang (2016)), to characterize the irrationality of investor in a tractable and simple way, we solve the following two-term optimization (in the first term "loss of imperfect decision", $A = 1$, $\mu = 1$, and the second term "cost of rationality").

Lemma 1. *Take $\tau_v^{S_2}$ (the precision of the date 2 private signal, i.e. the reciprocal value of its variance) as the volatility term, and assume $\mu = 1$ for simplicity.*

Then, the solution of

$$\arg\min_{\phi}\left[\frac{1}{2}\tau_v^{S_2}(\phi-1)^2 + \xi\sqrt{\tau_v^{S_2}}|\phi|\right]\Bigg|_{\phi\in[0,1]} \tag{9}$$

is

$$\phi^{BR} = \left[1 - \frac{\xi}{\sqrt{\tau_v^{S_2}}}\cdot sign(\phi)\right]\Bigg|_{\phi\in[0,1]} = 1 - \frac{\xi}{\sqrt{\tau_v^{S_2}}} \tag{10}$$

Where in the Lemma 1 above, ξ also satisfies the properties as follows:

(1) If $\xi = 0$, $\phi^{BR} = 1$, investor i has a relatively high level of rationality. The investor reacts to the signal (such as realized prices on a previous date), sometimes looks more like sophisticated investors.
(2) If $\xi \to \sqrt{\tau_v^{S_2}}$, $\phi^{BR} \approx 0$, the investor will make a decision using a "rule of thumb" and then attribute the cost to a low capacity of signal processing. This process is more reflects the behavior of novice retail investors.
(3) If $\xi \in (0, \sqrt{\tau_v^{S_2}})$, $\phi^{BR} \in (0, 1)$, the investor was in the bounded rationality state.

Clearly, $\phi^{BR} \in (0, 1]$ generates under-reaction of the variable which is in sparse view.

Here, a simplified numerical example illustrates the relationship between the variables in Eq. (10). The parameters are chosen as follows: coefficient of irrationality $\xi \in (0, 0.3]$, and signal precision $\tau_v^{S_2} \in (0, 4]$ (Choosing these values for more significantly, follow Yang and Liang (2016)). Fig. 2 indicates a positive correlation between bounded rationality operator ϕ^{BR} and the private signal precision $\tau_v^{S_2}$, which further indicates that as the private signal become more accurate

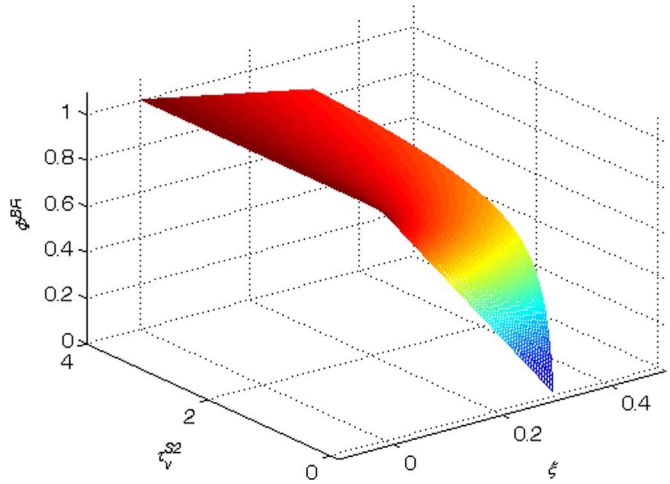

Fig. 2. The bounded rationality operator.

(a lower volatility), the investor will remain at a relatively higher level of rationality. In the above function, ϕ^{BR} will increase when the coefficient of irrationality ξ decreases, which implies that a more rational investor will have more resources for decision-making.

2.3. Equilibrium

In contrast to the theory of "hold until liquidation", we, similar to Allen et al. (2006), suggest a new theory regarding the "hedging demand" of long-lived investors. Investors' anticipations of asset purchases on date 2 create "hedging demand". Long-lived investors are concerned with short-run price movements, as well as the underlying fundamental value of assets at the point of ultimate liquidation.

We define an investor in the first period as the bounded information (or bounded rationality) period. As mentioned previously, the investor is unable to accurately determine the date 2 private signal and its information precision. The investor must utilize advance information in the market, and replace the date 2 expectation of private signal variance with its bounded rationality version ($\phi^{BR}\tau_v^{S_2}$).

Subsequently, an alternative model of P_2 is utilized with respect to the bounded rationality information set

$$P_2^{\mathrm{E}} = \frac{\tau_v^{S_1}}{\tau^E}(\theta + IS_1) + \frac{\phi^{BR}\tau_v^{S_2}}{\tau^{\mathrm{E}}}(\theta + IS_1) + \frac{\sigma}{\tau^E}(\theta + IS_1) = \frac{\tau_v^{S_1} + \phi^{BR}\tau_v^{S_2} + \sigma}{\tau^E}(\theta + IS_1) \tag{11}$$

where $\tau^E = \tau_0 + \tau_v^{S_1} + \phi^{BR}\tau_v^{S_2} + \sigma$.

Consider the dynamic hedging demand mentioned earlier and the sentiment risk tolerance demand together, and utilizing the consequences of the static model (see Yang and Cai (2014)), we derive the demand function of first-order investor as below,

$$x_{1i}^1 = \frac{(\mathrm{E}_{1i}^1(\theta) - P_1)}{\gamma \mathrm{var}_{1i}^1(\theta)} + \frac{(\mathrm{E}_{1i}^1(P_2^E) - P_1)}{\gamma \mathrm{var}_{1i}^1(P_2^E)} = \frac{\tau}{\gamma}\left(\frac{\tau_v^{S_1}}{\tau}v_{1i}^{S_1} - P_1\right) + \frac{\frac{(\tau_v^{S_1}+\phi^{BR}\tau_v^{S_2}+\sigma)}{\tau^E}\frac{\tau_v^{S_1}}{\tau}v_{1i}^{S_1} - P_1}{\gamma\left(\frac{\tau_v^{S_1}+\phi^{BR}\tau_v^{S_2}+\sigma}{\tau^E}\right)^2\left(\frac{1}{\tau}+\frac{1}{\sigma}\right)} = \frac{1}{\gamma}\left[\begin{array}{l}(\tau_v^{S_1}v_{1i}^{S_1} - \tau P_1) + \frac{\sigma\tau^E\tau_v^{S_1}}{(\tau+\sigma)(\tau_v^{S_1}+\phi^{BR}\tau_v^{S_2}+\sigma)}v_{1i}^{S_1}\\ -\left(\frac{\tau\sigma}{\tau+\sigma}\right)\left(\frac{\tau^E}{\tau_v^{S_1}+\phi^{BR}\tau_v^{S_2}+\sigma}\right)^2 P_1\end{array}\right]. \tag{12}$$

here the parameter γ is the absolute risk aversion.

Given the aggregate demands of all investors, we clearing the market

$$\sum x_{1i}^1 = \sum \frac{1}{\gamma}\left[\begin{array}{l}(\tau_v^{S_1} v_{1i}^{S_1} - \tau P_1) + \frac{\sigma\tau^E\tau_v^{S_1}}{(\tau+\sigma)(\tau_v^{S_1}+\phi^{BR}\tau_v^{S_2}+\sigma)} v_{1i}^{S_1} \\ -\left(\frac{\tau\sigma}{\tau+\sigma}\right)\left(\frac{\tau^E}{\tau_v^{S_1}+\phi^{BR}\tau_v^{S_2}+\sigma}\right)^2 P_1\end{array}\right] = \frac{\tau_v^{S_1}}{\gamma}(\theta + IS_1) - \frac{\tau}{\gamma}P_1 + \frac{1}{\gamma}\left[\begin{array}{l}\frac{\sigma\tau^E\tau_v^{S_1}}{(\tau+\sigma)(\tau_v^{S_1}+\phi^{BR}\tau_v^{S_2}+\sigma)}(\theta + IS_1) \\ -\left(\frac{\tau\sigma}{\tau+\sigma}\right)\left(\frac{\tau^E}{\tau_v^{S_1}+\phi^{BR}\tau_v^{S_2}+\sigma}\right)^2 P_1\end{array}\right] = 0. \tag{13}$$

Then, we obtain the equilibrium price at date 1, i.e.,

$$P_1^* = \frac{\left[1 + \frac{\sigma\tau^E}{(\tau+\sigma)(\tau_v^{S_1}+\phi^{BR}\tau_v^{S_2}+\sigma)}\right]}{\left[1 + \left(\frac{\sigma}{\tau+\sigma}\right)\left(\frac{\tau^E}{\tau_v^{S_1}+\phi^{BR}\tau_v^{S_2}+\sigma}\right)^2\right]} \cdot \frac{\tau_v^{S_1}}{\tau}(\theta + IS_1) = \frac{(\tau+\sigma)(\tau_v^{S_1} + \phi^{BR}\tau_v^{S_2} + \sigma) + \tau^{\mathrm{E}}\sigma}{(\tau+\sigma)(\tau_v^{S_1} + \phi^{BR}\tau_v^{S_2} + \sigma) + \frac{\sigma(\tau^E)^2}{\tau_v^{S_1}+\phi^{BR}\tau_v^{S_2}+\sigma}} \cdot \frac{\tau_v^{S_1}}{\tau}(\theta + IS_1). \tag{14}$$

Furthermore,

$$a = \frac{(\tau+\sigma)(\tau_v^{S_1} + \phi^{BR}\tau_v^{S_2} + \sigma) + \tau^{\mathrm{E}}\sigma}{(\tau+\sigma)(\tau_v^{S_1} + \phi^{BR}\tau_v^{S_2} + \sigma) + \frac{\sigma(\tau^E)^2}{\tau_v^{S_1}+\phi^{BR}\tau_v^{S_2}+\sigma}} \cdot \frac{\tau_v^{S_1}}{\tau}.$$

Utilizing the static model in Yang and Cai (2014), we form the term. $a = \frac{(\tau+\sigma)(\tau_v^{S_1} + \phi^{BR}\tau_v^{S_2} + \sigma) + \tau^{\mathrm{E}}\sigma}{(\tau+\sigma)(\tau_v^{S_1} + \phi^{BR}\tau_v^{S_2} + \sigma) + \frac{\sigma(\tau^E)^2}{\tau_v^{S_1}+\phi^{BR}\tau_v^{S_2}+\sigma}} \cdot \frac{\tau_v^{S_1}}{\tau}$ into date 1 price equation with aggregated private signal for hedging demand of long-lived investors in a dynamic model. Moreover, with a higher weight of private signal on date 1 price, the speculative demands will more significantly than the static model. This indicates that as market sentiment is optimistic, investors will increase their hedging demand, then aggregated demand increases, and prices rise dramatically, vice versa. Next we turn to the second-order expectations dynamic with bounded rationality.

3. A second-order expectations dynamic

3.1. Second-order expectations dynamic equilibrium

Kondor (2012) and other industry professionals state that an important aspect of forming higher-order expectations is that early investors must presuppose the information held by later investors. Higher-order expectations are significantly different when compared to first-order beliefs. In this model, for simplicity and based on intuition, we utilize second-order expectations to represent higher-order expectations.

Next, we set the second-order expectations dynamic to contrast with the previous section in the same economic environment. First, expectation and variance are conditional on second-order expectations investor (henceforth second-order investor) i's information set on date 2 may be denoted as follows

$$\begin{aligned}\mathrm{E}_{2i}^2(\theta) &= \left(\frac{\tau_v^{S_1} + \tau_v^{S_2}}{\tau'}\right)\mathrm{E}_{2i}(\theta) + \frac{\sigma P_1}{\tau' a'} \\ &= \left(\frac{\tau_v^{S_1} + \tau_v^{S_2}}{\tau'}\right)\left(\frac{\tau_v^{S_1}}{\tau'}v_{1i}^{S_1} + \frac{\tau_v^{S_2}}{\tau'}v_{2i}^{S_2} + \frac{\sigma P_1}{\tau' a}\right) + \frac{\sigma P_1}{\tau' a'} \\ &= \left(\frac{\tau_v^{S_1} + \tau_v^{S_2}}{\tau'}\right)\left(\frac{\tau_v^{S_1}}{\tau'}v_{1i}^{S_1} + \frac{\tau_v^{S_2}}{\tau'}v_{2i}^{S_2}\right) + \frac{\tau_v^{S_1} + \tau_v^{S_2} + \tau'}{\tau'}\frac{\sigma P_1}{\tau' a'}\end{aligned} \tag{15}$$

$$\begin{aligned}\mathrm{var}_{2i}^2(\theta) &= \mathrm{var}_{2i}(\overline{\mathrm{E}}^2(\theta)) \\ &= \mathrm{var}_{2i}\left[\frac{\tau_v^{S_1}}{\tau'}(\theta + IS_1) + \frac{\tau_v^{S_2}}{\tau'}(\theta + IS_2) + \frac{\sigma P_1}{\tau' a'}\right] \\ &= \mathrm{var}_{2i}\left(\frac{\tau_v^{S_1} + \tau_v^{S_2}}{\tau'}\cdot\theta\right) + \mathrm{var}_{2i}\left(\frac{\tau_v^{S_1}}{\tau'}IS_1 + \frac{\tau_v^{S_2}}{\tau'}IS_2\right) \\ &= \left(\frac{\tau_v^{S_1} + \tau_v^{S_2}}{\tau'}\right)^2 \mathrm{var}_{2i}(\theta) + \left(\frac{\tau_v^{S_1}}{\tau'}\right)^2 \mathrm{var}_{2i}(IS_1) \\ &\quad + \left(\frac{\tau_v^{S_2}}{\tau'}\right)^2 \mathrm{var}_{2i}(IS_2) = \frac{(\tau_v^{S_1} + \tau_v^{S_2})^2}{(\tau')^3} + \frac{(\tau_v^{S_1})^2 + (\tau_v^{S_2})^2}{(\tau')^2\sigma}\end{aligned} \tag{16}$$

then second-order investor i's demand on date 2 may be calculated below

$$x_{2i}^2 = \frac{(\mathrm{E}_{2i}^2(\theta) - P_2)}{\gamma\mathrm{var}_{2i}^2(\theta)} = \frac{\left[\begin{array}{l}\left(\frac{\tau_v^{S_1}+\tau_v^{S_2}}{\tau'}\right)\left(\frac{\tau_v^{S_1}}{\tau'}v_{1i}^{S_1} + \frac{\tau_v^{S_2}}{\tau'}v_{2i}^{S_2}\right) \\ + \frac{\tau_v^{S_1}+\tau_v^{S_2}+\tau'}{\tau'}\frac{\sigma P_1}{\tau' a'} - P_2\end{array}\right]}{\gamma\left[\frac{(\tau_v^{S_1}+\tau_v^{S_2})^2}{(\tau')^3} + \frac{(\tau_v^{S_1})^2+(\tau_v^{S_2})^2}{(\tau')^2\sigma}\right]} = \frac{\tau'\left[\begin{array}{l}(\tau_v^{S_1} + \tau_v^{S_2})(\tau_v^{S_1}v_{1i}^{S_1} + \tau_v^{S_2}v_{2i}^{S_2}) \\ + \sigma(\tau_v^{S_1} + \tau_v^{S_2} + \tau')\frac{P_1}{a'} - (\tau')^2 P_2\end{array}\right]}{\gamma(\tau_v^{S_1} + \tau_v^{S_2})^2 + \frac{\gamma\left[(\tau_v^{S_1})^2+(\tau_v^{S_2})^2\right]\tau'}{\sigma}}. \tag{17}$$

Imposing market clearing we calculate

$$\sum x_{2i}^2 = \frac{\tau'\left\{\begin{array}{l}(\tau_v^{S_1} + \tau_v^{S_2})[\tau_v^{S_1}(\theta + IS_1) + \tau_v^{S_2}(\theta + IS_2)] \\ + \sigma(\tau_v^{S_1} + \tau_v^{S_2} + \tau')\frac{P_1}{a'} - (\tau')^2 P_2\end{array}\right\}}{\gamma(\tau_v^{S_1} + \tau_v^{S_2})^2 + \frac{\gamma\left[(\tau_v^{S_1})^2+(\tau_v^{S_2})^2\right]\tau'}{\sigma}} = 0. \tag{18}$$

Reforming the above equation, then we calculate the date 2 equilibrium price with second-order as below

$$\begin{aligned}P_2^{**} &= \left(\frac{\tau_v^{S_1} + \tau_v^{S_2}}{\tau'}\right)\left[\frac{\tau_v^{S_1}}{\tau'}(\theta + IS_1) + \frac{\tau_v^{S_2}}{\tau'}(\theta + IS_2)\right] + \frac{\tau_v^{S_1} + \tau_v^{S_2} + \tau'}{\tau'}\frac{\sigma P_1}{\tau' a'} \\ &= \frac{(\tau_v^{S_1} + \tau_v^{S_2})\tau_v^{S_1} + (\tau_v^{S_1} + \tau_v^{S_2} + \tau')\sigma}{(\tau')^2}\frac{P_1}{a'} + \frac{(\tau_v^{S_1} + \tau_v^{S_2})\tau_v^{S_2}}{(\tau')^2}(\theta + IS_2) \\ &= \frac{(\tau_v^{S_1} + \tau_v^{S_2})\tau_v^{S_1} + (\tau_v^{S_1} + \tau_v^{S_2} + \tau')\sigma}{(\tau')^2}(\theta + IS_1) + \frac{(\tau_v^{S_1} + \tau_v^{S_2})\tau_v^{S_2}}{(\tau')^2}(\theta + IS_2)\end{aligned} \tag{19}$$

then the bounded rationality alternative price of second-order P_2 is denoted by

$$\begin{aligned}P_2^{E'} &= \left(\frac{\tau_v^{S_1} + \phi^{BR}\tau_v^{S_2}}{\tau^E}\right)\left[\frac{\tau_v^{S_1}}{\tau^E}(\theta + IS_1) + \frac{\phi^{BR}\tau_v^{S_2}}{\tau^E}(\theta + IS_1)\right] \\ &\quad + \frac{\tau_v^{S_1} + \phi^{BR}\tau_v^{S_2} + \tau^E}{\tau^E}\frac{\sigma}{\tau^E}(\theta + IS_1) \\ &= \left[\frac{(\tau_v^{S_1} + \phi^{BR}\tau_v^{S_2})^2 + (\tau_v^{S_1} + \phi^{BR}\tau_v^{S_2} + \tau^E)\sigma}{(\tau^E)^2}\right](\theta + IS_1).\end{aligned} \tag{20}$$

The bounded rationality alternative model of second-order expectation and variance in the same date are indicated as follows

$$\begin{aligned}\mathrm{E}_{1i}(P_2^{E'}) &= E_{1i}\left\{\left[\frac{(\tau_v^{S_1} + \phi^{BR}\tau_v^{S_2})^2 + (\tau_v^{S_1} + \phi^{BR}\tau_v^{S_2} + \tau^E)\sigma}{(\tau^E)^2}\right](\theta + IS_1)\right\} \\ &= \left[\frac{(\tau_v^{S_1} + \phi^{BR}\tau_v^{S_2})^2 + (\tau_v^{S_1} + \phi^{BR}\tau_v^{S_2} + \tau^E)\sigma}{(\tau^E)^2}\right]\frac{\tau_v^{S_1}}{\tau}v_{1i}^{S_1}\end{aligned} \tag{21}$$

$$\mathrm{var}_{1i}(P_2^{E'}) = \mathrm{var}_{1i}\left\{\left[\frac{(\tau_v^{S_1}+\phi^{BR}\tau_v^{S_2})^2+(\tau_v^{S_1}+\phi^{BR}\tau_v^{S_2}+\tau^E)\sigma}{(\tau^E)^2}\right](\theta+IS_1)\right\} = \left[\frac{(\tau_v^{S_1}+\phi^{BR}\tau_v^{S_2})^2+(\tau_v^{S_1}+\phi^{BR}\tau_v^{S_2}+\tau^E)\sigma}{(\tau^E)^2}\right]^2 \mathrm{var}_{1i}(\theta+IS_1) = \left[\frac{(\tau_v^{S_1}+\phi^{BR}\tau_v^{S_2})^2+(\tau_v^{S_1}+\phi^{BR}\tau_v^{S_2}+\tau^E)\sigma}{(\tau^E)^2}\right]^2\left(\frac{1}{\tau}+\frac{1}{\sigma}\right). \tag{22}$$

Now we can derive the demand function of second-order investor as shown below

$$x_{1i}^2 = \frac{(\mathrm{E}_{1i}^2(\theta)-P_1)}{\gamma \mathrm{var}_{1i}^2(\theta)} + \frac{(E_{1i}(P_2^{E'})-P_1)}{\gamma \mathrm{var}_v(P_2^{E'})} = \frac{\left(\frac{\tau_v^{S_1}}{\tau}\right)^2 v_{1i}^{S_1} - P_1}{\frac{\gamma(\tau_v^{S_1})^2}{\tau^3}} + \frac{\left[\frac{(\tau_v^{S_1}+\phi^{BR}\tau_v^{S_2})^2+(\tau_v^{S_1}+\phi^{BR}\tau_v^{S_2}+\tau^E)\sigma}{(\tau^E)^2}\right]\frac{\tau_v^{S_1}}{\tau}v_{1i}^{S_1} - P_1}{\gamma\left[\frac{(\tau_v^{S_1}+\phi^{BR}\tau_v^{S_2})^2+(\tau_v^{S_1}+\phi^{BR}\tau_v^{S_2}+\tau^E)\sigma}{(\tau^E)^2}\right]^2\left(\frac{1}{\tau}+\frac{1}{\sigma}\right)} = \frac{1}{\gamma}\left\{\begin{array}{l} \tau v_{1i}^{S_1} - \frac{\tau^3}{(\tau_v^{S_1})^2}P_1 \\ + \frac{(\tau^E)^2}{(\tau_v^{S_1}+\phi^{BR}\tau_v^{S_2})^2+(\tau_v^{S_1}+\phi^{BR}\tau_v^{S_2}+\tau^E)\sigma}\cdot\frac{\sigma}{\tau+\sigma}\tau_v^{S_1}v_{1i}^{S_1} \\ - \frac{\tau\sigma}{\tau+\sigma}\left[\frac{(\tau^E)^2}{(\tau_v^{S_1}+\phi^{BR}\tau_v^{S_2})^2+(\tau_v^{S_1}+\phi^{BR}\tau_v^{S_2}+\tau^E)\sigma}\right]^2 P_1 \end{array}\right\}. \tag{23}$$

Similarly, summing the demand and clearing the market, we show that

$$\sum x_{1i}^2 = \frac{1}{\gamma}\left[\tau(\theta+IS_1) - \frac{\tau^3}{(\tau_v^{S_1})^2}P_1\right] + \frac{1}{\gamma}\left\{\begin{array}{l} \frac{(\tau^E)^2}{(\tau_v^{S_1}+\phi^{BR}\tau_v^{S_2})^2+(\tau_v^{S_1}+\phi^{BR}\tau_v^{S_2}+\tau^E)\sigma}\cdot\frac{\sigma}{\tau+\sigma}\tau_v^{S_1}v_{1i}^{S_1} \\ - \frac{\tau\sigma}{\tau+\sigma}\left[\frac{(\tau^E)^2}{(\tau_v^{S_1}+\phi^{BR}\tau_v^{S_2})^2+(\tau_v^{S_1}+\phi^{BR}\tau_v^{S_2}+\tau^E)\sigma}\right]^2 P_1 \end{array}\right\} = 0 \tag{24}$$

then we formulate the bounded rationality second-order price equation as below

$$P_1^{**} = \frac{\tau + \frac{\sigma(\tau^E)^2\tau_v^{S_1}}{(\tau+\sigma)\left[\left(\tau_v^{S_1}+\phi^{BR}\tau_v^{S_2}\right)^2+\left(\tau_v^{S_1}+\phi^{BR}\tau_v^{S_2}+\tau^E\right)\sigma\right]}}{\frac{\tau^3}{\left(\tau_v^{S_1}\right)^2} + \frac{\tau(\tau^E)^4\sigma}{(\tau+\sigma)\left[\left(\tau_v^{S_1}+\phi^{BR}\tau_v^{S_2}\right)^2+\left(\tau_v^{S_1}+\phi^{BR}\tau_v^{S_2}+\tau^E\right)\sigma\right]^2}}\cdot(\theta+IS_1) \tag{25}$$

Furthermore,

$$a' = \frac{\tau + \frac{\sigma(\tau^E)^2\tau_v^{S_1}}{(\tau+\sigma)\left[(\tau_v^{S_1}+\phi^{BR}\tau_v^{S_2})^2+(\tau_v^{S_1}+\phi^{BR}\tau_v^{S_2}+\tau^E)\sigma\right]}}{\frac{\tau^3}{(\tau_v^{S_1})^2} + \frac{\tau(\tau^E)^4\sigma}{(\tau+\sigma)\left[(\tau_v^{S_1}+\phi^{BR}\tau_v^{S_2})^2+(\tau_v^{S_1}+\phi^{BR}\tau_v^{S_2}+\tau^E)\sigma\right]^2}}.$$

3.2. Properties of the equilibrium

To quantitatively illustrate the impact of sentiment on the equilibrium prices, we provide numerical examples with varying investor sentiment in Figs. 3–5. Considering the simple function $IS_t = \kappa_t S_t$, $\kappa_t > 0$, $t = 1, 2$ (in Yang and Liang (2016), we use the "sparsity-based" sentiment function; here, we use a constant sentiment S_t as the realized value of the normal distribution in data 1 or 2 to make the simulation more convenient), and $\tau_v^{S_1} = \tau_v^{S_2} = \exp^2(IS)$. Other parameters are designated as follows: the signal precisions (or variance) parameters are $\alpha = 1$, $\beta = 2$, and $\sigma = 1$, and the others are $\theta = 1$, $\kappa_1 = 0.6$, $\kappa_2 = 0.2$, $S \in [-6, 6]$. The bounded rationality operator ϕ^{BR} equals 0.7. In choosing the values of these parameters, we relied on our previous study, see Yang and Liang (2016) and Yang and Cai (2014).

Figs. 3 and 4 indicate that the equilibrium price on date 2 is a steadily increasing function of investor sentiment on both dates 1 and 2. The date 2 equilibrium price increases when the investor sentiments on date 1 and date 2 are jointly influenced by a bullish sentiment, while the second-order investors' equilibrium price on date 2 is lower than the first-order investors' equilibrium price. Therefore, the second-order investors' equilibrium price is less affected by the sentiment.

In this case, the equilibrium price P_1 is directly related to the date 1 sentiment S_1, which indicates that investor sentiment has a greater impact than the personal signal and is a more appropriate "beauty contest predictor" than the private signal. Thus, the equilibrium price curved surface of the first-order investor is more symmetrical to reflect that the investor sentiments on both dates 1 and 2 have similar effects on equilibrium price, whereas the equilibrium price curved surface of the second-order investor is less symmetrical (slightly leftward) and reflects the fact that the influences on equilibrium price of the date 2 sentiment S_2 are weaker than those of the date 1 sentiment S_1.

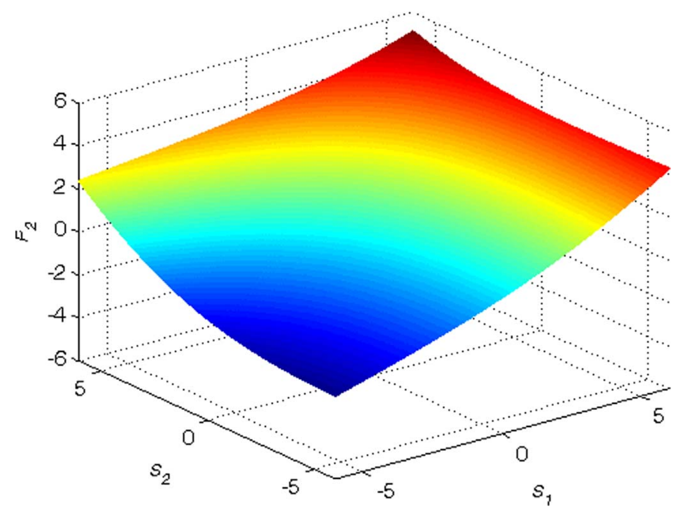

Fig. 3. The equilibrium price on date 2 with a first-order investor.

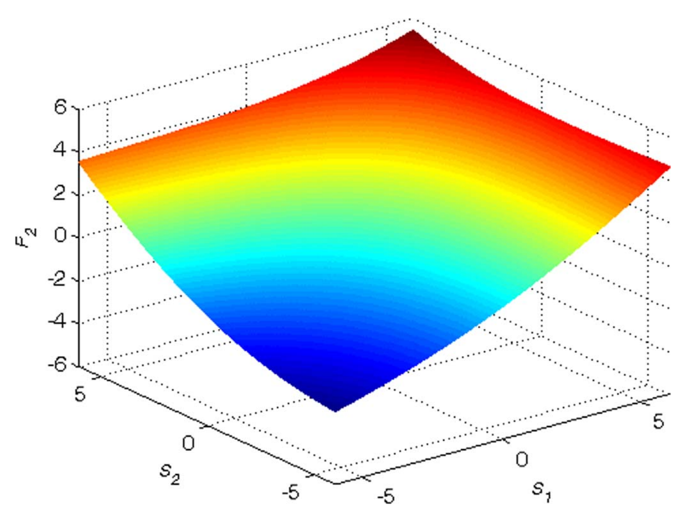

Fig. 4. The equilibrium price on date 2 with a second-order investor.

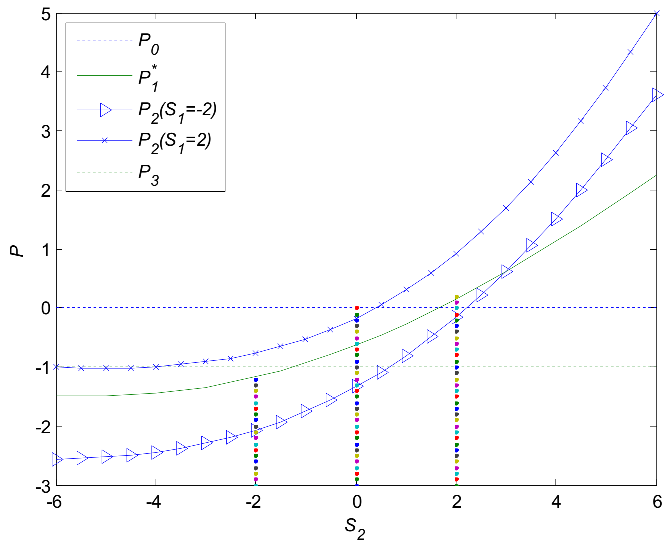

Fig. 5. Equilibrium prices with different sentiment levels.

We provide an equation in Fig. 5 to quantitatively illustrate the impact of investor sentiment on price at different points in time. Taking into account Eq. (14) and Eq. (25), where the date 1 investor sentiment is selected as $S_1 = -2$ or $S_1 = 2$, we found that the date 2 equilibrium price with bullish date 1 sentiment will increase more than with a bearish date 1 sentiment, however, an increase in date 2 sentiment increases the equilibrium price to the full extent. In addition, an investor who is first effected by bearish sentiment (the date 1 investor sentiment is fixed in $S_1 = -2$), will lead to soaring P at a faster rate. When an investor is first impacted by bullish sentiment (the data 1 investor sentiment is fixed in $S_1 = 2$), price changes appear slow. This behavior results in optimistic investor sentiment and leads to dramatic price increases. Nevertheless, if the investor sentiment becomes more pessimistic, asset price bubbles would be difficult to burst.

4. A heterogeneous-order expectations dynamic

In the previous two sections, we calculate for investors who form their expectations with the same order. A heterogeneous-order expectations dynamic model which focuses on inconsistent investor behavior in financial markets, will further extends the single-order expectations dynamic models. Now we shall study a model that builds on previous sections by incorporating information asymmetry and a more general process for investor expectations heterogeneity. These features are vital for generating an interactive case of heterogeneous-order expectations investors (henceforth heterogeneous-order investors) and help determine who gains and who loses from trading on different expectations order, an important issue that cannot be explained using ordinary single-order investor models.

Two types of investors exist in this economy, first-order and second-order. First-order and second-order investors differ in that they form expectations by adopting different orders. The percentage of first-order investors is λ ($\lambda \in (0, 1)$), which is denoted as investor i^1, while the percentage of second-order investors is $1 - \lambda$, which is denoted as investor i^2. (Here we use homogenous investor sentiment and homogenous bounded rationality parameters, leaving heterogeneous expectations order to highlight the primary deduction of this article.).

Summing Eq. (5) and Eq. (17) across first-order and second-order investors, respectively, the aggregated first-order investors' demands on date 2 is denoted by

$$X_2^1 = \sum x_{2i}^1 = \frac{\lambda\tau'}{\gamma}\left(\frac{\tau_v^{S_1}}{\tau'}(\theta + IS_1) + \frac{\tau_v^{S_2}}{\tau'}(\theta + IS_2) + \frac{\sigma P_1}{\tau' a''} - P_2\right). \tag{26}$$

Similarly, the aggregated second-order investors' demands on date 2 are denoted by

$$X_2^2 = \sum x_{2i}^2 = \frac{(1-\lambda)\tau'\left\{\begin{array}{l}(\tau_v^{S_1} + \tau_v^{S_2})[\tau_v^{S_1}(\theta + IS_1) + \tau_v^{S_2}(\theta + IS_2)] \\ + \sigma(\tau_v^{S_1} + \tau_v^{S_2} + \tau')\frac{P_1}{a''} - (\tau')^2 P_2\end{array}\right\}}{\gamma(\tau_v^{S_1} + \tau_v^{S_2})^2 + \frac{\gamma\left[(\tau_v^{S_1})^2 + (\tau_v^{S_2})^2\right]\tau'}{\sigma}}. \tag{27}$$

The market clearing condition is denoted by

$$X_2^1 + X_2^2 = 0$$

This implies

$$\begin{aligned}&\left\{\lambda\tau_v^{S_1} + \frac{(1-\lambda)\tau'(\tau_v^{S_1} + \tau_v^{S_2})\tau_v^{S_1}}{(\tau_v^{S_1} + \tau_v^{S_2})^2 + \frac{\left[\left(\tau_v^{S_1}\right)^2 + \left(\tau_v^{S_2}\right)^2\right]\tau'}{\sigma}}\right\}(\theta + IS_1) \\ &+ \left\{\lambda\tau_v^{S_2} + \frac{(1-\lambda)\tau'(\tau_v^{S_1} + \tau_v^{S_2})\tau_v^{S_2}}{(\tau_v^{S_1} + \tau_v^{S_2})^2 + \frac{\left[\left(\tau_v^{S_1}\right)^2 + \left(\tau_v^{S_2}\right)^2\right]\tau'}{\sigma}}\right\}(\theta + IS_2) \\ &+ \left\{\lambda + \frac{(1-\lambda)\tau'(\tau_v^{S_1} + \tau_v^{S_2} + \tau')}{(\tau_v^{S_1} + \tau_v^{S_2})^2 + \frac{\left[\left(\tau_v^{S_1}\right)^2 + \left(\tau_v^{S_2}\right)^2\right]\tau'}{\sigma}}\right\}\frac{\sigma P_1}{a''} \\ &- \left\{\lambda\tau' + \frac{(1-\lambda)(\tau')^3}{(\tau_v^{S_1} + \tau_v^{S_2})^2 + \frac{\left[\left(\tau_v^{S_1}\right)^2 + \left(\tau_v^{S_2}\right)^2\right]\tau'}{\sigma}}\right\}P_2 \\ &= 0. \end{aligned} \tag{28}$$

Therefore, the equilibrium price on date 2 with heterogeneous-order is denoted as below

$$\begin{aligned}P_2^{***} &= \frac{\lambda\eta(\tau_v^{S_1} + \sigma) + (1-\lambda)\tau'[(\tau_v^{S_1} + \tau_v^{S_2})(\tau_v^{S_1} + \sigma) + \tau'\sigma]}{[\lambda\tau'\eta + (1-\lambda)(\tau')^3]}\frac{P_1}{a} \\ &\quad + \frac{\tau_v^{S_2}[\lambda\eta + (1-\lambda)\tau'(\tau_v^{S_1} + \tau_v^{S_2})]}{[\lambda\tau'\eta + (1-\lambda)(\tau')^3]}(\theta + IS_2) \\ &= \frac{\lambda\eta(\tau_v^{S_1} + \sigma) + (1-\lambda)\tau'[(\tau_v^{S_1} + \tau_v^{S_2})(\tau_v^{S_1} + \sigma) + \tau'\sigma]}{[\lambda\tau'\eta + (1-\lambda)(\tau')^3]}(\theta + IS_1) \\ &\quad + \frac{\tau_v^{S_2}[\lambda\eta + (1-\lambda)\tau'(\tau_v^{S_1} + \tau_v^{S_2})]}{[\lambda\tau'\eta + (1-\lambda)(\tau')^3]}(\theta + IS_2)\end{aligned} \tag{29}$$

where

$$\eta = (\tau_v^{S_1} + \tau_v^{S_2})^2 + \frac{[(\tau_v^{S_1})^2 + (\tau_v^{S_2})^2]\tau'}{\sigma}.$$

Therefore, the bounded rationality alternative model under heterogeneous-order condition is explained as follows

$$P_2^{E''} = \rho^E(\theta + IS_1) \tag{30}$$

where

$$\rho^E = \frac{\lambda\eta^E(\tau_v^{S_1} + \phi^{BR}\tau_v^{S_2} + \sigma) + (1-\lambda)\tau^E[(\tau_v^{S_1} + \phi^{BR}\tau_v^{S_2})(1 + \sigma + \tau_v^{S_1}) + \tau^E\sigma]}{\lambda\tau^E\eta^E + (1-\lambda)(\tau^E)^3}$$

and

$$\eta^E = (\tau_v^{S_1} + \phi^{BR}\tau_v^{S_2})^2 + \frac{(\tau_v^{S_1})^2 + (\phi^{BR}\tau_v^{S_2})^2}{\sigma}\tau^E.$$

Accordingly, the heterogeneous-order expectations and variance on date 2 is explained as follows

$$E_{1i}^{S_1}(P_2^{E''}) = E_{1i}^{S_1}(\rho^E(\theta + IS_1)) = \rho^E \frac{\tau_v^{S_1}}{\tau} v_{1i}^{S_1} \tag{31}$$

$$var_{1i}^{S_1}(P_2^{E''}) = (\rho^E)^2\left(\frac{1}{\tau} + \frac{1}{\sigma}\right). \tag{32}$$

Hence, the demand of first-order investor on date 1 is indicated by

$$x_{1i}^1 = \frac{(E_{1i}^1(\theta) - P_1)}{\gamma var_{1i}^1(\theta)} + \frac{(E_{1i}(P_2^{E''}) - P_1)}{\gamma var_{1i}(P_2^{E''})} = \frac{\tau}{\gamma}\left(\frac{\tau_v^{S_1}}{\tau} v_{1i}^{S_1} - P_1\right) + \frac{\rho^E \frac{\tau_v^{S_1}}{\tau} v_{1i}^{S_1} - P_1}{\gamma(\rho^E)^2(\frac{1}{\tau} + \frac{1}{\sigma})}. \tag{33}$$

Thus, the demand of second-order investor on date 1 is indicated by

$$x_{1i}^2 = \frac{\left(E_{1i}^2(\theta) - P_1\right)}{\gamma var_{1i}^2(\theta)} + \frac{(E_{1i}(P_2^{E''}) - P_1)}{\gamma var_{1i}(P_2^{E''})} = \frac{1}{\gamma}\left[\tau v_{1i}^{S_1} - \frac{\tau^3}{(\tau_v^{S_1})^2} P_1\right] + \frac{\rho^E \frac{\tau_v^{S_1}}{\tau} v_{1i}^{S_1} - P_1}{\gamma(\rho^E)^2\left(\frac{1}{\tau} + \frac{1}{\sigma}\right)}. \tag{34}$$

Moreover, we aggregate the investors' demands and clear the market under the same condition as the previous model in this section

$X_1^1 + X_1^2 = 0.$

Then, we conclude that

$$\lambda[\tau_v^{S_1}(\theta + IS_1) - \tau P_1] + (1 - \lambda)\left[\tau(\theta + IS_1) - \frac{\tau^3}{(\tau_v^{S_1})^2} P_1\right] + \frac{\rho^E \frac{\tau_v^{S_1}}{\tau}(\theta + IS_1) - P_1}{(\rho^E)^2\left(\frac{1}{\tau} + \frac{1}{\sigma}\right)} = 0. \tag{35}$$

The equilibrium price on date 1 may be derived as below

$$P_1^{***} = \frac{\lambda\tau_v^{S_1} + (1 - \lambda)\tau + \frac{\tau_v^{S_1}\sigma}{\rho^E(\tau + \sigma)}}{\lambda\tau + (1 - \lambda)\frac{\tau^3}{(\tau_v^{S_1})^2} + \frac{\tau\sigma}{(\rho^E)^2(\tau + \sigma)}}(\theta + IS_1) \tag{36}$$

Furthermore,

$$a'' = \frac{\lambda\tau_v^{S_1} + (1 - \lambda)\tau + \frac{\tau_v^{S_1}\sigma}{\rho^E(\tau + \sigma)}}{\lambda\tau + (1 - \lambda)\frac{\tau^3}{(\tau_v^{S_1})^2} + \frac{\tau\sigma}{(\rho^E)^2(\tau + \sigma)}}$$

It has been noted that Eq. (36) is equivalent to Eq. (20) when $\lambda = 0$ and is also equivalent to Eq.(11) when $\lambda = 1$. Therefore, the fraction parameter λ in equilibrium pricing has fully illustrated the interaction of first-order and second-order investors.

Parameters are indicated as follows: $S_1 = 0$, $S_2 = 0$ and $\sigma \in (0, 3)$, and the remaining parameters are identical to those of Figs. 3 and 4. Fig. 6 illustrates the relationship between the precisions of investor sentiment and volatility of equilibrium price with a different percentage of first-order investors, such as 0.1, 0.5 and 0.9. We find that the volatility of the date 2 equilibrium price decreases by the investor sentiment precision. Furthermore, when other conditions are fixed, the volatility of the date 2 equilibrium price will increase in proportion to first-order investors and decrease proportional to second-order investors.

5. Comparative statics

We utilize the differential of the date 2 equilibrium price P_2 to the date 1 equilibrium price P_1 to represent the equilibrium price sensitivity of the investor sentiment or bounded rationality.

As indicated in Fig. 7a and b, increasing investor sentiment or bounded rationality operator decreases the proportion of P_1. In addition, if investor sentiment is high enough, then the weight of date 1 equilibrium price P_1 in P_2 will gradually decrease to zero. Nevertheless, if the bounded rationality operator is high enough (rising to 1, here we show $\phi \in (-1, 1)$ to better indicate the results), then the weight of date 1 equilibrium price P_1 in P_2 will decrease to a constant value. In particularly, in this numerical example, the proportion of first-order investors gradually decreases to 0.7. However, the proportion of second-order investors decreases to almost 1. This indicates that, for the existence of bounded rationality, the mispricing of the date 1 price will increase and, in turn, reduce the effects of P_1 in P_2.

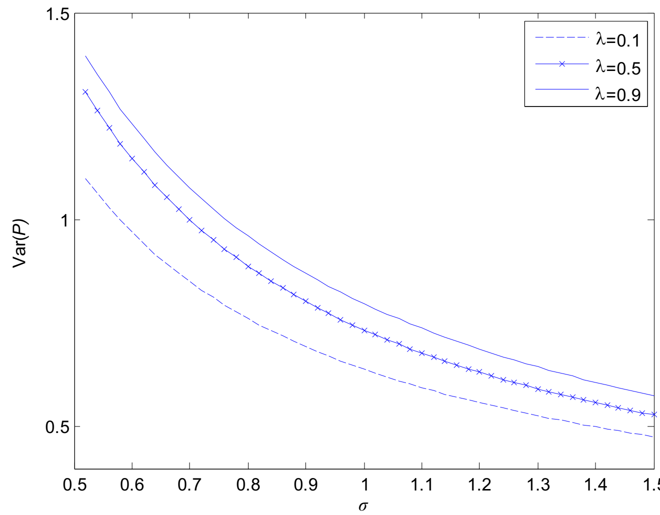

Fig. 6. Sentiment precisions-equilibrium price volatility relationship with heterogeneous investor fraction.

Fig. 8a and b illustrates the relationship between signal ratio and investor sentiment (or bounded rationality). The signal ratio defined here refers to the ratio of price signal to aggregated private signal, which decreases along with increasing investor sentiment (or bounded rationality). Clearly, the signal ratio of the date 1 equilibrium price by second-order expectations is significantly higher than by first-order expectations. This implies that investor strategies are more influenced by the date 1 equilibrium price with higher-order expectations. It follows that higher-order investors are more attentive to the strategies of other participants in the market, rather than their own private signals. This leads to a higher proportion of common signal P_1 in the date 2 price P_2. In particular, when investor sentiment becomes high enough, this ratio decrease to zero; when the bounded rationality operator becomes high enough, this ratio will be reduced to a constant value. Therefore, the second-order investors remain on a relatively high level of rationality, in other words, first-order investors are more typical of bounded rationality.

Fig. 9 indicates that the volatility of the equilibrium price changes by the precision of investor sentiment. When the variance of investor sentiment is large (or rather, the precisions of investor sentiment are small), the volatility of equilibrium price increases. This illustrates the positive relationship between the variance in investor sentiment and the volatility of the equilibrium price, which is consistent with the empirical results of Yu and Yuan (2011).

Compared with the equilibrium price equations on date 2 (Eqs. (14) and (25)), the price equations on date 1 (Eqs. (7) and (19)) showed greater equilibrium price volatility. This indicates that a robust dampening effect of higher-order expectations to price volatility.

6. Conclusions

Many financial market anomalies are not well explained by traditional financial theory. The financial market is not always informa-

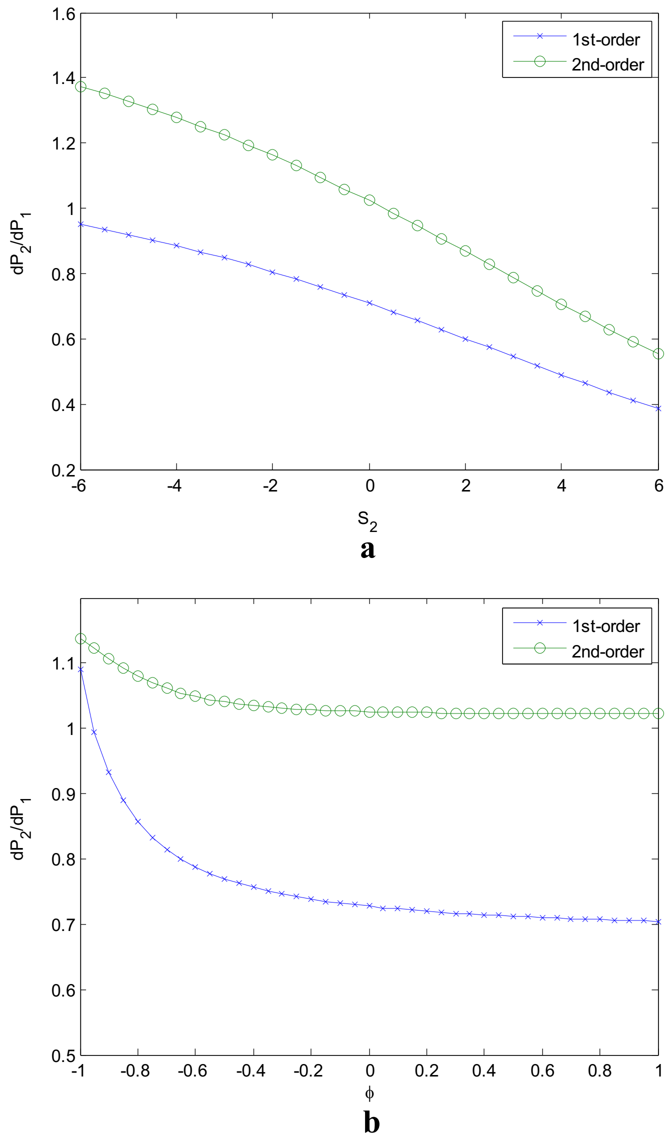

Fig. 7. **a.** The proportion of P_1 in P_2 (investor sentiment). **b.** The proportion of P_1 in P_2 (bounded rationality).

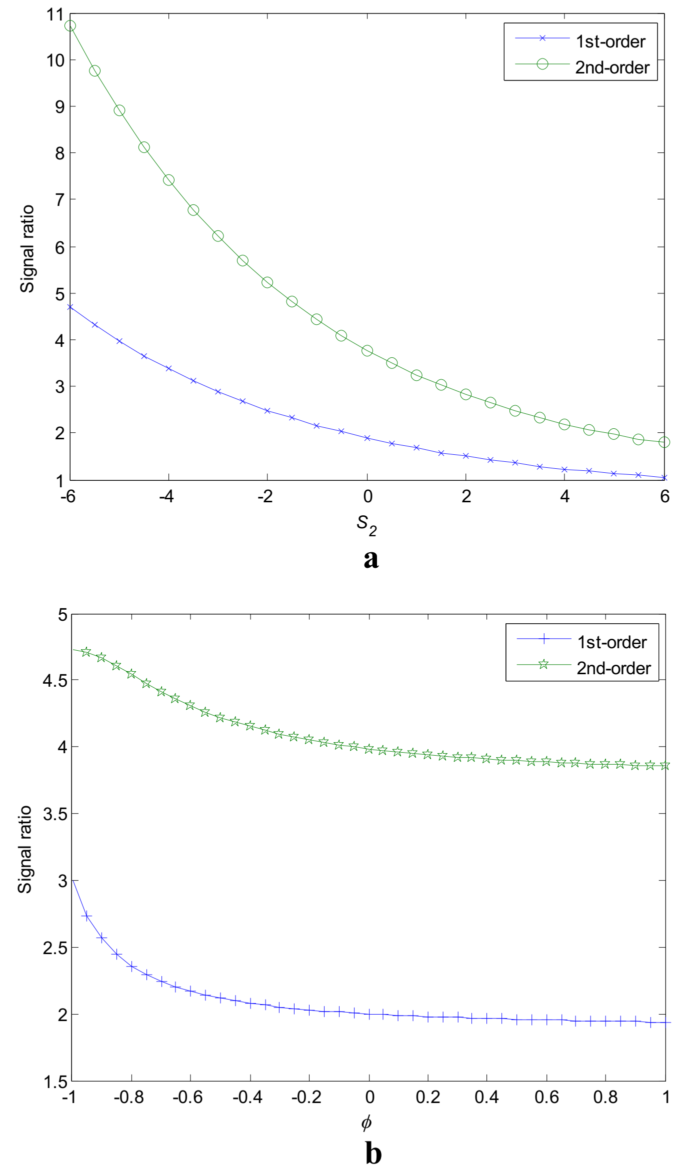

Fig. 8. **a.** Signal ratio of investor sentiment. **b.** Signal ratio of bounded rationality.

tionally efficient, and rational arbitrage usually could not completely eliminate the irrational effects on asset prices. Numerous scholars have suggested that stock price is a combined result of higher-order expectations, investor sentiment, and some forms of bounded rationality.

We examine this issue in a three-period market where investors are risk-averse, privately informed, heterogeneous expectations order, trade on private signal, make decisions by investor sentiment and sparsity-based bounded rationality jointly. Furthermore, as the extension to the dynamic setting which compared to the one-shot trading (static models such as Yang and Cai (2014)), investors often make multiple transactions in the real financial market, and determine current trading strategies while taking into account future trading opportunities. Moreover, the equilibrium prices result from multi-period games among different types of investors; hence, we adopted the heterogeneous-order expectations dynamic model for investors. Then, in the first section of this paper, we propose a "sparsity-based" bounded rationality operator to depict bounded rationality by solving a two-term optimization with the first term: loss of imperfect decision, and the second term: cost of rationality (for details, see Yang and Liang (2016)). Summaries and conclusions for the characteristics of our model are described below.

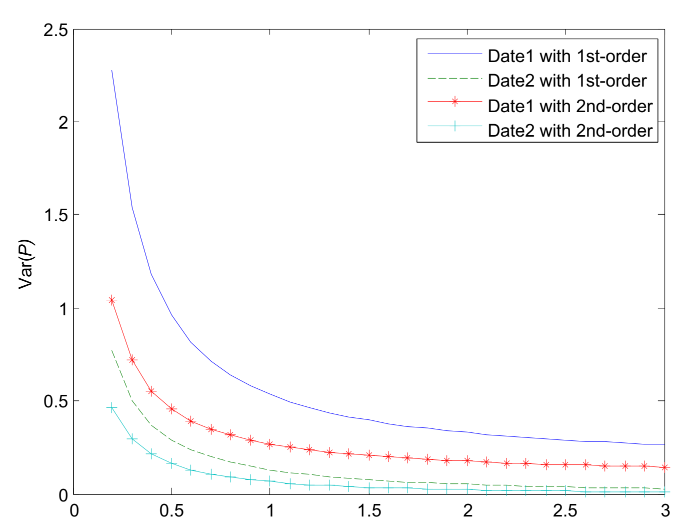

Fig. 9. Sentiment precisions-equilibrium price volatility relationship with different date.

First, by providing a benchmark for first-order expectations equilibrium, our analysis reveals that, optimistic investor sentiment will result in equilibrium price increases, which leads to investors increasing their hedging demand, and then aggregated demand increases, the prices radically increase, and the converse (pessimistic sentiment conditions) is also true. A bounded rationality operator and relevant algorithm have been described in this section.

Second, a second-order expectations dynamic reveals that the date 2 equilibrium price increases when the investor is influenced by the bullish sentiment on two dates jointly. Furthermore, the second-order investors' equilibrium price is less affected by the sentiment than first-order investors. Moreover, the equilibrium price curved surface is more symmetrical of the first-order investor and is less symmetrical than a second-order investor; this reflects that the influences to equilibrium price of date 2 sentiment is weaker than sentiment on date 1. Therefore, investor sentiment has more significant effects than personal signal and is also a more effective "beauty contest predictor". In addition, equilibrium price with bullish date 1 sentiment will increase more than with a bearish date 1 sentiment, and an increase in date 2 sentiment would increase the equilibrium price all of the way. This indicates that when investor sentiment becomes more optimistic, then asset price bubbles will rise faster. Nevertheless, if the investor sentiment becomes more pessimistic, asset price bubbles would be difficult to burst.

Third, a heterogeneous-order expectations dynamic indicates that the volatility of date 2 equilibrium price decreases by the sentiment precision, and increases by proportion to the first-order investors when the other conditions are fixed.

In conclusion, we emphasize four findings as follows. First, increasing the investor sentiment or bounded rationality operator decreases the weight of the date 1 price. In the case of bounded rationality, mispricing of the date 1 price will increase and, in turn, reduce its effects on date 2. Second, the second-order investors stay on a relatively high level of rationality, where first-order investors stay on a relatively low level. To clarify, first-order investors are more typical of bounded rationality. Third, investor strategies are more influenced by the date 1 equilibrium price with higher-order expectations. This indicates that higher-order investors are more attentive to the strategies of other participants in the market rather than their own private signals. Lastly, the positive correlation between the variance of investor sentiment and the volatility of equilibrium price indicates that the dampening effect of higher-order expectations on price volatility is robust.

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